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A SHORT PROOF OF BURNSIDE'S FORMULA FOR THE GAMMA FUNCTION

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Abstract. We present simple proofs for Burnside's asymptotic formula and for its extension to positive real numbers.

1. Introduction

Burnside's asymptotic formula for factorial n asserts that

$$n! \sim \sqrt{2\pi}\left(\frac{n+1/2}{e}\right)^{n+1/2}, \tag{B}$$

in the sense that the ratio of the two sides tends to 1 as $n \to \infty$. This provides a more efficient estimation of the factorial, comparing to Stirling's formula,

$$n! \sim \sqrt{2\pi} n^{n+1/2} e^{-n}. \tag{S}$$

Indeed, for $n = 100$, the exact value of 100! with 24 digits is

$$9.332\,621\,544\,394\,415\,268\,169\,924 \times 10^{157}.$$

Burnside's formula yields the approximation

$$100! \approx 9.336\,491\,570\,312\,414\,838\,264\,959 \times 10^{157},$$

while Stirling's formula is less precise, offering only the approximation

$$100! \approx 9.324\,847\,625\,269\,343\,247\,764\,756 \times 10^{157}.$$

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The aim of the present paper is to present a short (and elementary) proof of Burnside's asymptotic formula and to extend it to positive real numbers. The main ingredients are Wallis' s product formula for π and the property of log-convexity of the Gamma function.

2. The proof of Burnside's formula

The starting point is the following result concerning the monotonicity of the function $\left(1+\frac{1}{x}\right)^{x+\alpha}$ on the interval $[1,\infty)$.

Lemma 2.1. *(I. Schur [6], Problem 168, page 38). Let $\alpha \in \mathbb{R}$. The sequence $a_\alpha(n) = \left(1+\frac{1}{n}\right)^{n+\alpha}$ is decreasing if $\alpha \in [\frac{1}{2},\infty)$, and increasing for $n \geq N(\alpha)$ if $\alpha \in (-\infty, 1/2)$.*

According to Lemma 1 above, for $\alpha \in (0,1/2)$ arbitrarily fixed, there is a positive integer $N(\alpha)$ such that

$$\left(1+\frac{1}{k}\right)^{k+\alpha} < e < \left(1+\frac{1}{k}\right)^{k+1/2}$$

for all $k \geq N(\alpha)$. As a consequence,

$$\prod_{k=n}^{2n}\left(\frac{k+1}{k}\right)^{k+\alpha} < e^{n+1} < \prod_{k=n}^{2n}\left(\frac{k+1}{k}\right)^{k+1/2},$$

for all $n \geq N(\alpha)$, equivalently,

$$\frac{(2n+1)^{2n+\alpha}}{n^{n+\alpha}}\cdot\frac{1}{(n+1)\cdots(2n)} < e^{n+1} < \frac{(2n+1)^{2n+1/2}}{n^{n+1/2}}\cdot\frac{1}{(n+1)\cdots(2n)}.$$

This can be restated as

$$\frac{2^{2n+\alpha}\left(n+\frac{1}{2}\right)^{n+1/2}\left(1+\frac{1}{2n}\right)^{n+\alpha}}{\sqrt{n+\frac{1}{2}}}\cdot\frac{n!}{(2n)!} < e^{n+1}$$
$$< \frac{2^{2n+1/2}\left(n+\frac{1}{2}\right)^{n+1/2}\left(1+\frac{1}{2n}\right)^{n+1/2}}{\sqrt{n+\frac{1}{2}}}\cdot\frac{n!}{(2n)!},$$

whence

$$\frac{1}{\sqrt{2n+1}}\cdot\frac{(2n)!!}{(2n-1)!!}\cdot\frac{2^{\alpha+1/2}\left(1+\frac{1}{2n}\right)^{n+\alpha}}{\sqrt{e}} < n!\left(\frac{e}{n+\frac{1}{2}}\right)^{n+1/2}$$
$$< \frac{1}{\sqrt{2n+1}}\cdot\frac{(2n)!!}{(2n-1)!!}\cdot\frac{2\left(1+\frac{1}{2n}\right)^{n+1/2}}{\sqrt{e}}$$

for all $n \geq N(\alpha)$. Here $n!! = n\cdot(n-2)\cdots 4\cdot 2$ if n is even, and $n\cdot(n-2)\cdots 3\cdot 1$ if n is odd.

Taking into account Wallis's formula,

$$\lim_{n\to\infty} \frac{2\cdot 2\cdot 4\cdot 4\cdots (2n)\cdot (2n)}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdots (2n-1)\cdot (2n-1)\cdot (2n+1)} = \frac{\pi}{2},$$

that is,

$$\lim_{n\to\infty} \frac{1}{\sqrt{2n+1}}\cdot\frac{(2n)!!}{(2n-1)!!} = \sqrt{\frac{\pi}{2}},$$

we arrive easily at Burnside's formula for factorial n :

$$n! \sim \sqrt{2\pi}\left(\frac{n+1/2}{e}\right)^{n+1/2}.$$

3. The extension of Burnside's formula for the Gamma function

Our next goal is to derive from Burnside's formula the following asymptotic formula for the Gamma function:

Theorem 3.1. *(R. J. Wilton [7]).* $\Gamma(x+1) \sim \sqrt{2\pi}\left(\frac{x+1/2}{e}\right)^{x+1/2}$ *as* $x\to\infty$.

The proof of the above theorem will be done by estimating the function

$$f(x) = \Gamma(x+1)\left(\frac{e}{x+1/2}\right)^{x+1/2},$$

for large values of x. We shall need the following double inequality:

Lemma 3.2. $\lfloor x\rfloor!\, x^{\{x\}} \le \Gamma(x+1) \le \lfloor x\rfloor!\,(\lfloor x\rfloor+1)^{\{x\}}$ *for all* $x\ge 1$.

Here $\lfloor x\rfloor$ denotes the largest integer less than or equal to x and $\{x\} = x - \lfloor x\rfloor$.

Proof. Our argument is based on the property of log-convexity of the Gamma function:

$$\Gamma((1-\lambda)x+\lambda y) \le \Gamma(x)^{1-\lambda}\Gamma(y)^{\lambda},$$

for all $x, y > 0$ and $\lambda\in[0,1]$. See [5], Theorem 2.2.4, pp. 69-70.

If x is a positive number, then $\lfloor x\rfloor + 1 \le x+1 < \lfloor x\rfloor + 2$, which yields

$$x+1 = (1-\{x\})(\lfloor x\rfloor+1) + \{x\}(\lfloor x\rfloor+2).$$

Therefore,

$$\begin{aligned}\Gamma(x+1) &\le \Gamma(\lfloor x\rfloor+1)^{1-\{x\}}\Gamma(\lfloor x\rfloor+2)^{\{x\}}\\ &= \lfloor x\rfloor!^{1-\{x\}}\,(\lfloor x\rfloor+1)!^{\{x\}}\\ &\le \lfloor x\rfloor!\,(\lfloor x\rfloor+1)^{x-\lfloor x\rfloor}.\end{aligned}$$

In a similar way, taking into account that $\lfloor x\rfloor + 1 = \{x\}\,x + (1-\{x\})(x+1)$, we obtain

$$\lfloor x\rfloor! = \Gamma(\lfloor x\rfloor+1) \le \Gamma(x)^{\{x\}}\Gamma(x+1)^{1-\{x\}} = \frac{\Gamma(x+1)}{x^{x-\lfloor x\rfloor}},$$

whence $\lfloor x\rfloor!x^{x-\lfloor x\rfloor} \le \Gamma(x+1)$. The proof is done. □

According to Lemma 2,

$$\begin{aligned} f(x) &\geq \lfloor x\rfloor!\, x^{\{x\}} \cdot \frac{e^{x+1/2}}{(x+1/2)^{x+1/2}} \\ &= \Gamma(\lfloor x\rfloor+1)\cdot \frac{e^{\lfloor x\rfloor+1/2}}{(\lfloor x\rfloor+1/2)^{\lfloor x\rfloor+1/2}} \cdot \frac{e^{\{x\}}(\lfloor x\rfloor+1/2)^{\lfloor x\rfloor+1/2} x^{\{x\}}}{(x+1/2)^{x+1/2}} \\ &= f(\lfloor x\rfloor)\cdot \left(\frac{\lfloor x\rfloor+1/2}{x+1/2}\right)^{\lfloor x\rfloor+1/2} \cdot e^{\{x\}} \cdot \left(\frac{x}{x+1/2}\right)^{\{x\}} \\ &= f(\lfloor x\rfloor)\cdot \left(\frac{x}{x+1/2}\right)^{\{x\}} \cdot \left(\frac{e}{\left(1+\frac{\{x\}}{\lfloor x\rfloor+1/2}\right)^{\frac{\lfloor x\rfloor+1/2}{\{x\}}}}\right)^{\{x\}} \\ &\geq f(\lfloor x\rfloor)\cdot \left(\frac{x}{x+1/2}\right)^{\{x\}}. \end{aligned} \tag{LW}$$

Similarly,

$$\begin{aligned} f(x) &= \Gamma(x+1)\left(\frac{e}{x+1/2}\right)^{x+1/2} \\ &\leq \lfloor x\rfloor! \left(\frac{e}{x+1/2}\right)^{x+1/2} (\lfloor x\rfloor+1)^{\{x\}} \\ &= f(\lfloor x\rfloor)\left(\frac{\lfloor x\rfloor+1/2}{x+1/2}\right)^{\lfloor x\rfloor+1/2} \left(\frac{\lfloor x\rfloor+1}{x+1/2}\right)^{\{x\}} e^{\{x\}} \\ &= f(\lfloor x\rfloor)\left(\frac{e}{\left(1+\frac{\{x\}}{\lfloor x\rfloor+1/2}\right)^{\frac{\lfloor x\rfloor+1/2}{\{x\}}}}\right)^{\{x\}} \left(\frac{\lfloor x\rfloor+1}{x+1/2}\right)^{\{x\}}. \end{aligned} \tag{RW}$$

The formulas (LW) and (RW) show that

$$\lim_{x\to\infty} f(x) = \lim_{n\to\infty} f(n),$$

and this fact combined with Burnside's formula (B) allows us to conclude that the limit of f at infinity is $\sqrt{2\pi}$, that is,

$$\lim_{x\to\infty} \Gamma(x+1)\left(\frac{e}{x+1/2}\right)^{x+1/2} = \sqrt{2\pi}.$$

This ends the proof of Wilton's asymptotic formula.

It seems very likely that the above technique can be adapted to cover more accurate asymptotic formulas such as that of Gosper [4],

$$n! \sim \sqrt{2\pi\left(n+\frac{1}{6}\right)}\left(\frac{n}{e}\right)^n,$$

and of its extension to real numbers. This is also supported by our joint paper with D. E. Dutkay [3].

Additional information concerning the approximation of the Gamma function may be found in the recent paper of G. D. Anderson, M. Vuorinen and X. Zhang [1].

References

1. G.D. Anderson, M. Vuorinen and X. Zhang, *Topics in Special Functions* III. In vol. *Analytic Number Theory, Approximation Theory and Special Functions* (G.V. Milovanović and M.Th. Rassias eds.), pp. 297–345, Springer, 2014.
2. W. Burnside, *A rapidly convergent series for* $\log N!$, Messenger Math. **46** (1917), 157–159.
3. D.E. Dutkay, C.P. Niculescu and F. Popovici, *A note on Stirling's formula for the Gamma function*, Journal of Prime Research in Mathematics **8** (2012), 1–4.
4. R.W. Gosper, *Decision procedure for indefinite hypergeometric summation*, Proc. Natl. Acad. Sci. USA **75** (1978) 40–42.
5. C.P. Niculescu and L.-E. Persson, *Convex Functions and Their Applications. A Contemporary Approach*, CMS Books in Mathematics, Vol. **23**, Springer-Verlag, New York, 2006.
6. G. Pólya and G. Szegö, *Problems and Theorems in Analysis* I: *Series. Integral Calculus. Theory of Functions*. Reprint of the 1978 edition. Classics in Mathematics, Springer-Verlag, Berlin-Heidelberg-New York, 1998.
7. J.R. Wilton, *A proof of Burnside's formula for* $\log \Gamma(x+1)$ *and certain allied properties of the Riemann* ς*-function*, Messenger Math. **52** (1922), 90–93.

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