Chaos and Fine Observables

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Abstract. The aim of this paper is to indicate sufficient conditions under which a topologically transitive non-minimmal dynamical system acting on a locally connected metric space shows sensitive dependence on initial conditions. That extends previous work due to L. S. Block and W. A. Coppel [3] and I. Melbourne, M. Dellnitz and M. Golubitsky [6].

1991 *Mathematics Subject Classification*: 58F13.
Key words: *topological dynamical system,* $\omega-$*limit set*, *topologic transitivity, sensitive dependence on initial conditions.*
Conference given at the *Workshop on Dynamical Systems*, Craiova, October 18-19, 1996.

A nice result due to J. Banks, J. Brooks, G. Cairns, G. Davis and P. Stacey [2] asserts that a non-minimal dynamical system shows sensitive dependence on initial data (abbreviated, (S)) provided that the following two conditions are verified:

(T) The system is topologically transitive;
(P) The subset of all periodic points is dense.

Simple examples show that in general (T) is not strong enough to imply (P) or even (S). However, this is the case if the state space is an interval, or a union of intervals. See L. S. Block and W. A. Coppel [3] and respectively I. Melbourne, M. Dellnitz and M. Golubitsky [6].

The purpose of this paper is to prove that $(T) \Rightarrow (S)$ whenever the trajectories of the system can be *observed*. The basic idea is that under additional hypotheses (T) yields a substitute of (P), which makes possible to apply one of the criteria of sensitivity in [2], [4] or [8].

To make the things clear, we need some preparation.

The term of a *continuous dynamical system* will be understood as any continuous action $\Phi : \mathcal{S} \times M \to M$ (of one the semigroups **N, Z,** $\mathbf{R}_{+}$, or **R,** on a metric space); the corresponding time$-t$ mapping will be denoted as Φ_t.

[0] *Received November 20, 1996.*
Analele Universităţii din Craiova, Seria Matematică-Informatică, **XXIII** (1996), 1-8.

Φ *shows sensitive dependence on initial conditions* (equivalently, Φ is *sensitive*) if there exists a $\delta > 0$ such that for every $x \in M$ and every neighbourhood V of x one can find a point $y \in V$ and a number $t \in \mathcal{S}$, $t > 0$, for which

$$d(\Phi_t x, \Phi_t y) \geq \delta.$$

Only perfect metric spaces can carry sensitive dynamical systems. That's why we restrict ourselves throughout this paper to the case where M is perfect.

Φ is said to be *topologically transitive* if there exists a point $p \in M$ whose $\omega-$limit set is M. Usually the topological transitivity appears as a condition of irreducibility and also as a source of *complicated trajectories* i.e., of trajectories which are not eventually periodic. Recall that the trajectory of a point a is said to be *eventually periodic* provided that

$$\Phi_{t+T} a = \Phi_t a$$

for some $t, T \in \mathcal{S}_+$, $T > 0$.

The *observables* of a continuous dynamical system $\Phi : \mathcal{S} \times M \to M$ are the continuous mappings $V : M \to \mathbf{R}$. We shall say that V *can see* the positive orbit of a point $a \in M$ *if*

$$V(\Phi_s a) = V(\Phi_t a) \text{ with } s, t \in \mathcal{S}_+ \text{ implies } \Phi_s a = \Phi_t a.$$

If M is homeomorphic to a subspace of $\mathbf{R}$, then each continuous dynamical system acting on M admits an observable which can see all the orbits of the system. However, in the general context of metric spaces that nice picture is merely an exception than a rule. Think at the case where $M = S^1$. Consequently, what we can reasonably ask to an observable is to see most of the complicated trajectories.

To be more specific, let us denote by $EPer\,\Phi$ the subset of all eventually periodic points of Φ and consider a positively invariant subset $C \subset M$ such that

$$EPer\,\Phi \cap C = \emptyset.$$

An observable V of Φ *can see* C if V can see all orbits issued at the points of C. We shall call V a *fine observable* if V can see the entire complement of $EPer\,\Phi$. To illustrate this notion, let us consider the case of the doubling angles mapping

$$T : S^1 \to S^1, \ T(z) = z^2$$

(also known as the *Shub expansive mapping*). An easy computation shows that

$$V = pr_1$$

is a fine observable of T. In fact, a point $w = e^{2\pi i \cdot t}$ is eventually periodic for T if and only if $t = \dfrac{\text{integer}}{2^m(2^n-1)}$, with $m \in \mathbf{N}$ and $n \in \mathbf{N}^\star$. On the other hand, for t not in this range, an equality of the form

$$\cos(2\pi \cdot 2^p t) = \cos(2\pi \cdot 2^q t)$$

with $p, q \in \mathbf{N}$, forces $p = q$.

The notion of a fine observable could remind us that of a Lyapunov function. In fact, both distinguish among the points of certain orbits. However, in the presence of topological transitivity (the usual context in our paper) the only Lyapunov functions are the constant ones and thus they are irrelevant for the study of dynamics. See [1] for details.

In what follows we shall describe the topological implications of the existence of a fine observable.

Lemma 1. *Let $F : M \to M$ be a continuous mapping, let C be a positively invariant subset of M such that $C \cap EPer\, \Phi = \emptyset$ and let V be an observable of Φ which can see C. Then V is monotonic along each positive orbit issued at a point a of C, provided that a and $F(a)$ belong to a connected subset of C.*

Proof. First notice that an equality of the form

$$V(F^m(a)) = V(F^n(a))$$

with $m, n \in \mathbf{N}$ is not possible unless $m = n$.

Suppose that V is not monotonic along the orbit of a. Without loss of generality we may assume the existence of an $n \in \mathbf{N}^\star$ such that

$$V(a) < V(F(a)) < ... < V(F^n(a))$$

and

$$V(F^n(a)) > V(F^{n+1}(a)).$$

Then the continuous function $\varphi(x) = V(F^n(x)) - V(F^{n-1}(x))$ verifies the condition

$$\varphi(a) \cdot \varphi(F(a)) < 0$$

which yields a z in C such that $V(F^n(z)) = V(F^{n-1}(z))$, i.e., an eventually periodic point in N. Or, by our hypotheses, $C \cap Per\, \Phi = \emptyset$. Consequently V is monotonic along the trajectory of a. ∎

Corollary 2. *Let $\Phi : \mathcal{S} \times M \to M$ be a continuous dynamical system and let U be a connected open subset of M which contains no eventually periodic point of Φ. Suppose that a, $\Phi_s(a)$, $\Phi_t(a)$ are points of U with $0 < s < t$, and $V : M \to \mathbf{R}$ is an observable of Φ, which can see all positive orbits issued at U. Then either*

$$V(a) < V(\Phi_s(a)) < V(\Phi_t(a))$$

or

$$V(a) > V(\Phi_s(a)) > V(\Phi_t(a)).$$

Proof. Clearly, we can restrict ourselves to the case where $\mathcal{S}$ is continuous and $s = p_1/q$ and $t = p_2/q$, with $p_1, \ p_2, \ q \in \mathbf{N}^\star$ and $p_1 < p_2$. Suppose that

$$V(a) < V(\Phi_{p_1\,/\,q}(a)) \text{ and } V(\Phi_{p_2\,/\,q}(a)) > V(\Phi_{p_1\,/\,q}(a)).$$

Then, by Lemma 1,

$$V(a) < V(\Phi_{p_1\,/\,q}(a)) < V(\Phi_{(p_2\,-\,p_1)\,p_1\,/\,q}(a)) \tag{1}$$

and

$$\begin{aligned} V(\Phi_{(p_2\,-\,p_1)\,p_1\,/\,q}\Phi_{p_1\,/\,q}(a)) &< V(\Phi_{(p_2\,-\,p_1)\,/q}\Phi_{p_1\,/\,q}(a)) = \\ &= V(\Phi_{p_2\,/\,q}(a) < V(\Phi_{p_1\,/\,q}(a)). \end{aligned} \tag{2}$$

By (1) and (2), the continuous function

$$V(\Phi_{(p_2\,-\,p_1)\,p_1\,/\,q}(x)) - V(x)$$

has opposite signs at a and $\Phi_{p_1\,/\,q}(a)$. That produces a periodic point for Φ in U, a contradiction. ■

We are now in a position to indicate a topological consequence of the existence of observables:

Theorem 3. *Let $\Phi : \mathcal{S} \times M \to M$ be a topologically transitive dynamical system acting on a locally connected perfect metric space and let C be a positively invariant subset of M such that $C \cap EPer\Phi = \emptyset$. If Φ admits an observable V which can see C, then*

$$\overline{M \setminus C} = M.$$

Proof. Suppose that the contrary is true. Then one can choose a non-empty connected open subset N of C. Also, we can choose non-empty open subsets A and B of N with $A \cap B = \emptyset$. Because Φ is topologically transitive,

$$\Phi_s(A) \cap B \neq \emptyset$$

for some $s > 0$, which gives us an $x \in A$ with $\Phi_s(x) \in B$. Clearly, $x \neq \Phi_s(x)$, so there exists an open neighbourhood U of x such that $U \subset N$, $\Phi_s(U) \subset N$ and

$$V(\Phi_s(U)) \cap V(U) = \emptyset.$$

Again by the topological transitivity, we can find a $t > s$ such that

$$\Phi_t(U) \cap U \neq \emptyset$$

and that yields a $z \in U$ with $\Phi_t(z) \in U$. According to Corollary 1 above, $V(\Phi_s(z))$ must be between $V(z)$ and $V(\Phi_t(z))$. Since $V(U)$ is an interval, that leads to

$$V(\Phi_s(z)) \in V(\Phi_s(U)) \cap V(U),$$

a contradiction. ■

The above analysis on the presence of a fine observable will be accompanied by a criterion of sensitive dependence on initial conditions, a variant of the main result in [2]:

Lemma 4. *Suppose that Φ is a topologically transitive dynamical system acting on a metric space M. If Φ is not minimal and the union of all its eventually periodic orbits is dense, then Φ shows sensitive dependence on initial conditions.*

A formal proof can be obtained by refining the argument given in [4] or [8]. That is done for example in [7].

By combining Theorem 1 and Lemma 2 above we infer easily the main result of this paper:

Theorem 5. *Suppose that Φ is a topologically transitive non-minimal dynamical system, acting on a locally connected metric space. If Φ admits a fine observable, then Φ shows sensitive dependence on initial conditions.*

Corollary 6. (*I. Melbourne, M. Dellnitz and M. Golubitsky* [6]) *Every topologically transitive dynamical system acting on a union of nondegenerate intervals shows sensitive dependence on initial conditions.*

When the state space consists of an interval, the result of Corollary 2 was first noticed by L. S. Block and W. A. Coppel [3]. As was mentionned in [6], only dynamical systems acting on finite unions of intervals can fulfil the hypothesis of Corollary 2. An example illustrating the possibility to take into account non-connected state spaces can be easily exhibited by starting with the logistic mapping

$$F_4(x) = 4x(1 - x), \quad x \in [0, 1].$$

Due to its connection with the doubling mapping, F_4 and all its iterates are topologically transitive. As a consequence we infer that the mapping

$$F(x) = \begin{cases} 2 + F_4(x), & \text{if} \quad x \in [0,1] \\ F_4(x-2), & \text{if} \quad x \in [2,3]. \end{cases}$$

is topologically transitive on $[0,1] \cup [2,3]$ and thus Corollary 2 above applies to it.

Theorem 1 has a counterpart relative to the dynamics on the global attractor of a system. According to Haraux [5], by a *global attractor* we shall mean any compact, invariant and attracting subset of the state space.

Theorem 7. *Let Φ be a continuous dynamical system acting on a metric space M. Suppose that Φ admits a fine observable and has a global attractor $\mathcal{A}$ such that*:

(1) *$\Phi|\mathcal{A}$ is topologically transitive and non-minimal*;

(2) *$\mathcal{A}$ is locally connected.*

Then $\mathcal{A}$ is a strange attractor (*i.e., the dynamics on it is sensitive*).

Some open problems are in order:

Problem 1. Characterize the metric spaces on which every topologically transitive dynamical system shows sensitive dependence on initial conditions.

Problem 2. Theorem 2 was modelled over the criterion of chaoticity of Lemma 2. Or, Lemma 2 has a companion where the abundance of periodic orbits is replaced by the denseness of other types of well behaved orbits (such as the almost periodic, algebraically recurrent etc). See [4] or [8]. Reformulate Theorem 2 to take into account that fact.

Problem 2 has a practical interest because important strange attractors fails the abundance of periodic orbits.

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