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An ergodic characterization of uniformly exponentially stable evolution families

by
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Abstract

Suppose that $\varphi : [0, \infty) \to [0, \infty)$ is a nondecreasing function such that $\lim_{t\to\infty} \varphi(t) = \infty$ and $\mathcal{U} = \{U(t, s)\}_{t\geq s\geq 0}$ is an exponentially bounded evolution family on a Banach space X having the property that for each $x \in X$ and each $s \geq 0$ the map $t \to ||U(s + t, s)x||$ is continuous on $(0, \infty)$. Then $\mathcal{U}$ is uniformly exponentially stable if and only if there exist two positive constants α and C such that

$$\frac{1}{t}\int_0^t \varphi(e^{\alpha\tau}||U(s + \tau, s)x||)d\tau \leq \varphi(C||x||).$$

for all $t > 0$, $s \geq 0$ and $x \in X$.

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The aim of this paper is to discuss the long time behavior of some evolution families of bounded linear operators.

Given a Banach space X, we will denote by $L(X)$ the Banach algebra of all bounded linear operators $A : X \to X$. We say that A is *power bounded* if

$$\sup_{n\in\mathbb{N}} \|A^n\| < \infty,$$

and that A is *power-stable* if

$$\|A^n\| \leq Ce^{-\omega n} \quad \text{for every } n \in \mathbb{N},$$

where C and ω are suitable positive constants. A well known result (that can be traced back to J. A. van Casteren [2]) asserts that A is power bounded if and only if for some pair (equivalently, for all pairs) of numbers $p, q \in (1, \infty)$ with $1/p + 1/q = 1$, the following two conditions are fulfilled:

$$\sup_{n \geq 1} \frac{1}{n} \sum_{k=0}^{n-1} \left\| A^k x \right\|^p < \infty \quad \text{for every } x \in X \tag{1}$$

and

$$\sup_{n \geq 1} \frac{1}{n} \sum_{k=0}^{n-1} \left\| A^{\star k} x^\star \right\|^q < \infty \quad \text{for every } x^\star \in X^\star. \tag{2}$$

Simple examples show that the conditions (1) and (2) are independent. See [2]. A consequence of the above result is that an operator $A \in L(X)$ is power-stable if and only if for some pair (equivalently, for all pairs) of numbers $p, q \in (1, \infty)$ with $1/p + 1/q = 1$, and some $\alpha > 0$ the following two conditions are fulfilled:

$$\sup_{n \geq 1} \frac{1}{n} \sum_{k=0}^{n-1} e^{\alpha p k} \left\| A^k x \right\|^p < \infty \quad \text{for every } x \in X, \tag{3}$$

and

$$\sup_{n \geq 1} \frac{1}{n} \sum_{k=0}^{n-1} e^{\alpha q k} \left\| A^{\star k} x^\star \right\|^q < \infty \quad \text{for every } x^\star \in X^\star. \tag{4}$$

Surprisingly, this remark can be considerably improved. Indeed, the condition (3) implies the condition (4), and a much more general result works for the evolution families of bounded linear operators. To see that, we need a preparation.

A *semigroup* $\mathcal{T} = (T(t))_{t \geq 0}$ is a family of operators $T(t) \in L(X)$ such that $T(0) = I$ (the identity of $L(X)$) and $T(t+s) = T(t) \circ T(s)$ for all $s, t \geq 0$. $\mathcal{T}$ is said to be *uniformly exponentially stable* if

$$\omega_0(\mathcal{T}) := \inf_{t>0} \frac{\ln ||T(t)||}{t} < 0.$$

In this case there exist positive constants M and ν such that

$$||T(t)|| \leq M e^{-\nu t} \quad \text{for all } t \geq 0.$$

An important result due to R. Datko [4], [5] and A. Pazy [11] asserts that a semigroup $\mathcal{T} = (T(t))_{t \geq 0}$ is uniformly exponentially stable provided that $\mathcal{T}$ verifies the following two conditions:

(SC) $\mathcal{T}$ is strongly continuous, that is, for each $x \in X$ the map $t \mapsto T(t)x$ is continuous on $[0, \infty)$;

(DP) there exists a number $p \in [1, \infty)$ such that for every $x \in X$,

$$M(p,x) := \int_0^\infty ||T(t)x||^p dt < \infty.$$

The property of strong continuity assures the finitude of the exponential growth bound $\omega_0(\mathcal{T})$. The role of (DP) is to provide a fast decay of orbits at infinity. If $M(p,x) \leq C||x||^p$ for some positive constant C, then $\omega_0(\mathcal{T}) < -1/pC$. See [10], Theorem 3.1.8.

Remind that a family $\mathcal{U} = \{U(t,s)\}_{t \geq s \geq 0}$ of bounded linear operators on a Banach space X is said to be an *evolution family* if $U(t,t) = I$ and $U(t,s) = U(t,r)U(r,s)$ for all $t \geq r \geq s \geq 0$.

An evolution family $\mathcal{U}$ is called *exponentially bounded* if there exist two constants $\omega \in \mathbb{R}$ and $M_\omega \geq 0$ such that

$$||U(t,s)|| \leq M_\omega e^{\omega(t-s)} \quad \text{for all } t \geq s \geq 0. \tag{5}$$

The family $\mathcal{U}$ is called *uniformly exponentially stable* if the inequality (5) works for a negative ω.

To each semigroup $\mathcal{T} = (T(t))_{t \geq 0}$ we can attach an evolution family via the formula $U(t,s) = T(t-s)$. Conversely, if $\{U(t,s)\}_{t \geq s \geq 0}$ is an evolution family that verifies the convolution formula

$$U(t,s) = U(t-s,0),$$

then the family $(U(t,0))_{t \geq 0}$ represents a semigroup of operators. Under this correspondence, a semigroup is exponentially bounded (respectively uniformly exponentially stable) if and only if the corresponding evolution family has that property.

Let $N : [0,\infty) \to [0,\infty)$ be a nondecreasing function satisfying $N(0+) = 0$ and $N(t) > 0$ for $t > 0$. A well-known result due to S. Rolewicz, [13], [12] (see also W. Littman [8] or van Neerven [10] in the semigroup case) asserts that an exponentially bounded evolution family $\mathcal{U} = \{U(t,s)\}_{t \geq s \geq 0}$ on a Banach space X is uniformly exponentially stable if for each $x \in X$ one has that

$$R(x) := \sup_{s \geq 0} \int_0^\infty N(||U(s+\tau,s)x||) d\tau < \infty.$$

In particular, (DP) will do the same job even for p in the range $(0,1)$.

It is worth to mention that there exist a function N as above, an uniformly exponentially stable evolution family $\mathcal{U}$, and a $x_0 \in X$ such that $R(x_0) = \infty$. See [13] for further details.

We will prove that Rolewicz's aforementioned result has an ergodic analogue that works in the framework of evolution families. However, the class of functions N is slightly different.

The following result provides an useful criterion of uniform exponential stability:

Lemma 1. *Let $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ be an exponentially bounded evolution family on a Banach space X, such that for each $x \in X$ and each $s \geq 0$, the map $t \to ||U(s+t,s)x||$ is continuous on $(0,\infty)$. Then the family $\mathcal{U}$ is uniformly exponentially stable provided that there exist two real numbers $h > 0$ and $q \in (0,1)$ such that for every $x \in X$ one can find a $t(x) \in (0,h]$ for which*

$$\sup_{s\geq 0} ||U(s+t(x),s)x|| \leq q||x||.$$

Proof: Let $x \in X$ be arbitrarily fixed. According to our hypotheses we can find a $t_1 \in (0,h]$ such that $||U(s+t_1,s)x|| \leq q||x||$. The same argument yields a $t_2 \in (0,h]$ (which depends on $U(s+t_1,t_1)x$) and verifies the condition

$$\begin{aligned} ||U(s+t_2+t_1,s)x|| &= ||U(s+t_2+t_1,s+t_1)U(s+t_1,t_1)x|| \\ &\leq q||U(s+t_1,s)x|| \leq q^2||x||. \end{aligned}$$

By mathematical induction we infer the existence of a sequence $(t_n)_n$ of elements in $(0,h]$ such that

$$||U(s+t_1+t_2+\cdots+t_n,s)x|| \leq q^n||x||,$$

for all $s \geq 0$. Put $s_n := t_1+t_2+\cdots+t_n$. If $s_n \to \infty$, then for each $t \in [s_n, s_{n+1}]$ we have $t \leq (n+1)h$ and thus

$$\begin{aligned} ||U(s+t,s)x|| &\leq ||U(s+t,s+s_n)|| \cdot ||U(s+s_n,s)x|| \leq Mq^n||x|| \\ &\leq Me^{-\ln q}e^{\ln q\cdot t/h}||x||, \end{aligned}$$

where $M := \sup_{t\in[0,h]}\sup_{s\geq 0}||U(t+s,s)||$.

If the sequence $(s_n)_n$ is bounded, then it must be convergent, say to σ. Since $||U(s+s_n,s)x|| \leq q^n||x||$ and the map $t \to ||U(s+t,s)x||$ is continuous, it follows that

$$U(s+\sigma,s)x = 0,$$

for every $s \geq 0$. Consequently $U(t+\sigma,s)x = U(t+\sigma,s+\sigma)U(s+\sigma,s)x = 0$ for every $t \geq \sigma$ and every $s \geq 0$, a fact that assures the uniform exponential stability of $\mathcal{U}$. □

A result similar to Lemma 1 for semigroups has been proved in [1], while the case of reversible evolution families was noticed in [3], pages 190-192. See also [12] for further information.

Theorem 1. *Let $\varphi, \psi : [0,\infty) \to [0,\infty)$ be two nondecreasing functions with*

$$\lim_{t\to\infty}\varphi(t) = \lim_{t\to\infty}\psi(t) = \infty$$

and let $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ be an exponentially bounded evolution family on a Banach space X such that for each $x \in X$ and each $s \geq 0$ the map $t \to ||U(s+t,s)x||$ is continuous on $(0,\infty)$. Then $\mathcal{U}$ is uniformly exponentially stable if

$$\sup_{t>0}\sup_{s\geq 0}\frac{1}{t}\int_0^t \varphi\left(\psi(\tau)||U(s+\tau,s)x||\right)d\tau<\infty \quad \textit{for all } x\in X.$$

Proof: If the condition in Lemma 1 doesn't work, then for each $h>0$ and each $0<q<1$ there must exist a norm-1 vector x_0 and a number $s_0\geq 0$ such that

$$||U(s_0+\tau,s_0)x_0||\geq q||x_0||=q \quad \text{for all } \tau\in[0,h].$$

Therefore

$$\begin{aligned} M &= \sup_{t>0}\frac{1}{t}\int_0^t \varphi(\psi(\tau)||U(s_0+\tau,s_0)x_0||)d\tau \\ &\geq \sup_{t\in(0,h]}\frac{1}{t}\int_0^t \varphi(q\psi(\tau))d\tau \end{aligned}$$

for all $h>0$, whence

$$\begin{aligned} M &\geq \sup_{t>0}\frac{1}{t}\int_0^t \varphi(q\psi(\tau))d\tau \\ &\geq \limsup_{t\to\infty}\frac{1}{t}\int_0^t \varphi(q\psi(\tau))d\tau \\ &\geq \lim_{t\to\infty}\varphi(q\psi(t))=\infty, \end{aligned}$$

by the l'Hospital Rule. This contradiction shows that Lemma 1 applies and thus $\mathcal{U}$ is uniformly exponentially stable. ∎

Corollary 1. *Let ϕ and $\mathcal{U}$ be as in Theorem 1. Then $\mathcal{U}$ is uniformly exponentially stable if and only if there exist two positive constants α and C such that*

$$\sup_{t>0}\sup_{s\geq 0}\frac{1}{t}\int_0^t \varphi(e^{\alpha\tau}||U(s+\tau,s)x||)d\tau\leq\varphi(C||x||) \quad \textit{for all } x\in X.$$

Proof: Let $\nu>0$ and $N>0$ such that

$$||U(s+\tau,s)||\leq Ne^{-\nu\tau} \quad \text{for all } s\geq 0,\ \tau\geq 0.$$

and fix arbitrarily $\alpha\in(0,\nu]$ and $C\geq N$. Then for each $x\in X$ we have

$$\frac{1}{t}\int_0^t \phi(e^{\alpha\tau}||U(s+\tau,s)x||)d\tau\leq\frac{1}{t}\int_0^t \phi(C||x||)d\tau=\phi(C||x||)$$

whenever $t>0$ and $s\geq 0$. The conclusion follows now from Theorem 1. ∎

It is well-known that the strong measurability of a semigroup $\mathcal{T}$ is equivalent with its strong continuity on $(0,\infty)$. See [7]. Then the above Theorem 1 can be applied to a large class of semigroups. The next theorem shows that the assumption on the continuity on $(0,\infty)$ of all the maps $t \to \|T(t)x\|$ $(x \in X)$ can be replaced by the assumption on exponential boundedness.

Theorem 2. *Suppose that $\varphi : [0,\infty) \to [0,\infty)$ is a nondecreasing function with $\lim_{t\to\infty} \varphi(t) = \infty$, and $\mathcal{T} = (T(t))_{t\geq 0}$ is an exponentially bounded semigroup of bounded linear operators. Then $\mathcal{T}$ is uniformly exponentially stable if and only if there exist two positive real numbers α and C such that*

$$\sup_{t>0} \frac{1}{t} \int_0^t \varphi\left(e^{\alpha\tau} \|T(\tau)x\|\right) d\tau \leq \varphi\left(C\|x\|\right) \quad \text{for all } x \in X. \tag{6}$$

The measurability of the map $t \to \|T(t)x\|$ (for each $x \in X$) follows from the exponential boundedness of the semigroup. See [9] for details. In connection with this remark, let us recall here that the dual of a strongly continuous semigroup $\mathcal{T} = \{T(t)\}_{t\geq 0}$ is not necessarily strongly continuous. However, since $\mathcal{T}$ verifies an estimate of the form

$$||T(t)|| \leq Me^{\omega t} \quad \text{for all } t \geq 0,$$

(for suitable constants $M \geq 1$ and $\omega \in \mathbb{R}$), then a similar estimate works for $\mathcal{T}^* = \{T(t)^*\}_{t\geq 0}$, which assures that the map $t \mapsto ||T(t)^*x^*||$ is measurable for each $x^* \in X^*$.

The proof of Theorem 2 is based on the following fact that can be found in [9]: *If $\mathcal{T} = \{T(t)\}_{t\geq 0}$ is exponentially bounded and is not uniformly exponentially stable, then there exists a positive constant K such that for all complex-valued function $\gamma \in C_0(\mathbb{R}_+)$ with $\sup_{t\geq 0} |\gamma(t)| = 1$ one can find a norm-1 vector $x_0 \in X$ with*

$$||T(t)x_0|| \geq K\,|\gamma(t)| \quad \text{for all } t \geq 0.$$

In this case, by (6),

$$\begin{aligned}
\varphi\left(C\|x_0\|\right) &= \varphi(C) \geq \sup_{t>0} \frac{1}{t} \int_0^t \varphi\left(e^{\alpha\tau} \|T(\tau)x_0\|\right) d\tau \\
&\geq \sup_{t>0} \frac{1}{t} \int_0^t \varphi\left(Ke^{\alpha\tau} |\gamma(\tau)|\right) d\tau \\
&\geq \limsup_{t\to\infty} \frac{1}{t} \int_0^t \varphi\left(Ke^{\alpha\tau} |\gamma(\tau)|\right) d\tau \\
&\geq \lim_{t\to\infty} \varphi\left(Ke^{\alpha t} |\gamma(t)|\right),
\end{aligned}$$

by l'Hospital Rule. Or, for a suitable choice of γ, we may have

$$\lim_{t\to\infty} \varphi\left(Ke^{\alpha t} |\gamma(t)|\right) = \infty.$$

This shows that $\mathcal{T}$ should be uniformly exponentially stable.

Corollary 2. *Suppose that $\varphi : [0,\infty) \to [0,\infty)$ is a nondecreasing function with $\lim_{t\to\infty}\varphi(t) = \infty$, and $\mathcal{T} = (T(t))_{t\geq 0}$ is an exponentially bounded semigroup of bounded linear operators. Then $\mathcal{T}$ is uniformly exponentially stable if and only if there exists a positive real number α such that*

$$\sup_{t>0}\frac{1}{t}\int_0^t \varphi\left(e^{\alpha\tau}\left\|T^\star(\tau)x^\star\right\|\right)d\tau \leq \varphi\left(C\left\|x^\star\right\|\right) \quad \textit{for all } x^\star \in X^\star.$$

The discrete versions of Theorem 2 and of Corollary 2 yield the fact that an operator $A \in L(X)$ is power stable if and only if it verifies one of the conditions (3) and (4).

The natural assumption in Theorem 2 is the measurability of all the maps $t \mapsto ||T(t)x||$, for $x \in X$. We leave open the question whether Theorem 2 still works when the condition of exponential boundedness is replaced by this apparently weaker condition.

In this connection it is worth to mention the following easy consequence of the Hölder inequality:

Let $\mathcal{T} = \{T(t)\}_{t\geq 0}$ be a semigroup of operators acting on X, and let $p, q \in (1,\infty)$ with $1/p + 1/q = 1$. We assume that for each $x \in X$ and each $x^* \in X^*$ the maps $t \mapsto \|T(t)x\|$ and $t \mapsto \|T(t)^*x^*\|$ are measurable and moreover there exists a positive real number α such that

$$\sup_{t>0}\frac{1}{t}\int_0^t e^{p\alpha s}||T(s)x||^p ds < \infty$$

and

$$\sup_{t>0}\frac{1}{t}\int_0^t e^{q\alpha s}||T(s)^*x^*||^q ds < \infty.$$

Then the semigroup $\mathcal{T}$ is uniformly exponentially stable.

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