CONVERSES OF THE CHAUCHY-SCHWARZ INEQUALITY IN THE $C^{\star}$-FRAMEWORK

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Abstract. We present several multiplicative and addtive converses of the Cauchy-Schwarz inequality in the framework of $C^{\star}-$algebra theory. Our results complements those obtained by M. Fujii, T. Furuta, R. Nakamoto and Sin-Ei Takahasi [4] and S. Izumino, H. Mori and Y. Seo [6].

The classical Cauchy-Schwarz inequality asserts that

$$|< x,y >|^2 \leq < x,x > \cdot < y,y >$$

for every x, y in a vector space E endowed with a hermitian product $< .,. >$. There are two ways to formulate a converse to it. In the *multiplicative* approach (initiated by G. Polya and G. Szegö [11]), we are looking for a positive constant k such that

$$|< x,y >|^2 \geq k \ < x,x > \cdot < y,y >$$

for all x, y in a suitable cone. The restriction to cones is motivated by the formula

$$\cos(x,y) = \frac{< x,y >}{< x,x >^{1/2} \cdot < y,y >^{1/2}}.$$

The *additive* approach (initiated by N. Ozeki [6]) refers to inequalities such as

$$k + |< x,y >|^2 \geq < x,x > \cdot < y,y >$$

with $k > 0$.

The aim of our paper is to discuss both these types of converses in the framework of $C^{\star}-$algebra theory and complements recent papers by M. Fujii, T. Furuta, R. Nakamoto and Sin-Ei Takahasi [4] and S. Izumino, H. Mori and Y. Seo [6].

1. Multiplicative converses

Let $\mathfrak{A}$ be a $C^{\star}-$algebra and let φ be a positive functional on $\mathfrak{A}$. Then the formula

$$< A,B > = \varphi(B^{\star}A)$$

defines a hermitian product on $\mathfrak{A}$ (first considered by Gelfand, Naimark and Segal), so that

$$| < A,B > | \leq \langle A,A\rangle^{1/2} \cdot \langle B,B\rangle^{1/2}$$

for every $A, B \in \mathfrak{A}$.

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A partial (multiplicative) converse of this Cauchy-Schwarz inequality is as follows:

Theorem 1.1. *Suppose that* $A, B \in \operatorname{Re}\mathfrak{A}$ *and*

$$\omega B \leq A \leq \Omega B$$

for some scalars $\omega, \Omega > 0$. *Then*

$$\frac{\operatorname{Re} < A, B >}{\langle A, A\rangle^{1/2} \cdot \langle B, B\rangle^{1/2}} \geq \frac{2}{\sqrt{\frac{\omega}{\Omega}} + \sqrt{\frac{\Omega}{\omega}}}$$

in each of the following two cases:

i) $AB = BA$ *(i.e., A and B commute);*

ii) φ *verifies the condition* $\varphi(XY) = \overline{\varphi(YX)}$ *for every* $X, Y \in \mathfrak{A}$ *(particularly, this is the case if* φ *is a trace).*

Proof. We start noticing the inequality

$$(*) \qquad \operatorname{Re} \varphi \left((A - \Omega B)(A - \omega B)\right) \leq 0.$$

For, when A and B commute, $A - \omega B$ and $\Omega B - A$ are commutative positive elements and thus their square roots commute too. Consequently

$$(A - \omega B)(\Omega B - A) = (A - \omega B)^{1/2} (\Omega B - A) (A - \omega B)^{1/2} \geq 0.$$

In the case ii), we have

$$\begin{aligned} \varphi\left((\Omega B - A)(A - \omega B)\right) &= \varphi\Big((\Omega B - A)^{1/2}(\Omega B - A)^{1/2}(A - \omega B)\Big) = \\ &= \varphi\Big((\Omega B - A)^{1/2}(A - \omega B)(\Omega B - A)^{1/2}\Big) \geq 0. \end{aligned}$$

Once $(*)$ established we have

$$\begin{aligned} 0 &\geq \operatorname{Re}\langle A - \omega B, A - \Omega B\rangle = \\ &= \langle A, A\rangle - (\omega + \Omega) \operatorname{Re}\langle A, B\rangle + \omega\Omega \langle B, B\rangle \end{aligned}$$

which yields

$$\begin{aligned} \left(\sqrt{\frac{\omega}{\Omega}} + \sqrt{\frac{\Omega}{\omega}}\right) \operatorname{Re}\langle A, B\rangle &\geq \frac{1}{\sqrt{\omega\Omega}}\langle A, A\rangle + \sqrt{\omega\Omega}\langle B, B\rangle \geq \\ &\geq \langle A, A\rangle^{1/2} \cdot \langle B, B\rangle^{1/2} + \langle B, B\rangle^{1/2} \cdot \langle A, A\rangle^{1/2}. \blacksquare \end{aligned}$$

Given a self-adjoint element $C \in \mathfrak{A}$, its *spectral bounds* are defined by the formulas

$$\omega_C = \inf \sigma(C), \quad \Omega_C = \sup \sigma(C);$$

accordingly, C is called *strictly positive* (i.e., $C > 0$) if $\omega_C > 0$. If $\mathfrak{A}$ is unital (with unit I) and A and B are strictly positive then

$$\omega_A I \leq A \leq \Omega_A I \quad \text{and} \quad \omega_B I \leq B \leq \Omega_B I$$

which yields

$$\frac{\omega_A}{\Omega_B} B \leq A \leq \frac{\Omega_A}{\omega_B} B.$$

Corollary 1.2. (*W. Greub and W. Rheinboldt* [5]). *If H is Hilbert space and $A, B \in L(H,H)$ are two strictly positive operators such that $AB = BA$, then*

$$\frac{< Ax, Bx >}{< Ax, Ax >^{1/2} < Bx, Bx >^{1/2}} \geq \frac{2}{\sqrt{\frac{\omega_A \, \omega_B}{\Omega_A \, \Omega_B}} + \sqrt{\frac{\Omega_A \, \Omega_B}{\omega_A \, \omega_B}}}$$

for every $x \in \mathbb{R}^n$, $x \neq 0$.

This inequality corresponds to the case where $\mathfrak{A} = L(H,H)$ and φ is the positive functional given by

$$\varphi(A) = < Ax, x > .$$

Notice that $\varphi(AB) = \overline{\varphi(BA)}$ for every self-adjoint operators $A, B \in L(H,H)$.

In turn, the inequality of Greub and Rheinboldt extends many other classical inequalities such as that of Polya and Szegö (which represents the case of diagonal matrices) and that of L. V. Kantorovich (which represents the case where $A, B \in \operatorname{Re} M_n(\mathbb{C})$ and $B = A^{-1}$) :

Corollary 1.3. (*G. Polya and G. Szegö* [11]). *Suppose that $0 < a \leq a_1, ..., a_n \leq A$ and $0 < b \leq b_1, ..., b_n \leq B$. Then*

$$\frac{\sum_{k=1}^{n} a_k b_k}{\left(\sum_{k=1}^{n} a_k^2\right)^{1/2} \left(\sum_{k=1}^{n} b_k^2\right)^{1/2}} \geq \frac{2}{\sqrt{\frac{ab}{AB}} + \sqrt{\frac{AB}{ab}}}.$$

The particular case where $a_k b_k = 1$ for all k has been previously settled by P. Schweitzer. This later case can be further improved on as follows:

$$\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)\left(\frac{1}{n}\sum_{k=1}^{n} \frac{1}{a_k}\right) \leq \frac{(A+a)^2}{4Aa} + \frac{[1+(-1)^{n-1}](A-a)^2}{8Aan^2}$$

for every $0 < a \leq a_1, ..., a_n \leq A$.

The corresponding continuous analogue (as well as the weighted analogue) also works. More generally, if (X, Σ, μ) is a probability space and $f, g \in L^{\infty}(\mu)$, with $0 \leq a \leq f \leq A$, $0 \leq b \leq g \leq B$, then

$$\int_X fg \, d\mu \geq \frac{2}{\sqrt{\frac{ab}{AB}} + \sqrt{\frac{AB}{ab}}} \left(\int_X f^2 \, d\mu\right)^{1/2} \left(\int_X g^2 \, d\mu\right)^{1/2}.$$

This fact corresponds to the commutative $C^{\star}-$algebra $L^{\infty}(\mu)$ and the positive functional

$$\varphi(f) = \int_X f \, d\mu.$$

Corollary 1.4.

$$\frac{Trace\ AB}{(Trace\ A^2)^{1/2}\,(Trace\ B^2)^{1/2}} \geq \frac{2}{\sqrt{\frac{\omega_A \, \omega_B}{\Omega_A \, \Omega_B}} + \sqrt{\frac{\Omega_A \, \Omega_B}{\omega_A \, \omega_B}}}$$

for every strictly positive matrices $A, B \in \operatorname{Re} M_n(\mathbb{C})$.

This result corresponds to the case where $\mathfrak{A} = M_n(\mathbb{C})$ and $\varphi = Trace$. Of course, we can replace $M_n(\mathbb{C})$ by the ideal of all Hilbert-Schmidt operators on a Hilbert space, due to the fact that the product of any two such operators is of trace class.

2. An additive converse

In the $C^{\star}-$algebra framework the $AM-QM$ inequality works as follows:

$$\left|\frac{1}{n}\sum_{k=1}^{n} A_k\right|^2 \leq \frac{1}{n}\sum_{k=1}^{n} |A_k|^2 \tag{2.1}$$

for all families $A_1, ..., A_n$ of elements in a unital $C^{\star}-$algebra; as usually the modulus is defined by the formula $|T|^2 = T^{\star}T$.

We can formulate a partial additive converse to it, which for $\mathfrak{A} = \mathbb{C}$ is due to L. G. Khanin [8]:

Proposition 2.1. *Let $\mathfrak{A}$ be a unital $C^{\star}-$algebra, with unit I and let $A_1, ..., A_n$ be positive elements in $\mathfrak{A}$, with $0 \leq m \cdot I \leq A_1, ..., A_n \leq M \cdot I$. Then*

$$\frac{1}{n}\sum_{k=1}^{n} A_k^2 - \left(\frac{1}{n}\sum_{k=1}^{n} A_k\right)^2 \leq \frac{(M-m)^2}{4} \cdot I.$$

The equality occurs when n *is odd, half of the A_n's are $m \cdot I$ and half are $M \cdot I$.*

Proof. In fact, the functional calculus with self-adjoint elements assures us that

$$0 \leq (M \cdot I - A)(A - m \cdot I) \leq \frac{(M-m)^2}{4} \cdot I \tag{2.2}$$

for every $A \in \mathfrak{A}$ such that $m \cdot I \leq A \leq M \cdot I$. The left side inequality in (2.2) yields $A_k^2 \leq (M+m)A_k - Mm \cdot I$ and thus

$$\begin{aligned} \frac{1}{n}\sum_{k=1}^{n} A_k^2 - \left(\frac{1}{n}\sum_{k=1}^{n} A_k\right)^2 &\leq (M+m)\left(\frac{1}{n}\sum_{k=1}^{n} A_k\right) - Mm \cdot I - \left(\frac{1}{n}\sum_{k=1}^{n} A_k\right)^2 \\ &\leq \left(M \cdot I - \frac{1}{n}\sum_{k=1}^{n} A_k\right)\left(\frac{1}{n}\sum_{k=1}^{n} A_k - m \cdot I\right) \\ &\leq \frac{(M-m)^2}{4} \cdot I \end{aligned}$$

the last step being motivated by the right side inequality in (2.2). ■

Based on the variance inequality in the noncommutative probability theory, S. Izumino, H. Mori and Y. Seo [6], have obtained another additive converse of the Cauchy-Schwarz inequality in the noncommutative setting:

Proposition 2.2. *Let A and B be positive operators on the Hilbert space H, satisfying $0 < m_1 I \leq A \leq M_1 I$ and $0 < m_1 I \leq A \leq M_1 I$ respectively. Then for any unit vector $x \in H$,*

$$\langle A^2 x, x\rangle\langle B^2 x, x\rangle - \langle A^2 \natural_{1/2} B^2 x, x\rangle^2 \leq \frac{1}{4\gamma^2}\left(M_1 M_2 - m_1 m_2\right)^2$$

where $\gamma = \max\{m_1/M_1, m_2/M_2\}$ and $A^2 \natural_{1/2} B^2$ denotes the Kubo-Ando geometric mean of A^2 and B^2 i.e.,

$$A^2 \natural_{1/2} B^2 = A(A^{-1}B^2 A^{-1})^{1/2} A.$$

3. Hilbert $C^\star-$Modules and Cauchy-Schwarz Inequality

Let $\mathcal{B}$ be a $C^\star-$ algebra with norm $||\cdot||$.

A *pre-Hilbert* $\mathcal{B}-$*module* is a complex vector space E which is also a right $\mathcal{B}-$module equipped with a map $< .,. >: E \times E \to \mathcal{B}$, which is linear in the first variable and satisfies the following relations for all $x, y \in E$ and all $b \in \mathcal{B}$:

i) $< x, x > \geq 0$
ii) $< x, y >^\star = < y, x >$
iii) $< xb, y > = < x, y > b$.

It is easy to see that the scalar multiplication and the right $\mathcal{B}-$module structure of E are compatible in the sense that

$$(\lambda x)b = \lambda(xb) = x(\lambda b)$$

for every $\lambda \in \mathbb{C}$, $x \in E$, $b \in \mathcal{B}$.

Every $C^\star-$ algebra can be seen as a pre-Hilbert module over itself letting

$$< A, B > = B^\star A.$$

A more sophisticated example is $E = \mathcal{H}_B$, the space of all sequences $(A_n)_n$ of elements of $\mathcal{B}$ such that $\sum_n A_n^\star A_n$ converges. In this case,

$$< (A_n)_n, (B_n)_n > = \sum_n B_n^\star A_n.$$

Let us mention also that every complex vector space endowed with a hermitian product constitutes a pre-Hilbert $\mathbb{C}-$module.

Lemma 3.1. (*Paschke's extension of the the Cauchy-Schwarz inequality*). *Let E be a pre-Hilbert $\mathcal{B}-$module and set*

$$||x|| = ||< x, x >||^{1/2}, \quad x \in E.$$

Then $E = (E, ||\cdot||)$ is a normed vector space and the following inequalities hold:

$$\begin{aligned} ||xb|| &\leq ||x|| \cdot ||b|| \\ ||< x, y >|| &\leq ||x|| \cdot ||y|| \end{aligned}$$

for every $x, y \in E$ and every $b \in \mathcal{B}$.

See [10], or [7], for details.

However, it is conceivable that like in the case of the triangle inequality, a stronger form of the Cauchy-Schwarz inequality (avoiding the presence of the norms) works in the setting of pre-Hilbert $\mathcal{B}-$modules. During the 17th Conference on Operator Theory in Timişoara (June 22-26, 1998) we proposed several candidates such as:

$$|< x, y >| \leq \frac{1}{2}\left(u^\star < x, x >^{1/2} u + v^\star < y, y >^{1/2} v\right) \tag{3.1}$$

where u and v are suitable elements of $\mathcal{B}$ with $||u|| \leq ||y||^{1/2}$ and $||v|| \leq ||x||^{1/2}$.

Notice that (3.1) is straightforward in the commutative case.

Leaving open the problem mentionned above, we end this paper with the following result, representing a converse Cauchy-Schwarz type inequality:

Proposition 3.2. *Let E be a pre-Hilbert $\mathcal{B}-$module. Then*

$$\operatorname{Re} < x, y > \geq \frac{1}{\sqrt{\frac{\omega}{\Omega}} + \sqrt{\frac{\Omega}{\omega}}}\left(< x, x >^{1/2} \cdot < y, y >^{1/2} + < y, y >^{1/2} \cdot < x, x >^{1/2}\right)$$

for every $x, y \in E$ *and every* $\omega, \Omega > 0$ *for which* $\mathrm{Re} < x - \omega y, x - \Omega y > \leq 0$.

Proof. In fact, by our hypothesis,

$$\begin{aligned} 0 &\geq \mathrm{Re} < x - \omega y, x - \Omega y > = \\ &= < x, x > - (\omega + \Omega)\,\mathrm{Re} < x, y > + \omega\Omega < y, y > \end{aligned}$$

which yields

$$\begin{aligned} \left(\sqrt{\frac{\omega}{\Omega}} + \sqrt{\frac{\Omega}{\omega}}\right)\mathrm{Re} &< x, y > \geq \frac{1}{\sqrt{\omega\Omega}} < x, x > + \sqrt{\omega\Omega} < y, y > \geq \\ &\geq < x, x >^{1/2} \cdot < y, y >^{1/2} + < y, y >^{1/2} \cdot < x, x >^{1/2}. \blacksquare \end{aligned}$$

Corollary 3.3. *Let* E *be a vector space endowed with a hermitian product* $< ., . >$. *Then*

$$\frac{\mathrm{Re} < x, y >}{< x, x >^{1/2} \cdot < y, y >^{1/2}} \geq \frac{2}{\sqrt{\frac{\omega}{\Omega}} + \sqrt{\frac{\Omega}{\omega}}}$$

for every $x, y \in E$ *and every* $\omega, \Omega > 0$ *for which* $\mathrm{Re} < x - \omega y, x - \Omega y > \leq 0$.

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