FAN'S INEQUALITY IN GEODESIC SPACES

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Abstract. Fan's minimax inequality is extended to the context of metric spaces with global nonpositive curvature. As a consequence, a much more general result on the existence of a Nash equilibrium is obtained.

1. Preliminaries

Suppose that C is a nonempty compact and convex subset of a linear topological space. Fan's minimax inequality asserts that any function $f : C \times C \to \mathbb{R}_+$ which is quasi-concave in the first variable and lower semicontinuous in the second variable verifies the minimax inequality,

$$(F) \qquad \min_{y \in C} \sup_{x \in C} f(x, y) \leq \sup_{z \in C} f(z, z).$$

As is well known, this result is equivalent to the Brouwer Fixed Point Theorem. See [2], pp. 205-206.

The aim of this work is to extend Fan's minimax inequality to the framework of global NPC spaces, that is, to the complete metric spaces with global nonpositive curvature.

Definition 1. *A global NPC space is a complete metric space $E = (E, d)$ for which the following inequality holds true: for each pair of points $x_0, x_1 \in E$ there exists a point $y \in E$ such that for all points $z \in E$,*

$$(NPC) \qquad d^2(z, y) \leq \frac{1}{2} d^2(z, x_0) + \frac{1}{2} d^2(z, x_1) - \frac{1}{4} d^2(x_0, x_1).$$

In a global NPC space each pair of points $x_0, x_1 \in E$ can be connected by a geodesic (that is, by a rectifiable curve $\gamma : [0, 1] \to E$ such that the length of $\gamma|_{[s,t]}$ is $d(\gamma(s), \gamma(t))$ for all $0 \leq s \leq t \leq 1$). Moreover, this geodesic is unique. The point y that appears in the inequality (NPC) is the *midpoint* of x_0 and x_1 and has the property

$$d(x_0, y) = d(y, x_1) = \frac{1}{2} d(x_0, x_1).$$

Every Hilbert space is a global NPC space. In this case the geodesics are the line segments.

A Riemannian manifold (M, g) is a global NPC space if and only if it is complete, simply connected and of nonpositive sectional curvature. Besides manifolds, other

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important examples of global NPC spaces are the Bruhat-Tits buildings (in particular, the trees). See [3]. More information on the global NPC spaces is available in [1] and [4].

In what follows E will denote a global NPC space.

Definition 2. *A set $C \subset E$ is called convex if $\gamma([0,1]) \subset C$ for each geodesic $\gamma : [0,1] \to C$ joining $\gamma(0), \gamma(1) \in C$.*

A function $\varphi : C \to \mathbb{R}$ is called convex if the function $\varphi \circ \gamma : [0,1] \to \mathbb{R}$ is convex for each geodesic $\gamma : [0,1] \to C$, $\gamma(t) = \gamma_t$, that is,

$$\varphi(\gamma_t) \leq (1-t)\varphi(\gamma_0) + t\varphi(\gamma_1)$$

for all $t \in [0,1]$. The function φ is called concave if $-\varphi$ is convex.

All closed convex subsets of a global NPC space are in turn spaces of the same nature. In a global NPC space, the distance function is convex with respect to both variables, a fact which implies that every ball is convex in the sense of Definition 2.

An important feature of global NPC spaces is the possibility of introducing a well behaved concept of a barycenter of a probability measure. See [7] for details. For the convenience of the reader, we shall recall here some basic facts.

$\mathcal{P}^1(E)$ denotes the set of all Borel probability measures μ on E with separable support, which verify the condition

$$\int_E d(x,y)d\mu(y) < \infty$$

for some (and hence for all) $x \in E$. The *barycenter* of a measure $\mu \in \mathcal{P}^1(E)$ is the unique point $z \in E$ which minimizes the uniformly convex function $F_y : z \to \int_E \left[d^2(z,x) - d^2(y,x)\right] d\mu(x)$; this point is independent of $y \in E$ and is also denoted as $b(\mu)$.

If the support of μ is included in a convex closed set K, then $b(\mu) \in K$.

$\mathcal{P}^1(E)$ can be made a metric space with respect to the *Wasserstein distance,*

$$d^W(\mu,\nu) = \inf \iint_{E\times E} d(x,y)d\lambda(x,y),$$

where the infimum is taken over all $\lambda \in \mathcal{P}^1(E \times E)$ with marginals μ and ν. With respect to this metric the barycenter map is nonexpansive, that is,

$$d(b(\mu), b(\nu)) \leq d^W(\mu,\nu)$$

for all $\mu, \nu \in \mathcal{P}^1(E)$.

In what follows we shall be interested also in a more general class of convex like functions, based on their behavior under the action of means.

The weighted M_p-mean is defined for pairs of positive numbers x, y by the formula

$$M_p(x,y;1-t,t) = \begin{cases} ((1-t)x^p + ty^p)^{1/p}, & \text{if } p \in \mathbb{R}\backslash\{0\} \\ x^{1-t}y^t, & \text{if } p = 0 \\ \min\{x,y\}, & \text{if } p = -\infty \\ \max\{x,y\}, & \text{if } p = \infty, \end{cases}$$

where $t \in [0,1]$. If p is an odd number, we can extend M_p to pairs of real numbers.

The unweighted means $M_p(x,y)$ correspond to the case where $\lambda = 1/2$.

Definition 3. *We say that a function* $\varphi : C \longrightarrow \mathbb{R}$ *is* M_p*-concave if for each geodesic* $\gamma : [0,1] \to C$,

$$\varphi(\gamma_t) \geq M_p(\varphi(\gamma_0), \varphi(\gamma_1); 1-t, t), \quad \textit{for all } t \in [0,1].$$

Thus the M_1-concave functions are the usual concave functions, while the M_∞-concave functions are precisely the quasi-concave functions.

The aim of this work is to prove the following analogue of Fan's inequality:

Theorem 1. *Let* C *be a compact convex subset of a global NPC space* E.

i) If $f : C \times C \to \mathbb{R}_+$ *is quasi-concave in the first variable and lower semicontinuous in the second variable, then*

$$(F) \qquad \min_{y \in C} \sup_{x \in C} f(x,y) \leq \sup_{z \in C} f(z,z).$$

ii) If $p \in \mathbb{R}$ *and* $f : C \times C \to \mathbb{R}_+$ *is* M_p*-concave and lower semicontinuous in each variable, then*

$$(pF) \qquad \min_{y \in C} \sup_{x \in C} M_p^p(f(x,y), f(y,x); 1-t, t) \leq \sup_{z \in C} f^p(z,z),$$

for all $t \in (0,1)$.

For p an odd number, the function f may take negative values.

The "flat" version of Theorem 1 (ii) is discussed in our paper [6].

2. The KKM Lemma

The Knaster-Kuratowski-Mazurkiewicz Lemma (abbreviated, as the KKM-Lemma) is an important result in nonlinear analysis, equivalent to the Brouwer Fixed Point Theorem. Recall here its statement:

Lemma 1. (*Knaster-Kuratowski-Mazurkiewicz*). *Suppose that for every point* x *in a nonempty set* X, *of a linear Hausdorff topological space* E, *there is an associated closed subset* $M(x) \subset X$ *such that*

$$\text{co } F \subset \bigcup\nolimits_{x \in F} M(x)$$

holds for all finite subsets $F \subset X$. *Then for any finite subset* $F \subset X$ *we have*

$$\bigcap\nolimits_{x \in F} M(x) \neq \emptyset.$$

Hence if some subset $M(z)$ *is compact, we have*

$$\bigcap\nolimits_{x \in X} M(x) \neq \emptyset.$$

The proof of the KKM Lemma follows from the basic fact that the convex hull $\operatorname{co} F$, of any finite set F, lies in a finite dimensional space and thus it is also compact. This makes possible to apply the Brouwer fixed point theorem and to conclude that $\operatorname{co} F$ has the fixed point property. See [2], pp. 185-186. Recall that a topological space K has the *fixed point property* if every continuous map $f : K \to K$ has a fixed point.

In the context of global NPC spaces we will adopt a similar strategy, based on the remark that in a locally compact global NPC space, the closed convex hull of each finite family of points has the fixed point property. As a consequence, in a global NPC space *every compact convex set has the fixed point property* (and this fact can be used to prove the analogue of the Schauder Fixed Point Theorem).

Recall that the notion of a *convex hull* is introduced via the formula

$$\text{co } F = \bigcup\nolimits_{n=0}^{\infty} F_n,$$

where $F_0 = F$ and for $n \geq 1$ the set F_n consists of all points in E which lie on geodesics which start and end in F_{n-1}.

Lemma 2. *The KKM Lemma extends to any global NPC space E, provided that the closed convex hull of every nonempty finite family of points of E has the fixed point property.*

Proof. We will concentrate here on the case where some of the sets $M(x)$ are compact.

Assuming $\bigcap_{x\in X} M(x) = \emptyset$, this yields the existence of a finite family of points $x_1, \ldots, x_N \in X$ such that

$$\bigcap\nolimits_{i=1}^{N} M(x_i) = \emptyset.$$

Then the map $x \to \mu_x = \sum_{i=1}^{N} d(x, M(x_i))\delta_{x_i} / \sum_{i=1}^{N} d(x, M(x_i))$ is continuous (from E into $\mathcal{P}^1(E)$) and $\operatorname{supp} \mu_x \subset K = \overline{\text{co}}\ \{x_1, \ldots, x_N\}$.

According to our hypothesis, the composite map $P : x \to \mu_x \to b_{\mu_x}$ should have a fixed point $\bar{x} \in K$. Via a permutation, we may assume that $d(\overline{x}, M(x_i)) > 0$ for $i = 1, ..., j$ and $d(\overline{x}, M(x_i)) = 0$ for $i > j$. This shows that actually $\overline{x} \in \overline{\text{co}}\ \{x_1, \ldots, x_j\} \subset \bigcup_{i=1}^{j} M(x_i)$. Equivalently, $\overline{x} \in M(x_i)$ for some $i \leq j$, a fact that contradicts the choice of j. Therefore the intersection $\bigcap_{x\in X} M(x)$ is nonempty. □

3. Proof of the main result

We actually prove a much more general result:

Theorem 2. *Suppose that $p \in \mathbb{R}$ and $f : C \times C \to \mathbb{R}_+$ is a function which is M_p-concave and lower semicontinuous in each variable. Then for every continuous affine onto function $g : C \to C$ and every $t \in (0, 1)$,*

$$\min_{y\in C} \sup_{x\in C} M_p^p(f(x, y), f(y, x); 1 - t, t) \leq \sup_{z\in C} M_p^p(f(z, g(z)), f(g(z), z); 1 - t, t).$$

This result has a straightforward variant for the M_p-convex functions which are upper semicontinuous with respect to each variable.

Recall that a function $g : X \to Y$ between geodesic metric spaces is called *affine* if it maps the geodesics to geodesics. For more details, see [5].

Theorem 1 represents the particular case where g is the identity of C.

Proof. We attach to each $t \in [0, 1]$ a family of sets $(M(g(x))_{x\in C}$, where $M(g(x))$ consists of all $y \in C$ such that

$$M_p^p(f(x, y), f(y, x); 1 - t, t) \leq \sup_{z} M_p^p(f(z, g(z)), f(g(z), z); 1 - t, t).$$

We will show that this family satisfies the hypothesis of Lemma 2. In fact, $g(x) \in M(g(x))$ for every $x \in C$ and

$$\text{co } F \subset \bigcup_{x\in F} M(g(x))$$

for every finite subset $F \subset C$. For example, if F consists of two elements x_1 and x_2, we have to show that the geodesic β joining the points $g(x_1)$ and $g(x_2)$ verifies

$$\beta_\theta \in M(g(x_1)) \cup M(g(x_2)) \tag{3.1}$$

for every $\theta \in (0,1)$. Our argument is by reductio ad absurdum.

If (3.1) fails, then for some $\theta \in (0,1)$ we have

$$M_p^p(f(x_1,\beta_\theta), f(\beta_\theta, x_1); 1-t, t) > \sup_z M_p^p(f(z, g(z)), f(g(z), z); 1-t, t), \tag{3.2}$$

and

$$M_p^p(f(x_2,\beta_\theta), f(\beta_\theta, x_2); 1-t, t) > \sup_z M_p^p(f(z, g(z)), f(g(z), z); 1-t, t). \tag{3.3}$$

The Intermediate Value Theorem yields an element γ_{θ_1} of the geodesic γ, joining x_1 and x_2, such that

$$g(\gamma_{\theta_1}) = \beta_\theta.$$

Since f is M_p-concave in each variable, it follows that the number

$$M_p^p(f(\gamma_{\theta_1}, g(\gamma_{\theta_1})), f(g(\gamma_{\theta_1}), \gamma_{\theta_1}); 1-t, t)$$

exceeds

$$\begin{aligned}
&(1-t)((1-\theta_1)f^p(x_1,\beta_\theta) + \theta_1 f^p(x_2,\beta_\theta)) + t((1-\theta_1)f^p(\beta_\theta,x_1) + \theta_1 f^p(\beta_\theta,x_2)) \\
&= (1-\theta_1)((1-t)f^p(x_1,\beta_\theta) + tf^p(\beta_\theta,x_1)) + \theta_1((1-t)f^p(x_2,\beta_\theta) + tf^p(\beta_\theta,x_2)) \\
&= (1-\theta_1)M_p^p(f(x_1,\beta_\theta), f(\beta_\theta,x_1); 1-t,t) + \theta_1 M_p^p(f(x_2,\beta_\theta), f(\beta_\theta,x_2); 1-t,t) \\
&\qquad > (1-\theta_1)\sup_z M_p^p(f(z,g(z)), f(g(z),z); 1-t,t) \\
&\qquad\quad + \theta_1 \sup_z M_p^p(f(z,g(z)), f(g(z),\alpha_t); 1-t,t) \\
&\qquad\qquad = \sup_z M_p^p(f(z,g(z)), f(g(z),z); 1-t,t),
\end{aligned}$$

which is a contradiction. Thus (3.1) follows.

By Lemma 2 we infer that $\bigcap_x M(g(x)) \neq \emptyset$, which yields the existence of $y \in C$ such that

$$M_p^p(f(x,y), f(y,x); 1-t,t) \le \sup_z M_p^p(f(z,g(z)), f(g(z),z); 1-t,t),$$

for every $x \in C$, or equivalently,

$$\sup_x M_p^p(f(x,y), f(y,x); 1-t,t) \le \sup_z M_p^p(f(z,g(z)), f(g(z),z); 1-t,t).$$

In conclusion,

$$\min_y \sup_x M_p^p(f(x,y), f(y,x); 1-t,t) \le \sup_z M_p^p(f(z,g(z)), f(g(z),z); 1-t,t).$$

□

4. Further results

As above, E denotes a global NPC space.

The following nonsymmetric version of Theorem 2 can be proved in a similar manner:

Theorem 3. *Let C_1 and C_2 be two nonempty compact and convex subsets of E, and let g be a continuous affine onto function $g : C_1 \to C_2$. Then for every function $f : C_1 \times C_2 \to \mathbb{R}_+$ which is quasi-concave in the first variable and lower semicontinuous in the second variable, the following inequality holds:*

$$\min_{x\in C_1} \sup_{y\in C_2} f(x,y) \le \sup_{z\in C_1} f(z, g(z)).$$

If $f : C_1 \times C_2 \to \mathbb{R}_+$ is quasi-convex with respect to the second variable and upper semicontinuous in the first variable, then

$$\max_{x \in C_1} \inf_{y \in C_2} f(x, y) \geq \inf_{z \in C_1} f(z, g(z)).$$

Proof. In the first case, apply Lemma 2 to the following family of sets:

$$M(g(x)) = \{y \in C_2 : f(x, y) \leq \sup_{z \in C_1} f(z, g(z))\}, \quad \text{for all } x \in C_1.$$

□

An important application of Theorem 3 is the existence of a g-equilibrium, a fact that generalizes the well known result on the Nash equilibrium:

Theorem 4. *Let $C = C_1 \times C_2 \times ... \times C_n$ be a Cartesian product of n nonempty compact and convex subsets of E, let $g = (g_1, g_2, ..., g_n) : C \to C$ be a continuous affine onto function and let $f_1, \ldots, f_n : C \to C$ be lower semicontinuous functions such that each of the maps $x_i \to f_i(y_1, ..., g_i(x_i), ..., y_n)$ $(i = 1, ..., n)$ is quasi-convex for every $y \in C$. Then there exists an $\bar{y} \in C$ such that*

$$f_i(\bar{y}) \leq f_i(\bar{y}_1, ..., g_i(x_i), ..., \bar{y}_n)),$$

for every $x_i \in C_i$, $i = 1, ..., n$.

Proof. Let $f(x, y) = \sum_{i=1}^{n} (f_i(y) - f_i(y_1, ..., g_i(x_i), ..., y_n))$. It is easy to see that f satisfies the assumptions of Theorem 3. This yields an $\bar{y} \in C$ such that

$$\sup_{x \in C} f(x, \bar{y}) \leq \sup_{z \in C} f(z, g(z)) = 0.$$

Letting $x = (\bar{y}_1, ..., x_i, ..., \bar{y}_n)$ $(i = 1, ..., n)$ in the last inequality we conclude that

$$f_i(\bar{y}) - f_i(\bar{y}_1, ..., g_i(x_i), ..., \bar{y}_n)) \leq 0$$

for every $x_i \in C_i$, $i = 1, ..., n$. □

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References

[1] W. Ballmann, Lectures on spaces with nonpositive curvature, DMV Seminar Band **25**, Birkhäuser Verlag, Basel, 2005.

[2] J. M. Borwein and A. S. Lewis, Convex Analysis and Nonlinear Optimization. Theory and Examples. Springer-Verlag, 2000.

[3] M. R. Bridson and A. Haefliger, Metric spaces of non-positive curvature, Grundlehren der Mathematischen Wissenschaften vol. 319, Springer-Verlag, 1999.

[4] J. Jost, Nonpositive curvature: geometric and analytic aspects, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 1997.

[5] A. Lytchak and V. Schroeder, Affine functions on Cat(k) spaces, Math. Z. 255 (2007), 231-244.

[6] C. P. Niculescu and I. Rovenţa, Fan's inequality in the context of M_p-convexity, in vol. Applied Analysis and Differential Equations, Proc. ICAADE 2006 (Ovidiu Carja and Ioan I. Vrabie editors), pp. 267-274, World Scientific, Singapore, 2007.

[7] K. T. Sturm, Probability measures on metric spaces of nonpositive curvature, in vol.: Heat kernels and analysis on manifolds, graphs, and metric spaces (Pascal Auscher et al. editors). Lecture notes from a quarter program on heat kernels, random walks, and analysis on manifolds and graphs, April 16–July 13, 2002, Paris, France. In: Contemp. Math., vol. 338, Amer. Math. Soc., Providence, RI, 2003, pp. 357–390.

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