THE BEHAVIOR AT INFINITY OF AN INTEGRABLE FUNCTION

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Abstract. Given a density d defined on the Borel subsets of $[0,\infty)$, the limit at infinity in density of a function $f:[0,\infty)\to\mathbb{R}$ is zero if each of the sets $\{t:|f(t)|\geq\varepsilon\}$ has zero density whenever $\varepsilon>0$. It is proved that every Lebesgue integrable function $f:[0,\infty)\to\mathbb{R}$ verifies this type of behavior at infinity with respect to a scale of densities including the usual one, $d(A)=\lim_{r\to\infty}\frac{m(A\cap[0,r))}{r}$.

The analogy between convergent series and integrals over the positive semi-axis was an elegant and fruitful subject present in all major treatises of mathematical analysis published during the 20th Century. As was noted by G. H. Hardy in his *Course of Pure Mathematics* [4], p. 324, there is one fundamental property of a convergent infinite series in regard to which the analogy between infinite series and infinite integrals breaks down. If $\sum a_n$ is convergent then $a_n\to0$, but it is not always true, even when $f:[0,\infty)\to\mathbb{R}$ is positive, that if $\int_0^\infty f(x)dx$ is convergent then $f(x)\to0$ as $x\to\infty$. Due to the prominent role played by negligible sets one might expect that a conclusion of the type

$$f(x)\to0\quad\text{as } x \text{ runs to } \infty \text{ outside a negligible set}$$

must be working. That this is not the case is shown by the integrable function

$$f(x)=\sum_{n=1}^{\infty}\chi_{[n,n+2^{-n})}(x),\quad x\in[0,\infty).$$

Surprisingly, the analogy can be re-established if the usual limit is replaced by limit in density. This fact is implicit in a famous paper by B. O. Koopman and J. von Neumann [6] dedicated to weakly mixing transformations, and was recently made explicit and extended by us [10] to a scale of densities measuring how thin are the various Borel subsets of $\mathbb{R}$.

The aim of the present note is to provide a short argument for this general result along Koopman-von Neumann's ideas.

In what follows we shall adopt the convention used in dynamical system theory for the iterates of a function $f=f(x)$,

$$f^{(0)}(x)=x\quad\text{and}\quad f^{(n)}(x)=(\underbrace{f\circ f\circ\cdots\circ f}_{n\text{ times}})(x)\text{ for } n\geq1.$$

Note that $f^{(n)}(x)$ does not mean the nth derivative of $f(x)$, a function that we never use in this paper.

Date: August 15, 2011. Revised, December 6, 2011.
2000 *Mathematics Subject Classification.* Primary 26A42; Secondary 37A45.
Key words and phrases. Lebesgue integral, density, convergence in density.
Published in *Expo. Math.* **30** (2012) 277–282.

The *density of order* 0 (the usual density) is defined by the formula

$$\begin{aligned} d_0(A) &= \lim_{r\to\infty} \frac{1}{r}\int_{A\cap[0,r)} dt \\ &= \lim_{r\to\infty} \frac{m\left(A\cap[0,r)\right)}{r}, \end{aligned}$$

where m denotes the Lebesgue measure. This applies whenever the limit exists and corresponds to the limiting relative frequency in probability theory. Notice that every set of finite measure has zero density. The union of intervals $\bigcup_{n=1}^{\infty}(n,n+1/n)$ provides an example of a set of infinite measure having zero density.

The next density in our scale, the *density of order* 1, is nothing but the continuous analogue of *harmonic density* from number theory:

$$d_1(A) = \lim_{r\to\infty} \frac{1}{\ln r}\int_{A\cap[1,r)} \frac{dt}{t}.$$

See [3], p. 241. The higher order densities are introduced via the formulas

$$d_n(A) = \lim_{r\to\infty} \frac{1}{\ln^{(n)} r}\int_{A\cap[\exp^{(n-1)} 1,r)} d\alpha_n, \quad n\geq 1,$$

where

$$d\alpha_n = \frac{dt}{\prod_{k=0}^{n-1}\ln^{(k)} t}$$

represents the *Abel measure* of order $n\geq 1$.

Given a real-valued function f defined on an interval $[\alpha,\infty)$, its *limit in density of order* n at infinity,

$$\ell = (d_n)\text{-}\lim_{x\to\infty} f(x),$$

is defined by the condition that each of the sets $\{t\geq\alpha : |f(t)-\ell|\geq\varepsilon\}$ has zero density of order n, whenever $\varepsilon>0$.

Lemma 1. $d_0(A)=0$ *implies* $d_1(A)=0$.

Proof. Since $d_0(A)=0$, for $\varepsilon>0$ arbitrarily fixed there is an $s\geq 1$ such that

$$\frac{m\left(A\cap[1,t)\right)}{t} < \frac{\varepsilon}{3}$$

whenever $t\geq s$. Then, according to the formula of integration by parts for absolutely continuous functions ([5], Corollary 18.20, p. 287), for every $r>\max\{s,e\}$ we have

$$\begin{aligned} \frac{1}{\ln r}\int_{A\cap[1,r)} \frac{dt}{t} &= \frac{1}{\ln r}\int_1^s \frac{\chi_{A\cap[1,r)}(t)}{t}dt + \frac{1}{\ln r}\int_s^r \frac{\chi_{A\cap[s,r)}(t)}{t}dt \\ &\leq \frac{\ln s}{\ln r} + \frac{1}{\ln r}\int_s^r \frac{1}{t}\frac{d}{dt}\left(\int_s^t \chi_{A\cap[s,r)}(\tau)d\tau\right)dt \\ &< \frac{\varepsilon}{3} + \frac{1}{\ln r}\left(\frac{1}{r}\int_s^r \chi_{A\cap[s,r)}(\tau)d\tau + \int_s^r\left(\frac{1}{t}\int_s^t \chi_{A\cap[s,r)}(\tau)d\tau\right)\frac{dt}{t}\right) \\ &= \frac{\varepsilon}{3} + \frac{m\left(A\cap[s,r)\right)}{r\ln r} + \frac{1}{\ln r}\int_s^r\left(\frac{m\left(A\cap[s,t)\right)}{t}\right)\frac{dt}{t} \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3\ln r} + \frac{\varepsilon}{3} < \varepsilon, \end{aligned}$$

whence

$$\lim_{r\to\infty}\frac{1}{\ln r}\int_{A\cap[1,r)}\frac{dt}{t}=0.$$

□

A similar argument shows that in general

$$d_n(A)=0 \text{ implies } d_{n+1}(A)=0,$$

and thus the existence of limit in density of order n assures the existence of limit in density of order $n+1$.

As we mentioned above, the notion of limit in density can be traced back to the paper [6] by B. O. Koopman and J. von Neumann. Their basic remark concerns the connection between convergence in density of order 0 and convergence of certain arithmetic means:

Lemma 2. *Suppose that $f:[0,\infty)\to\mathbb{R}$ is a nonnegative locally integrable function. Then*

$$\lim_{x\to\infty}\frac{1}{x}\int_0^x f(t)dt=0 \textit{ implies } (d_0)\text{-}\lim_{x\to\infty} f(x)=0,$$

and the converse holds if in addition f is bounded.

Corollary 1. *If $f\in L^1(0,\infty)$, then $(d_0)\text{-}\lim_{x\to\infty} f(x)=0$.*

Even when f is also continuous the conclusion of Corollary 1 cannot be improved to usual convergence to 0. However this happens in two important particular cases: *a*) f is uniformly continuous (this case is known as Barbălat's Lemma [9]); and *b*) $f\in L^1(0,\infty)$ is a nonnegative nonincreasing function (see [4], p. 324, who attributed this result to Abel). The monotonicity assumption in case *b*) can be relaxed considerably by asking only the existence of a constant $C>0$ such that $f(t)\le Cf(x)$ for any $t\in[x,2x]$ and any $x>0$. See [8].

In the case of series, Olivier and Abel have proved that the sequence of terms of any convergent positive series $\sum a_n$ verifies the condition $na_n\to 0$ provided that it is nonincreasing. See the paper by M. Goar [2] for the story of this nice result. In 2003, T. Šalát and V. Toma [12] made the important remark that the monotonicity condition can be dropped if the convergence of $(na_n)_n$ is weakened to convergence in density. The corresponding result for integrals appeared in our recent paper [10]:

Theorem 1. *If $f\in L^1(0,\infty)$, then*

$$(d_0)\text{-}\lim_{x\to\infty} xf(x)=0.$$

Surprisingly, this result can be extended to the entire scale of densities mentioned above:

Theorem 2. *If $f\in L^1(0,\infty)$, then*

$$(d_n)-\lim_{x\to\infty}\left(\prod\nolimits_{k=0}^{n}\ln^{(k)}x\right)f(x)=0$$

for every $n\in\mathbb{N}$.

The statement of Theorem 2 was mentioned in [10], p. 746, without any proof. Here we show how this result can be obtained from the following analogue of the Koopman-von Neumann Lemma for densities of higher order.

Lemma 3. *Suppose that $f : [0, \infty) \to \mathbb{R}$ is a nonnegative function which is locally integrable with respect to the Abel measure of order $n \geq 1$. Then*

$$\lim_{x\to\infty} \frac{1}{\ln^{(n)} x} \int_a^x f(t) d\alpha_n = 0 \text{ implies } (d_n)\text{-}\lim_{x\to\infty} f(x) = 0,$$

and the converse holds if in addition f belongs to one of the spaces $L^p(d\alpha_n)$, for $p \in [1, \infty]$.

Proof. If

$$\lim_{x\to\infty} \frac{1}{\ln^{(n)} x} \int_0^x f(t) d\alpha_n = 0,$$

then the sets $A_\varepsilon = \{x > 0 : f(x) \geq \varepsilon\}$ associated to $\varepsilon > 0$ verify the condition $d_n (A_\varepsilon) = 0$ due to the fact that

$$\frac{1}{\ln^{(n)} x} \int_{A_\varepsilon \cap [\exp^{(n-1)} 1, x)} d\alpha_n \leq \frac{1}{\varepsilon \ln^{(n)} x} \int_{\exp^{(n-1)} 1}^x f(t) d\alpha_n \to 0$$

as $x \to \infty$. Therefore $(d_n) - \lim_{x\to\infty} f(x) = 0$.

Conversely, if $(d_n) - \lim_{x\to\infty} f(x) = 0$, then for $\varepsilon > 0$ arbitrarily fixed there is a set J of zero density of order n, outside which $f < \varepsilon$. For $x > 0$ sufficiently large we have

$$\begin{aligned} 0 \leq \frac{1}{\ln^{(n)} x} \int_{\exp^{(n-1)} 1}^x f(t) d\alpha_n &= \frac{1}{\ln^{(n)} x} \int_{[\exp^{(n-1)} 1, x) \cap J} f(t) d\alpha_n \\ &\quad + \frac{1}{\ln^{(n)} x} \int_{[\exp^{(n-1)} 1, x) \setminus J} f(t) d\alpha_n \\ &\leq \frac{1}{\ln^{(n)} x} \int_{[\exp^{(n-1)} 1, x) \cap J} f(t) d\alpha_n + \varepsilon \end{aligned}$$

and it remains to prove that

$$\lim_{x\to\infty} \frac{1}{\ln^{(n)} x} \int_{[\exp^{(n-1)} 1, x) \cap J} f(t) d\alpha_n = 0.$$

If $f \in L^1(d\alpha_n)$, this follows from the inequality

$$\frac{1}{\ln^{(n)} x} \int_{A \cap [\exp^{(n-1)} 1, x) \cap J} f(t) d\alpha_n \leq \frac{1}{\ln^{(n)} x} \int_{\exp^{(n-1)} 1}^\infty f(t) d\alpha_n,$$

while if $f \in L^\infty(d\alpha_n)$ we have to notice that

$$0 \leq \frac{1}{\ln^{(n)} x} \int_{[\exp^{(n-1)} 1, x) \cap J} f(t) d\alpha_n \leq \left(\frac{1}{\ln^{(n)} x} \int_{[\exp^{(n-1)} 1, x) \cap J} d\alpha_n \right) \|f\|_{L^\infty(d\alpha_n)}.$$

If $f \in L^p(d\alpha_n)$ for some $p \in (1, \infty)$, then

$$\begin{aligned} 0 &\leq \frac{1}{\ln^{(n)} x} \int_{[\exp^{(n-1)} 1, x) \cap J} f(t) d\alpha_n \\ &\leq \left(\frac{1}{\ln^{(n)} x} \int_{A \cap [\exp^{(n-1)} 1, x) \cap J} d\alpha_n \right)^{1-1/p} \left(\frac{1}{\ln^{(n)} x} \int_{\exp^{(n-1)} 1}^x f^p(t) d\alpha_n \right)^{1/p} \\ &\leq \left(\frac{1}{\ln^{(n)} x} \right)^{1/p} \left(\frac{1}{\ln^{(n)} x} \int_{A \cap [\exp^{(n-1)} 1, x) \cap J} d\alpha_n \right)^{1-1/p} \|f\|_{L^p(d\alpha_n)}, \end{aligned}$$

according to Hölder's inequality. The conclusion is now clear. □

Proof of Theorem 2. Notice first that

$$\lim_{x\to\infty}\frac{1}{\ln^{(n)}x}\int_{\exp^{(n-1)}1}^{x}|f(t)|\ln^{(n)}t\,dt=0. \tag{$*$}$$

In fact, for $\varepsilon > 0$ arbitrarily fixed there is a number $\delta > \exp^{(n-1)} 1$ such that $\int_\delta^\infty |f(t)|dt < \varepsilon/2$. Then for $x > \delta$ we have

$$\begin{aligned}0 &\leq \frac{1}{\ln^{(n)}x}\int_{\exp^{(n-1)}1}^{x}|f(t)|\ln^{(n)}t\,dt \\ &= \frac{1}{\ln^{(n)}x}\int_{\exp^{(n-1)}1}^{\delta}|f(t)|\ln^{(n)}t\,dt+\int_{\delta}^{x}|f(t)|\frac{\ln^{(n)}t}{\ln^{(n)}x}dt \\ &\leq \frac{\ln^{(n)}\delta}{\ln^{(n)}x}\|f\|_{L^1}+\int_{\delta}^{\infty}|f(t)|dt<\frac{\ln^{(n)}\delta}{\ln^{(n)}x}\|f\|_{L^1}+\frac{\varepsilon}{2},\end{aligned}$$

and we can choose x_ε large enough such that $\ln^{(n)}\delta/\ln^{(n)}x < \varepsilon/\left(2\|f\|_{L^1}\right)$ for every $x \geq x_\varepsilon$. This ends the proof of the formula $(*)$. Now the conclusion of Theorem 2 follows from Lemma 3 when applied to the locally integrable function $\left(\prod_{k=0}^{n}\ln^{(k)}x\right)f(x)$. □

A different approach of the behavior at infinity of an integrable function has been recently described by E. Lesigne [7].

In the discrete case (that is, in the case of series), important applications of convergence in density can be found in the monograph of Furstenberg [1], dedicated to the ergodic approach of Szemerédi's theorem. See also the preprint [11]. However, as far as we know, the continuous case is still largely unexplored.

Acknowledgement. This work is supported by a grant of the Romanian National Authority for Scientific Research, CNCS – UEFISCDI, project number PN-II-ID-PCE-2011-3-0257. The authors thank the referee for his many valuable suggestions.

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