SOLVABILITY OF AN ELLIPTIC SYSTEM WITH DISCONTINUOUS NONLINEARITY AND L^1 DATA

Constantin P. Niculescu and Nicolae Ţarfulea*
Department of Mathematics, University of Craiova, Romania

Abstract

In this paper we prove that there exists a solution for the semilinear elliptic system -$\triangle u = f(x,u) - v + p$ in Ω, -$\triangle v = \delta u - \gamma v + h$ in Ω, $u = v = 0$ on $\partial\Omega$, where $\Omega \subset \mathbf{R}^N$ $(2 \leq N \leq 7)$ is a bounded domain with smooth boundary, δ and γ are positive constants, f is a discontinuous sublinear nonlinearity with some specific properties and p and h belong to $L^1(\Omega)$.

Key words: elliptic systems, variational methods, discontinuous nonlinearities.
AMS Subject Classification: 35 R 05, 35 R 50.

1 Introduction

The purpose of this note is to study the elliptic system:

$$(P)\quad \begin{cases} -\Delta u = f(x,u) - v + p & \text{in } \Omega \\ -\Delta v = \delta u - \gamma v + h & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbf{R}^N$ $(2 \leq N \leq 7)$, δ and γ are positive constants such that $\gamma + \lambda_1 > \sqrt{\delta}$, p and h belong to $L^1(\Omega)$ and $f : \Omega \times \mathbf{R} \to \mathbf{R}$ satisfies some properties, which will be mentioned latter.

*Appeared in *Commun. Appl. Nonlinear Anal.* **6** (1999), no. 3, pp. 49-58.

The solutions (u, v) of this system represent steady state solutions of reaction diffusion systems derived from several applications, such as mathematical biology, chemical reactions and combustion theory. There is an extensive bibliography concerning this subject (see [CF], [FM], [LM], [Ro], [Si] and references therein). The review, even partial, of their results is out of the scope of this note.

Notice that the second equation in (P) yields v in terms of u. Then (P) is equivalent to the integro-differential equation

$$(P') \begin{cases} -\Delta u + Bu = f(x, u) + q(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Bu is the solution of the problem

$$\begin{cases} -\Delta v + \gamma v = \delta u & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

and $q = p - \dfrac{1}{\delta} B(h) \in L^1(\Omega)$. In other words $B = \delta(-\Delta + \gamma)^{-1}$, under zero Dirichlet boundary conditions on $\partial\Omega$.

By the L^p theory of linear elliptic equations, B can be viewed as a bounded linear operator from $L^p(\Omega)$ into $W^{2,p}(\Omega) \cap H_0^1(\Omega)$; also, by the Schauder theory, B maps the Hölder space $C^\alpha(\overline{\Omega})$ into $C^{2+\alpha}(\overline{\Omega})$.

Let us define the operator

$$T \equiv -\Delta + B : L^2(\Omega) \to L^2(\Omega), \text{ with } D(T) = H^2(\Omega) \cap H_0^1(\Omega).$$

It is easy to observe that T is symmetric on its domain $D(T)$ i.e., $\langle Tu_1, u_2 \rangle = \langle u_1, Tu_2 \rangle$ for all $u_1, u_2 \in D(T)$, where $\langle \cdot, \cdot \rangle$ denotes the L^2- inner product.

If $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \dots$ and $(\varphi_k)_k$ denote respectiely the eigenvalues and the eigenfunctions of $-\Delta$ in Ω under zero Dirichlet boundary conditions, then one can verify easily that the ${\varphi_k}'s$ are also eigenfunctions of T corresponding to the modified eigenvalues

$$\tilde{\lambda}_k = \lambda_k + \frac{\delta}{\gamma + \lambda_k}, \; k = 1, 2, \dots .$$

A more detailed analysis shows that the spectrum $\sigma(T)$ of T consists precisely of these eigenvalues; this is a simple consequence of the fact that for every $\lambda \in \rho(T) = \mathbf{C} \setminus \sigma(T)$, the resolvent operator $T_\lambda = (T - \lambda I)^{-1}$ is compact (see [FM], Corollary 1.2).

We know that T_λ is a positive operator if $\gamma + \lambda_1 > \sqrt{\delta}$ and $2\sqrt{\delta} - \gamma \leq \lambda < \tilde{\lambda}_1$ (see [FM], Corollary 1.3). This is a maximum principle for the equation

$$\begin{cases} -\Delta u + Bu - \lambda u = g(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, it says that a strong maximum principle holds: if $g \in C(\Omega)$ and $g \geq 0$ in Ω, then $u > 0$ in Ω and the outward normal derivative satisfies the inequality $\dfrac{\partial u}{\partial \nu} < 0$.

Suppose that $f : \Omega \times \mathbf{R} \to \mathbf{R}$ satisfies the following assumptions:

(f1) f is a Caratheodory function on $\Omega \times (\mathbf{R} \setminus A)$, where $A \subset \mathbf{R}$ is a set with no finite point of accumulation (in fact, we will take $A = \{a\}$ since that will simplify the notations. The arguments in the general case are quite similar.);

(f2) There exists $m \in \rho(T) \cap (-\tilde{\lambda}_1, \gamma - 2\sqrt{\delta}]$ such that the function $\omega : \Omega \times \mathbf{R} \to \mathbf{R}$, $\omega(x,t) = mt + f(x,t)$, is strictly monotone in the second variable;

(f3) There exist $c, k \in \mathbf{R}$, with $0 < k < \tilde{\lambda}_1$, such that: $|f(x,t)| \leq c + k|t|$, for every $(x,t) \in \Omega \times \mathbf{R}$.

Put

$$\sigma_+(x) = \limsup_{t \to a} f(x,t) - f(x,a) \;\text{ and }\; \sigma_-(x) = f(x,a) - \liminf_{t \to a} f(x,t).$$

Definition 1 *A pair of functions $(u,v) \in E \times E$, where $E = \bigcap\limits_{1 \leq p < \frac{N}{N-1}} W_0^{1,p}(\Omega)$, is said to be a solution of the problem (P) if*

$$\int_\Omega \nabla u \cdot \nabla \varphi \, dx - \int_\Omega f(x,u)\,\varphi \, dx + \int_\Omega v\varphi \, dx - \int_\Omega p\varphi \, dx \in$$

$$\in \Big[\int_{[u=a,\,\varphi<0]} \sigma_+(x)\,\varphi \, dx - \int_{[u=a,\,\varphi>0]} \sigma_-(x)\,\varphi \, dx, \int_{[u=a,\,\varphi>0]} \sigma_+(x)\,\varphi \, dx -$$

$$- \int_{[u=a,\,\varphi<0]} \sigma_-(x)\,\varphi \, dx \Big]$$

and
$$\int_\Omega \nabla v \cdot \nabla \varphi \, dx - \delta \int_\Omega u\varphi \, dx + \gamma \int_\Omega v\varphi \, dx - \int_\Omega h\varphi \, dx = 0,$$
for every $\varphi \in C^2\left(\overline{\Omega}\right)$, *with* $\varphi = 0$ *on* $\partial\Omega$.

Definition 2 *A function* $u \in E = \bigcap_{1 \le p < \frac{N}{N-1}} W_0^{1,p}(\Omega)$ *is said to be a solution of the problem* (P') *if*
$$\int_\Omega \nabla u \cdot \nabla \varphi \, dx + \int_\Omega \varphi B u \, dx - \int_\Omega f(x,u)\, \varphi \, dx - \int_\Omega q\varphi \, dx \in$$
$$\in \Big[\int_{[u=a,\, \varphi<0]} \sigma_+(x)\, \varphi \, dx - \int_{[u=a,\, \varphi>0]} \sigma_-(x)\, \varphi \, dx, \int_{[u=a,\, \varphi>0]} \sigma_+(x)\, \varphi \, dx -$$
$$- \int_{[u=a,\, \varphi<0]} \sigma_-(x)\, \varphi \, dx \Big],$$
for every $\varphi \in C^2\left(\overline{\Omega}\right)$, *with* $\varphi = 0$ *on* $\partial\Omega$.

It is easy to observe that u is a solution of the problem (P') iff the pair $\left(u, v = Bu + \frac{1}{\delta} Bh\right)$ is a solution of the problem (P).

The main result of this paper is as follows:

Theorem 3 *The problem* (P) *has a solution.*

2 The L^2 case

In this section we consider the problem (P) under the above conditions, except the fact that p and h belong to $L^2(\Omega)$. Then, also $q \in L^2(\Omega)$.

As in [AB], we set
$$T_a(x) = \left[\liminf_{t\to a} f(x,t), \limsup_{t\to a} f(x,t)\right]$$
and
$$\tilde{f}(x,s) = \begin{cases} f(x,s) & \text{if} \quad (x,s) \notin \Omega \times \{a\} \\ T_a(x) & \text{if} \quad (x,s) \in \Omega \times \{a\} \end{cases}.$$

Definition 4 *A pair* $(u,v) \in \left(H_0^1(\Omega) \cap H^2(\Omega)\right)^2$ *is said to be a solution of the problem* (P) *if*

$$-\triangle u(x) + v(x) - p(x) \in \tilde{f}(x, u(x)) \ \ a.e. \ \ in \ \Omega;$$

$$-\triangle v(x) + \gamma v(x) = \delta u(x) \ \ a.e. \ \ in \ \Omega.$$

Definition 5 *A function* $u \in H_0^1(\Omega) \cap H^2(\Omega)$ *is said to be a solution of the problem* (P') *if*

$$-\triangle u(x) + Bu(x) - q(x) \in \tilde{f}(x, u(x)) \ \ a.e. \ \ in \ \Omega.$$

Clearly: u is a solution of the problem (P') iff the pair $\left(u, v = Bu + \dfrac{1}{\delta} Bh\right)$ is a solution of the problem (P).

By **(f2)** it is possible to define a single-valued function $g : \Omega \times \mathbf{R} \to \mathbf{R}$ by letting

$$g(x,t) = \begin{cases} a, & \text{if } t - ma \in T_a(x) \\ s, & \text{with } \omega(x,s) = t, \ \text{if } t - ma \notin T_a(x) \end{cases}.$$

Set $G(x,t) = \int\limits_0^t g(x,\tau)\, d\tau$. Since $|g(x,t)| \le c_1 + c_2|t|$, then $|G(x,t)| \le c_3|t| + c_4|t|^2$. Hence

$$G(x,u) \in L^1(\Omega), \ \text{if } u \in L^2(\Omega).$$

As in [AB], we consider the functional $J : L^2(\Omega) \to \mathbf{R}$, given by the formula

$$J(u) = \int\limits_{\Omega} \left\{ G(x,u) - \frac{1}{2} u T_{-m} u - u T_{-m} q \right\} dx.$$

Clearly, J is well defined on $L^2(\Omega)$ and, in a standard way, one can prove that $J \in C^1\left(L^2(\Omega), \mathbf{R}\right)$, with

$$dJ(u)\varphi = \int\limits_{\Omega} \left(g(x,u) - T_{-m}u - T_{-m}q\right) \varphi \, dx, \ \ \forall \varphi \in L^2(\Omega).$$

The following result uses some ideas and techniques of Ambrosetti and Badiale (see [AB],Thm. 1).

Theorem 6 *Under the aforementioned conditions, there exists a solution* $u \in H_0^1(\Omega) \cap H^2(\Omega)$ *of the problem* (P'). *Moreover, the set*

$$\Omega_a = \{x \in \Omega : u(x) = a\}$$

has Lebesgue measure $|\Omega_a| = 0$ *and therefore* u *satisfies*

$$-\triangle u(x) + Bu(x) - q(x) = f(x, u(x)) \ \text{ a.e. in } \Omega.$$

Proof. From **(f2),** we have that

$$G(x,u) \geq \frac{1}{2(k+m)} u^2 - c_5|u|. \tag{2.1}$$

Let $u \in L^2(\Omega)$ be an arbitrary element and let $\varphi = T_{-m}u \in H_0^1(\Omega) \cap H^2(\Omega)$. Then

$$-\triangle \varphi + B\varphi + m\varphi = u.$$

Multiplying both sides by φ and integrating over Ω we get

$$\int_\Omega (|\nabla\varphi|^2 + \varphi B\varphi)\, dx + m \int_\Omega \varphi^2 dx = \int_\Omega u\varphi\, dx \leq \| u \|_2 \cdot \| \varphi \|_2 . \tag{2.2}$$

According to [FM], Remark 1.6, we have the inequality

$$\int_\Omega (|\nabla\varphi|^2 + \varphi B\varphi) dx \geq \tilde{\lambda}_1 \| \varphi \|_2, \ (\forall)\ \varphi \in H_0^1(\Omega). \tag{2.3}$$

and thus by (2.2) and (2.3) we infer that

$$\int_\Omega uT_{-m}u\, dx \leq \| u \|_2 \| T_{-m}u \|_2 \leq \frac{1}{\tilde{\lambda}_1 + m} \| u \|_2^2, \ (\forall)\ u \in L^2(\Omega). \tag{2.4}$$

From (2.1) and (2.4) it follows that

$$J(u) \geq \left(\frac{1}{2(k+m)} - \frac{1}{2(\tilde{\lambda}_1 + m)} \right) \| u \|_2^2 - c_6 \| u \|_2, \ (\forall)\ u \in L^2(\Omega).$$

Since $0 < k < \tilde{\lambda}_1$, J is bounded from below and coercive. Then by [St], Thm. 1.2, it follows that there exists a $u_0 \in L^2(\Omega)$ such that

$$J(u_0) = \min_{L^2(\Omega)} J(u). \tag{2.5}$$

Set $u = T_{-m}(u_0 + q) \in H_0^1(\Omega) \cap H^2(\Omega)$. Since u_0 is a critical point for J, it follows that

$$\int_{\Omega} \left(g(x, u_0) - T_{-m}u_0 - T_{-m}q\right)\varphi \, dx = 0, \ (\forall) \ \varphi \in L^2(\Omega)$$

so that $g(x, u_0) \overset{a.e.}{=} T_{-m}(u_0 + q)$. Hence

$$-\triangle u(x) + Bu(x) - q(x) = u_0 - mu. \tag{2.6}$$

Since $u_0(x) - mu(x) \in \tilde{f}(x, u(x))$ a.e. in Ω, it follows that u is a solution of the problem (P').

Our next goal is to prove that $|\Omega_a| = 0$.

Because $u \in H^2(\Omega)$, a well known result of Stampacchia shows that $\triangle u(x) = 0$ a.e. in Ω_a. Then

$$Bu(x) - q(x) \in \tilde{f}(x, u(x)) \text{ a.e. in } \Omega$$

which implies

$$Bu(x) - q(x) \in T_a(x) \text{ a.e. in } \Omega_a.$$

For $\varepsilon > 0$ small enough, we have

$$Bu(x) - q(x) + \varepsilon\chi(x) \in T_a(x) \text{ a.e. in } \Omega_a,$$

where $\chi \in L^2(\Omega)$ is the function

$$\chi(x) = \begin{cases} -1, & \text{if } Bu(x) - q(x) \geq \frac{1}{2}\left(\liminf_{t\to a} f(x,t) + \limsup_{t\to a} f(x,t)\right) \\ 1, & \text{if } Bu(x) - q(x) < \frac{1}{2}\left(\liminf_{t\to a} f(x,t) + \limsup_{t\to a} f(x,t)\right) \\ 0, & \text{if } x \in \Omega \setminus \Omega_a \end{cases}.$$

Clearly,

$$0 \geq \frac{d}{d\varepsilon}J(u_0 + \varepsilon\chi) = dJ(u_0 + \varepsilon\chi)(\chi) = \int_{\Omega_a} \left[g(x, u_0 + \varepsilon\chi) - \varepsilon T_{-m}\chi - u\right]\chi \, dx =$$

$$= \int_{\Omega_a} g(x, u_0 + \varepsilon\chi)\chi \, dx - \varepsilon \int_{\Omega_a} \chi T_{-m}\chi dx - a \int_{\Omega_a} \chi \, dx.$$

From (2.6) we obtain

$$u_0 + \varepsilon\chi = Bu - q + ma + \varepsilon\chi \text{ a.e. in } \Omega.$$

This implies that

$$g\left(x, u_0 + \varepsilon\chi\right) = a \text{ a.e. in } \Omega.$$

Consequently

$$\frac{d}{d\varepsilon} J\left(u_0 + \varepsilon\chi\right) = a \int_{\Omega_a} \chi \, dx - \varepsilon \int_{\Omega_a} \chi T_{-m}\chi \, dx - a \int_{\Omega_a} \chi \, dx = -\varepsilon \int_{\Omega_a} \chi T_{-m}\chi \, dx \leq 0.$$

Now, from the positivity of T_{-m}, we conclude that $\chi \equiv 0$ a.e. in Ω_a. Then $|\Omega_a| = 0$ and this ends the proof. ■

3 Proof of Theorem 1.1

Let $(q_n)_{n\geq 1}$ be an arbitrary sequence in $L^2(\Omega)$ such that

$$q_n \to q \text{ in } L^1(\Omega), \text{ as } n \to \infty,. \tag{3.1}$$

According to Theorem 2.1, for each $n \in \mathbf{N}$, the problem

$$(P_n) \quad \begin{cases} -\triangle u + Bu = f(x, u) + q_n & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

admits a solution $u_n \in H_0^1(\Omega) \cap H^2(\Omega)$ with the following properties:

1) $u_n = T_{-m}(u_0 + q_n) = g(x, u_{0n})$, where u_{0n} is the global minimizer of the functional $J_n : L^2(\Omega) \to \mathbf{R}$, given by

$$J_n(u) = \int_{\Omega} \{G(x, u) - \frac{1}{2} u T_{-m} u - u T_{-m} q_n\} \, dx;$$

2) $|\Omega_{a,n}| = |\{x \in \Omega : u_n(x) = a\}| = 0$.

We shall prove that the sequence $\{u_n\}_{n\geq 1}$ is bounded in $L^1(\Omega)$. For, observe that

$$0 = J_n(0) \geq J_n(u_n) \geq \left(\frac{1}{2(k+m)} - \frac{1}{2(\tilde{\lambda}_1 + m)}\right) \| u_n \|_2^2 - \int_{\Omega} u_n T_{-m} q_n \, dx.$$

Since T_{-m} is a self-adjoint operator on $L^2(\Omega)$, we have

$$0 \geq \left(\frac{1}{2(k+m)} - \frac{1}{2(\tilde{\lambda}_1 + m)}\right) \| u_n \|_2^2 - \int_{\Omega} q_n T_{-m} u_n \, dx. \tag{3.2}$$

Because $u_n \in H_0^1(\Omega) \cap H^2(\Omega)$, we can infer, via standard regularity results (see [B], Thm. IX.25), that $T_{-m}u_n \in H^4(\Omega)$. Or,

$$H^4(\Omega) \hookrightarrow C\left(\overline{\Omega}\right), \tag{3.3}$$

so that $T_{-m}u_n \in C\left(\overline{\Omega}\right)$. Here we need the assumption that $N \leq 7$. From this fact and (3.2), we obtain

$$\| T_{-m}u_n \|_\infty \int_\Omega q_n \, dx \geq \left(\frac{1}{2(k+m)} - \frac{1}{2(\tilde{\lambda}_1 + m)} \right) \| u_n \|_2^2 . \tag{3.4}$$

By (3.1), (3.3) and (3.4) we infer the existence of a constant $c_7 > 0$ such that

$$c_7 \| u_n \|_2^2 \leq \| T_{-m}u_n \|_{H^4} .$$

Because of the continuity of T_{-m} and an estimate given in [B], Thm. IX.25, we can find a constant $c_8 > 0$ such that

$$\| u_n \|_2 \leq c_8, \; (\forall) \, n \geq 1.$$

Moreover, the sequence $(u_n)_{n \geq 1}$ is bounded in $W_0^{1,p}(\Omega)$, for every $1 \leq p < \dfrac{N}{N-1}$. In fact,

$$\| u_n \|_{1,p} = \sup_{\|w\|_{1,p'}=1} \int_\Omega \nabla u_n \cdot \nabla w dx \leq$$

$$\leq \sup_{\|w\|_{1,p'}=1} \int_\Omega f(x, u_n) \, w \, dx + \sup_{\|w\|_{1,p'}=1} \int_\Omega q_n w \, dx + \sup_{\|w\|_{1,p'}=1} \int_\Omega w B u_n \, dx \leq$$

$$\leq \sup_{\|w\|_{1,p'}=1} \| w \|_\infty \int_\Omega (c + k|u_n|) \; dx + \sup_{\|w\|_{1,p'}=1} \| w \|_\infty \int_\Omega |q_n| \, dx + c_9 \| u_n \|_p \leq C.$$

Here we have used the fact that $p' > N$ and thus $W^{1,p'}(\Omega) \overset{\text{compact}}{\hookrightarrow} C\left(\overline{\Omega}\right)$.

By the compactness of the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^1(\Omega)$, there exists a $u \in \bigcap_{1 \leq p < \frac{N}{N-1}} W_0^{1,p}(\Omega)$ such that

$$u_n \longrightarrow u \text{ a.e. and strongly in } L^1(\Omega);$$

$$u_n \rightharpoonup u \text{ weakly in } W_0^{1,p}(\Omega).$$

Then from **(f2)** it follows that

$$f(x, u_n(x)) \to f(x, u(x)) \text{ a.e. on } [u \neq a]$$

and the Dominated Convergence Theorem yields

$$\int\limits_{[u\neq a]} f(x, u_n(x))\, dx \longrightarrow \int\limits_{[u\neq a]} f(x, u(x))\, dx.$$

Multiplying (P_n) with $\varphi \in C^2(\Omega)$, $\varphi = 0$ on $\partial\Omega$, and integrating over Ω, we obtain

$$\int\limits_{\Omega} \nabla u_n \cdot \nabla\varphi\, dx + \int\limits_{\Omega} \varphi B u_n\, dx - \int\limits_{[u\neq a]} f(x, u_n(x))\, \varphi\, dx - \int\limits_{[u=a]} f(x, u(x))\, \varphi\, dx -$$

$$- \int\limits_{\Omega} q_n \varphi\, dx = \int\limits_{[u=a]} f(x, u_n(x))\, \varphi\, dx - \int\limits_{[u=a]} f(x, u(x))\, \varphi\, dx. \tag{3.5}$$

It is easy to observe that the left hand of the above equality tends to

$$\int\limits_{\Omega} \nabla u \cdot \nabla\varphi\, dx + \int\limits_{\Omega} \varphi B u\, dx - \int\limits_{\Omega} f(x, u(x))\, \varphi\, dx - \int\limits_{\Omega} q\varphi\, dx. \tag{3.6}$$

For the right hand, notice that by Fatou Lemma we have

$$\liminf_{n\to\infty} \int\limits_{[u=a]} \left(f(x, u_n(x)) - f(x, u(x))\right) \varphi\, dx \geq$$

$$\geq \liminf_{n\to\infty} \int\limits_{[u=a,\varphi>0]} \left(f(x, u_n(x)) - f(x, u(x))\right) \varphi\, dx +$$

$$+ \liminf_{n\to\infty} \int\limits_{[u=a,\varphi<0]} \left(f(x, u_n(x)) - f(x, u(x))\right) \varphi\, dx \geq$$

$$\geq \int\limits_{[u=a,\varphi<0]} \sigma_+(x)\, \varphi(x)\ dx - \int\limits_{[u=a,\varphi>0]} \sigma_-(x)\, \varphi(x)\ dx. \tag{3.7}$$

Similarly,

$$\limsup_{n\to\infty} \int\limits_{[u=a]} \left(f\left(x,u_n\left(x\right)\right)-f\left(x,u\left(x\right)\right)\right)\varphi\,dx \leq \int\limits_{[u=a,\varphi>0]} \sigma_+\left(x\right)\varphi\left(x\right)\,dx-$$

$$-\int\limits_{[u=a,\varphi<0]} \sigma_-\left(x\right)\varphi\left(x\right)\,dx. \tag{3.8}$$

Clearly, from $(3.5)-(3.8)$, it follows that u is a solution of the problem (P'). Thus, the pair $\left(u, v = Bu + \frac{1}{\delta}Bh\right)$ is a solution of the problem (P) and the claim is proved. ■

Remark 1 *If $\gamma = \delta = 0$ and $h = 0$, then the result above shows that the semilinear elliptic problem*

$$\begin{cases} -\triangle u = f\left(x,u\right)+p & in\ \Omega \\ \quad u = 0 & on\ \partial\Omega \end{cases},$$

has a solution $u \in E = \bigcap\limits_{1\leq p<\frac{N}{N-1}} W_0^{1,p}\left(\Omega\right)$, in the sense that

$$\int\limits_{\Omega} \nabla u\cdot\nabla\varphi dx - \int\limits_{\Omega} f\left(x,u\right)\varphi\,dx - \int\limits_{\Omega} p\varphi\,dx \in$$

$$\in [\int\limits_{[u=a,\,\varphi<0]} \sigma_+\left(x\right)\varphi\,dx - \int\limits_{[u=a,\,\varphi>0]} \sigma_-\left(x\right)\varphi\,dx, \int\limits_{[u=a,\,\varphi>0]} \sigma_+\left(x\right)\varphi\,d-$$

$$-\int\limits_{[u=a,\,\varphi<0]} \sigma_-\left(x\right)\varphi\,dx],$$

for every $\varphi \in C^2\left(\overline{\Omega}\right)$, with $\varphi = 0$ on $\partial\Omega$.

References

[AB] A. Ambrosetti and M. Badiale, *The Dual Variational Principle and Elliptic Problems with Discontinuous Nonlinearities*, J. Math. Anal. Appl., **140** (1989), 363-373.

[B] H. Brezis, *Analyse Fonctionnelle*, Masson, Paris, 1983.

[BG] L. Boccardo and Th. Gallouët, *Nonlinear Elliptic and Parabolic Equations Involving Measure Data*, J. Funct. Anal., **87** (1989), 149-169.

[CF] R. Chiappinelli and D. G. de Figueiredo, *Bifurcation from infinity and multiple solutions for an elliptic system*, Relatório de Pesquisa (junho-1992), Univ. Estadual de Campinas, Brasil.

[FM] D. G. de Figueiredo and E. Mitidieri, *A maximum principle for an elliptic system and applications to semilinear problems*, SIAM J. Math. Anal., **17** (1986), 836-849.

[LM] A. C. Lazer and P. J. McKenna, *On steady state solutions of a system of reaction-diffusion equations from biology*, Nonlinear Anal. T. M. A., **6** (1982), 523-530.

[O] L. Orsina, *Solvability of Linear and Semilinear Eigenvalues Problems with L^1 Data*, Rend. Sem. Math. Univ. Padova, **90** (1993), 207-238.

[Ro] F. Rothe, *Global existence of branches of stationary solutions for a system of reaction diffusion equations from biology*, Nonlinear Anal. T. M. A., **5** (1981), 487-498.

[Si] E. A. de B. e Silva, *Existence and multiplicity of solutions for semilinear elliptic systems*, NoDEA, **1** (1994), 339-363.

[St] M. Struwe, *Variational Methods. Applications to Nonlinear Partial Differential Equations and Hamiltonian systems,* Springer-Verlag, 1990.