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AN EXTENSION OF THE HERMITE-HADAMARD INEQUALITY THROUGH SUBHARMONIC FUNCTIONS*

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Abstract. In this paper we obtain a Hermite-Hadamard type inequality for a class of subharmonic functions. Our proofs rely essentially on the properties of elliptic partial differential equations of second order. Our study extends some recent results from [**1**], [**2**] and [**6**].

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1. Introduction and main result. The classical Hermite-Hadamard inequality provides a valuable two-sided estimate of the mean value of a continuous convex function $f : [a, b] \to \mathbb{R}$:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

This fact was extended within the Choquet theory to the general framework of continuous convex functions on a compact convex subset K (of a metrizable locally convex space) and of Borel probability measures μ on K. See [**7**] for details. Is it possible to extend Choquet's theory to the more general case of signed measures? Recently, A. Florea and C. P. Niculescu [**2**] solved completely the case of compact intervals, based on earlier work due to A. M. Fink [**1**]. More precisely, they provided a full characterization of those signed Borel measures μ on $[a, b]$ such that $\mu([a, b]) > 0$ and

$$\begin{aligned} f(x_\mu) &\leq \frac{1}{\mu([a,b])}\int_{[a,b]} f(t)\,d\mu(t) \\ &\leq \frac{b-x_\mu}{b-a}\cdot f(a) + \frac{x_\mu - a}{b-a}\cdot f(b), \end{aligned}$$

for all continuous convex functions $f : [a, b] \to \mathbb{R}$, where

$$x_\mu = \frac{1}{\mu([a,b])}\int_{[a,b]} t\,d\mu(t)$$

is the *barycenter* of μ. Besides the case of Borel probability measures, other examples are offered by the family $d\mu = (x^2 + \lambda)\,dx$ on $[-1, 1]$, when $\lambda \geq -1/6$. See [**2**].

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A natural method to extend results regarding convex functions of one real variable to several variable functions is due to P. Montel **[5]** and appeals to subharmonic functions. By a *subharmonic function* u defined on a domain $D \subset \mathbb{R}^N$ ($N \geq 2$), we understand a C^2-differentiable function on D with the property that

$$\Delta u \geq 0, \quad \text{in } D,$$

where $\Delta = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$ denotes the Laplace operator.

C. P. Niculescu and L.-E. Persson gave in **[6]** an extension of the Hermite-Hadamard inequality to this context. They proved that if $\Omega \subset \mathbb{R}^N$ is a bounded open subset with smooth boundary, $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is subharmonic and $\varphi \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is a solution of the problem

$$\begin{cases} \Delta\varphi = 1, & \text{for } x \in \Omega \\ \varphi = 0, & \text{for } x \in \partial\Omega \end{cases}$$

then

$$\int_\Omega u \, dV < \int_{\partial\Omega} u(\nabla\varphi \cdot n) \, dS \tag{2}$$

except for harmonic functions (when equality occurs).

In the particular case when Ω is the open ball $B_R(a)$ (centered in a and of radius R) in $\mathbb{R}^3$, the maximum principle for elliptic problems combined with the above result yield the following Hermite-Hadamard type inequality for subharmonic functions (which are not harmonic):

$$u(a) \leq \frac{1}{\text{Vol } \overline{B}_R(a)} \int\int\int_{\overline{B}_R(a)} u(x) \, dV < \frac{1}{\text{Area } S_R(a)} \int\int_{S_R(a)} u(x) \, dS. \tag{3}$$

Formula (3) shows that for the measure $d\mu = \frac{1}{\text{Vol } \overline{B}_R(a)} \, dV$ there exists a measure $d\nu = \frac{1}{\text{Area } S_R(a)} \, dS$ concentrated on the boundary of $\Omega = \overline{B}_R(a)$ such that

$$\int_\Omega f \, d\mu \leq \int_{\partial\Omega} f \, d\nu,$$

for all subharmonic functions f.

In this paper we prove that a similar result works when the Laplace operator is replaced by a strictly elliptic self-adjoint linear differential operator of second order which admits a Green function.

More precisely, we shall deal with operators $L : C^2(\Omega) \to C(\Omega)$ defined by

$$Lu = \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u, \tag{4}$$

where $a_{ij}(x) = a_{ji}(x) \in C^1(\Omega)$, $b_i(x) \in C(\Omega)$ and $c(x) \in C(\Omega)$ is a negative function in Ω.

As above, $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) will be a bounded domain with smooth boundary.

We assume that L is *strictly elliptic* on Ω, i.e.

$$\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \quad \forall\, x \in \Omega,\ \forall\,(\xi_1,\ldots,\xi_N) \in \mathbb{R}^N,$$

for some positive constant λ and *self-adjoint*, i.e.

$$b_i(x) = \sum_{j=1}^{N} \frac{\partial a_{ij}(x)}{\partial x_j}, \quad \forall\, i = 1,\ldots,N, \ \ \forall\, x \in \Omega.$$

For the strictly elliptic, self-adjoint, linear second order differential operator L on the domain Ω we introduce *the Green function* $G : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R}$ as a function having the following three properties:

(G1) $G(x,\xi)|x-\xi|^{N-2}$ is a bounded function of ξ and has a positive lower bound for ξ near x;

(G2) $L_\xi[G(x,\xi)] = 0$ in Ω for $\xi \neq x$. The notation L_ξ means that we apply the operator L to the coordinates $(\xi_1,\ldots,\xi_N)$ of ξ in $G(x,\xi)$ and keep $x = (x_1,\ldots,x_N)$ fixed;

(G3) $G(x,\xi) = 0$ for $\xi \in \partial\Omega$ and $x \in \Omega$.

Since L is self-adjoint, Green's function is symmetric, in the sense that

$$G(x,\xi) = G(\xi,x), \quad \forall\, x,\xi \in \Omega.$$

As noticed in [**9**, pp. 87–88], a Green function with properties (G1)–(G3) exists for an operator L as above if the coefficients of L and the boundary of Ω are sufficiently smooth and in addition the problem

$$\begin{cases} Lu(x) = h(x), & \text{for } x \in \Omega \\ u(x) = g(x), & \text{for } x \in \partial\Omega \end{cases} \tag{5}$$

has a unique solution for suitable data. Under these circumstances a solution u of equation (5) is given by the formula:

$$u(\xi) = -\int_\Omega G(x,\xi)h(x)\,dx - \int_{\partial\Omega} g(x)\frac{\partial G(x,\xi)}{\partial \nu_x}\,d\sigma(x). \tag{6}$$

The main result of this paper is the following theorem.

THEOREM 1. *Assume* $p \in C^{0,\alpha}(\overline{\Omega})$*, for some* $\alpha \in (0,1)$*. Then a necessary and sufficient condition for the inequality*

$$\int_\Omega f(x)p(x)\,dx \leq \int_{\partial\Omega} f(\xi)\cdot\left[-\int_\Omega \frac{\partial G(\xi,x)}{\partial \gamma_\xi}p(x)\,dx\right]d\sigma(\xi) \tag{7}$$

to hold for all $f \in C^{2,\alpha}(\overline{\Omega})$ *with*

$$Lf(x) \geq 0, \quad \forall\, x \in \Omega, \tag{8}$$

is that the solution of the Dirichlet problem

$$\begin{cases} Lv(x) = p(x), & \textit{for } x \in \Omega \\ v(x) = 0, & \textit{for } x \in \partial\Omega \end{cases} \tag{9}$$

satisfies $v(x) \leq 0$ *for all* $x \in \Omega$.

Here, $G(x,\xi)$ is Green's function for the operator L on the domain Ω, and $\frac{\partial}{\partial\gamma}$ denotes the derivative in direction $\gamma=(\gamma_1,\ldots,\gamma_N)$.

REMARK 1. Problem (9) has a unique solution via [**3**, Theorems 6.8 and 4.3]. Furthermore, if $p(x)\geq 0$ for all $x\in\Omega$, then the maximum principle (see [**3**, Corollary 3.2]) implies that solution v is negative in Ω.

REMARK 2. There exist functions $p(x)$ which may take negative values in Ω and such that problem (9) still has a negative solution. Indeed, in the particular case when Ω is the unit ball centered in the origin of $\mathbb{R}^N$, $L=\Delta$ (the Laplace operator), and $p(x)=|x|^2-\frac{N}{6}$, the solution of problem (9) is given by

$$v(x)=\frac{|x|^2(|x|^2-1)}{12}<0,\quad \forall\, x\in B_1(0).$$

REMARK 3. Theorem 1 extends both the right hand side inequalities in (1) and (3). The boundary measure associated to $p(x)\,dx$ appears to be $[-\int_\Omega \frac{\partial G(\xi,x)}{\partial\gamma_\xi}p(x)\,dx]\,d\sigma(\xi)$.

REMARK 4. It is worth noticing that Theorem 1 can be easily extended to the general framework of signed Borel measures. For this it suffices to replace the Dirichlet problem (9) by a similar problem having the right-hand side a measure.

2. Proof of Theorem 1. Inequality (7) is equivalent to

$$0\geq\int_\Omega\left[f(x)+\int_{\partial\Omega}\frac{\partial G(\xi,x)}{\partial\gamma_\xi}f(\xi)\,d\sigma(\xi)\right]p(x)\,dx. \tag{10}$$

Since $f\in C^{2,\alpha}(\overline{\Omega})$ it follows that $Lf\in C^{0,\alpha}(\overline{\Omega})$ and thus by [**3**, Theorems 6.8 and 4.3] we infer that f is the unique solution of the problem

$$\begin{cases} Lw(x)=Lf(x), & \text{for } x\in\Omega\\ w(x)=f(x), & \text{for } x\in\partial\Omega.\end{cases}$$

Hence, by (6), we get

$$f(x)=-\int_\Omega G(\xi,x)Lf(\xi)\,d\xi-\int_{\partial\Omega}\frac{\partial G(\xi,x)}{\partial\gamma_\xi}f(\xi)\,d\sigma(\xi), \tag{11}$$

and (10) is equivalent to

$$0\geq\int_\Omega\left[-\int_\Omega G(\xi,x)p(x)\,dx\right]d\xi. \tag{12}$$

A new appeal to formula (6) yields

$$\begin{aligned} v(\xi)&=-\int_\Omega G(x,\xi)p(x)\,dx\\ &=-\int_\Omega G(\xi,x)p(x)\,dx,\end{aligned}$$

taking into account the symmetry of G.

Consequently, relation (12) can be restated as

$$0 \geq \int_{\Omega} v(\xi) L f(\xi)\, d\xi.$$

Or, $Lf \geq 0$ in Ω and Lf runs over $C^{0,\alpha}(\overline{\Omega})(\supset C_0(\Omega))$ when f runs over $C^{2,\alpha}(\overline{\Omega})$. Thus, the last inequality holds true if and only if $v \leq 0$ over Ω.

The proof of Theorem 1 is complete.

3. A particular case. In this section we point out once more the connection between Theorem 1 and the Hermite-Hadamard inequality. To do that we consider the particular case where $L = \Delta$, $\Omega = B_R(0)$ (the ball of radius R centred in the origin in $\mathbb{R}^N$ $(N \geq 2)$) and $p(x) \equiv 1$ in $\overline{\Omega}$.

We denote by $E(x)$ the *fundamental solution* of the Laplace equation on $\mathbb{R}^N$ (see [**4**, p. 8]), that is

$$E(x) = \begin{cases} \dfrac{1}{(2-N)\omega_N} \cdot \dfrac{1}{|x|^{N-2}}, & \text{if } N \geq 3,\ x \neq 0 \\ \dfrac{1}{2\pi} \cdot \ln(|x|), & \text{if } N = 2,\ x \neq 0 \end{cases}$$

where ω_N represents the area of the unit ball in $\mathbb{R}^N$.

Then it is known (see [**4**]) that Green's function for $N \geq 3$ is given by the formula

$$G(x,\xi) = \begin{cases} \left(\dfrac{R}{|x|}\right)^{N-2} \cdot E(x^\star - \xi) - E(x - \xi), & \text{for } x \in B_R(0) \setminus \{0\} \\ \dfrac{1}{(2-N)\omega_N R^{N-2}} - E(\xi), & \text{for } x = 0 \end{cases}$$

while Green's function for $N = 2$ is given by the formula

$$G(x,\xi) = \begin{cases} \dfrac{1}{2\pi} \cdot (-\ln(|x-\xi|) + \ln(|x^\star - \xi| \cdot |x|/R)), & \text{for } x \in B_R(0) \setminus \{0\} \\ \dfrac{1}{2\pi}(-\ln(|\xi| + \ln(R))), & \text{for } x = 0 \end{cases}$$

where $x^\star = R^2/|x|^2 x$, for all $x \in B_R(0) \setminus \{0\}$.

A simple computation (see [**4**, p. 13]) shows that the normal derivative of Green's function is given by

$$\frac{\partial G(x,\xi)}{\partial \nu_\xi} = \frac{|x|^2 - R^2}{R\omega_N |x-\xi|^N}, \tag{13}$$

for all $N \geq 2$.

By Theorem 1 we infer that for any function $f \in C^{2,\alpha}(\overline{\Omega})$ with $\Delta f \geq 0$ in Ω the following inequality holds:

$$\frac{1}{\mathrm{Vol}\overline{B}_R(0)} \int_{B_R(0)} f(x)\, dx \leq \int_{\partial B_R(0)} f(\xi) \cdot \left[\frac{1}{\mathrm{Vol}\overline{B}_R(0)} \int_{B_R(0)} \left(\frac{R^2 - |x|^2}{\omega_N R}\right) \frac{1}{|x-\xi|^N}\, dx\right] d\sigma(\xi). \tag{14}$$

The above inequality is a Hermite-Hadamard type inequality since for any $x \in B_R(0)$ we have

$$\frac{R^2 - |x|^2}{\omega_N R} \int_{\partial B_R(0)} \frac{1}{|x - \xi|^N}\, d\sigma(\xi) = 1.$$

The last equality is an immediate consequence of Poisson's formula (see [**4**, p. 14–15]) and of the fact that the unique solution of the Dirichlet problem

$$\begin{cases} \Delta u(x) = 0, & \text{for } x \in B_R(0) \\ u(x) = 1, & \text{for } x \in \partial B_R(0), \end{cases}$$

is $u \equiv 1$ via the maximum principle.

More generally, relation (14) still works for a weighted Lebesgue measure, $p(x)\, dx$, where $p(x)$ satisfies a Dirichlet problem of the type (9). In that situation $\mathrm{Vol}(\overline{B}_R(0))$ must be replaced by $\int_{B_R(0)} p(x)\, dx$.

AN OPEN PROBLEM. Based on the above considerations, it seems very likely that the main result of this paper remains valid for all operators L that possess a Green function. In particular, for the *biharmonic operator* in $\mathbb{R}^2$ (see, e.g. [**8**, p. 194]),

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}.$$

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