THE HERMITE-HADAMARD INEQUALITY FOR LOG-CONVEX FUNCTIONS

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Abstract. We discuss the existence of a strengthening of Hermite-Hadamard inequality in the case of log-convex functions. Unlike the classical case, which belongs to the field of linear functional analysis, this analogue involves nonlinear means such as the geometric mean and the logarithmic mean.

1. Introduction

A good mathematical result is one which continues to surprise us by pointing out new entanglements, new implications and also new open questions. No doubt, everyone has his/her favorite list of such beauties of mathematics. My list includes the Hermite-Hadamard inequality, a result first noticed by Ch. Hermite in 1883 and rediscovered ten years later by J. Hadamard. A complete account on the history of this inequality may be found in the paper by D. S. Mitrinović and I. B. Lacković [6], or in the recent book by me and L.-E. Persson [13].

The *Hermite-Hadamard inequality* asserts that the mean value of a continuous convex function $f : [a,b] \to \mathbb{R}$ lies between the value of f at the midpoint of the interval $[a,b]$ and the arithmetic mean of the values of f at the endpoints of this interval, that is,

$$(HH) \qquad f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}\,.$$

Moreover, each side of this double inequality characterizes convexity in the sense that a real-valued continuous function f defined on an interval I is convex if its restriction to each compact subinterval $[a,b] \subset I$ verifies the left hand side of (HH) (equivalently, the right hand side of (HH)). See [1] and [13] for details.

A remarkable fact is the connection of Hermite-Hadamard inequality with Choquet's theory, briefly recalled here for the convenience of the reader. Full details are available in [13] and [15].

Choquet's theory deals with compact convex subsets of a locally convex Hausdorff space E. Given a Borel probability measure μ on such a subset K, one can prove the existence of a unique point $b_\mu \in K$ (called the *barycenter* of μ) such that

$$(B) \qquad x'(b_\mu) = \int_K x'(x)\,d\mu(x)$$

for all continuous linear functionals x' on E.

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Since (B) works also for all continuous affine functions on K, it follows that

$$(LHH) \qquad f(b_\mu) \leq \int_K f(x)\, d\mu(x)$$

for every continuous convex function $f : K \to \mathbb{R}$. See [13], Lemma 4.1.8, pp. 181-182. This extends the left hand side of the classical Hermite-Hadamard inequality (which corresponds to the case where K is the interval $[a, b]$ and μ is the normalized Lebesgue measure $\frac{dx}{b-a}$; the barycenter of μ is in this case the midpoint $(a + b)/2$).

Remark 1. *Using the technique of pushing-forward measures, one can put the inequality (LHH) in a more general form that includes Jensen's inequality. In fact, if (X, Σ, ν) is a finite measure space (on an abstract set X) and $T : X \to K$ is a ν-integrable map, then we may consider the push-forward measure $\mu = T\#\nu$, which is given by the formula $\mu(A) = \nu(T^{-1}(A)))$. The barycenter of μ is*

$$\overline{T} = \frac{1}{\nu(X)} \int_X T(x) d\nu(x)$$

and the formula (LHH) becomes

$$\begin{aligned} f(\overline{T}) &\leq \frac{1}{\mu(K)} \int_K f(t) d\mu(t) \\ &= \frac{1}{\nu(X)} \int_X f(T(x)) d\nu(x). \end{aligned}$$

The right hand side of (HH) represents the mean value of f with respect to a probability measure, $\lambda = \frac{1}{2}\delta_a + \frac{1}{2}\delta_b$, supported on the extreme points of the interval $[a, b]$. As usually, δ_c denotes here the Dirac measure concentrated at c.

The extension of the right hand side of (HH) to the general setting of continuous convex functions defined on metrizable compact convex sets is accomplished by the following theorem due to G. Choquet, which relates the geometry of K to a given mass distribution.

Theorem 1. *Let μ be a Borel probability measure on a metrizable compact convex subset K of a locally convex Hausdorff space. Then there exists a probability measure λ on K which has the same barycenter as μ, is null outside the set $\operatorname{Ext} K$, of all extreme points of K, and verifies the inequality*

$$(RHH) \qquad \int_K f(x)\, d\mu(x) \leq \int_{\operatorname{Ext} K} f(x)\, d\lambda(x),$$

for all continuous convex functions $f : K \to \mathbb{R}$.

In the case of functions defined on intervals, the combination of the inequalities (LHH) and (RHH) reads as follows:

Corollary 1. *If μ is a Borel probability measure on an interval $[a, b]$, then*

$$f(b_\mu) \leq \int_a^b f(x)\, d\mu(x) \leq \frac{b - b_\mu}{b - a} f(a) + \frac{b_\mu - a}{b - a} f(b)$$

for all continuous convex functions $f : [a, b] \to \mathbb{R}$.

An extension of Theorem 1 to the general case of compact convex sets non-necessarily metrizable can be found in [15].

A stronger property of convexity is log-convexity. A positive function defined on an interval (or, more generally, on a convex subset of some vector space) is called log-*convex* if $\log f(x)$ is a convex function of x, equivalently, if for any two points x and y in its domain and any t in $[0,1]$ we have

$$f((1-t)x+ty) \leq f(x)^{1-t} f(y)^t; \qquad (\log\text{-}C)$$

f is called log-*concave* if the inequality above works in the reversed way (that is, when $1/f$ is log-convex). The arithmetic mean-geometric mean inequality easily yields that every log-convex function is also convex.

Remark 2. *For example,* $f(x) = x^2$ *is a convex function on* $(0,\infty)$*, which is not is not* log-*convex. On the other hand,* $f(x) = e^{x^2}$ *is* log-*convex on* $\mathbb{R}$ *since* $\log e^{x^2} = x^2$ *is convex. A less trivial example of a log-convex function is the gamma function, if restricted to the positive reals* (*see the Bohr–Mollerup theorem* [13]). *Some examples of* log-*concave functions:*

i) x^α *on* $(0,\infty)$ *is* log-*concave for* $\alpha \geq 0$ (*and* log-*convex for* $\alpha \leq 0$);

ii) *The normal probability density* $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ *and the cumulative Gaussian distribution function*

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt,$$

are both log-*concave;*

iii) *The determinant function* det *is* log-*concave on the set of all positively defined matrices of dimension* $n \times n$;

iv) *If* $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ *is* log-*concave, then*

$$g(x) = \int_{\mathbb{R}^m} f(x,y) dy$$

is log-*concave too. In particular, the convolution of two log-concave functions* f *and* g,

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y) g(y) dy,$$

is log-*concave.*

v) *The product of* log-*concave functions is log-concave.*

As log-convexity implies convexity, one may expect that stronger variants of the inequalities (LHH) and (RHH) are available in the case of log-convex functions f. Indeed, a direct application of these results to $\log f$ yields the double inequality

$$f(b_\mu) \leq \exp\left(\int_K \log\, f(x)\, d\mu(x)\right) \leq \exp\left(\int_{\operatorname{Ext} K} \log f(x)\, d\lambda(x)\right), \qquad (HH')$$

where λ is related to μ as in Theorem 1. This inequality says that the geometric mean of f lies between the value of f at the barycenter of μ and the geometric mean of the restriction of f to $\operatorname{Ext} K$ with respect to a Borel probability measure λ induced by μ. Due to the geometric mean-arithmetic mean inequality,

$$f(b_\mu) \leq \exp\left(\int_a^b \log\, f(x)\, d\mu(x)\right) \leq \int_a^b f(x)\, d\mu(x),$$

so at least the left hand side of (HH') is stronger than (LHH).

Surprisingly, the existence of an analogue of (RHH) in the context of log-convexity proves to be an open problem. The essence of (RHH) is the existence of Borel probability measure on $\operatorname{Ext} K$ that gives rise to an induced arithmetic mean that exceeds the integral arithmetic mean on K associated to μ. What should be this induced mean in the log-convex case?

A possible answer is suggested by the following remark due to P. M. Gill, C. E. M. Pearce, and J. Pečarić [5]: if f is a log-convex function defined on an interval $[a, b]$, then

$$\frac{1}{b-a}\int_a^b f(x)\,dx \leq L(f(a), f(b)),$$

where

$$L(x,y) = \begin{cases} \frac{x-y}{\log x - \log y} & \text{if } x \neq y \\ x & \text{if } x = y \end{cases} \tag{L}$$

represents the so called *logarithmic mean*. This mean lies between the geometric mean and the arithmetic mean of x and y. See [2]. The logarithmic mean proves useful in engineering problems involving heat and mass transfer.

Most of the usual means, including the arithmetic mean,

$$A\left(a,b\right) = \frac{a+b}{2},$$

and the geometric mean,

$$G\left(a,b\right) = \sqrt{ab},$$

comes through formulae involving pairs of numbers with equal weights and extends canonically to the case of finite family of numbers with unequal weights, and next to the general framework of probability fields. For example, if (X, Σ, μ) is a probability field, then the arithmetic mean of a function $f \in L^1(\mu)$ is

$$A(f;\mu) = \int_X f d\mu$$

and the geometric mean of a positive function $f \in L^1(\mu)$ is

$$G(f;\mu) = \exp\left(\int_X \log f\, d\mu\right).$$

Problem 1. *What is $L(f;\mu)$?*

The existing literature concerning the logarithmic mean associated to a probability measure contains several (rather different) approaches. See [8], [9], [10], [16], [17], and [18]. Unfortunately, none of them seems able to offer a full solution to the problem of extending the aforementioned result of Gill, Pearce and Pečarić to the framework of Theorem 1.

The aim of the present paper is to show how a solution can be built in three special cases: the case of intervals, the case of balls and the case of simplices. Of course, the above problem can be extended to the more involving framework of generalized convexity, and our paper ends with the case of multiplicatively convex functions. That case is interesting because it emphasizes a certain asymmetry in the formation of Hermite-Hadamard type formulae. In fact, while the upper bound of the (integral) arithmetic mean depends on both means defining the type of convexity of the given function, the lower bound depends only on the mean acting on the domain of definition.

2. The case of intervals

We start with the case where K is a compact interval $[a,b]$ endowed with a Borel probability measure μ. In this case, the measure

$$\lambda = \frac{b-b_\mu}{b-a}\delta_a + \frac{b_\mu - a}{b-a}\delta_b,$$

represents the *arithmetic displacement* of μ at the endpoints of the interval $[a,b]$. We will denote λ also $\operatorname{disp}_A(\mu// \operatorname{Ext}[a,b])$. The coefficients

$$c_1(\lambda) = \frac{b-b_\mu}{b-a} \text{ and } c_2(\lambda) = \frac{b_\mu - a}{b-a},$$

represent the *weights* of λ. A key remark is the formula

$$A\left(f; \frac{b-b_\mu}{b-a}\delta_a + \frac{b_\mu - a}{b-a}\delta_b\right) = \int_a^b (mx+n)d\mu, \tag{AD}$$

where and $y = mx+n$ represents the chord joining the endpoints of f. Indeed, the equation of this chord is $y = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$ and the proof of (AD) turns easily into a straightforward computation. It is worth to restate (AD) as

$$A\left(f; \operatorname{disp}_A(\mu// \operatorname{Ext}[a,b])\right) = \int_a^b (mx+n)d\mu$$

and to view the left hand side as the arithmetic mean induced by μ at $\operatorname{Ext}[a,b]$.

Now it is natural to introduce the *logarithmic mean induced by* μ at $\operatorname{Ext}[a,b]$ via the formula

$$L(f;\mu// \operatorname{Ext}[a,b]) = \int_a^b \exp(mx+n)d\mu,$$

where $y = mx+n$ represents the chord joining the endpoints of $\log f$.

The logarithmic mean can also be interpreted as the area under an exponential curve. Therefore

$$\begin{aligned}
&L(f;\mu// \operatorname{Ext}[a,b]) \\
&= \int_a^b \exp(\log f(a) + \frac{\log f(b) - \log f(a)}{b-a}(x-a))d\mu \\
&= \int_a^b f(a)^{(b-x)/(b-a)} f(b)^{(x-a)/(b-a)} d\mu \\
&= \int_a^b \exp\left(\frac{b-x}{b-a}\log f(a) + \frac{x-a}{b-a}\log f(b)\right)d\mu.
\end{aligned}$$

A special case is that of the normalized Lebesgue measure,

$$\begin{aligned}
L(f;\frac{dx}{b-a}// \operatorname{Ext}[a,b]) &= \frac{1}{b-a}f(a)\int_a^b \left(\frac{f(b)}{f(a)}\right)^{(x-a)/(b-a)} dx \\
&= L(f(a), f(b)),
\end{aligned}$$

which agrees with the usual definition of logarithmic mean.

The fact that the induced logarithmic mean is indeed a mean is straightforward. Indeed

$$f \le g \text{ implies } L(f;\mu// \operatorname{Ext}[a,b]) \le L(g;\mu// \operatorname{Ext}[a,b]),$$

and

$$L(1;\mu// \operatorname{Ext}[a,b]) = 1.$$

Clearly, this mean depends only on the values (of the function under attention) at the endpoints.

Notice that unlike the arithmetic mean, the induced logarithmic mean is not linear. However, it is positively homogeneous.

Simple examples shows that two different Borel probability measures on $[a,b]$ having the same arithmetic displacement at the endpoints may lead to different induced logarithmic means. This is not surprising because even in the statement of Theorem 1 the measure λ it is not unique.

It is important to notice a basic difference between the induced arithmetic mean at the endpoints and the induced logarithmic mean. The later comes in a direct way, skipping the computation of $L(f;\lambda)$ for some measure λ playing the role of "logarithmic displacement" of μ at the endpoints.

The following result provides a log-convex analogue of Theorem 1 in the case of functions defined on compact intervals.

Theorem 2. *Let f be a continuous positive function defined on an interval I and let μ be a positive Borel measure on I.*

i) If f is log-*convex, then for every compact subinterval $[a,b]$ of I,*

$$
\begin{aligned}
A\left(f;\frac{\mu}{\mu\left([a,b]\right)}\right) &\leq L\left(f;\frac{\mu}{\mu\left([a,b]\right)}//\operatorname{Ext}[a,b]\right)\\
&\leq A\left(f;\operatorname{disp}_A\frac{\mu}{\mu\left([a,b]\right)}//\operatorname{Ext}[a,b]\right).
\end{aligned}
$$

ii) Conversely, if μ is a non-atomic measure and

$$
A\left(f;\frac{\mu}{\mu\left([a,b]\right)}\right) \leq L\left(f;\frac{\mu}{\mu\left([a,b]\right)}//\operatorname{Ext}[a,b]\right)
$$

for every compact subinterval $[a,b]$ of I, then f is log-*convex.*

The assertion $i)$ of Theorem 2 improves on the conclusion of Theorem 1 in the case of log-convex functions.

Theorem 2 has a companion for the log-concave functions (all inequalities working in the reversed way).

Proof. $i)$ Consider the homeomorphism $T:[0,1]\to[a,b],\ T(t)=(1-t)\,a+tb.$ Then

$$
T\#\left(T^{-1}\#\frac{\mu}{\mu\left([a,b]\right)}\right)=\frac{\mu}{\mu\left([a,b]\right)}.
$$

If f is log-convex,

$$
\begin{aligned}
A\left(f;\frac{\mu}{\mu\left([a,b]\right)}\right) &= \frac{1}{\mu\left([a,b]\right)}\int_a^b f(x)\,d\mu \\
&= \int_0^1 f\left((1-t)\,a+tb\right)d\left(T^{-1}\#\frac{\mu}{\mu\left([a,b]\right)}\right) \\
&\leq \int_0^1 f(a)^{1-t}f(b)^t d\left(T^{-1}\#\frac{\mu}{\mu\left([a,b]\right)}\right) \\
&= \int_a^b f(a)^{(b-x)/(b-a)}f(b)^{(x-a)/(b-a)}d\left(T\#\left(T^{-1}\#\frac{\mu}{\mu\left([a,b]\right)}\right)\right) \\
&= L\left(f;\frac{\mu}{\mu\left([a,b]\right)}//\operatorname{Ext}[a,b]\right),
\end{aligned}
$$

for all $a<b$ in I. On the other hand, by the geometric mean-arithmetic mean inequality

$$
f(a)^{(b-x)/(b-a)}f(b)^{(x-a)/(b-a)} \leq \frac{b-x}{b-a}f(a)+\frac{x-a}{b-a}f(b),
$$

which yields

$$
L\left(f;\frac{\mu}{\mu\left([a,b]\right)}//\operatorname{Ext}[a,b]\right) \leq A\left(f;\operatorname{disp}_A\frac{\mu}{\mu\left([a,b]\right)}//\operatorname{Ext}[a,b]\right).
$$

ii) Assume, by reductio ad absurdum, that $f(x)$ is not log-convex. Then there must exist a subinterval $[x,y]\subset I$ and a number $\varepsilon\in(0,1)$ such that

$$
f\left((1-\varepsilon)\,x+\varepsilon y\right) > f(x)^{1-\varepsilon}f(y)^{\varepsilon},
$$

Since f is continuous, the above inequality holds on an entire neighborhood $[\varepsilon_1,\varepsilon_2]$ of ε. We choose $[\varepsilon_1,\varepsilon_2]$ the biggest neighborhood with that property. Put $a=(1-\varepsilon_1)\,x+\varepsilon_1 y$ and $b=(1-\varepsilon_2)\,x+\varepsilon_2 y$. Because of continuity we have

$$
f\left(a\right) = f(x)^{1-\varepsilon_1}f(y)^{\varepsilon_1}
$$

and

$$
f\left(b\right) = f(x)^{1-\varepsilon_2}f(y)^{\varepsilon_2}.
$$

Then $a<b$ are points in I and for every $t\in(0,1)$ we have:

$$
\begin{aligned}
f\left((1-t)\,a+tb\right) &= f\left((1-t)\left((1-\varepsilon_1)\,x+\varepsilon_1 y\right)+t\left((1-\varepsilon_2)\,x+\varepsilon_2 y\right)\right) \\
&= f\left(x\left[(1-t)\,(1-\varepsilon_1)+t\,(1-\varepsilon_2)\right]+y\left((1-t)\,\varepsilon_1+t\varepsilon_2\right)\right) \\
&> f\left(x\right)^{(1-t)(1-\varepsilon_1)+t(1-\varepsilon_2)}\,f\left(y\right)^{(1-t)\varepsilon_1+t\varepsilon_2} \\
&= \left(f\left(x\right)^{1-\varepsilon_1}f\left(y\right)^{\varepsilon_1}\right)^{1-t}\left(f\left(x\right)^{1-\varepsilon_2}f\left(y\right)^{\varepsilon_2}\right)^{t} \\
&= f\left(a\right)^{1-t}f\left(b\right)^{t}.
\end{aligned}
$$

By integrating this inequality with respect to t, we obtain

$$\begin{aligned}
A(f;\mu) &= \frac{1}{\mu([a,b])}\int_a^b f(x)\,d\mu \\
&= \int_0^1 f((1-t)\,a+tb)\,d\left(T^{-1}\#\frac{\mu}{\mu([a,b])}\right) \\
&> \int_0^1 f(a)^{1-t}f(b)^t d\left(T^{-1}\#\frac{\mu}{\mu([a,b])}\right) \\
&= \frac{1}{\mu([a,b])}\int_a^b f(a)^{(b-x)/(b-a)}f(b)^{(x-a)/(b-a)}d\mu \\
&= L\left(f;\frac{\mu}{\mu([a,b])}//\,\mathrm{Ext}[a,b]\right),
\end{aligned}$$

a fact that contradicts *iii*). □

Corollary 2. (*The* (log-HH) *formula*). *If f is a* log*-convex function defined on an interval* $[a,b]$, *then*

$$\begin{aligned}
&f\left(\frac{a+b}{2}\right) \\
&\quad\le \left(\frac{1}{b-a}\int_a^b \log\, f(x)\,dx\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le L(f(a),f(b)) \\
&\quad\le \frac{f(a)+f(b)}{2}.
\end{aligned}$$

Recently the Hermite-Hadamard inequality was extended outside the setting of Borel probability measures. Indeed, A. Florea and C. P. Niculescu [4] provided a complete characterization of those signed Borel measures for which Corollary 1 holds true (the so called Hermite-Hadamard measures). The inequalities (HH') also work in the context of Hermite-Hadamard measures (since they are a direct consequence of Corollary 1).

Problem 2. *Characterize those signed Borel measures μ on the interval $[a,b]$ such that $\mu([a,b])=1$ and*

$$\int_a^b f(x)d\mu \le \int_a^b \exp\left(\frac{b-x}{b-a}\log f(a)+\frac{x-a}{b-a}\log f(b)\right)d\mu$$

for all log*-convex functions* $f:[a,b]\to\mathbb{R}$.

A. M. Fink [3] has some partial results related to this problem.

3. The Higher Dimensional Case

Most of the above discussion was restricted to functions on intervals, but the framework of log-convexity is much larger.

Problem 3. *Does Theorem* 2 *admit a higher dimensional analogue?*

At present this is a tantalizing problem. We can answer it in some special cases, for example, for balls and simplices in $\mathbb{R}^n$, when endowed with the normalized Lebesgue measure. For the shake of simplicity we will restrict here to the

2-dimensional case. The set of extreme points of the compact disc $\overline{D}(a, R)$ (of center a and radius r) is the circle $S^1(a, R)$, while in the case of a (nondegenerate) triangle ΔABC of vertices A, B, C this set is $\{A, B, C\}$.

Theorem 3. *Suppose that f is a continuous function defined on a convex subset $\Omega \subset \mathbb{R}^2$.*

i) If f is log*-convex, then for every compact disc $\overline{D}_R(a, R)$ contained in Ω,*

$$\frac{1}{\pi R^2} \iint_{\overline{D}(a,R)} f(x, y)\, dxdy \le \frac{1}{\pi R} \int_{S^1(a,R)} L(f(x, y), f(x, -y)) y^2\, ds.$$

ii) If the above inequality works for all compact discs contained in Ω, then f is log*-convex.*

Proof. $i)$ Clearly, it suffices to consider the case of the unit disc. In this case, by performing the change of coordinates

$$T : [-1, 1] \times [0, 1], \quad T(x, t) = \left(x, (2t - 1)\sqrt{1 - x^2}\right),$$

we obtain

$$\begin{aligned}
A\left(f; \frac{dxdy}{\pi}\right) &= \frac{1}{\pi} \iint_{\{x^2+y^2\le 1\}} f(x, y)\, dxdy \\
&= \frac{2}{\pi} \int_{-1}^{1} \int_{0}^{1} f\left((1 - t)\left(x, -\sqrt{1 - x^2}\right) + t\left(x, \sqrt{1 - x^2}\right)\right) \sqrt{1 - x^2} dtdx \\
&\le \frac{2}{\pi} \int_{-1}^{1} \int_{0}^{1} f^{1-t}\left(x, -\sqrt{1 - x^2}\right) f^{t}\left(x, \sqrt{1 - x^2}\right) \sqrt{1 - x^2} dtdx \\
&= \frac{2}{\pi} \int_{-1}^{1} \frac{f\left(x, \sqrt{1 - x^2}\right) - f\left(x, -\sqrt{1 - x^2}\right)}{\log f\left(x, \sqrt{1 - x^2}\right) - \log f\left(x, -\sqrt{1 - x^2}\right)} \sqrt{1 - x^2} dx \\
&= \frac{1}{\pi} \int_{S^1} L(f(x, y), f(x, -y)) y^2\, ds.
\end{aligned}$$

$ii)$ Adapt the argument for Theorem 2 $ii)$. $\square$

The case of triangles is very similar, the role of (RHH) being played by the inequality

$$\begin{aligned}
&\frac{1}{\operatorname{area}(\Delta ABC)} \iint_{\Delta ABC} f(x, y)\, dxdy \\
&\qquad \le 2 \int\int_{\substack{\lambda_1+\lambda_2\le 1 \\ \lambda_1,\lambda_2 \ge 0}} f(A)^{\lambda_1} f(B)^{\lambda_2} f(C)^{1-\lambda_1-\lambda_2}\, d\lambda_1 d\lambda_2 \\
&= \frac{-2\sum f(A)(\log f(B) - \log f(C))}{(\log f(A) - \log f(B))(\log f(B) - \log f(C))(\log f(C) - \log f(A))}.
\end{aligned}$$

Here the right hand side equals the value of the logarithmic mean of three variables proposed by Neuman [10] and Mustonen [8], [9].

4. Convexity Associated to Means

Log-convexity is only one of the many variants of convexity now in use, and the problems raised above should be viewed in the more general context of convexity with respect to a pair of means. Given a continuous function $f : E \to F$ and a pair of means, M on E, and N on F, we say that f is (M, N)-*convex* if

$$f\left(M\left(x,y\right)\right) \leq N\left(f(x), f(y)\right)$$

for all $x, y \in E$. Under this terminology the usual convex functions are the same as the (A, A)-convex functions, while the log-convex functions coincide with the (A, G)-functions.

See [11], [12] and [13] for a discussion of convexity from this point of view.

We end our paper by noticing the analogue of the Hermite-Hadamard inequality (HH) in the case of (G, G)-convex functions (also known as the *multiplicatively convex* functions). These are the positive continuous functions $f : I \to \mathbb{R}$ (defined on subintervals of $(0, \infty)$) such that

$$f\left(\sqrt{xy}\right) \leq \sqrt{f(x)f(y)}$$

for all $x, y \in I$. See [11] and [13], for examples and main results.

If $f : [a, b] \to \mathbb{R}$ is (G, G)-convex, then $f \circ \exp : [\log a, \log b] \to \mathbb{R}$ is log-convex and Theorem 2 applies to it. Therefore, for every Borel probability measure μ on $[a, b]$,

$$\begin{aligned}\int_a^b f(x)d\mu &= \int_{\log a}^{\log b} f\left(\exp t\right) d\left(\log \#\mu\right)\\ &\leq \int_{\log a}^{\log b} f(a)^{\frac{\log b - t}{\log b - \log a}} f(b)^{\frac{t - \log a}{\log b - \log a}} d\left(\log \#\mu\right)\\ &= \int_a^b f(a)^{\frac{\log b - \log x}{\log b - \log a}} f(b)^{\frac{\log x - \log a}{\log b - \log a}} d\mu.\end{aligned}$$

On the other hand, by (log-LHH),

$$f(e^{b_{\log \#\mu}}) \leq \exp\left(\int_K \log\, f(x)\, d\mu(x)\right) \leq \int_a^b f(x)d\mu,$$

where

$$b_{\log \#\mu} = \int_{\log a}^{\log b} td\left(\log \#\mu\right) = \int_a^b \log x d\mu$$

is the barycenter of $\log \#\mu$.

For $\mu = \frac{1}{b-a}dx$ this barycenter is precisely the logarithm of the *identric mean*,

$$I(a, b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)},$$

and thus the analogue of the Hermite-Hadamard inequality (HH) in the case of (GG)-convex functions has the form

$$\text{(log-}GG\text{)} \quad \begin{aligned} f(I(a,b)) &\leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{1}{b-a}\int_a^b f(a)^{\frac{\log b - \log x}{\log b - \log a}} f(b)^{\frac{\log x - \log a}{\log b - \log a}} dx\\ &= \frac{bf(b) - af(a)}{\log\left(bf(b)\right) - \log\left(af(a)\right)} \cdot \frac{\log b - \log a}{b - a}\\ &= L\left(af(a), bf(b)\right)/L(a, b).\end{aligned}$$

The second inequality was first noticed in [19].

As in the case of logarithmic mean, the problem of finding the formula of $I(f;\mu)$ in full generality is still open.

An extension of the two-sided inequality (log-GG) to the case of pairs of quasi-arithmetic means will appear in [7].

Last but not least, the Hermite-Hadamard inequality was recently extended by the author of this paper to the setting of spaces with a curved geometry, where the geodesics play the role of segments. See [14]. The problem of establishing a log-convex companion remains open.

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