AN ANTIPLANE CONTACT PROBLEM FOR VISCOELASTIC MATERIALS WITH LONG-TERM MEMORY [1]

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Abstract. We study a mechanical problem modeling the antiplane shear deformation of a cylinder in frictional contact with a rigid foundation. The material is assumed to be viscoelastic with long-term memory, the process is quasistatic, and the friction is modeled with Tresca's law. The problem leads to an evolutionary variational inequality that we study in an abstract framework. We then use the abstract result to prove the existence of a unique weak solution to the mechanical model. Moreover, we study its behavior with respect to the memory term and establish a convergence result.

Key words: antiplane problem, variational inequality, viscoelasticity with long-term memory, Tresca's friction law, approach to elasticity

1. Introduction

Some people think that Mathematics is an arbitrary creation and the job of mathematicians is to develop abstract theories. In their view, the more abstract and demanding is a theory, the bigger is its value. However, everyday experience shows a somewhat different picture, namely the interest for intellectual activities comes primarily from their practical implication. Nowadays a considerable progress has been achieved in the different areas of Mathematics and most of it was motivated by applications. Indeed, the abstract character

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of mathematical tools allows to solve wide classes of problems which present a common feature and to provide significant applications in the study of problems arising in sciences, engineering and everyday life.

The Theory of Variational Inequalities has not been an exception. Indeed, the cross fertilization between modeling and applications on the one hand and nonlinear mathematical analysis on the other hand was an important aspect which contributed to its development in the last four decades. Currently, the Theory of Variational Inequalities became a fully mature discipline which deals with existence, uniqueness or nonuniqueness, regularity and continuous dependence results, together with numerical approximations and optimal control of the solutions. It provides results which are of considerable theoretical and applied interest.

The aim of this paper is to recall the attention to the great potential of inequalities in Mechanics and Physics. In the spirit of the classical book of G. Duvaut and J. L. Lions [3], we show how a concrete viscoelastic contact problem leads to a mathematical model which can be solved by using the methods of the Theory of Variational Inequalities.

The contact between a deformable body and a foundation is a very frequent and important phenomenon. For this reason the literature covering this phenomenon is extensive, both in Applied Mathematics and in Engineering or Geophysics. The Mathematical Theory of Contact Mechanics is concerned with the mathematical structures which underline general problems of contact with different constitutive laws and different contact conditions, see for instance [4, 12] and the references therein. Its aim is to provide a sound, clear and rigorous background to the specific applied problems. A number of recent publications is dedicated to the study of quasistatic contact models involving viscoelastic materials with long-term memory. The variational analysis of such problems leads to evolutionary variational inequalities with integral term or with integro-differential term, see for instance [9, 10, 11] for details.

In this paper we study the frictional contact between a deformable cylinder and a rigid foundation. We consider the case of antiplane shear deformation i.e., the displacement is parallel to the generators of the cylinder and is independent of the axial coordinate. Such kind of problems were studied in a number of papers, in the context of various constitutive laws and contact conditions, see, e.g. [1, 5, 8] and the references therein. The novelty in this paper consists in the fact that we model the friction with Tresca's law and the material's behavior with a viscoelastic constitutive law with long-term memory. We neglect the inertial term in the equation of motion to obtain a quasistatic approximation of the process. In the variational formulation, this mechanical problem leads to an integro-differential variational inequality. The main result we provide concerns the existence of a unique weak solution to the model. Its proof is carried out in several steps, and is based on arguments of evolutionary variational inequalities and Banach's fixed point theorem.

Our paper is structured as follows. In Section 2 we present the mechanical problem of antiplane contact, derive its variational formulation and state the main existence and uniqueness result, Theorem 1. In Section 3 we prove an abstract existence and uniqueness result, Theorem 2, and we apply it in the

proof of Theorem 1. Finally, in Section 4 we study the behavior of the solution as the memory term converges to zero and provide a convergence result, Theorem 4.

2. The antiplane contact problem and its variational formulation

We consider a body $\mathcal{B}$ identified with a region in $\mathbb{R}^3$ it occupies in a fixed and undistorted reference configuration. We assume that $\mathcal{B}$ is a cylinder with generators parallel to the x_3-axes with a cross-section which is a regular region Ω in the x_1, x_2-plane, $Ox_1x_2x_3$ being a Cartesian coordinate system. The cylinder is assumed to be sufficiently long so that the end effects in the axial direction are negligible. Thus, $\mathcal{B} = \Omega \times (-\infty, +\infty)$. Let $\partial\Omega = \Gamma$. We assume that Γ is divided into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 such that the one-dimensional measure of Γ_1, denoted *meas* Γ_1, is strictly positive. Let $T > 0$ and let $[0, T]$ denote the time interval of interest. The cylinder is clamped on $\Gamma_1 \times (-\infty, +\infty)$ and is in contact with a rigid foundation on $\Gamma_3 \times (-\infty, +\infty)$ during the process. Moreover, the cylinder is subjected to time dependent volume forces of density $\boldsymbol{f}_0$ on $\mathcal{B}$ and to time dependent surface tractions of density $\boldsymbol{f}_2$ on $\Gamma_2 \times (-\infty, +\infty)$. We assume that

$$\boldsymbol{f}_0 = (0, 0, f_0) \quad \text{with} \quad f_0 = f_0(x_1, x_2, t) : \Omega \times [0, T] \to \mathbb{R}, \qquad (2.1)$$

$$\boldsymbol{f}_2 = (0, 0, f_2) \quad \text{with} \quad f_2 = f_2(x_1, x_2, t) : \Gamma_2 \times [0, T] \to \mathbb{R}. \qquad (2.2)$$

The body forces (2.1) and the surface tractions (2.2) would be expected to give rise to a deformation of the cylinder whose displacement, denoted by $\boldsymbol{u}$, is independent on x_3 and has the form

$$\boldsymbol{u} = (0, 0, u) \quad \text{with} \quad u = u(x_1, x_2, t) : \Omega \times [0, T] \to \mathbb{R}. \qquad (2.3)$$

Such kind of deformation is called an *antiplane shear*, see for instance [6, 7] for details.

The infinitesimal strain tensor is denoted by $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u}))$ and the stress field by $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_{ij})$. Here and below, in order to simplify the notation, we do not indicate the dependence of various functions on x_1, x_2 or t.

The material is modeled by the following viscoelastic constitutive law with long-term memory,

$$\boldsymbol{\sigma} = \lambda(\text{tr } \boldsymbol{\varepsilon}(\boldsymbol{u}))\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) + 2 \int_0^t \theta(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))ds, \qquad (2.4)$$

where $\lambda > 0$ and $\mu > 0$ are the Lamé coefficients, $\text{tr } \boldsymbol{\varepsilon}(\boldsymbol{u}) = \sum_{i=1}^{3} \varepsilon_{ii}(\boldsymbol{u})$, $\boldsymbol{I}$ is the unit tensor in $\mathbb{R}^3$ and $\theta : [0, T] \to \mathbb{R}$ is the relaxation function. In the antiplane context (2.3), keeping in mind (2.4), the stress field becomes

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & 0 & \boldsymbol{\sigma}_{13} \\ 0 & 0 & \boldsymbol{\sigma}_{23} \\ \boldsymbol{\sigma}_{31} & \boldsymbol{\sigma}_{32} & 0 \end{pmatrix} \tag{2.5}$$

where

$$\boldsymbol{\sigma}_{13} = \boldsymbol{\sigma}_{31} = \mu\,\partial_{x_1}u + \int_0^t \theta(t-s)\,\partial_{x_1}u(s)ds,$$

$$\boldsymbol{\sigma}_{23} = \boldsymbol{\sigma}_{32} = \mu\,\partial_{x_2}u + \int_0^t \theta(t-s)\,\partial_{x_2}u(s)ds.$$

Neglecting the inertial term in the equation of motion we obtain the quasistatic approximation for the process. Thus, taking into account (2.5), (2.1) and the previous equalities, the equation of equilibrium reduces to the scalar equation

$$\mu\Delta u + \int_0^t \theta(t-s)\Delta\,u(s)ds + f_0 = 0 \quad \text{in} \quad \Omega \times (0, T).$$

As the cylinder is clamped on $\Gamma_1 \times (-\infty, +\infty) \times (0, T)$, the displacement field vanishes there. Thus, (2.3) implies that

$$u = 0 \quad \text{on} \quad \Gamma_1 \times (0, T).$$

Let $\boldsymbol{\nu}$ denote the unit normal on $\Gamma \times (-\infty, +\infty)$. We have

$$\boldsymbol{\nu} = (\nu_1, \nu_2, 0) \quad \text{with} \quad \nu_i = \nu_i(x_1, x_2) : \Gamma \to \mathbb{R}, \ i = 1, 2. \tag{2.6}$$

For a vector $\boldsymbol{v}$ we denote by v_ν and $\boldsymbol{v}_\tau$ its normal and tangential components on the boundary, given by

$$v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}, \quad \boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu\boldsymbol{\nu}. \tag{2.7}$$

In (2.7) and everywhere in this paper " $\cdot$ " represents the inner product on the space $\mathbb{R}^d$ ($d = 2$ or 3). Moreover, throughout this paper the notation $|\cdot|$ will designate the Euclidean norm on $\mathbb{R}^d$, and a dot above a function will represent its derivative with respect to the time variable. For a given stress field $\boldsymbol{\sigma}$ we denote by σ_ν and $\boldsymbol{\sigma}_\tau$ the normal and the tangential components on the boundary, that is

$$\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}, \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}. \tag{2.8}$$

From (2.5) and (2.6) we deduce that the Cauchy stress vector is given by

$$\boldsymbol{\sigma}\boldsymbol{\nu} = (0, 0, \mu\,\partial_\nu u + \int_0^t \theta(t-s)\,\partial_\nu u(s)ds). \tag{2.9}$$

Here and below we use the notation $\partial_\nu u = \partial_{x_1}u\,\nu_1 + \partial_{x_2}u\,\nu_2$. Taking into account the traction boundary condition $\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_2$ on $\Gamma_2 \times (0, T)$, it follows from (2.2) and (2.9) that

$$\mu \, \partial_\nu u + \int_0^t \theta(t-s) \, \partial_\nu u(s)ds = f_2 \quad \text{on} \quad \Gamma_2 \times (0,T).$$

We now describe the contact condition on $\Gamma_3 \times (-\infty, +\infty)$. First, from (2.3) and (2.6) we infer that $u_\nu = 0$, which shows that the contact is bilateral, that is, the contact is kept during all the process. Using now (2.3), (2.6)–(2.8) we conclude that

$$\boldsymbol{u}_\tau = (0,0,u), \quad \boldsymbol{\sigma}_\tau = (0,0,\mu \, \partial_\nu u + \int_0^t \theta(t-s) \, \partial_\nu u(s)ds). \tag{2.10}$$

We assume that the friction is invariant with respect to the x_3 axis and is modeled with Tresca's friction law, that is

$$\begin{cases} |\boldsymbol{\sigma}_\tau| \le g, \\ |\boldsymbol{\sigma}_\tau| < g \Rightarrow \dot{\boldsymbol{u}}_\tau = 0, & \text{on } \Gamma_3 \times (0,T). \\ |\boldsymbol{\sigma}_\tau| = g \Rightarrow \exists \beta \ge 0 \text{ such that } \boldsymbol{\sigma}_\tau = -\beta \dot{\boldsymbol{u}}_\tau \end{cases} \tag{2.11}$$

Here $g : \Gamma_3 \to \mathbb{R}_+$ is a given function, the friction bound, and $\dot{\boldsymbol{u}}_\tau$ represents the tangential velocity on the contact boundary. The strict inequality holds in the stick zone and the equality in the slip zone. Using now (2.10) it is straightforward to see that the conditions (2.11) imply

$$\begin{cases} |\mu \, \partial_\nu u + \int_0^t \theta(t-s) \, \partial_\nu u(s)ds| \le g, \\ |\mu \, \partial_\nu u + \int_0^t \theta(t-s) \, \partial_\nu u(s)ds| < g \Rightarrow \dot{u} = 0, \\ |\mu \, \partial_\nu u + \int_0^t \theta(t-s) \, \partial_\nu u(s)ds| = g & \text{on } \Gamma_3 \times (0,T). \\ \quad \Rightarrow \exists \beta \ge 0 \text{ such that } \mu \, \partial_\nu u + \int_0^t \theta(t-s) \, \partial_\nu u(s)ds = -\beta \dot{u} \end{cases}$$

Finally, we prescribe the initial displacement,

$$u(0) = u_0 \quad \text{in} \quad \Omega,$$

where u_0 is a given function on Ω.

We collect the above equations and conditions to obtain the classical formulation of the antiplane problem for viscoelastic materials with long-term memory, in frictional contact with a foundation.

Problem $\mathcal{P}$. *Find the displacement field $u : \Omega \times [0,T] \to \mathbb{R}$ such that*

$$\mu \Delta u + \int_0^t \theta(t-s) \Delta u(s)ds + f_0 = 0 \quad \text{in} \quad \Omega \times (0,T), \tag{2.12}$$

$$u = 0 \quad \text{on} \quad \Gamma_1 \times (0,T), \tag{2.13}$$

$$\mu \, \partial_\nu u + \int_0^t \theta(t-s) \, \partial_\nu u(s)ds = f_2 \quad \text{on} \quad \Gamma_2 \times (0,T), \tag{2.14}$$

$$\begin{cases} |\mu\, \partial_\nu u + \int_0^t \theta(t-s)\, \partial_\nu u(s)ds| \le g, \\[2mm] |\mu\, \partial_\nu u + \int_0^t \theta(t-s)\, \partial_\nu u(s)ds| < g \Rightarrow \dot{u} = 0, \\[2mm] |\mu\, \partial_\nu u + \int_0^t \theta(t-s)\, \partial_\nu u(s)ds| = g \qquad\quad \text{on}\quad \Gamma_3 \times (0,T), \quad (2.15) \\[2mm] \Rightarrow \exists\, \beta \ge 0 \text{ such that} \\[2mm] \mu\, \partial_\nu u + \int_0^t \theta(t-s)\, \partial_\nu u(s)ds = -\beta \dot{u} \end{cases}$$

$$u(0) = u_0 \quad \text{in} \quad \Omega. \tag{2.16}$$

Notice that once the displacement field u which solves Problem $\mathcal{P}$ is known, then the stress tensor can be calculated using (2.4).

We derive now the variational formulation of the Problem $\mathcal{P}$. To this end we introduce the functional space

$$V = \{v \in H^1(\Omega)\,|\, v = 0 \ \text{ on } \ \Gamma_1\}.$$

Here and below we write v for the trace γv of v on Γ. Since $meas\ \Gamma_1 > 0$, the Friedrichs-Poincaré inequality holds, that is there exists a positive constant C_P (which depends only on Ω and Γ_1) such that

$$\|u\|_{H^1(\Omega)} \le C_P \|\nabla u\|_{L^2(\Omega)^2} \qquad \forall u \in V. \tag{2.17}$$

We consider on V the inner product given by

$$(u,v)_V = \int_\Omega \nabla u \cdot \nabla v\, dx \qquad \forall u, v \in V,$$

and let $\|\cdot\|_V$ be the associated norm, i.e.

$$\|v\|_V = \|\nabla v\|_{L^2(\Omega)^2} \qquad \forall v \in V.$$

Using (2.17), it follows that $\|\cdot\|_{H^1(\Omega)}$ and $\|\cdot\|_V$ are equivalent norms on V and therefore $(V, \|\cdot\|_V)$ is a real Hilbert space. By Sobolev's trace theorem we deduce that there exists $C_0 > 0$ (depending only on Ω, Γ_1 and Γ_3) such that

$$\|v\|_{L^2(\Gamma_3)} \le C_0 \|v\|_V \qquad \forall v \in V. \tag{2.18}$$

If $(X, \|\cdot\|_X)$ represents a real Banach space, we denote by $C([0,T]; X)$ the space of continuous functions from $[0,T]$ to X, with the norm

$$\|x\|_{C([0,T];X)} = \max_{t \in [0,T]} \|x(t)\|_X$$

and we use standard notations for the Lebesgue space $L^2(0,T;X)$ as well as for the Sobolev space $W^{1,2}(0,T;X)$. In particular, recall that the norm on the space $L^2(0,T;X)$ is given by

$$\|u\|_{L^2(0,T;X)} = \Big(\int_0^T \|u(t)\|_X^2 dt \Big)^{\frac{1}{2}}$$

and the norm on the space $W^{1,2}(0,T;X)$ is given by

$$\|u\|_{W^{1,2}(0,T;X)} = \Big(\int_0^T \|u(t)\|_X^2 dt + \int_0^T \|\dot u(t)\|_X^2 dt \Big)^{\frac{1}{2}}.$$

Finally, we suppress the argument X when $X = \mathbb{R}$; thus, for example, we use the notation $W^{1,2}(0,T)$ for the space $W^{1,2}(0,T;\mathbb{R})$ and the notation $\|\cdot\|_{W^{1,2}(0,T)}$ for the norm $\|\cdot\|_{W^{1,2}(0,T;\mathbb{R})}$.

In the study of the mechanical Problem $\mathcal{P}$ we assume that the friction bound function g satisfies

$$g \in L^\infty(\Gamma_3) \quad \text{and} \quad g(x) \geq 0 \quad \text{a.e. } x \in \Gamma_3. \tag{2.19}$$

The forces and tractions are assumed to have the regularity

$$f_0 \in W^{1,2}(0,T;L^2(\Omega)), \qquad f_2 \in W^{1,2}(0,T;L^2(\Gamma_2)) \tag{2.20}$$

and, for the relaxation function, we assume

$$\theta \in W^{1,2}(0,T). \tag{2.21}$$

We consider the functional $j : V \to \mathbb{R}_+$ given by

$$j(v) = \int_{\Gamma_3} g\,|v|da \qquad \forall v \in V \tag{2.22}$$

and let $f : [0,T] \to V$ be defined by

$$(f(t),v)_V = \int_\Omega f_0(t)v\,dx + \int_{\Gamma_2} f_2(t)v\,da \qquad \forall v \in V,\ t \in [0,T]. \tag{2.23}$$

The definition of f is based on Riesz's representation theorem and, by (2.20) and (2.23), we infer that

$$f \in W^{1,2}(0,T;V). \tag{2.24}$$

Finally, we assume that the initial data verifies

$$u_0 \in V \tag{2.25}$$

and, moreover,

$$\mu\,(u_0,v)_V + j(v) \geq (f(0),v)_V \qquad \forall v \in V. \tag{2.26}$$

Using Green's formula it is straightforward to derive the following variational formulation of Problem $\mathcal{P}$.

Problem $\mathcal{P}_V$. *Find a displacement field $u : [0,T] \to V$ such that*

$$\mu\,(u(t),v - \dot u(t))_V + \Big(\int_0^t \theta\,(t-s)u(s)ds, v - \dot u(t)\Big)_V \tag{2.27}$$

$$+j(v) - j(\dot u(t)) \geq (f(t),v - \dot u(t))_V \qquad \forall v \in V,\quad a.e.\ t \in (0,T),$$

$$u(0) = u_0. \tag{2.28}$$

Our main existence and uniqueness result, which we establish in Section 3, is the following.

Theorem 1. *Assume that* (2.19)–(2.21), (2.25) *and* (2.26) *hold. Then there exists a unique solution of problem* $\mathcal{P}_V$. *Moreover, the solution satisfies* $u \in W^{1,2}(0, T; V)$.

An element u which solves Problem $\mathcal{P}_V$ is called a *weak solution* of the mechanical problem $\mathcal{P}$. We conclude by Theorem 1 that the antiplane contact problem $\mathcal{P}$ has a unique weak solution, provided that (2.19)–(2.21), (2.25) and (2.26) hold.

3. An abstract existence and uniqueness result

The following abstract problem includes Problem $\mathcal{P}_V$ as a special case.

Problem $\mathcal{P}_V^A$**.** *Find* $u : [0, T] \to X$ *such that*

$$a(u(t), v - \dot{u}(t)) + (\int_0^t \theta\,(t - s)u(s)ds, v - \dot{u}(t))_X + j(v) - j(\dot{u}(t)) \quad (3.1)$$
$$\geq (f(t), v - \dot{u}(t))_X \qquad \forall v \in X, \quad a.e. \ \ t \in (0, T),$$

$$u(0) = u_0. \tag{3.2}$$

Here X is a real Hilbert space endowed with the inner product $(\cdot, \cdot)_X$ and the data a, θ, j, f, u_0 will be described below. We denote by $\| \cdot \|_X$ the associated norm in X and by 0_X the zero element of X.

We assume that:

$$\begin{cases} a : X \times X \to R \text{ is a bilinear symmetric form and} \\ \text{(a) there exists } m > 0 \text{ such that } |a(u, v)| \leq \|u\|_X \|v\|_X \ \forall u, v \in X, \quad (3.3) \\ \text{(b) there exists } \alpha > 0 \text{ such that } a(u, u) \geq \alpha \|u\|_X^2 \ \forall u \in X. \end{cases}$$

$$\theta \in W^{1,2}(0, T). \tag{3.4}$$

$$j : X \to (-\infty, +\infty] \ \text{ is a convex, lower semicontinuous function.} \tag{3.5}$$

$$f \in W^{1,2}(0, T; X). \tag{3.6}$$

$$u_0 \in X. \tag{3.7}$$

$$a(u_0, v) + j(v) \geq (f(0), v)_X \qquad \forall v \in X. \tag{3.8}$$

We have the following existence and uniqueness result.

Theorem 2. *Assume that* (3.3)–(3.8) *hold. Then, there exists a unique solution* $u \in W^{1,2}(0, T; X)$ *to the problem* $\mathcal{P}_V^A$.

We turn to the proof of Theorem 2 which will be carried out in several steps. To this end, in the rest of this section we assume that (3.3)–(3.8) hold and we recall the following result.

Theorem 3. *([2], p. 117). Let $(X, (\cdot, \cdot)_X)$ be a real Hilbert space and let $j : X \to (-\infty, +\infty]$ be a convex lower semicontinuous functional. Assume that $j \not\equiv +\infty$, that is*

$$D(j) = \{v \in X \mid j(v) < \infty\} \neq \emptyset.$$

Let $f \in W^{1,2}(0, T; X)$ and $u_0 \in X$ be such that

$$\sup_{v \in D(j)} \{(f(0), v)_X - (u_0, v)_X - j(v)\} < +\infty.$$

Then, there exists a unique element $u \in W^{1,2}(0, T; X)$ which satisfies $u(0) = u_0$ and
$$(u(t), v - \dot{u}(t))_X + j(v) - j(\dot{u}(t)) \geq (f(t), v - \dot{u}(t))_X,$$

for all $v \in X$, and a.e. $t \in (0, T)$.

In the first step of the proof we introduce the set

$$\mathcal{W} = \{ \eta \in W^{1,2}(0, T; X) \mid \eta(0) = 0_X \} \tag{3.9}$$

and we prove the following existence and uniqueness result.

Lemma 1. *For all $\eta \in \mathcal{W}$ there exists a unique element $u_\eta \in W^{1,2}(0, T; X)$ such that*

$$a(u_\eta(t), v - \dot{u}_\eta(t)) + (\eta(t), v - \dot{u}_\eta(t))_X + j(v) - j(\dot{u}_\eta(t)) \tag{3.10}$$
$$\geq (f(t), v - \dot{u}_\eta(t))_X \qquad \forall v \in X, \quad a.e. \ \ t \in (0, T),$$

$$u_\eta(0) = u_0. \tag{3.11}$$

Proof. Let

$$(u, v)_a = a(u, v) \qquad \forall u, v \in X. \tag{3.12}$$

It follows that $(\cdot, \cdot)_a$ is an inner product on the space X and the associated norm, denoted $\| \cdot \|_a$, is equivalent to the norm $\| \cdot \|_X$ on X. Consequently, $(X, (\cdot, \cdot)_a)$ is a real Hilbert space.

Let $f_\eta : [0, T] \to X$ be the function defined by

$$(f_\eta(t), v)_a = (f(t), v)_X - (\eta(t), v)_X \qquad \forall v \in X, \ t \in [0, T]. \tag{3.13}$$

Then, (3.6) and the regularity $\eta \in \mathcal{W}$ imply that

$$f_\eta \in W^{1,2}(0, T; X). \tag{3.14}$$

Moreover, using (3.9), (3.12) and (3.13), we obtain

$$(f_\eta(0), v)_a - (u_0, v)_a - j(v) = (f(0), v)_X - a(u_0, v) - j(v) \quad \forall v \in X$$

and, taking into account (3.8), we find

$$\sup_{v \in D(j)} \{(f_\eta(0), v)_A - (u_0, v)_A - j(v)\} < +\infty. \tag{3.15}$$

Keeping in mind (3.5), (3.7), (3.14) and (3.15), we can use Theorem 3 on the space $(X, (\cdot, \cdot)_a)$ to obtain the existence of a unique element $u_\eta \in W^{1,2}(0, T; X)$ such that $u_\eta(0) = u_0$ and

$$(u_\eta(t), v - \dot{u}_\eta(t))_a + j(v) - j(\dot{u}_\eta(t)) \geq (f_\eta(t), v - \dot{u}_\eta(t))_a \quad \forall v \in X, \text{ a.e. } t \in (0, T).$$

It follows from (3.12) and (3.13) that u_η is the unique solution to the problem (3.10)–(3.11) which concludes the proof. ∎

In the next step we consider the operator $\Lambda : \mathcal{W} \to \mathcal{W}$ defined by

$$\Lambda \eta(t) = \int_0^t \theta(t - s) u_\eta(s) \, ds \qquad \forall \eta \in \mathcal{W}, \ t \in [0, T]. \tag{3.16}$$

It follows from (3.4) that the operator Λ is well defined, since $\eta \in \mathcal{W}$ implies $\Lambda \eta \in \mathcal{W}$. We also note that

$$\left(\frac{d}{dt} \Lambda \eta \right)(t) = \theta(0) u_\eta(t) + \int_0^t \dot{\theta}(t - s) u_\eta(s) \, ds \quad \forall \eta \in \mathcal{W}, \quad \text{a.e. } t \in (0, T). \tag{3.17}$$

We have the following result.

Lemma 2. *The operator Λ has a unique fixed point $\eta^* \in \mathcal{W}$.*

Proof. Let $\eta_1, \eta_2 \in \mathcal{W}$ and, for the sake of simplicity, denote $u_1 = u_{\eta_1}$ and $u_2 = u_{\eta_2}$. Using (3.16) and (3.4) it follows that

$$\| \Lambda \eta_1(t) - \Lambda \eta_2(t) \|_X^2 \leq c \int_0^t \| u_1(s) - u_2(s) \|_X^2 \, ds \qquad \forall t \in [0, T]. \tag{3.18}$$

Here and in what follows c represents a generic positive constant which may depend on $\|\theta\|_{W^{1,2}(0,T)}$, a and T, and whose value may change from place to place.

Moreover, from (3.17) we infer that

$$\left\| \left(\frac{d}{dt} \Lambda \eta_1 \right)(t) - \left(\frac{d}{dt} \Lambda \eta_2 \right)(t) \right\|_X \leq |\theta(0)| \| u_1(t) - u_2(t) \|_X$$
$$+ \int_0^t |\dot{\theta}(t - s)| \, \| u_1(s) - u_2(s) \|_X \, ds$$
$$\leq c \left(\| u_1(t) - u_2(t) \|_X + \left(\int_0^t \| u_1(s) - u_2(s) \|_X^2 \, ds \right)^{\frac{1}{2}} \right) \quad \text{a.e. } t \in (0, T),$$

which yields

$$\left\| \left(\frac{d}{dt} \Lambda \eta_1 \right)(t) - \left(\frac{d}{dt} \Lambda \eta_2 \right)(t) \right\|_X^2 \leq c \Big(\| u_1(t) - u_2(t) \|_X^2 \tag{3.19}$$
$$+ \int_0^t \| u_1(s) - u_2(s) \|_X^2 \, ds \Big) \quad \text{a.e. } t \in (0, T).$$

On the other hand, taking into account (3.10), we have the inequalities

$$a(u_1(s), v - \dot{u}_1(s)) + (\eta_1(s), v - \dot{u}_1(s))_X + j(v) - j(\dot{u}_1(s)) \geq (f(s), v - \dot{u}_1(s))_X,$$

$$a(u_2(s), v - \dot{u}_2(s)) + (\eta_2(s), v - \dot{u}_2(s))_X + j(v) - j(\dot{u}_2(s)) \geq (f(s), v - \dot{u}_2(s))_X,$$

for all $v \in X$, a.e. $s \in (0, T)$. We choose $v = \dot{u}_2(s)$ in the first inequality, $v = \dot{u}_1(s)$ in the second inequality, add the results and use (3.12) to obtain

$$\frac{1}{2} \frac{d}{ds} \|u_1(s) - u_2(s)\|_a^2 \leq -(\eta_1(s) - \eta_2(s), \dot{u}_1(s) - \dot{u}_2(s))_X \quad \text{a.e. } s \in (0, T).$$

Let $t \in [0, T]$. Integrating the previous inequality from 0 to t and using (3.11), we get

$$\frac{1}{2} \|u_1(t) - u_2(t)\|_a^2 \leq -(\eta_1(t) - \eta_2(t), u_1(t)) - u_2(t)))_X$$
$$+ \int_0^t (\dot{\eta}_1(s) - \dot{\eta}_2(s), u_1(s) - u_2(s))_X \, ds.$$

It follows now that

$$c \|u_1(t) - u_2(t)\|_X^2 \leq \|\eta_1(t) - \eta_2(t)\|_X \|u_1(t) - u_2(t)\|_X$$
$$+ \int_0^t \|\dot{\eta}_1(s) - \dot{\eta}_2(s)\|_X \|u_1(s) - u_2(s)\|_X ds$$

and, using the inequality

$$ab \leq \frac{a^2}{2\alpha} + 2\alpha b^2 \qquad \text{for } a, b, \alpha > 0,$$

we find

$$\|u_1(t) - u_2(t)\|_X^2 \leq c \left(\|\eta_1(t) - \eta_2(t)\|_X^2 \right. \tag{3.20}$$
$$+ \int_0^t \|\dot{\eta}_1(s) - \dot{\eta}_2(s)\|_X^2 ds + \left. \int_0^t \|u_1(s) - u_2(s)\|_X^2 \, ds \right).$$

As

$$\eta_1(t) - \eta_2(t) = \int_0^t (\dot{\eta}_1(s) - \dot{\eta}_2(s)) \, ds,$$

we deduce that

$$\|\eta_1(t) - \eta_2(t)\|_X^2 \leq c \int_0^t \|\dot{\eta}_1(s) - \dot{\eta}_2(s)\|_X^2 \, ds.$$

Plugging this inequality in (3.20) we obtain

$$\|u_1(t) - u_2(t)\|_X^2 \leq c \left(\int_0^t \|\dot{\eta}_1(s) - \dot{\eta}_2(s)\|_X^2 \, ds + \int_0^t \|u_1(s) - u_2(s)\|_X^2 \, ds \right).$$

Applying now Gronwall's inequality we deduce

$$\|u_1(t) - u_2(t)\|_X^2 \le c \int_0^t \|\dot\eta_1(s) - \dot\eta_2(s)\|_X^2 \, ds \qquad (3.21)$$

which yields

$$\int_0^t \|u_1(s) - u_2(s)\|_X^2 \, ds \le c \int_0^t \|\dot\eta_1(s) - \dot\eta_2(s)\|_X^2 \, ds. \qquad (3.22)$$

Combining now (3.18), (3.19), (3.21) and (3.22) we obtain

$$\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_X^2 + \left\|\left(\frac{d}{dt}\Lambda\eta_1\right)(t) - \left(\frac{d}{dt}\Lambda\eta_2\right)(t)\right\|_X^2 \le c \int_0^t \|\dot\eta_1(s) - \dot\eta_2(s)\|_X^2 \, ds.$$

Iterating the last inequality p times we infer

$$\|\Lambda^p\eta_1(t) - \Lambda^p\eta_2(t)\|_X^2 + \left\|\left(\frac{d}{dt}\Lambda^p\eta_1\right)(t) - \left(\frac{d}{dt}\Lambda^p\eta_2\right)(t)\right\|_X^2$$
$$\le c^p \int_0^t \int_0^{s_1} \cdots \int_0^{s_{p-1}} \|\dot\eta_1(s_p) - \dot\eta_2(s_p)\|_X^2 \, ds_p \cdots ds_1,$$

where Λ^p denotes the power of the operator Λ. The last inequality implies

$$\|\Lambda^p\eta_1 - \Lambda^p\eta_2\|_{W^{1,2}(0,T;X)}^2 \le \frac{c^p T^p}{p!} \|\eta_1 - \eta_2\|_{W^{1,2}(0,T;X)}^2.$$

Since $\lim_{p\to\infty} \dfrac{c^p T^p}{p!} = 0$, the previous inequality implies that a power Λ^p of Λ is a contraction in $\mathcal{W}$ for p large enough. It follows now from Banach's fixed point theorem that there exists a unique element $\eta^* \in \mathcal{W}$ such that $\Lambda^p \eta^* = \eta^*$. Moreover, since $\Lambda^p(\Lambda \eta^*) = \Lambda(\Lambda^p \eta^*) = \Lambda\eta^*$, we deduce that $\Lambda\eta^*$ is also a fixed point of the operator Λ^p. By the uniqueness of the fixed point, we conclude that $\Lambda\eta^* = \eta^*$, which shows that η^* is a fixed point of Λ. The uniqueness of the fixed point of the operator Λ follows from the uniqueness of the fixed point of the operator Λ^p. ∎

We have now all the ingredients to prove the theorem.

Proof of Theorem 2. Existence. Let $\eta^* \in \mathcal{W}$ be the fixed point of Λ and let u_{η^*} be the function defined by Lemma 1 for $\eta = \eta^*$. Since $\Lambda\eta^* = \eta^*$, it follows from (3.16), (3.10) and (3.11) that u_{η^*} is a solution to the problem (3.1)–(3.2). Moreover, the regularity $u_{\eta^*} \in W^{1,2}(0,T;X)$ is obtained from Lemma 1.

Uniqueness. The uniqueness follows by arguments similar to those used in [9] and is a consequence of the uniqueness of the fixed point of the operator Λ. Indeed, let u be a solution of problem (3.1)–(3.2) with regularity $u \in W^{1,2}(0,T;X)$ and consider the element $\eta \in \mathcal{W}$ defined by

$$\eta(t) = \int_0^t \theta(t-s)u(s) \, ds \qquad \forall t \in [0,T]. \qquad (3.23)$$

Then u is also the solution of problem (3.10)–(3.11), so by Lemma 1 we have

$$u = u_\eta. \tag{3.24}$$

By (3.16), (3.23) and (3.24) we obtain that $\Lambda\eta = \eta$. As Λ has a unique fixed point, we conclude that

$$\eta = \eta^*. \tag{3.25}$$

The uniqueness part of Theorem 2 is now a consequence of equalities (3.24) and (3.25). ∎

We now apply Theorem 2 to provide the well-posedness of problem $\mathcal{P}_V$.

Proof of Theorem 1. Consider the form $a : V \times V \to \mathbb{R}$ defined by

$$a(u,v) = \mu \int_\Omega \nabla u \cdot \nabla v \, dx \qquad \forall u, v \in V.$$

Clearly this form is bilinear, continuous, coercive and symmetric; moreover, using (2.18) and (2.19) it follows that the functional j defined by (2.22) is convex, lower semicontinuous and proper. Therefore, we conclude that conditions (3.3) and (3.5) are satisfied in this case. Taking into account (2.21) and (2.24)–(2.26), the conclusion of Theorem 1 follows from Theorem 3. ∎

4. A convergence result

In this section we investigate the behavior of the weak solution of the antiplane contact problem as the relaxation function converges to zero. To this end, in order to underline the dependence with respect to the function θ, we reformulate the problem $\mathcal{P}_V$ as follows.

Problem $\mathcal{P}_V^\theta$**.** *Find a displacement field* $u_\theta : [0, T] \to V$ *such that*

$$\mu\,(u_\theta(t), v - \dot{u}_\theta(t))_V + (\int_0^t \theta\,(t-s)u_\theta(s)ds, v - \dot{u}_\theta(t))_V + j(v) \tag{4.1}$$

$$-j(\dot{u}_\theta(t)) \geq (f(t), v - \dot{u}_\theta(t))_V \quad \forall v \in V, \quad a.e. \ \ t \in (0, T),$$

$$u_\theta(0) = u_0. \tag{4.2}$$

We also consider the problem obtained for $\theta = 0$ that is:

Problem $\mathcal{P}_V^0$**.** *Find a displacement field* $u : [0, T] \to V$ *such that*

$$\mu\,(u(t), v - \dot{u}(t))_V + j(v) - j(\dot{u}(t)) \geq (f(t), v - \dot{u}(t))_V \tag{4.3}$$

$$\forall v \in V, \quad a.e. \ \ t \in (0, T),$$

$$u(0) = u_0. \tag{4.4}$$

Clearly problem $\mathcal{P}_V^0$ represents the variational formulation of the antiplane frictional contact problem for linear elastic materials, i.e. the problem obtained when (2.4) is replaced by the elastic constitutive law

$$\sigma = \lambda(tr(\varepsilon(\boldsymbol{u})))\boldsymbol{I} + 2\mu\varepsilon(\boldsymbol{u}).$$

Everywhere in this section we assume that (2.19)–(2.21), (2.25) and (2.26) hold. It follows from Theorem 1 that Problem $\mathcal{P}_V^\theta$ has a unique solution $u_\theta \in W^{1,2}(0,T;V)$ and Problem $\mathcal{P}_V^0$ has a unique solution $u \in W^{1,2}(0,T;V)$.

Consider now the assumption

$$\|\theta\|_{W^{1,2}(0,T)} \to 0. \tag{4.5}$$

Our main result in this section is the following.

Theorem 4. *Assume that* (2.19)–(2.21), (2.25), (2.26) *and* (4.5) *hold. Then the solution* u_θ *of the Problem* $\mathcal{P}_V^\theta$ *converge to the solution* u *of the Problem* $\mathcal{P}_V^0$, *i.e.*

$$\|u_\theta - u\|_{C([0,T];V)} \to 0. \tag{4.6}$$

Proof. Denote

$$\eta_\theta(t) = \int_0^t \theta(t-s)u_\theta(s)ds \qquad \forall t \in [0,T]. \tag{4.7}$$

Using (4.1), (4.3), (4.7) and arguments similar to those used to obtain (3.21), yields

$$\|u_\theta(t) - u(t)\|_V^2 \le c \int_0^t \|\dot{\eta}_\theta(s)\|_V^2 ds \qquad \forall t \in [0,T]. \tag{4.8}$$

Here and below c represent a positive constant which may depend on u but is independent of θ and whose value may change from line to line.

It follows from (4.7) that

$$\dot{\eta}_\theta(t) = \theta(0)u_\theta(t) + \int_0^t \dot{\theta}(t-s)u_\theta(s)ds, \qquad \text{a.e. } t \in (0,T)$$

which implies the inequality

$$\|\dot{\eta}_\theta(t)\|_V \le |\theta(0)|\|u_\theta(t) - u(t)\|_V + |\theta(0)|\|u(t)\|_V \tag{4.9}$$

$$+ \int_0^t |\dot{\theta}(t-s)|\|u_\theta(s) - u(s)\|_V ds + \int_0^t |\dot{\theta}(t-s)|\|u(s)\|_V ds \qquad \text{a.e. } t \in (0,T).$$

Moreover (4.5) implies that there exists $c > 0$ such that

$$\|\theta\|_{C(0,T)} \le c, \qquad \|\dot{\theta}\|_{L^2(0,T)} \le c. \tag{4.10}$$

Keeping in mind (4.9) and (4.10), after some algebra yields

$$\|\dot{\eta}_\theta(t)\|_V^2 \le c(\|\theta\|_{W^{1,2}(0,T)}^2 + \|u_\theta(t) - u(t)\|_V^2 + \int_0^t \|u_\theta(s) - u(s)\|_V^2 ds)$$
$$\text{a.e. } t \in (0,T).$$

We integrate the previous inequality to find

$$\int_0^t \|\dot{\eta}_\theta(s)\|_V^2 \, ds \le c(\|\theta\|_{W^{1,2}(0,T)}^2 + \int_0^t \|u_\theta(s) - u(s)\|_V^2 ds) \quad \forall t \in [0,T]. \quad (4.11)$$

We combine (4.8) and (4.11) to obtain

$$\|u_\theta(t) - u(t)\|_V^2 \le c(\|\theta\|_{W^{1,2}(0,T)}^2 + \int_0^t \|u_\theta(s) - u(s)\|_V^2 ds) \quad \forall t \in [0,T],$$

and, using Gronwall's inequality, we find

$$\|u_\theta(t) - u(t)\|_V \le c \|\theta\|_{W^{1,2}(0,T)} \quad \forall t \in [0,T]. \quad (4.12)$$

The convergence result (4.6) is now a consequence of (4.5) and (4.12). ∎

From Theorem 4 we conclude that the weak solution to the antiplane viscoelastic problem with Tresca's friction law may be approached by the weak solution to the antiplane elastic problem with Tresca's friction law as the relaxation function is small enough. In addition to the mathematical interest in this result, this is of importance from mechanical point of view, as it indicates that the elasticity with friction may be considered as a limit case of viscoelasticity with friction as the memory decreases.

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