STIRLING'S FORMULA AND ITS EXTENSION FOR THE GAMMA FUNCTION

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ABSTRACT. We present new short proofs for both Stirling's formula and Stirling's formula for the Gamma function. Our approach in the first case relies upon analysis of Wallis' formula, while the second result follows from the log-convexity property of the Gamma function.

The well known formula of Stirling asserts that

$$(1) \qquad n! \approx \sqrt{2\pi}n^{n+1/2}e^{-n} \quad \text{as } n \to \infty,$$

in the sense that the ratio of the two sides tends to 1. This provides an efficient estimation to the factorial, used widely in probability theory and in statistical physics.

Articles treating Stirling's formula account for hundreds of items in JSTOR. A few of the most relevant references may be found in [2], [3], [5], and [7].

As was noticed by Stirling himself, the presence of π in the formula (1) is motivated by the Wallis formula,

$$\frac{\pi}{2} = \lim_{n\to\infty} \frac{2\cdot 2\cdot 4\cdot 4\cdots(2n)\cdot(2n)}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdots(2n-1)\cdot(2n-1)\cdot(2n+1)},$$

that is,

$$\lim_{n\to\infty} \frac{1}{\sqrt{2n+1}}\cdot\frac{(2n)!!}{(2n-1)!!} = \sqrt{\frac{\pi}{2}},$$

where $n!! = n\cdot(n-2)\cdots 4\cdot 2$ if n is even, and $n\cdot(n-2)\cdots 3\cdot 1$ if n is odd.

The aim of our note is to provide a short proof of formula (1) based on the Wallis formula and the following remark due to I. Schur concerning the standard sequences defining e.

Lemma (I. Schur; Cf. [6], Problem 168, page 38, solution on page 215). *If $\alpha \in \mathbb{R}$, then the sequence $a_\alpha(n) = \left(1 + \frac{1}{n}\right)^{n+\alpha}$ is decreasing if $\alpha \in [\frac{1}{2}, \infty)$, and increasing for $n \geq N(\alpha)$ if $\alpha \in (-\infty, 1/2)$.*

Proof. The derivative of the function $f(x) = \left(1+\frac{1}{x}\right)^{x+\alpha}$ (defined on $[1,\infty)$) is of the form $f'(x) = \left(1+\frac{1}{x}\right)^{x+\alpha} g(x)$, where $g(x) = \ln\left(1+\frac{1}{x}\right) - \frac{x+\alpha}{x(x+1)}$. Since $g'(x) = \frac{(2\alpha-1)x+\alpha}{x^2(x+1)^2}$ and $\lim_{x\to\infty} g(x) = 0$, it follows that $f'(x) < 0$ when $\alpha \geq 1/2$, and $x \geq 1$, and $f'(x) > 0$ when $\alpha < 1/2$, and $x \geq \max\left\{1, \frac{\alpha}{1-2\alpha}\right\}$. The monotonicity of the sequence $a_\alpha(n)$ is now clear. $\qquad\square$

According to Lemma above, for every $\alpha \in (0, 1/2)$ there is a positive integer $N(\alpha)$ such that

$$\left(1 + \frac{1}{k}\right)^{k+\alpha} < e < \left(1 + \frac{1}{k}\right)^{k+1/2}$$

for all $k \geq N(\alpha)$. As a consequence, we obtain

$$(2) \qquad \prod_{k=n}^{2n-1} \left(1 + \frac{1}{k}\right)^{k+\alpha} < e^n < \prod_{k=n}^{2n-1} \left(1 + \frac{1}{k}\right)^{k+1/2}.$$

That is,

$$\frac{2^\alpha}{\sqrt{n}} \cdot \frac{(2n)!!}{(2n-1)!!} < \frac{n!e^n}{n^{n+1/2}} < \frac{2^{1/2}}{\sqrt{n}} \cdot \frac{(2n)!!}{(2n-1)!!}$$

for all $n \geq N(\alpha)$. Taking into account the Wallis formula, we infer that

$$2^\alpha \sqrt{\pi} \leq \liminf_{n\to\infty} \frac{n!e^n}{n^{n+1/2}} \leq \limsup_{n\to\infty} \frac{n!e^n}{n^{n+1/2}} \leq \sqrt{2\pi},$$

and Stirling's formula follows by passing to the limit as $\alpha \to 1/2$.

Our next goal is to derive from Stirling's formula the following asymptotic formula for the Gamma function:

$$(3) \qquad \Gamma(x+1) \approx \sqrt{2\pi} x^{x+1/2} e^{-x} \quad \text{as } x \to \infty.$$

Recall that

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \quad \text{for } x > 0,$$

which yields $\Gamma(n+1) = n!$ for positive integers n.

Our argument is based on the property of log-convexity of the Gamma function:

$$\Gamma((1-\lambda)x + \lambda y) \leq \Gamma(x)^{1-\lambda} \Gamma(y)^\lambda \text{ for all } x, y > 0 \text{ and } \lambda \in [0,1]$$

(see [4], Theorem 2.2.4, pp. 69-70). This is used to estimate the function

$$f(x) = \frac{\Gamma(x+1)e^x}{x^{x+1/2}}$$

for large values of x.

Indeed, if x is a positive number, then $\lfloor x \rfloor + 1 \leq x + 1 < \lfloor x \rfloor + 2$, which allows us to represent $x + 1$ as a convex combination,

$$x + 1 = (1 - \{x\})(\lfloor x \rfloor + 1) + \{x\}(\lfloor x \rfloor + 2),$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x and $\{x\} = x - \lfloor x \rfloor$. As a consequence,

$$\Gamma(x+1) \leq \Gamma(\lfloor x \rfloor + 1)^{1-\{x\}} \Gamma(\lfloor x \rfloor + 2)^{\{x\}}$$
$$= \lfloor x \rfloor!^{1-\{x\}} (\lfloor x \rfloor + 1)!^{\{x\}}$$
$$\leq \lfloor x \rfloor! (\lfloor x \rfloor + 1)^{x - \lfloor x \rfloor},$$

which yields,

$$f(x) \le \frac{\lfloor x \rfloor! \left(\lfloor x \rfloor + 1 \right)^{x - \lfloor x \rfloor} e^{\lfloor x \rfloor} e^{\{x\}}}{x^{x+1/2}}$$

$$\le f(\lfloor x \rfloor) \frac{e^{\{x\}}}{\left(\frac{1 + \lfloor x \rfloor}{\lfloor x \rfloor} \right)^{\lfloor x \rfloor}} \left(\left(1 + \frac{1 - \{x\}}{x} \right)^{x/(1 - \{x\})} \right)^{1 - \{x\}}$$

(4)
$$\le f(\lfloor x \rfloor) \frac{e^{\{x\}} e^{1 - \{x\}}}{\left(1 + \frac{1}{\lfloor x \rfloor} \right)^{\lfloor x \rfloor}} = f(\lfloor x \rfloor) \frac{e}{\left(1 + \frac{1}{\lfloor x \rfloor} \right)^{\lfloor x \rfloor}}$$

for every $x \ge 1$. In the same way, taking into account that $\lfloor x \rfloor + 1 = \{x\} x + (1 - \{x\})(x+1)$, we obtain

$$\lfloor x \rfloor! = \Gamma(\lfloor x \rfloor + 1) \le \Gamma(x)^{\{x\}} \Gamma(x+1)^{1 - \{x\}} = \frac{\Gamma(x+1)}{x^{x - \lfloor x \rfloor}}.$$

That is, $\lfloor x \rfloor! x^{x - \lfloor x \rfloor} \le \Gamma(x+1)$, whence

$$f(x) = \frac{\Gamma(x+1)e^x}{x^{x+1/2}} \ge f(\lfloor x \rfloor) \left(\frac{\lfloor x \rfloor}{x} \right)^{1/2} \left(\frac{e}{\left(1 + \frac{\{x\}}{\lfloor x \rfloor} \right)^{\lfloor x \rfloor / \{x\}}} \right)^{\{x\}}$$

(5)
$$\ge f(\lfloor x \rfloor) \left(\frac{\lfloor x \rfloor}{x} \right)^{1/2}$$

for every $x \ge 1$.

By Stirling's formula, $\lim_{x \to \infty} f(\lfloor x \rfloor) = \sqrt{2\pi}$, so by (4) and (5) we conclude that

$$\lim_{x \to \infty} \frac{\Gamma(x+1)e^x}{x^{x+1/2}} = \sqrt{2\pi}.$$

This ends the proof of the formula (3).

Remark. Stirling's formula is not the only asymptotic formula for factorial n. Less known, but actually more accurate is Burnside's asymptotic formula [1],

(6)
$$n! \approx \sqrt{2\pi} \left(\frac{n + 1/2}{e} \right)^{n+1/2}.$$

An extension of (6) to positive real values was obtained by Wilton [8]. It is worth mentioning that the arguments used above to prove the formulas (1) and (3) can be adapted with minor changes to derive Burnside's formula (6) and its companion for Gamma function. Indeed, in the first case it suffices to replace (2) by a formula with longer products,

$$\prod_{k=n}^{2n} \left(\frac{k+1}{k} \right)^{k+\alpha} < e^{n+1} < \prod_{k=n}^{2n} \left(\frac{k+1}{k} \right)^{k+1/2},$$

while in the second case we have to combine formula (6) with the double inequality $\lfloor x \rfloor! x^{x - \lfloor x \rfloor} \le \Gamma(x+1) \le \lfloor x \rfloor! \left(\lfloor x \rfloor + 1 \right)^{x - \lfloor x \rfloor}$.

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