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A Note on Jensen's inequality for $2D$-Convex Functions

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Abstract. Recently, M. Bencze, C. P. Niculescu and F. Popovici [2] have introduced a notion of 2-dimensional convexity for functions of several variables, together with an analogue of Jensen's inequality. We present here an alternative argument, based on a result due to D. D. Adamović and P. M. Vasić [1].

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In 1965, T. Popoviciu [8] (see also [4], [7, p.171]) proved the following characterization of convex functions:

Theorem 1. *Suppose that f is a real-valued continuous function defined on a nondegenerate interval and n and k are two positive integers such that $n \geq 3$ and $2 \leq k \leq n-1$. Then f is convex if and only if*

$$\sum_{1\leq i_1<\ldots<i_k\leq n} f\left(\frac{1}{k}\sum_{j=1}^{k} x_{i_j}\right) \leq \frac{1}{k} C_{k-2}^{n-2}\left(\frac{n-k}{k-1}\sum_{i=1}^{n} f(x_i) + n\, f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\right),$$

for all $x_1, ..., x_n \in I$.

Corollary 1. *Let $f : I \to \mathbb{R}$ be a continuous function. Then f is convex if, and only if,*

$$\frac{f(x_1)+f(x_2)+f(x_3)}{3} + f\left(\frac{x_1+x_2+x_3}{3}\right) \geq \frac{2}{3}\left[f\left(\frac{x_1+x_2}{2}\right) + f\left(\frac{x_2+x_3}{2}\right) + f\left(\frac{x_3+x_1}{2}\right)\right]$$

for all $x_1, x_2, x_3 \in I$.

A refinement of Corollary 1 was obtained in 2006 by C. P. Niculescu and F. Popovici [5] and in 2009, C. P. Niculescu [3] established the integral form of this corollary.

In 1974, J. C. Burkill gave the weighted version of Corollary 1, under the assumption of twice differentiability of the function f. In 1976, P. M. Vasić, Lj. R. Stanković and V. J. Baston replaced the assumption of twice differentiability by that of continuity. In 1982, A. Lupas provided another proof of Burkill's inequality, while in 1986, J. Pečarić noticed that the weighted case is actually covered by the original argument of Popoviciu.

Motivated by Hlawka's inequality (see [4], p. 100), M. Bencze, C. P. Niculescu and F. Popovici [2] extended Popoviciu's inequality for functions of several variables. For this purpose they introduced a new concept of convex function, that proves to be stronger than the usual one:

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Definition 1. Let U be a convex subset of a real linear space L. A function $f : U \to \mathbb{R}$ is called $2D$-*convex* (that is, 2 *dimensional convex*) if it verifies the inequality

$$\frac{p_1 f(x_1) + p_2 f(x_2) + p_3 f(x_3)}{p_1 + p_2 + p_3} + f\left(\frac{p_1 x_1 + p_2 x_2 + p_3 x_3}{p_1 + p_2 + p_3}\right)$$
$$\geq \frac{1}{p_1 + p_2 + p_3}\left[(p_1 + p_2) f\left(\frac{p_1 x_1 + p_2 x_2}{p_1 + p_2}\right)\right.$$
$$\left. + (p_2 + p_3) f\left(\frac{p_2 x_2 + p_3 x_3}{p_2 + p_3}\right) + (p_3 + p_1) f\left(\frac{p_3 x_3 + p_1 x_1}{p_3 + p_1}\right)\right]$$

for all $x_1, x_2, x_3 \in U$ and all $p_1, p_2, p_3 \geq 0$, with $p_1 + p_2 + p_3 > 0$.

Every $2D$-convex function is convex in the usual sense. The converse works in the case of continuous functions defined on intervals, but not in general. The norm of every 2-dimensional Banach space is 2D-convex. Also, the absolute value of every affine function and all functions of the form

$$f(x_1, x_2) = (ax_1 + bx_2)^2, \quad (x_1, x_2) \in \mathbb{R}^2,$$

are $2D$-convex. Among the many nice features of $2D$-convex functions is the existence of a $2D$ analogue of Jensen's inequality:

Theorem 2. (See [2]). *Let U be a convex subset of a real linear space L. If f is a $2D$-convex function defined on U, then f verifies the following family of inequalities,*

$$(C_{k,n}): \quad f_{k,n}(\mathbf{x}, \mathbf{p}) \leq \frac{n-k}{n-1} f_{1,n}(\mathbf{x}, \mathbf{p}) + \frac{k-1}{n-1} f_{n,n}(\mathbf{x}, \mathbf{p}),$$

where

$$f_{k,n}(\mathbf{x}, \mathbf{p}) := \frac{1}{C_{k-1}^{n-1}} \sum_{1 \leq i_1 < \ldots < i_k \leq n} \left(\sum_{j=1}^{k} p_{i_j}\right) f\left(\frac{\sum_{j=1}^{k} p_{i_j} x_{i_j}}{\sum_{j=1}^{k} p_{i_j}}\right),$$

and $\mathbf{x} = (x_1, ..., x_n) \in U^n$, $n \geq 3$, $k \in \{2, ..., n\}$, *and* $\mathbf{p} = (p_1, ..., p_n)$ *is a positive n-tuple.*

In [2], Theorem 2 is proved by mathematical induction. The aim of the present paper is to provide an alternative argument, via a general result of Adamović and Vasić [1], which we recall here in the variant given in [6] (see also [7, p.176]).

We assume that E is a nonempty subset of a commutative additive semigroup D that verifies the following condition:

If $a_i \in E$ for $i = 1, ..., n$ and $\sum_{i=1}^{n} a_i \in E$, then $\sum_{i \in F} a_i \in E$ for every $F \subset \{1, ..., n\}$, $F \neq \emptyset$.

In addition, we consider a commutative additive group G endowed with a total ordering $\leq$ such that

$$a < b \text{ in } G \text{ implies } a + c < b + c \text{ for every } c \in G.$$

Theorem 3. *Given a function $g : E \to G$ and integers $2 \leq k \leq n$, we consider the condition*

$$(Q_{k,n}): \qquad g_{k,n}(\mathbf{a}) \leq \frac{n-k}{n-1} g_{1,n}(\mathbf{a}) + \frac{k-1}{n-1} g_{n,n}(\mathbf{a}),$$

where

$$g_{k,n}(\mathbf{a}) = g_{k,n}(a_1, ..., a_n) := \frac{1}{C_{k-1}^{n-1}} \sum_{1 \le i_1 < ... < i_k \le n} g\left(\sum_{j=1}^{k} a_{i_j}\right),$$

and $\mathbf{a} \in E^n$ *is an n-tuple such that* $\sum_{i=1}^{n} a_i \in E$. *Then:*

a) $(Q_{2,3}) \Rightarrow (Q_{k,n})$;

b) $(Q_{k,n}) \Rightarrow (Q_{2,3})$ *if* $2 \le k < n$ *and* D *admits a neutral element* 0, $0 \in E$ *and* $f(0) = 0$.

In order to apply Theorem 3, we shall need to turn the linear space L (that appears in Theorem 2) into a weighted space, $D = L \times \mathbb{R}_+$. A point $X = (x,p)$ of D is a point located at x and having mass p. Archimedes' Law of the Lever (from Statics) makes D a commutative semigroup with respect to the following rule of addition:

$$X + Y = (x,p) + (y,p) = \left(\frac{px+qy}{p+q}, p+q\right) \quad \text{for } X, Y \in D.$$

Here $\frac{px+qy}{p+q}$ is precisely the position of the equilibrium of the two mass points (x,p) and (y,p).

If we apply Theorem 3 to $D = L \times \mathbb{R}_+$, $E = U \times \mathbb{R}_+$ and the function $g : E \to \mathbb{R}$ defined by $g(x,p) = pf(x)$, the condition $(Q_{k,n})$ becomes $(C_{k,n})$.

Under these circumstances, it is clear that Theorem 3 (a) yields the conclusion of Theorem 2. But we can prove more. Assuming that $0 \in U$, we obtain that the neutral element (0,0) of D belongs to E. According to Theorem 3 (b), $(C_{k,n}) \Rightarrow (C_{2,3})$, which represents the converse of Theorem 2.

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