A NOTE ON THE BEHAVIOR OF INTEGRABLE FUNCTIONS AT INFINITY

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ABSTRACT. We discuss a scale of necessary conditions for the integrability of a function $f : [a, \infty) \to \mathbb{R}$, based on the concept of limit in density.

In 1827, L. Olivier [12] published a paper claiming that the harmonic series represents a sort of "boundary" case with which other potentially convergent series of positive terms could be compared. More precisely, he asserted that a positive series $\sum a_n$ whose terms are monotone decreasing is convergent if and only if $na_n \to 0$. One year later, Abel [1] disproved this convergence test by considering the case of the (divergent) positive series $\sum_{n\geq 2} \frac{1}{n \ln n}$. However the necessity part survived the scrutiny of Abel and became known as Olivier's Theorem:

Theorem 1. *If $\sum a_n$ is a convergent positive series and $(a_n)_n$ is monotone decreasing, then $na_n \to 0$.*

A nice account on Abel's contribution to the nonexistence of "boundary" positive series can be found in the paper of M. Goar [5].

Simple examples show that the monotonicity condition is vital for Olivier's Theorem. See the case of the series $\sum a_n$, where $a_n = \frac{\log n}{n}$ if n is a square, and $a_n = \frac{1}{n^2}$ otherwise.

In 2003, T. Šalát and V. Toma [13] made the interesting remark that the monotonicity condition in Theorem 1 can be dropped if the convergence of the sequence $(na_n)_n$ is weakened to convergence in density.

In order to explain the terminology, recall that a subset A of $\mathbb{N}$ has *zero density* if

$$d(A) = \lim_{n\to\infty} \frac{|A \cap [1, n]|}{n} = 0.$$

Here $|\cdot|$ stands for cardinality.

A sequence $(x_n)_n$ of real numbers *converges in density* to a number x (abbreviated, $(d)\text{-}\lim_{n\to\infty} x_n = x$) if for every $\varepsilon > 0$ the set $A(\varepsilon) = \{n : |x_n - x| \geq \varepsilon\}$ has zero density. Šalát and Toma [13] called this *statistical convergence*, but we adopted here the terminology of H. Furstenberg [4].

Of course, the above concepts have natural integral analogues. For simplicity, we will consider here only the case of Lebesgue measure m on $\mathbb{R}$.

A measurable subset A of $\mathbb{R}$ has *zero density* (at infinity) if

$$\lim_{r\to\infty} \frac{m\,(A \cap (-r, r))}{r} = 0.$$

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Clearly, the intervals $(a-r, a+r)$ centered at any other point a of $\mathbb{R}$ will do the same job. In fact, every set of finite measure has zero density. The union of intervals $\bigcup_{n=1}^{\infty}(n, n+1/n)$ provides an example of a set of infinite measure having zero density.

Given a real-valued function f defined on an interval $[\alpha, \infty)$, its *limit in density* at infinity,

$$\ell = (d)\text{-}\lim_{x\to\infty} f(x),$$

is defined by the condition that each of the sets $\{t \geq \alpha : |f(t) - \ell| \geq \varepsilon\}$ has zero density, whenever $\varepsilon > 0$. Equivalently, $\ell = (d)\text{-}\lim_{x\to\infty} f(x)$ means the existence of a subset $I \subset \mathbb{R}$ of zero density such that for every $\varepsilon > 0$ there is a positive number δ for which $|f(x) - \ell| < \varepsilon$ whenever $x \in (\delta, \infty)\backslash I$.

The above notions can be traced back to a famous paper by B. O. Koopman and J. von Neumann [8], dedicated to weakly mixing transformations. Their basic remark concerns the connection between convergence in density and convergence of certain arithmetic means. We recall it here in a slightly more general formulation (and with a simplified argument).

Theorem 2. *Suppose that* $f : [0, \infty) \to \mathbb{R}$ *is a nonnegative locally integrable function. Then*

$$\lim_{x\to\infty} \frac{1}{x}\int_0^x f(t)dt = 0 \text{ implies } (d)\text{-}\lim_{x\to\infty} f(x) = 0.$$

and the converse holds if in addition f *belongs to one of the spaces* $L^p(0, \infty)$, *for* $p \in [1, \infty]$.[1]

Proof. Assuming $\lim_{x\to\infty} \frac{1}{x}\int_0^x f(t)dt = 0$, we consider for each $\varepsilon > 0$ the set $A_\varepsilon = \{x > 0 : f(x) \geq \varepsilon\}$. Each of these sets has zero density since

$$\begin{aligned}\frac{m([0, x] \cap A_\varepsilon)}{x} &\leq \frac{1}{x}\int_0^x \frac{f(t)}{\varepsilon}dt \\ &\leq \frac{1}{\varepsilon x}\int_0^x f(t)dt \to 0\end{aligned}$$

as $x \to \infty$. Therefore $(d)\text{-}\lim_{x\to\infty} f(x) = 0$.

Conversely, if $(d)\text{-}\lim_{x\to\infty} f(x) = 0$, then for $\varepsilon > 0$ arbitrarily fixed there is a set J of zero density outside which $f < \varepsilon$. For every $x > 0$,

$$\begin{aligned}\frac{1}{x}\int_0^x f(t)dt &= \frac{1}{x}\int_{[0,x]\cap J} f(t)dt + \frac{1}{x}\int_{[0,x]\backslash J} f(t)dt \\ &\leq \frac{1}{x}\int_{[0,x]\cap J} f(t)dt + \varepsilon.\end{aligned}$$

If $f \in L^p(0, \infty)$ for some $p \in (1, \infty)$, then

$$\frac{1}{x}\int_{[0,x]\cap J} f(t)dt \leq \frac{1}{x^{1-1/q}}\left(\frac{m([0, x] \cap J)}{x}\right)^{1/q}\left(\int_0^\infty f^p(t)dt\right)^{1/p},$$

by Hölder's inequality; here $1/p + 1/q = 1$. Since J is a set of zero density, the limit $\lim_{x\to\infty} \frac{m([0,x]\cap J)}{x}$ equals 0, which forces $\lim_{x\to\infty} \frac{1}{x}\int_0^x f(t)dt = 0$. For the other

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two cases, notice that

$$\frac{1}{x}\int_{[0,x]\cap J} f(t)dt \le \frac{1}{x}\int_0^\infty f(t)dt,$$

when $f \in L^1(0,\infty)$, and

$$\frac{1}{x}\int_{[0,x]\cap J} f(t)dt \le \frac{m\left([0,x]\cap J\right)}{x}\left\|f\right\|_{L^\infty},$$

when $f \in L^\infty(0,\infty)$. □

Corollary 1. *If $f \in L^1(0,\infty)$, then (d)-$\lim_{x\to\infty} f(x) = 0$.*

Remark 1. *If $f \in L^1(0,\infty)$ and $T:(0,\infty)\to(0,\infty)$ is a measurable map which preserves the Lebesgue measure, then also $f\circ T \in L^1(0,\infty)$ and thus (d)-$\lim_{x\to\infty} f(T(x)) = 0$.*

Even when f is also continuous the conclusion of Corollary 1 cannot be improved to usual convergence to 0. However this happens in two important particular cases: *a*) f is uniformly continuous (this case is covered by Barbălat's Lemma [2]); and *b*) $f \in L^1(0,\infty)$ is a nonnegative nonincreasing function (since $\lim_{x\to\infty} xf(x) = 0$, according to the integral analogue of Olivier's Theorem). The monotonicity assumption in case *b*) can be slightly relaxed by asking only the existence of a constant $C > 0$ such that $f(t) \le Cf(x)$ for any $t \in [x, 2x]$ and any $x > 0$. See [10].

It is worth to notice that the aforementioned result of Šalát and Toma can be obtained easily from the discrete version of Theorem 2. Details are to be found in [11] (which contains also an account on the history of convergence in density). Apparently that short argument cannot be adapted in the integral. However we will be able to establish the integral analogue of the result of Šalát and Toma by a different strategy, which has the advantage to cover (with obvious modifications) both the integral and the discrete case.

Theorem 3. *If $f \in L^1(0,\infty)$, then*

$$(d)\text{-}\lim_{x\to\infty} xf(x) = 0.$$

Theorem 3 allows us easily to conclude that certain oscillatory continuous functions such as $\frac{\sin x}{x}$ are not Lebesgue integrable on $(0,\infty)$.

In the variant of Lebesgue integrable functions defined on cones in $\mathbb{R}^n$, the conclusion of Theorem 3 reads as

$$(d)\text{-}\lim_{|x|\to\infty} xf(x) = 0.$$

The main ingredient in the proof of Theorem 3 is the following technical result:

Lemma 1. *If $g:(0,\infty)\to\mathbb{R}$ is a decreasing positive function such that $\alpha = \inf\{xg(x) : x > 0\} > 0$, then every measurable subset A on which g is integrable has zero density.*

Proof. Indeed, if $m(A) < \infty$, then clearly A has zero density. Suppose now that $m(A) = \infty$. Since g is decreasing and integrable on A, then necessarily

$$\lim_{x\to\infty} g(x) = 0. \tag{1}$$

This conclusion can be strengthened to

$$\lim_{x\to\infty} m\left(A\cap(0,x]\right)g(x) = 0. \tag{2}$$

In fact for $\varepsilon > 0$ arbitrarily fixed we can find $x_1 > 0$ such that

$$\int_{A\cap[x_1,\infty)} g(x)dx < \varepsilon/2,$$

while (1) yields an $x_2 \in (x_1, \infty)$ for which

$$g(x) < \frac{\varepsilon}{2x_1} \quad \text{whenever } x > x_2.$$

Thus for $x > x_2$ we get

$$\begin{aligned} m\left(A \cap (0,x]\right) g(x) &= m\left(A \cap (0,x_1]\right) g(x) + m(A \cap (x_1,x])g(x) \\ &< x_1 g(x) + \int_{A\cap(x_1,\infty)} g(x)dx \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

and the proof of (2) is done.

The proof of Lemma 1 ends by noticing that

$$\begin{aligned} 0 \leq \frac{m\left(A \cap (0,x]\right)}{x} &= \frac{m\left(A \cap (0,x]\right) g(x)}{xg(x)} \\ &\leq \frac{1}{\alpha} m\left(A \cap (0,x]\right) g(x) \to 0, \end{aligned}$$

as $x \to \infty$. □

Once Lemma 1 is established, the proof of Theorem 3 can be completed easily by considering the measurable sets

$$S_\varepsilon = \left\{x : x\left|f(x)\right| \geq \varepsilon\right\},$$

associated to $\varepsilon > 0$. Since

$$\varepsilon \int_{S_\varepsilon} \frac{dx}{x} \leq \int_{S_\varepsilon} \left|f(x)\right| dx < \infty,$$

by Lemma 1 applied to $g(x) = 1/x$, we infer that S_ε has zero density. Consequently $(d)\text{-}\lim_{x\to\infty} xf(x) = 0$. □

In order to discuss the higher order analogues of the above results we will adopt the notation used in dynamical system theory for the iterates of a function $f = f(x)$:

$$f^{(0)}(x) = x \quad \text{and} \quad f^{(n)}(x) = (\underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}})(x) \text{ for } n \geq 1.$$

Proposition 1. (*The higher order Olivier criterion*). *If $f : [a, \infty) \to \mathbb{R}$ is a non-negative integrable function such that $\left(\prod_{k=0}^{n} \ln^{(k)} x\right) f(x)$ is decreasing, then*

$$\lim_{x\to\infty} \left(\prod_{k=0}^{n+1} \ln^{(k)} x\right) f(x) = 0.$$

Proof. The case where $n = 0$ is an immediate consequence of the following estimate of $(x \ln x) f(x)$,

$$\int_{\sqrt{x}}^{x} f(t)dt = \int_{\sqrt{x}}^{x} tf(t)\frac{dt}{t} \geq xf(x) \int_{\sqrt{x}}^{x} \frac{dt}{t} = \frac{1}{2}\left(x \ln x\right) f(x),$$

valid for all $x \geq 2$. The proof can now be completed by mathematical induction. □

If we discard the hypothesis on monotonicity, then the conclusion of Proposition 1 is no longer true. Instead one may use the higher order analogues of Theorem 3.

Theorem 4. *If* $f \in L^1(0,\infty)$, *then*

$$(d_h)\text{-}\lim_{x\to\infty} (x\ln x)\, f(x) = 0.$$

Here d_h stands for the *harmonic density*,

$$d_h(A) = \lim_{r\to\infty} \frac{1}{\ln r} \int_{A\cap[1,r)} \frac{dt}{t},$$

and the limit in harmonic density, $(d_h)\text{-}\lim_{x\to\infty} g(x) = \ell$, means that each of the sets $\{t : |g(t) - \ell| \geq \varepsilon\}$ has zero harmonic density, whenever $\varepsilon > 0$.

Proof. We start by noticing the following analogue of Lemma 1: If $g : (1,\infty) \to \mathbb{R}$ is a measurable positive function such that $xg(x)$ is decreasing and

$$\inf\{(x\ln x)\, g(x) : x > 1\} = \alpha > 0,$$

then every measurable subset A of $(1,\infty)$ on which g is integrable has zero harmonic density.

To prove this assertion, it suffices to consider the case where $m(A) = \infty$ and to show that

$$\lim_{x\to\infty} \left(\int_{A\cap[1,x]} \frac{dt}{t} \right) xg(x) = 0. \tag{3}$$

The details are very similar to those used in the proof of Lemma 1 and thus they are omitted.

Having (3) at hand, the proof of Theorem 4 can be completed by considering for each $\varepsilon > 0$ the measurable set

$$S_\varepsilon = \{x \geq 1 : (x\ln x)\,|f(x)| \geq \varepsilon\},$$

Since

$$\varepsilon \int_{S_\varepsilon} \frac{dx}{x\ln x} \leq \int_{S_\varepsilon} |f(x)|\, dx < \infty,$$

then by the aforementioned analogue of Lemma 1, applied to $g(x) = 1/(x\ln x)$, we infer that S_ε has zero harmonic density. Consequently $(d_h)\text{-}\lim_{x\to\infty} (x\ln x)\, f(x) = 0$, and the proof is done. □

Since

$$d(A) = 0 \text{ implies } d_h(A) = 0,$$

(see [6], Lemma 1, p. 241), it follows that the existence of limit in density assures the existence of limit in harmonic density.

A result known in measure theory as the *layer cake representation* (see [9], Theorem 1.13, p. 26) asserts that every nonnegative function $f \in L^1(0,\infty)$ verifies the formula

$$\int_0^\infty f(x)dx = \int_0^\infty m(\{x : f(x) > t\})\, dt.$$

According to the integral form of Olivier's Theorem,

$$\lim_{t\to\infty} tm(\{x : f(x) > t\}) = 0,$$

while Theorem 4 yields the apparently better conclusion,

$$(d_h)\text{-}\lim_{t\to\infty}(t\ln t)\, m(\{x : f(x) > t\}) = 0.$$

A similar remark works for the *decreasing rearrangement* of any function $f \in L^1(0,\infty)$. Details concerning the rearrangement of sets and functions may be found in the book of Lieb and Loss [9], Section 3.1.

Iterating the idea behind Theorem 4, we arrive at the conclusion that actually the membership to $L^1(0,\infty)$ imposes a sequence of necessary conditions in terms of limit in density,

$$(d_n)\text{-}\lim_{x\to\infty}\left[\left(\prod\nolimits_{k=0}^{n}\ln^{(k)} x\right) f(x)\right] = 0,$$

where d_n stands for the density of order n. Precisely, $d_0 = d$, $d_1 = d_h$ and, in general,

$$d_n(A) = \lim_{r\to\infty}\frac{1}{\ln^{(n)} r}\int_{A\cap[\exp^{(n-1)}1,r)}\frac{dt}{\prod_{k=0}^{n-1}\ln^{(k)} t},$$

for every measurable subset A and every $n \geq 1$.

Under these circumstances it is natural to ask whether any continuous positive function $g : [0,\infty) \to \mathbb{R}$ such that

$$\lim_{x\to\infty}\left(\prod\nolimits_{k=0}^{n}\ln^{(k)} x\right) g(x) = 0 \quad \text{for every } n \geq 1$$

is necessarily integrable. The answer is negative as follows from an old paper by P. Du Bois-Reymond [3] (see also [7]).

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