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The Krein-Milman Theorem in Global NPC Spaces

by
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Abstract

We extend the Krein-Milman Theorem to the context of global NPC spaces.

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An important result concerning the behavior of a continuous convex function defined on a compact convex subset K (of a locally convex Hausdorff space) asserts that its maximum is attained at an extreme point. The proof is an easy consequence of the Krein-Milman Theorem. The aim of this note is to prove that the Krein-Milman Theorem (and thus the aforementioned property of convex functions) still works in the context of some spaces with a curved geometry.

We need some preparation.

Definition 1. *A global NPC space is a complete metric space $M = (M, d)$ for which the following inequality holds true: for each pair of points $x_0, x_1 \in M$ there exists a point $y \in M$ such that for all points $z \in E$,*

$$d^2(z, y) \leq \frac{1}{2}d^2(z, x_0) + \frac{1}{2}d^2(z, x_1) - \frac{1}{4}d^2(x_0, x_1). \qquad (NPC)$$

In a global NPC space each pair of points $x_0, x_1 \in E$ can be connected by a geodesic (that is, by a rectifiable curve $\gamma : [0, 1] \to E$ such that the length of $\gamma|_{[s,t]}$ is $d(\gamma(s), \gamma(t))$ for all $0 \leq s \leq t \leq 1$). Moreover, this geodesic is unique.

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The point y that appears in the inequality (NPC) is called the *midpoint* of x_0 and x_1 and has the property

$$d(x_0, y) = d(y, x_1) = \frac{1}{2} d(x_0, x_1).$$

Every Hilbert space is a global NPC space. In this case the geodesics are the line segments.

The upper half-plane $\mathbf{H}= \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, endowed with the Poincaré metric,

$$ds^2 = \frac{dx^2 + dy^2}{y^2},$$

constitutes a very instructive example of a global NPC space, where the geodesics are the semicircles in $\mathbf{H}$ perpendicular to the real axis and the straight vertical lines ending on the real axis.

A Riemannian manifold (M, g) is a global NPC space if and only if it is complete, simply connected and of nonpositive sectional curvature. Besides manifolds, other important examples of global NPC spaces are the Bruhat-Tits buildings (in particular, the trees). See [2]. More information on global NPC spaces is available in [1] and [4].

In what follows M will denote a global NPC space.

Definition 2. *A subset $C \subset M$ is called* convex *if $\gamma([0,1]) \subset C$ for each geodesic $\gamma : [0,1] \to C$ joining two points in C.*

A function $\varphi : C \to \mathbb{R}$ is called convex if the function $\varphi \circ \gamma : [0,1] \to \mathbb{R}$ is convex whenever $\gamma : [0,1] \to C$, $\gamma(t) = \gamma_t$, is a geodesic, that is,

$$\varphi(\gamma_t) \leq (1-t)\varphi(\gamma_0) + t\varphi(\gamma_1)$$

for all $t \in [0,1]$. The function φ is called concave if $-\varphi$ is convex.

All closed convex subsets of a global NPC space are in turn spaces of the same nature. In a global NPC space, the distance from a point x_0,

$$\varphi : x \to d(x, x_0),$$

provides a basic example of a convex function. Moreover, its square is strictly convex. See [6], Proposition 2.3 and Corollary 2.5. As a consequence, the balls in a global NPC space are convex sets in the sense of Definition 2.

In the case of flat spaces, the Krein-Milman Theorem is derived as a consequence of the geometric form of the Hahn-Banach Theorem. See [3], [5]. In the framework of global NPC spaces we shall use a separation argument based on convex functions.

Let K be a convex subset of a global NPC space M. A subset $A \subset K$ is called an *extremal subset* if it is nonempty, closed and verifies the following property: If $x, y \in K$ and the geodesic γ joining x and y meets A at a point γ_t for some $t \in (0,1)$, then both endpoints x, y should be in A.

A point $z \in K$ is called an *extreme* point of K if $\{z\}$ is an extremal subset (equivalently, if z is not interior for any geodesic with endpoints in K).

Theorem 1. *Let K be a nonempty compact convex subset of a global NPC space M. Then K is the closed convex hull of the set* $\operatorname{Ext} K$, *of all extreme points of K.*

Proof: We start by noticing that every extremal subset of K contains at least one extreme point. In particular, $\operatorname{Ext} K \neq \emptyset$.

In fact, let us consider an extremal subset S that contains at least two points, say x_0 and y_0. The function $x \to d^2(x, x_0)$ is strictly convex and not constant on S, therefore

$$S_0 = \{x : d^2(x, x_0) = M := \sup_{k \in S} d^2(k, x_0)\}$$

is a closed proper subset of S. S_0 is also an extremal subset; if $x', x'' \in S$ and S_0 contains a point x_t, interior to the geodesic joining x' and x'', then

$$\begin{aligned} M = d^2(x_t, x_0) &\leq (1-t)d^2(x', x_0) + td^2(x'', x_0) - t(1-t)d^2(x', x'') \\ &\leq M - t(1-t)d^2(x', x''), \end{aligned}$$

which forces $x' = x'' \in S_0$.

By Zorn's Lemma, every extremal subset includes a minimal extremal subset. The discussion above shows that the minimal extremal subsets are one-point sets.

To end the proof we have to show that $\overline{\operatorname{conv}}(\operatorname{Ext} K) = K$. Clearly, $\overline{\operatorname{conv}}(\operatorname{Ext} K) \subset K$. If this inclusion were strict, then the set

$$T_0 = \{x \in K : d(x, \overline{\operatorname{conv}}(\operatorname{Ext} K)) = \sup_{y \in K} d(y, \overline{\operatorname{conv}}(\operatorname{Ext} K))\}$$

is an extremal one. The argument is based on the fact that the function $x \to d(x, \overline{\operatorname{conv}}(\operatorname{Ext} K))$ is convex (which follows from the geodesic comparison, Corollary 2.5 in [6]).

According to a remark above, T_0 must intersect $\operatorname{Ext} K$. Or,

$$T_0 \cap \overline{\operatorname{conv}}(\operatorname{Ext} K) = \emptyset.$$

Consequently $\overline{\operatorname{conv}}(\operatorname{Ext} K) = K$. □

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