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# The Hermite–Hadamard inequality for convex functions on a global NPC space

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ABSTRACT

We prove an extension of Choquet's theorem to the framework of compact metric spaces with a global nonpositive curvature. Together with Sturm's extension [K.T. Sturm, Probability measures on metric spaces of nonpositive curvature, in: Pascal Auscher, et al. (Eds.), Heat Kernels and Analysis on Manifolds, Graphs, and Metric Spaces, Lecture Notes from a Quarter Program on Heat Kernels, Random Walks, and Analysis on Manifolds and Graphs April 16–July 13, 2002, Paris, France, in: Contemp. Math., vol. 338, Amer. Math. Soc., Providence, RI, 2003, pp. 357–390] of Jensen's inequality, this provides a full analogue of the Hermite–Hadamard inequality for the convex functions defined on such spaces.



## 1. Introduction

According to the classical Hermite–Hadamard inequality, the mean value of a continuous convex function $f:[a,b]\to\mathbb{R}$ lies between the value of $f$ at the midpoint of the interval $[a,b]$ and the arithmetic mean of the values of $f$ at the endpoints of this interval, that is,

$$f\left(\frac{a+b}{2}\right)\leqslant\frac{1}{b-a}\int_a^b f(x)\,dx\leqslant\frac{f(a)+f(b)}{2}. \tag{HH}$$

Choquet's theory offers a considerable insight into this matter, based on the fact that $(a+b)/2$ is the barycenter of the interval $[a,b]$ (with respect to the uniform distribution of mass $\frac{dx}{b-a}$), and the right-hand side of (HH) represents the mean value of $f$ with respect to a probability measure $\lambda=\frac{1}{2}\delta_a+\frac{1}{2}\delta_b$, supported on the extreme points of the interval $[a,b]$. For the convenience of the reader we briefly recall here the main facts concerning this theory. Full details are available in [9].

Suppose that $K$ is a compact convex subset of a locally convex Hausdorff space $E$. The *barycenter* of a probability measure $\mu$ on $K$ (that is, of a nonnegative regular Borel measure on $K$ for which $\mu(K)=1$), is defined as the unique point $b_\mu$ of $K$ such that

$$x'(b_\mu)=\int_K x'(x)\,d\mu(x) \tag{B}$$

for all continuous linear functionals $x'$ on $E$. Since (B) still works for all continuous affine functions on $K$, it follows that

$$f(b_\mu)\leqslant\int_K f(x)\,d\mu(x),$$

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for all continuous convex functions $f : K \to \mathbb{R}$ (a fact which extends the left-hand side of the classical Hermite–Hadamard inequality). The right-hand side is generalized by the following theorem due to Choquet, which relates the geometry of $K$ to the mass distribution.

**Theorem 1.** *Let $\mu$ be a probability measure on a metrizable compact convex subset $K$ of a locally convex Hausdorff space. Then there exists a probability measure $\lambda$ on $K$ which has the same barycenter as $\mu$, is null outside* Ext $K$ *and verifies the inequality*

$$\int_K f(x)\,d\mu(x) \leqslant \int_{\operatorname{Ext} K} f(x)\,d\lambda(x),$$

*for all continuous convex functions $f : K \to \mathbb{R}$.*

*Here* Ext $K$ *denotes the set of all extreme points of $K$.*

Using the technique of pushing-forward measures, we can put the Hermite–Hadamard inequality in a more general form, that encompasses Jensen's inequality. In fact, if $(X, \Sigma, \nu)$ is a finite measure space (on an abstract set $X$) and $T : X \to K$ is a $\nu$-integrable map, then the push-forward measure $\mu = T\#\nu$ is given by the formula $\mu(A) = \nu(T^{-1}(A))$ and the formula (HH) becomes

$$f(\overline{T}) \leqslant \frac{1}{\nu(X)} \int_X f\bigl(T(x)\bigr)\,d\nu(x) = \frac{1}{\mu(K)} \int_K f(t)\,d\mu(t) \leqslant \int_{\operatorname{Ext} K} f(x)\,d\lambda(x). \tag{1.1}$$

Here

$$\overline{T} = \frac{1}{\nu(X)} \int_X T(x)\,d\nu(x)$$

represents the barycenter of $\mu$.

The aim of this paper is to discuss an analogue of the Hermite–Hadamard inequality for the continuous convex functions defined on a space with curved geometry, more precisely on a metric space with global nonpositive curvature.

**Definition 1.** A global NPC space is a complete metric space $M = (M, d)$ for which the following inequality holds true: for each pair of points $x_0, x_1 \in M$ there exists a point $y \in M$ such that for all points $z \in E$,

$$d^2(z, y) \leqslant \frac{1}{2}d^2(z, x_0) + \frac{1}{2}d^2(z, x_1) - \frac{1}{4}d^2(x_0, x_1). \tag{NPC}$$

In a global NPC space each pair of points $x_0, x_1 \in E$ can be connected by a geodesic (that is, by a rectifiable curve $\gamma : [0, 1] \to E$ such that the length of $\gamma|_{[s,t]}$ is $d(\gamma(s), \gamma(t))$ for all $0 \leqslant s \leqslant t \leqslant 1$). Moreover, this geodesic is unique. The point $y$ that appears in the inequality (NPC) is called the *midpoint* of $x_0$ and $x_1$ and has the property

$$d(x_0, y) = d(y, x_1) = \frac{1}{2}d(x_0, x_1).$$

Every Hilbert space is a global NPC space. In this case the geodesics are the line segments.

A Riemannian manifold $(M, g)$ is a global NPC space if and only if it is complete, simply connected and of nonpositive sectional curvature. Besides manifolds, other important examples of global NPC spaces are the Bruhat–Tits buildings (in particular, the trees). See [2]. More information on global NPC spaces is available in [1] and [5].

In what follows $M$ will denote a global NPC space.

**Definition 2.** A subset $C \subset M$ is called *convex* if $\gamma([0, 1]) \subset C$ for each geodesic $\gamma : [0, 1] \to C$ joining two points in $C$.

A function $f : C \to \mathbb{R}$ is called convex if the function $f \circ \gamma : [0, 1] \to \mathbb{R}$ is convex whenever $\gamma : [0, 1] \to C$, $\gamma(t) = \gamma_t$, is a geodesic, that is,

$$f(\gamma_t) \leqslant (1 - t) f(\gamma_0) + t f(\gamma_1)$$

for all $t \in [0, 1]$. The function $f$ is called concave if $-f$ is convex.

All closed convex subsets of a global NPC space are in turn spaces of the same nature. In a global NPC space, the distance from a point $z$,

$$d_z(x) = d(x, z),$$

provides a basic example of a convex function. Moreover, its square is strictly convex. See [10, Proposition 2.3 and Corollary 2.5]. As a consequence, the balls in a global NPC space are convex sets in the sense of Definition 2.

Another important feature of global NPC spaces is the possibility to introduce a well behaved concept of barycenter for a large class of probability measures. In this respect we shall adopt the approach by K.-T. Sturm in [10], that concerns the class $\mathcal{P}^1(M)$, of all probability measures $\mu$ with separable support that verify the condition

$$\int_M d(x,y)\,d\mu(y) < \infty$$

for some (hence for all) $x \in M$.

The basic remark is the property of *uniform convexity* of the function $d_z^2$:

$$d^2(\gamma_t, z) \leqslant (1-t)d^2(\gamma_0, z) + td^2(\gamma_1, z) - t(1-t)d^2(x_0, x_1)$$

for all geodesics $\gamma : [0,1] \to C$, $\gamma(t) = \gamma_t$, all points $z \in M$ and all numbers $t \in [0,1]$. Technically this represents the extension of (NPC) from the case of midpoints to that of arbitrary convex combinations.

The *barycenter* of a probability measure $\mu \in \mathcal{P}^1(M)$ is the unique point $z \in M$ which minimizes the uniformly convex function

$$F_y : z \to \int_M \left[d^2(z,x) - d^2(y,x)\right] d\mu(x); \tag{1.2}$$

this point is independent of $y \in M$ and is also denoted $b_\mu$.

If the support of $\mu$ is included in a convex closed set $K$, then $b_\mu \in K$.

**Lemma 1** *(The variance inequality). (See [10, Proposition 4.4].) For any probability measure $\mu \in \mathcal{P}^1(M)$ and any $z \in M$,*

$$\int_M \left[d^2(z,x) - d^2(b_\mu, x)\right] d\mu(x) \geqslant d^2(z, b_\mu).$$

K.-T. Sturm [10] noticed the possibility to extend Jensen's inequality in the framework of global NPC spaces as follows:

**Theorem 2** *(Jensen's inequality in NPC spaces). For any lower semicontinuous convex function $f : M \to \mathbb{R}$ and any probability measure $\mu \in \mathcal{P}^1(M)$,*

$$f(b_\mu) \leqslant \int_M f(x)\,d\mu(x),$$

*provided the right-hand side is well-defined.*

If $K$ is a convex subset of a global NPC space, a point $z \in K$ is called an *extreme point* if $z$ is not interior for any geodesic with endpoints in $K$. The following result represents the generalization of the Krein–Milman theorem to the context of global NPC spaces:

**Theorem 3.** *(See [7].) Let $K$ be a nonempty compact convex subset of a global NPC space $M$. Then $K$ is the closed convex hull of the set* $\operatorname{Ext} K$, *of all extreme points of $K$.*

*In particular,* $\operatorname{Ext} K$ *is nonempty.*

Section 2 is devoted to the notion of majorization, that provides a useful tool for studying the mass transportation in a convex domain. Based on this notion, we prove in Section 3 an extension of Theorem 1 to the context of global NPC spaces. Our approach was inspired by the case of flat spaces, as presented in [8] and [9]. Together with Sturm's aforementioned result this extension provides a full analogue of the Hermite–Hadamard inequality to the same class of spaces.

## 2. Majorization in global NPC spaces

Given a compact NPC space $K$, the set $\operatorname{Conv}(K)$, of all continuous convex functions on $K$, is a cone in the Banach lattice $C(K)$, of all real-valued continuous functions on $K$, endowed with the sup norm and the pointwise ordering. Moreover, $\operatorname{Conv}(K) - \operatorname{Conv}(K)$ is a vector sublattice of $C(K)$ that contains the constant functions and separates the points of $K$; the later is a consequence of the fact that the functions $d^2(\cdot, x_0)$ are strictly convex. According to M.H. Stone's characterization of sublattices of a space $C(K)$, it follows that $\operatorname{Conv}(K) - \operatorname{Conv}(K)$ is dense into $C(K)$. See M.M. Day [3, Theorem 1, p. 133], for details.

If $\lambda$ and $\mu$ are nonnegative regular Borel measures on $K$, then we say that $\mu$ is *majorized* by $\lambda$ (equivalently, $\mu \prec \lambda$) if

$$\int_K f\,d\mu \leqslant \int_K f\,d\lambda$$

for all continuous convex functions $f$ on $M$. Since $\pm 1 \in \mathrm{Conv}(K)$, then necessarily $\mu(K) = \lambda(K)$ for $\mu \prec \lambda$.

Clearly, the relation $\prec$ is reflexive and transitive. It is also antisymmetric (and thus a partial ordering) by M.H. Stone's aforementioned result.

**Remark 1.** Unlike the case of flat spaces, it is possible that $\mu \prec \lambda$, though $\lambda$ and $\mu$ have different barycenters.

In fact, for every convex combination $\sum_{k=1}^n a_k x_k$ we have

$$\delta_{\sum_{k=1}^n a_k x_k} \prec \mu = \sum_{k=1}^n a_k \delta_{x_k}$$

while $b_\mu$ is not necessarily $\sum_{k=1}^n a_k x_k$. An example is provided by K.-T. Sturm in [10].

The following technical result allows us to approximate any probability measure $\mu$ by a weak-star convergent sequence of discrete probability measures which are majorized by $\mu$.

**Lemma 2.** *Every probability measure $\mu$ on $K$ is the pointwise limit of a sequence of discrete probability measures $\mu_n$ on $K$, such that $\mu_n \prec \mu$ for all $n$.*

**Proof.** We have to prove that for every $\varepsilon > 0$ and every finite family $f_1, \dots, f_n$ of continuous real functions on $K$ there exists a discrete probability measure $\nu$ such that

$$b_\nu = b_\mu \quad \text{and} \quad \sup_{1 \leqslant k \leqslant n} \left|\nu(f_k) - \mu(f_k)\right| < \varepsilon.$$

As $K$ is compact and convex and the $f_k$'s are continuous, there exists a finite covering $(D_\alpha)_\alpha$ of $K$ by open convex sets such that the oscillation of each of the functions $f_k$ on each set $D_\alpha$ is $< \varepsilon$. Let $(\varphi_\alpha)_\alpha$ be a partition of the unity, subordinated to the covering $(D_\alpha)_\alpha$ and put

$$\nu = \sum_\alpha \left( \int_K \varphi_\alpha\,d\mu \right) \delta_{x(\alpha)},$$

where $x(\alpha)$ is the barycenter of the measure $f \to (\int_K f\varphi_\alpha\,d\mu)/(\int_K \varphi_\alpha\,d\mu)$. Since $D_\alpha$ is convex and the support of $\varphi_\alpha$ is included in $D_\alpha$, then necessarily $x(\alpha) \in \overline{D}_\alpha$ (see [10, Proposition 6.1]). On the other hand,

$$\int_K h\,d\mu = \sum_\alpha \int_K h\varphi_\alpha\,d\mu = \sum_\alpha \frac{\int_K h\varphi_\alpha\,d\mu}{\int_K \varphi_\alpha\,d\mu} \int_K \varphi_\alpha\,d\mu \geqslant \sum_\alpha h\big(x(\alpha)\big) \int_K \varphi_\alpha\,d\mu = \int_K h\,d\nu$$

for every continuous convex function $h : K \to \mathbb{R}$, that is, $\nu \prec \mu$. In order to conclude the proof it remains to notice that for each index $k$ the following estimate holds true:

$$\begin{aligned}
\left| \int_K f_k\,d\nu - \int_K f_k\,d\mu \right| &= \left| \sum_\alpha f_k\big(x(\alpha)\big) \int_K \varphi_\alpha\,d\mu - \sum_\alpha \int_K f_k\varphi_\alpha\,d\mu \right| \\
&= \left| \sum_\alpha \left[ f_k\big(x(\alpha)\big) - \int_K f_k\varphi_\alpha\,d\mu \Big/ \int_K \varphi_\alpha\,d\mu \right] \int_K \varphi_\alpha\,d\mu \right| \\
&\leqslant \varepsilon \cdot \sum_\alpha \int_K \varphi_\alpha\,d\mu = \varepsilon. \qquad \square
\end{aligned}$$

In the case of flat spaces, all approximates $\mu_n$ have the same barycenter as $\mu$, a fact that yields Jensen's inequality. We do not know whether this property remains true or not in the general case.

In connection with Lemma 2 it arises the important problem of finding a simple characterization of the relation of majorization in the case of discrete probability measures. In the Euclidean case, an inequality of the form

$$\frac{1}{n} \sum_{k=1}^n \delta_{x_k} \prec \frac{1}{n} \sum_{k=1}^n \delta_{y_k} \tag{2.1}$$

means the existence of a double stochastic matrix $(a_{ij})_{i,j=1}^n$ such that

$$x_i = b_{\sum_{j=1}^n a_{ij}\delta_{y_j}} \quad \text{for all } i = 1, \dots, n \tag{2.2}$$

(modulo a permutation of the indices $i$). What we only know in the general case of global NPC spaces is that the existence of such a matrix for which (2.2) holds true forces (2.1).

## 3. The extension of Choquet's theorem

The core of Choquet's theorem admits an extension to the framework of NPC spaces. As in the case of locally convex spaces, we need to attach to each function $f \in C(K)$ its *upper envelope*,

$$\overline{f}(x) = \inf\{h(x):\ h \in C(K),\ h \text{ concave},\ h \geqslant f\},$$

which is concave, bounded and upper semicontinuous (that is, for every real number $\alpha$, the set $\{x:\ \overline{f}(x) < \alpha\}$ is open). Moreover,

(UE1) $f \leqslant \overline{f}$ and $f = \overline{f}$ if $f$ is concave;
(UE2) the map $f \to \overline{f}$ is sublinear.

**Theorem 4.** *Let $\mu$ be a probability measure on a compact convex subset $K$ of a global NPC space $M = (M, d)$. Then there exists a probability measure $\lambda$ on $K$ such that the following two conditions hold*:

(i) $\lambda \succ \mu$;
(ii) *the set* $\operatorname{Ext} K$*, of all extreme points of $K$, is a Borel set and $\lambda$ is supported by* $\operatorname{Ext} K$ (*that is,* $\lambda(K \setminus \operatorname{Ext} K) = 0$).

Under the hypotheses of Theorem 4 we get

$$f(x_\mu) \leqslant \frac{1}{\mu(K)} \int_K f(x)\,d\mu(x) \leqslant \int_{\operatorname{Ext} K} f(x)\,d\lambda(x) \tag{Ch}$$

for every continuous convex function $f : K \to \mathbb{R}$, a fact that represents a full extension of the Hermite–Hadamard inequality (HH) to the context of global *NPC* spaces.

The right-hand side of the formula (Ch) easily yields the Krein–Milman Theorem,

$$K = \overline{\operatorname{conv}}(\operatorname{Ext} K).$$

In fact, consider the continuous function $x \to d(x, \overline{\operatorname{conv}}(\operatorname{Ext} K))$, whose property of being convex is assured by the geodesic comparison, Corollary 2.5 in [10].

Unlike the case of flat spaces, it is possible that every measure $\lambda \succ \mu$ supported by $\operatorname{Ext} K$ have a barycenter different from the barycenter of $\mu$, see Remark 1.

**Proof.** *Step* 1. First notice that $\operatorname{Ext} K$ is a countable intersection of open sets in the relative topology of $K$ (in particular, it is a Borel set). In fact,

$$\operatorname{Ext} K = K \setminus \bigcup_{n=0}^{\infty} K_n,$$

where for each $n$, the set $K_n$ consists of all midpoints $x = \gamma(1/2)$ of the geodesics $\gamma$ joining points $y, z \in K$ with $d(y,z) \geqslant 1/2^n$. An easy compactness argument shows that the sets $K_n$ are closed. Therefore, $\operatorname{Ext} K = K \cap \bigcap_n \complement K_n$ is a Borel set.

*Step* 2. We may choose a maximal probability measure $\lambda \succ \mu$. To show that Zorn's lemma may be applied, consider a chain $\mathcal{C} = (\lambda_\alpha)_\alpha$ in

$$\mathcal{P} = \{\lambda:\ \mu \prec \lambda,\ \lambda \text{ probability measure on } K\}.$$

We may regard $\mathcal{C}$ as a net (the directed index set being the elements of $\mathcal{C}$). According to the Riesz representation theorem, $(\lambda_\alpha)_\alpha$ is contained in the weak-star compact set

$$W = \{\lambda:\ \lambda \in C(K)',\ \lambda \geqslant 0,\ \lambda(1) = 1\},$$

which assures the existence of a subnet $(\lambda_\beta)_\beta$ which converges to a probability measure $\tilde{\lambda} \in W$ (in the weak-star topology). Clearly, $\mu \prec \tilde{\lambda}$. If $\lambda_\alpha$ is any element in $\mathcal{C}$, it follows from the definition of a subnet that eventually $\lambda_\beta \succ \lambda_\alpha$ and hence $\tilde{\lambda} \succ \lambda_\alpha$. Thus $\tilde{\lambda}$ is an upper bound for $\mathcal{C}$. By Zorn's lemma, $\mathcal{P}$ contains a maximal element, say $\lambda$.

It remains to prove that $\lambda$ does the job.

*Step* 3. Since $M$ is an NPC space, for each $x_0 \in M$ arbitrarily fixed, the distance function

$$\varphi(\cdot) = d^2(\cdot, x_0)$$

is continuous and strictly convex, from which it follows that

$$\mathcal{E} = \{x \in K\colon\ \varphi(x) = \overline{\varphi}(x)\} \subset \operatorname{Ext} K.$$

In fact, if $x$ is the midpoint of a geodesic joining two distinct points $y$ and $z$ of $K$, then the strict convexity of $\varphi$ implies that

$$\varphi(x) < \varphi(y)/2 + \varphi(z)/2 \leqslant \overline{\varphi}(y)/2 + \overline{\varphi}(z)/2 \leqslant \overline{\varphi}(x).$$

*Step* 4. As a consequence of the maximality of $\lambda$, we shall show that

$$\int_K \varphi\, d\lambda = \int_K \overline{\varphi}\, d\lambda. \tag{3.1}$$

Then $\overline{\varphi} - \varphi \geqslant 0$ and $\int_K (\overline{\varphi} - \varphi)\, d\lambda = 0$, so that $\lambda$ is indeed supported by $\mathcal{E}$.

Consider the sublinear functional $q : C(K) \to \mathbb{R}$, given by $q(f) = \int_K \overline{f}\, d\lambda$, and the linear functional $L$ defined on $\mathbb{R} \cdot \varphi$ by $L(\alpha\varphi) = \alpha \int_K \overline{\varphi}\, d\lambda$. If $\alpha \geqslant 0$, then $L(\alpha\varphi) = q(\alpha\varphi)$, while if $\alpha < 0$, then

$$0 = \overline{\alpha\varphi - \alpha\varphi} \leqslant \overline{\alpha\varphi} + \overline{(-\alpha\varphi)} = \overline{\alpha\varphi} - \alpha\overline{\varphi},$$

which shows that

$$L(\alpha\varphi) = \int_K \alpha\overline{\varphi}\, d\lambda \leqslant \int_K \overline{\alpha\varphi}\, d\lambda = q(\alpha\varphi).$$

By the Hahn–Banach extension theorem, there exists a linear extension $\widetilde{L}$ of $L$ to $C(K)$ such that $\widetilde{L} \leqslant q$. If $f \leqslant 0$, then $\overline{f} \leqslant 0$, so that

$$\widetilde{L}(f) \leqslant q(f) = \int_K \overline{f}\, d\lambda \leqslant 0.$$

Therefore $\widetilde{L} \geqslant 0$ and the Riesz representation theorem shows that $\widetilde{L}$ is the integral associated to a nonnegative measure $\nu$ on $K$. If $f$ is a continuous convex function on $K$, then $-f$ is concave and

$$\int_K (-f)\, d\nu \leqslant q(-f) = \int_K (\overline{-f})\, d\lambda = \int_K (-f)\, d\lambda,$$

that is, $\lambda \prec \nu$. Since $\lambda$ is maximal, this forces $\lambda = \nu$. Consequently,

$$\int_K \varphi\, d\lambda = \int_K \varphi\, d\nu = L(\varphi) = \int_K \overline{\varphi}\, d\lambda,$$

which ends the proof. $\square$

The upper half-plane $\mathfrak{H} = \{z \in \mathbb{C}\colon\ \operatorname{Im} z > 0\}$, endowed with the Poincaré metric,

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

constitutes an example of a global NPC space. In this case the geodesics are the semicircles in $\mathfrak{H}$ perpendicular to the real axis and the straight vertical lines ending on the real axis. The geodesic triangle $\Delta ABC$, of vertices $A$, $B$ and $C$, is a metrizable compact convex set in this geometry and $\operatorname{Ext} \Delta ABC = \{A, B, C\}$. By Theorem 4 above we infer the existence of three nonnegative numbers $\alpha, \beta, \gamma$ with $\alpha + \beta + \gamma = 1$, such that

$$\iint_{\Delta ABC} \frac{f(x,y)}{y^2}\, dx\, dy \leqslant \big(\alpha f(A) + \beta f(B) + \gamma f(C)\big) \iint_{\Delta ABC} \frac{dx\, dy}{y^2}$$

for all continuous convex functions $f : \Delta ABC \to \mathbb{R}$. Letting $G$ denote the barycenter of the triangle $\Delta ABC$, the numbers $\alpha, \beta, \gamma$ are respectively the areas of the triangles $\Delta GBC$, $\Delta GCA$ and $\Delta GAB$.

In the case of subintervals of $\mathbb{R}$ there is a fully developed generalization of the Hermite–Hadamard inequality that encompasses the case of signed measured. See [4] for details. Some progress was done to extend this theory to the case of several variables. See [8] and [6]. However, the case of convex functions defined on curved spaces remains largely open from this point of view. In fact, since the signed measures are not monotonic, even the concept of barycenter needs a new approach.

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