A NOTE ON THE HERMITE-HADAMARD INEQUALITY

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It is well known that every convex function $f : [a,b] \to \mathbb{R}$ can be modified at the endpoints to become convex and continuous. An immediate consequence of this remark is the integrability of f. The integral of f can then be estimated by the *Hermite-Hadamard Inequality,*

$$(HH) \qquad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}\ .$$

(HH) can be easily derived via the midpoint and trapezoidal approximation to the middle term. Moreover, under the presence of continuity, equality occurs (in either side) only for linear functions. Interesting applications of (HH) are to be found in [4], pp. 137-151. See also [3].

The argument used by F. Burk [1] to prove the Geometric, Logarithmic and Arithmetic Mean Inequality can be embedded in an abstract scheme, leading us to a *dual Hermite-Hadamard Inequality.* Comparing to a straightforward application of (HH), a number of examples will illustrate that the latter often yields better results.

Theorem (The Dual Hermite-Hadamard Inequality)**.** *Suppose that I and J are two intervals and $F : I \to J$ is an invertible mapping such that F and F^{-1} both are differentiable. If moreover $\left(F^{-1}\right)'$ is nonlinear and convex, then*

$$(DHH) \qquad \frac{1}{F'(F^{-1}\left(\frac{F(a)+F(b)}{2}\right))} < \frac{b-a}{F(b)-F(a)} < \frac{1}{2}\left(\frac{1}{F'(a)} + \frac{1}{F'(b)}\right)$$

for every $a < b$ in I.

When $\left(F^{-1}\right)'$ is nonlinear and concave, the inequalities above should be reversed.

Proof. One applies the inequality (HH) to the derivative of F^{-1}, taking into account that

$$\frac{1}{F(b)-F(a)}\int_{F(a)}^{F(b)} \left(F^{-1}(y)\right)'\,dy = \frac{b-a}{F(b)-F(a)}$$

and $\left(F^{-1}(y)\right)' = 1/F'(F^{-1}(y))$. ■

Examples. i) For $F(x) = \ln x$, $x \in (0,\infty)$, (DHH) yields the Geometric, Logarithmic and Arithmetic Mean Inequality,

$$\sqrt{xy} < \frac{x-y}{\ln x - \ln y} < \frac{x+y}{2}.$$

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ii) The function $F(x) = \tan x$, establishes a diffeomorphism between $(0, \pi/2)$ and $(0, \infty)$, whose inverse is $\left(F^{-1}\right)(y) = \arctan y$. Because

$$\left(F^{-1}\right)'''(y) = \frac{d^2}{dy^2}\left(\frac{1}{1+y^2}\right) = \frac{6(y - 1/\sqrt{3})(y + 1/\sqrt{3})}{(1+y^2)^3},$$

it follows that $\left(F^{-1}\right)'$ is concave in $(0, 1/\sqrt{3}]$ and convex in $[1/\sqrt{3}, \infty)$. According to the Dual Hermite-Hadamard Inequality,

$$1 + \left(\frac{\tan x + \tan y}{2}\right)^2 < \frac{\tan x - \tan y}{x - y} < \frac{2}{\cos^2 x + \cos^2 y} \tag{1}$$

whenever $x < y$ in $(0,\ \pi/6]$ and

$$1 + \left(\frac{\tan x + \tan y}{2}\right)^2 > \frac{\tan x - \tan y}{x - y} > \frac{2}{\cos^2 x + \cos^2 y}, \tag{2}$$

whenever $x < y$ in $[\pi/6,\ \pi/2)$.

On the other hand, a direct application of the Hermite-Hadamard Inequality to $f(x) = 1/\cos^2 x$ gives us

$$\frac{1}{\cos^2\left(\frac{x+y}{2}\right)} < \frac{\tan x - \tan y}{x - y} < \frac{1}{2}\left(\frac{1}{\cos^2 x} + \frac{1}{\cos^2 y}\right), \tag{3}$$

whenever $x < y$ in $(0, \pi/2)$. Notice that each of the inequalities (1) and the first part of (2) are stronger than (3) in the corresponding domain; the second part of (2) is stronger than the first part of (3) only for $x < y$ in $[\pi/6,\ \pi/4]$.

iii) The function $F(x) = \ln \tan \frac{x}{2}$ establishes a diffeomorphism between $(0, \pi)$ and $\mathbb{R}$, whose inverse is $F^{-1}(y) = 2 \arctan e^y$. Because

$$\frac{d^3}{dy^3}(\arctan e^y) = \frac{e^y - 6e^{3y} + e^{5y}}{(1+e^{2y})^3} = \frac{e^y\left(e^{2y} - 2e^y - 1\right)\left(e^{2y} + 2e^y - 1\right)}{(1+e^{2y})^3},$$

it follows that $\left(F^{-1}\right)'$ is convex in each of the intervals

$$(-\infty,\ \ln(-1+\sqrt{2})] \quad \text{and} \quad [\ln(1+\sqrt{2}),\ \infty)$$

and concave in $\left[\ln(-1+\sqrt{2}),\ \ln(1+\sqrt{2})\right]$. According to the Dual Hermite-Hadamard Inequality,

$$\frac{2\left(\tan\frac{x}{2}\tan\frac{y}{2}\right)^{1/2}}{1 + \tan\frac{x}{2}\tan\frac{y}{2}} < \frac{x - y}{\ln\tan\frac{x}{2} - \ln\tan\frac{y}{2}} < \frac{\sin x + \sin y}{2}$$

for all $x < y$ in $(0, \pi/4)$ or in $[3\pi/4, \pi)$ and

$$\frac{2\left(\tan\frac{x}{2}\tan\frac{y}{2}\right)^{1/2}}{1 + \tan\frac{x}{2}\tan\frac{y}{2}} > \frac{x - y}{\ln\tan\frac{x}{2} - \ln\tan\frac{y}{2}} > \frac{\sin x + \sin y}{2}$$

for all $x < y$ in $[\pi/4, 3\pi/4]$.

On the other hand, the Hermite-Hadamard Inequality yields

$$\frac{1}{2}\left(\frac{1}{\sin x} + \frac{1}{\sin y}\right) > \frac{\ln\tan\frac{y}{2} - \ln\tan\frac{x}{2}}{y - x} = \frac{1}{y - x}\int_x^y \frac{1}{\sin x}\,dx > 1/\sin\frac{x+y}{2}$$

whenever $x < y$ in $(0, \pi)$. This fact is weaker than the conclusion of the Dual Hermite-Hadamard Inequality.

iv) The author is much indebted to the referee for calling to his attention that the inequality in Problem 82.J in the March 1998 issue of Gazette,

$$\left(\frac{\cos x - \cos y}{x-y}\right)^2 < 1 - \left(\frac{\cos x + \cos y}{2}\right)^2 \quad \text{for } x \neq y$$

can be also obtained as a consequence of the Dual Hermite-Hadamard Inequality.

It is worth noticing that any improvement of (HH) leads to a corresponding improvement of (DHH). For example, as

$$\frac{1}{b-a}\int_a^b f(x)\,dx < \frac{f(a) + 2f\left((a+b)/2\right) + f(b)}{4} < \frac{f(a)+f(b)}{2}$$

(for the nonlinear convex functions), the right hand side of (DHH) can be replaced by

$$\left[\frac{1}{F'\left(F^{-1}\left(\frac{F(a)+F(b)}{2}\right)\right)} + \frac{1}{2}\left(\frac{1}{F'(a)} + \frac{1}{F'(b)}\right)\right]/2.$$

Unfortunately, this new bound often looks clumsy. An exception is example i) above, which can cf. [5] be improved to

$$\sqrt{xy} < \frac{x-y}{\ln x - \ln y} < \frac{1}{2}\left(\frac{x+y}{2} + \sqrt{xy}\right).$$

References

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