FUNCTION SPACES ATTACHED TO ELLIPTIC OPERATORS

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1. Introduction

The aim of this paper is to extend classical results in Complex Analysis to the general setting of elliptic operators. This idea originates to the mathematicians of the XIXth century, especially to Cauchy and Riemann, who developed a parallel between analytic functions of one variable and harmonic functions in planar domains. Nowadays, books like those of L. Hörmander [6], R. Narasimhan [7] or R. O. Wells [14], go further and illustrate how the basic ideas of classical Analysis evolved into Analysis of Differential Operators.

Given a linear elliptic operator P, we can attach to it several function spaces. The first one is $\mathcal{A}(P) = Ker\, P$, the space of the so called *P-analytic functions.* That allows us to bring together several important classes of functions such as the usual analytic functions (the case where $P = \overline{\partial}$) and the harmonic functions (the case where $P = \Delta$). As we shall show in Section 3, the convergence of the sequences of P-analytic functions has some special features, noticed in particular cases by a number of mathematicians such as K. Weierstrass, G. Vitali, H. Harnack et al. A sample of the results in this area is the classical theorem due to K. Weierstrass, which asserts that uniform convergence on compacta preserves analyticity.

The *Bergman space* (of index 2 and weight w) associated to a linear elliptic operator P and a continuous weight w is the space of all square integrable P-analytic functions i.e.,

$$\mathcal{B}g^2(P, w\,dx) = \left\{ u \in \mathcal{A}(P) : \int_\Omega |u(x)|^2\, w(x)\, dx < \infty \right\}.$$

In Section 4 we shall show that $\mathcal{B}g^2(P, w\,dx)$ is a closed subspace in the corresponding L^2 space associated to $w\,dx$ and that will be used to infer the denseness of $\mathcal{A}(P_1) \otimes \mathcal{A}(P_2)$ into $\mathcal{A}(P_1 \boxtimes P_2)$; here $P_1 \boxtimes P_2$ represents the Fubini product (in the sense of [10]) of the two linear elliptic operators P_1 and P_2.

The paper ends by discussing some aspects of Bergman space theory based on Bergman kernel and Bergman projection.

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2. Preliminaries on elliptic operators

Throughout this section Ω will denote a bounded open subset of $\mathbb{R}^N$ and $C^\infty(\Omega, r)$ will denote the Fréchet space $C^\infty(\Omega, \mathbb{C}^r)$, endowed with the family of seminorms

$$||u||_n^K = \sum_{|\alpha| \leq n} \frac{1}{\alpha!} \sup_{x \in K} |D^\alpha u(x)| \,,$$

where n runs over $\mathbb{N}$ and K runs over the compact subsets of Ω.

We shall consider linear elliptic operators $P : C^\infty(\Omega, r) \to C^\infty(\Omega, s)$ of order m, i.e. operators of the form

$$(Pu)(x) = \sum_{|\alpha| \leq m} a_\alpha(x)(D^\alpha u)(x)$$

whose leading symbols

$$\sigma_P(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x)\xi^\alpha : \mathbb{C}^r \to \mathbb{C}^s$$

are injective, whenever $x \in \Omega$ and $\xi \in \mathbb{R}^r \setminus \{0\}$; all coefficients are supposed to be C^∞.

The simplest examples of linear elliptic operators are

$$\frac{d^p}{dx^p}, \quad \overline{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \quad \Delta = \sum_{k=1}^{N} \frac{\partial^2}{\partial x_k^2}\,, \quad \Delta^p$$

and their perturbations by lower order terms.

Much of the theory of elliptic operators depends upon the powerful methods of Functional Analysis and in this connection an important role is played by Sobolev spaces.

For $m \in \mathbb{N}$, the Sobolev space $H^m(\Omega, r)$ is the space of all functions $u \in L^2(\Omega, r) = L^2(\Omega, \mathbb{C}^r)$, whose distributional derivatives of order $\leq m$ are in $L^2(\Omega, r)$. This is a Hilbert space for the norm

$$||u||_{H^m(\Omega,r)} = \left(\sum_{|\alpha| \leq m} \int_\Omega |D^\alpha u(x)|^2 \, dx\right)^{1/2}.$$

Notice that $H^m(\Omega, r)$ can be described alternatively as the completion of

$$\left\{u \in C^\infty(\Omega, r) : \ ||u||_{H^m(\Omega,r)} < \infty\right\}$$

with respect to $||\cdot||_{H^m(\Omega,r)}$.

According to Sobolev Embedding Theorem [12], if $m > \frac{N}{2} + j$, then for every compact subset K of Ω there exists a constant $C_1 > 0$ such that

$$||u||_j^K \leq C_1 \, ||u||_{H^k(\Omega,r)}$$

whenever $u \in C^\infty(\Omega, r)$. This theorem shows that every $u \in H^m(\Omega, r)$ is a. e. equal to a function of class $C^{m-[N/2]-1}$.

A basic result on linear elliptic operators is the Friedrichs' Inequality [5]: *If P is as above, then for every relatively compact open subset Ω' of Ω there exists a constant $C_2 > 0$ such that*

$$||u||_{H^{m+k}(\Omega',r)} \leq C_2 \left(||Pu||_{H^k(\Omega,r)} + ||u||_{H^0(\Omega,r)}\right)$$

for every $k \in \mathbb{N}$ and every $u \in C^\infty(\Omega, r)$ for which the right hand side is finite.

Elliptic operators of the type considered above have nice regularity properties. Particularly they are *hypoelliptic,* i.e. if $u \in L^2_{loc}(\Omega, r)$ and $Pu = 0$ (in the sense of distributions), then u is a.e. equal to a C^∞-function.

3. C^∞- CONVERGENCE OF SEQUENCES OF P- ANALYTIC FUNCTIONS

As above, $P : C^\infty(\Omega, r) \to C^\infty(\Omega, s)$ will denote a linear elliptic operator of order m. Attached to it will be the vector space

$$\mathcal{A}(P) = Ker\, P$$

of the so called P-*analytic* functions. The usual analytic functions correspond to the case where $P = \overline{\partial}$. For $P = \frac{d^p}{dx^p}$, $\mathcal{A}(P)$ consists of all polynomials of degree $\leq p$, while for $P = \Delta$ we retrieve the case of harmonic functions.

The convergence of sequences of P-analytic functions has some special features, noticed in particular cases by a number of mathematicians such as K. Weierstrass, G. Vitali, H. Harnack et al. A sample of the results in this area is Weierstrass' theorem, which asserts that uniform convergence on compacta preserves analyticity.

In order to develop a unifying approach in the framework of P-analyticity, we have to make the following basic remark, which combines Friedrichs' inequality and Sobolev embedding theorem:

Let $(u_n)_n$ be a sequence of elements of $C^\infty(\Omega, r)$ such that:

i) $(Pu_n)_n$ is a converging sequence in $C^\infty(\Omega, s)$;

ii) $\lim\limits_{j,\, k \to \infty} \int_K |u_k - u_j|^2 \, dx = 0$

for every compact subset K of Ω.

Then $(u_n)_n$ is a converging sequence in $C^\infty(\Omega, r)$.

Lemma 3.1 yields a number of criteria of C^∞-convergence, which provide themselves very useful in concrete applications:

(Vitali's Criterion of C^∞-convergence). Suppose that $(u_n)_n$ is a sequence of P-analytic functions such that:

i) $(u_n)_n$ is pointwise convergent to a function $u : \Omega \to \mathbb{C}^r$;

ii) $(u_n)_n$ is uniformly bounded on each compact subset of Ω.

Then u is P-analytic and $u_n \to u$ in $C^\infty(\Omega, r)$.

Proof. Use the theorem of Lebesgue on dominated convergence. ■

(Weierstrass' Criterion of C^∞-convergence). If $(u_n)_n$ is a sequence of P-analytic functions and $u_n \to u$ uniformly on each compact subset of Ω, then u is P-analytic and $u_n \to u$ in $C^\infty(\Omega, r)$.

The discussion above shows that $\mathcal{A}(P)$ constitutes a Fréchet space (and also a closed subspace of $C^\infty(\Omega, r)$) when endowed with the family of seminorms

$$||u||_K = \sup_{x \in K} |u(x)|$$

where K runs over the compact subsets of Ω.

(Stieltjes-Vitali Criterion of Compactness). Every sequence of $P-$ analytic functions which is bounded on compacta contains a converging subsequence.

In other words, $\mathcal{A}(P)$ is a Fréchet-Montel space.

Proof. First notice that Ω can be represented as the union of an increasing sequence of compact subsets e.g., $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$ where

$$\Omega_n = \{x \in \Omega : |x| \leq n \text{ and } dist\,(x, \partial\Omega) \geq 1/n\}$$

for each $n \in \mathbb{N}^\star$.

By the Sobolev embedding theorem, we get uniform estimates for the derivatives of the u_n's on each subset Ω_n. In particular, the functions u_n are equicontinuous. By the Arzela-Ascoli theorem we can choose a uniformly converging subsequence on each Ω_n and using a diagonal argument we obtain a subsequence converging uniformly an each compact subset of Ω.

To end the proof it remains to apply to that subsequence the result of Corollary 3.3 above. ■

As a consequence of Theorem 3.4 we obtain that condition i) in Vitali's criterion of C^∞-convergence can be weakned as:

i') $(u_n(x))_n$ is convergent for x in a dense subset of Ω.

How far is pointwise convergence from C^∞-convergence in the case of $P-$analytic functions? The answer is given by the following theorem, which extends a result due to W. F. Osgood [9]:

Let $(u_n)_n$ be a sequence of P-analytic functions which is pointwise converging to a function $u : \Omega \to \mathbb{C}^r$. Then u is P-analytic in a dense open subset $\Omega_1 \subset \Omega$ and convergence is uniform on compact subsets of Ω_1.

Proof. Let K be an arbitrary closed ball included in Ω. Then $K = \bigcup_{n=1}^{\infty} K_n$, where the K_n's are closed subsets defined as

$$K_n = \{x \in K : |u_k(x)| \leq n \text{ for every } k\}.$$

By the Baire category theorem some K_m must have non-empty interior. For this m the sequence $(u_n)_n$ is uniformly bounded on $Int\,K_m$, hence by Corollary 3.2 above it converges uniformly on compact subsets of $Int\,K_m$. Thus u is P-analytic on $Int\,K_m$. Since the argument can be applied to any closed ball, it follows that u is P-analytic on a dense open subset $\Omega_1 \subset \Omega$. The fact that the convergence is uniform on compacta contained in Ω_1 is standard and we omit the details. ■

A natural question arising in connection with Theorem 3.5 is how *thin* can the subset Ω_1 be? One can prove easily that for each open subset Ω of $\mathbb{R}^N$ and each $\varepsilon > 0$ there must exist a dense open subset $\Omega_\varepsilon \subset \Omega$ whose Lebesgue measure is $< \varepsilon$. The problem is how to fix the convergence aspects as in Theorem 3.5.

The following example could be useful to settle that problem. Let $\lambda \in (0, 1/2)$. From the closed unit square $K_0 = [0,1] \times [0,1]$ delete $[0,1] \times (\lambda, 1-\lambda) \cup (\lambda, 1-\lambda) \times [0,1]$, thus leaving a set K_1 of 4 closed squares. Continue in a similar manner so that at the nth stage we are left with a set K_n of 4^n closed squares, whose centers we denote $z_{n,k}$ $(k = 1, ..., 4^n)$. Then $K_\infty = \bigcap_n K_n$ is a totally disconnected set, of planar Lebesgue measure zero. Letting

$$f(z) = \lim_{n\to\infty} \frac{1}{4^n} \sum_{k=1}^{4^n} \frac{1}{z - z_{n,k}}$$

we obtain a function continuous on K_0, which has no analytic continuation off $K_0 \setminus K_\infty$.

Notice that the Hausdorff dimension of the exceptional set K_∞ is $-\log 4/\log\lambda$, a quantity which goes to 2 as $\lambda \to 1/2$.

The example above shows that the implication

$$u = \text{continuous} \ \& \ P(u) = 0 \text{ a.e. } \ \Rightarrow P(u) = 0 \text{ everywhere}$$

fails even for $P = \overline{\partial}$. However, for $P = \overline{\partial}$ one can prove the following result on removable singularities:

(A. S. Besicovitch [2]) . If Ω is an open subset of $\mathbb{C}$ and $u : \Omega \to \mathbb{C}$ is a continuous function such that $\overline{\partial}u = 0$ except on a thin subset, then $\overline{\partial}u = 0$ everywhere i.e., u is analytic.

Recall that a subset of $\mathbb{R}^N$ is called *thin* if it has $\sigma-$finite $(N-1)-$dimensional Hausdorff measure.

Open Problem. *Does Theorem 3.6 above extend to all elliptic operators*?

4. Approximation of P-analytic functions of several variables

According to Weierstrass approximation theorem, if $\Omega_j \subset \mathbb{C}^{N_j}$ are open subsets and x_j are points in $\mathbb{C}^{N_j}$ $(j \in \{1,2\})$, then the finite linear combinations

$$\sum\nolimits_n u_n^{(1)}(x_1) \otimes u_n^{(2)}(x_2)$$

with $u_n^{(j)} \in C^\infty(\Omega_j, r_j)$ $(j \in \{1,2\})$ are dense in $C^\infty(\Omega_1 \times \Omega_2, r_1 r_2)$; notice that $\mathbb{C}^{r_1 r_2} = \mathbb{C}^{r_1} \otimes \mathbb{C}^{r_2}$. In other words, $C^\infty(\Omega_1 \times \Omega_2, r_1 r_2)$ is the completion of

$$C^\infty(\Omega_1, r_1) \otimes C^\infty(\Omega_2, r_2).$$

We shall show in the sequel that similar results are valid in the context of $P-$analytic functions. Our approach, inspired by the case of functions of several variables as treated in [7], makes use of some functional spaces.

Let $\Omega \subset \mathbb{R}^N$ be an open subset and let $w : \Omega \to (0,\infty)$ be a continuous function (reffered to as a *weight*). $L^2(w\,dx, r)$ denotes the Hilbert space of all functions $u : \Omega \to \mathbb{C}^r$, which are square integrable with respect to the Lebesgue weighted measure $w\,dx$; $L^2(w\,dx, r)$ is endowed with the norm

$$||u||_{L^2(w\,dx,\,r)} = \left(\int_\Omega |u(x)|^2\, w(x)\, dx\right)^{1/2}.$$

The *Bergman space* (of index 2 and weight w) associated to a linear elliptic operator $P : C^\infty(\Omega, r) \to C^\infty(\Omega, s)$ and a continuous weight w is the space of all square integrable $P-$analytic functions i.e.,

$$\mathcal{B}g^2(P, w\,dx) = \left\{u \in \mathcal{A}(P);\ \int_\Omega |u(x)|^2\, w(x)\, dx < \infty\right\}.$$

$\mathcal{B}g^2(P)$ will stand for $\mathcal{B}g^2(P, w\,dx)$, when $w = 1$.

$\mathcal{B}g^2(P, w\,dx)$ is closed in $L^2(wdx, r)$ and thus it constitutes a Hilbert space when endowed with the induced norm.

Proof. In fact, w is bounded from below on each compact space and thus the results in the section 2 show that for every compact subset $K \subset \Omega$ there exists a constant $C_K > 0$ such that

$$(*) \qquad \sup_{x\in K} |u(x)| \le C_K\, ||u||_{L^2(wdx,r)}$$

for every $u \in \mathcal{A}(P)$. ■

A similar argument yields the following result:

Let $(\varphi_n)_n$ be an orthonormal basis of $\mathcal{B}g^2(P, w\,dx)$. Then every $u \in \mathcal{B}g^2(P, w\,dx)$ can be approximated uniformly on compact subsets of Ω by finite linear combinations $\sum_n c_n \varphi_n$ with complex coefficients.

Suppose now there are given two elliptic operators

$$P_j : C^\infty(\Omega_j, r_j) \to C^\infty(\Omega_j, r_j) \quad (j \in \{1,2\})$$

of the same order m and consider their *Fubini product,*

$$P_1 \boxtimes P_2$$

which acts from

$$(C^\infty(\Omega_1, r_1) \otimes C^\infty(\Omega_2, r_2)) \oplus (C^\infty(\Omega_1, r_1) \otimes C^\infty(\Omega_2, r_2))$$

into

$$(C^\infty(\Omega_1, r_1) \otimes C^\infty(\Omega_2, r_2)) \oplus (C^\infty(\Omega_1, r_1) \otimes C^\infty(\Omega_2, r_2))$$

by the formula

$$\begin{aligned}(P_1 \boxtimes P_2)\,(u_1 \otimes u_2 \oplus v_1 \otimes v_2) &= \\ &= (P_1 u_1 \otimes u_2 - v_1 \otimes P_2^\star v_2) \oplus (u_1 \otimes P_2 u_2 + P_1^\star v_1 \otimes v_2)\,.\end{aligned}$$

(R. S. Palais [10], ch. IV, §8) . $P_1 \boxtimes P_2$ is an elliptic operator, of order m.
Proof. In fact, $P_1 \boxtimes P_2$ can be represented as

$$\begin{pmatrix} P_1 \otimes id_{r_2} & -id_{r_1} \otimes P_2^\star \\ id_{r_1} \otimes P_2 & P_1^\star \otimes id_{r_2} \end{pmatrix}$$

where id_{r_k} is the identity of $C^\infty(\Omega_k, r_k)$. Then

$$(P_1 \boxtimes P_2)^\star = (-1)^m \begin{pmatrix} P_1^\star \otimes id_{r_2} & id_{r_1} \otimes P_2^\star \\ -id_{r_1} \otimes P_2 & P_1 \otimes id_{r_2} \end{pmatrix}$$

which yields that $(P_1 \boxtimes P_2)^\star\,(P_1 \boxtimes P_2)$ has the form

$$(-1)^m \begin{pmatrix} P_1^\star P_1 \otimes id_{r_2} + id_{r_1} \otimes P_2^\star P_2 & 0 \\ 0 & id_{r_1} \otimes P_2 P_2^\star + P_1 P_1^\star \otimes id_{r_2} \end{pmatrix}$$

and thus it is a uniformly elliptic operator of order $2m$. Consequently $P_1 \boxtimes P_2$ is an elliptic operator of order m. ■

$\mathcal{A}(P_1) \otimes \mathcal{A}(P_2)$ is dense in $\mathcal{A}(P_1 \boxtimes P_2)$.
Proof. In fact, every function in $\mathcal{A}(P_1 \boxtimes P_2)$ belongs to a certain space $\mathcal{B}g^2(P_1 \boxtimes P_2, wdx)$. Using the splitting techniques described in [7], we may assume that

$$wdx = w_1 dx_1 + w_2 dx_2.$$

Or,

$$\mathcal{B}g^2(P_1 \boxtimes P_2, w_1 dx_1 + w_2 dx_2) = \mathcal{B}g^2(P_1, w_1 dx_1) \widehat{\otimes} \mathcal{B}g^2(P_2, w_2 dx_2)$$

where $\hat{}$ represents the completion under the projective tensor product topology. If we take orthonormal bases $(\varphi_n^{(k)})_n$ in $\mathcal{B}g^2(P_k, w_k dx_k)$ $(k \in \{1,2\})$ then

$$(\varphi_k^{(1)} \otimes \varphi_j^{(2)})_{k,j}$$

will constitute an orthonormal basis of $\mathcal{B}g^2(P_1, w_1 dx_1) \widehat{\otimes} \mathcal{B}g^2(P_2, w_2 dx_2)$. The conclusion follows now from Lemma 4.2. ■

5. The Bergman kernel

A useful tool for studying the Bergman space and the underlying geometry of the domain is a reproducing kernel, whose existence is motivated by the inequalities $(*)$.

In fact, the point evaluation at any $x \in \Omega$ is a finite rank operator on the Bergman space so that the Riesz representation theorem yields a unique function K_x in $\mathcal{B}g^2(P)$ such that

$$u(x) = \int_\Omega K_x(y)^\star\, u(y)\, dy$$

for every $u \in \mathcal{B}g^2(P)$. Let $K : \Omega \times \Omega \to L(\mathbb{C}^r, \mathbb{C}^r)$ be the function defined by

$$K(x, y) = K_x(y).$$

We call K the *Bergman kernel* (or the *reproducing kernel)* of $\mathcal{B}g^2(P)$ because the equality

$$u(x) = \int_\Omega K(x, y)u(y)\, dy$$

"reproduces" every $u \in \mathcal{B}g^2(P)$.

Let $(\varphi_n)_n$ be an orthonormal basis of $\mathcal{B}g^2(P)$. Then the Bergman kernel verifies the equality

$$K(x, y) = \sum\nolimits_n \varphi_n(x) \otimes \overline{\varphi_n(y)}.$$

Proof. For each compact subset H of Ω we have

$$\begin{aligned} & \sup\left\{\left(\sum\nolimits_n |\varphi_n(x)|^2\right)^{1/2} : x \in H\right\} = \\ = \quad & \sup\left\{\left|\sum\nolimits_n c_n\varphi_n(x)\right| : x \in H,\ \sum\nolimits_n |c_n|^2 \leq 1\right\} \\ = \quad & \sup\left\{|u(x)| : x \in H,\ ||u||_{L^2\Omega, r)} \leq 1\right\} \leq C_K \end{aligned}$$

and thus the series $\sum_n \varphi_n(x) \otimes \overline{\varphi_n(y)}$ converges uniformly for x and y in compact subsets. If $u \in \mathcal{B}g^2(P)$, then

$$u = \sum\nolimits_n < u, \varphi_n > \varphi_n$$

the series being uniformly convergent on the compact subsets of Ω. Particularly,

$$u(x) = \sum\nolimits_n < u, \varphi_n > \varphi_n(x) =< u, \sum\nolimits_n \overline{\varphi_n(x)} \otimes \varphi_n >$$

for every $x \in \Omega$. Since $\sum_n \overline{\varphi_n(x)} \otimes \varphi_n \in \mathcal{B}g^2(P)$, the uniqueness of the Riesz representation shows that

$$K_x(y) = \sum\nolimits_n \overline{\varphi_n(x)} \otimes \varphi_n(y)$$

i.e., $K(x, y) = \sum_n \overline{\varphi_n(x)} \otimes \varphi_n(y)$. ■

The original Bergman space corresponds to the case where

$$\Omega = D = \{z \in \mathbb{C};\ |z| < 1\} \text{ and } P = \overline{\partial}.$$

Because the functions $\sqrt{\frac{n+1}{\pi}} \cdot z^n$ $(n \in \mathbb{N})$ constitutes an orthonormal basis of the corresponding Bergman space $\mathcal{B}g^2(P)$, the reproducing kernel is

$$K(z,w) = \sum_{n=0}^{\infty} \frac{n+1}{\pi} z^n \overline{w}^n = \frac{1}{(1-z\overline{w})^2}.$$

As follows from the proof of Proposition 5.1, the Bergman kernel is always continuous and self-adjoint.

Because $\mathcal{B}g^2(P, wdx)$ is a closed subspace of the Hilbert space $L^2(wdx, \mathbb{C}^r)$, it must be the image of an orthogonal projection P from $L^2(wdx, r)$ onto $\mathcal{B}g^2(P, wdx)$; by evident reasons, we call π_{Bg} the *Bergman projection.* It is easy to verify that π_{Bg} is given by the formula

$$(\pi_{Bg} u)(x) = \int_\Omega K(x,y)u(y) w(y)\, dy.$$

The Bergman projection is connected to the Dirichlet Problem:

Suppose that Ω is a bounded open subset of $\mathbb{R}^N$, with smooth boun-dary, so that P admits a right inverse $G : C^\infty(\overline{\Omega}, r) \to C^\infty(\overline{\Omega}, r)$ with

$$\begin{aligned} P(Gu) &= u \\ Gu|\partial\Omega &= 0. \end{aligned}$$

Then

$$\pi_{Bg} = I - P^\star G P.$$

Proof. If $\varphi \in C^\infty(\overline{\Omega}, r)$ then $u = P^\star G \varphi$ is the unique solution of the problem $Pu = \varphi$ with u orthogonal to $\mathcal{B}g^2(P)$. ■

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