Proceedings of the Royal Society of Edinburgh, **145A**, 445–465, *2015*

Infinitely many solutions for a class of sublinear Schrödinger equations with indefinite potentials

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(MS received 10 July 2013; accepted 11 February 2014)

In this paper we are concerned with qualitative properties of entire solutions to a Schrödinger equation with sublinear nonlinearity and sign-changing potentials. Our analysis considers three distinct cases and we establish sufficient conditions for the existence of infinitely many solutions.

1. Historical perspective of the Schrödinger equation

The Schrödinger equation plays the role of Newton's laws and conservation of energy in classical mechanics, that is, it predicts the future behaviour of a dynamic system. The linear Schrödinger equation is a central tool of quantum mechanics, which provides a thorough description of a particle in a non-relativistic setting. Schrödinger's linear equation is

$$\Delta\psi + \frac{8\pi^2 m}{\hbar^2}(E - V(x))\psi = 0,$$

where ψ is the Schrödinger wave function, m is the mass, $\hbar$ denotes Planck's constant, E is the energy and V stands for the potential energy.

The structure of the nonlinear Schrödinger equation is much more complicated. This equation is a prototypical dispersive nonlinear partial differential equation that has been central for almost four decades to a variety of areas in mathematical physics. The relevant fields of application vary from Bose–Einstein condensates and nonlinear optics (see [15]), propagation of the electric field in optical fibres (see [26, 32]) to the self-focusing and collapse of Langmuir waves in plasma physics (see [43]) and the behaviour of deep water waves and freak waves (so-called rogue waves) in the ocean (see [34]). The nonlinear Schrödinger equation also describes various phenomena arising in the theory of Heisenberg ferromagnets and magnons, the self-channelling of a high-power ultra-short laser in matter, condensed matter theory, dissipative quantum mechanics, electromagnetic fields (see [5]) and plasma physics (for example, the Kurihara superfluid film equation). We refer to Ablowitz *et al*. [1] and Sulem [36] for a modern overview, including applications.

Schrödinger also established the classical derivation of his equation based upon the analogy between mechanics and optics and closer to de Broglie's ideas. He developed a perturbation method, inspired by the work of Lord Rayleigh in acoustics, proved the equivalence between his wave mechanics and Heisenberg's matrix mechanics and introduced the time-dependent Schrödinger equation

$$\mathrm{i}\hbar\psi_t = -\frac{\hbar^2}{2m}\Delta\psi + V(x)\psi - \gamma|\psi|^{p-1}\psi \quad \text{in } \mathbb{R}^N \ (N \geqslant 2), \tag{1.1}$$

where $p < 2N/(N-2)$ if $N \geqslant 3$ and $p < +\infty$ if $N = 2$. In physical problems a cubic nonlinearity corresponding to $p = 3$ is common; in this case (1.1) is called the Gross–Pitaevskii equation. In the study of (1.1), Floer and Weinstein [24] and Oh [33] supposed that the potential V is bounded and possesses a non-degenerate critical point at $x = 0$. More precisely, it is assumed that V belongs to the class (V_a) (for some real number a) introduced by Kato [29]. Taking $\gamma > 0$ and $\hbar > 0$ sufficiently small and using a Lyapunov–Schmidt type reduction, Oh [33] proved the existence of standing wave solutions of (1.1), that is, a solution of the form

$$\psi(x,t) = \mathrm{e}^{-\mathrm{i}Et/\hbar}u(x). \tag{1.2}$$

Using the ansatz (1.2), we reduce the nonlinear Schrödinger equation (1.1) to the semilinear elliptic equation

$$-\frac{\hbar^2}{2m}\Delta u + (V(x) - E)u = |u|^{p-1}u.$$

The change of variable $y = \hbar^{-1}x$ (and replacing y by x) yields

$$-\Delta u + 2m(V_\hbar(x) - E)u = |u|^{p-1}u \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

where $V_\hbar(x) = V(\hbar x)$.

If, for some $\xi \in \mathbb{R}^N \setminus \{0\}$, $V(x + s\xi) = V(x)$ for all $s \in \mathbb{R}$, then (1.1) is invariant under the Galilean transformation

$$\psi(x,t) \mapsto \psi(x - \xi t, t)\exp(\mathrm{i}\xi \cdot x/\hbar - \tfrac{1}{2}\mathrm{i}|\xi|^2 t/\hbar)\psi(x - \xi t, t).$$

Thus, in this case, standing waves reproduce solitary waves travelling in the direction of ξ. In other words, Schrödinger discovered that the standing waves are scalar waves rather than vector electromagnetic waves. This is an important difference: vector electromagnetic waves are mathematical waves that describe a direction (vector) of force, whereas the wave motions of space are scalar waves, which are simply described by their wave amplitude. The importance of this discovery was pointed out by Einstein [23], who wrote:

> The Schrödinger method, which has in a certain sense the character of a field theory, does indeed deduce the existence of only discrete states, in surprising agreement with empirical facts. It does so on the basis of differential equations applying a kind of resonance argument.

In a celebrated paper, Rabinowitz [35] proved that (1.3) has a ground-state solution (mountain pass solution) for $\hbar > 0$ small, under the assumption that $\inf_{x\in\mathbb{R}^N} V(x) > E$. After making a standing wave ansatz, Rabinowitz reduced the problem to that of studying the semilinear elliptic equation

$$-\Delta u + V(x)u = f(x,u) \quad \text{in } \mathbb{R}^N \tag{1.4}$$

under suitable conditions on V and assuming that f is smooth, superlinear and has a subcritical growth.

2. Introduction and main results

In the present paper we are concerned with the existence of infinitely many solutions of the semilinear Schrödinger equation

$$-\Delta u + V(x)u = a(x)g(u) \quad x \in \mathbb{R}^N \ (N \geqslant 3), \tag{2.1}$$

where V and a are functions changing sign and the nonlinearity g has a sublinear growth. Such problems in $\mathbb{R}^N$ arise naturally in various branches of physics and present challenging mathematical difficulties.

If (2.1) is considered in a bounded domain Ω, with the Dirichlet boundary condition, then there is a large literature on existence and a multiplicity of solutions (see [4,14,27,28,37,38,41]). In particular, Kajikiya [27] has considered such sublinear cases with sign-changing nonlinearity and has proved the existence of infinitely many solutions.

If Ω is an unbounded domain, and especially if $\Omega = \mathbb{R}^N$, then the existence and multiplicity of non-trivial solutions for (2.1) have been extensively investigated in the literature over the past several decades, both for sublinear and superlinear nonlinearities.

In the superlinear case, we can cite [2,3,6,8,17,19,21,22,25,35,42]. In particular, Costa and Tehrani [17] have considered the problem

$$-\Delta u - \lambda h(x)u = a(x)g(u), \quad u > 0, \text{ in } \mathbb{R}^N, \tag{2.2}$$

where $\lambda > 0$, h is a positive function, a changes the sign in $\mathbb{R}^N$, $N \geqslant 3$, and g is a superlinear function. With further assumptions on h, a and g, they proved the existence of $\lambda_1(h) > 0$ such that (2.2) admits one positive solution for $0 < \lambda < \lambda_1(h)$ and two positive solutions for $\lambda_1(h) < \lambda < \lambda_1(h) + \varepsilon$ for some $\varepsilon > 0$.

In recent years, many authors have studied the question of existence and multiplicity of solutions for (2.1) with sublinear nonlinearity (see [7,10–12,16,18,30,39]). In most of the problems studied in these papers, V and a are considered to be positive. In particular, Brezis and Kamin [12] gave a sufficient and necessary condition for the existence of bounded positive solutions of (2.1) with $V = 0$ and $a > 0$.

Balabane *et al*. [7] proved that for each integer k, (2.1) has a radially compactly supported solution that has k zeros in its support provided that $V = a = -1$ and $g(u) = |u|^{-2\theta}u$, where $\theta \in]0, \frac{1}{2}[$.

Zhang and Wang [44] proved the existence of infinitely many solutions for (2.1) with $g(u) = |u|^{p-1}u$, $0 < p < 1$, and the potentials $V > 0$, $a > 0$ satisfy the following assumptions:

(S_1) $V \in C(\mathbb{R}^N, \mathbb{R})$ and there exists $r > 0$ such that

$$m\{x \in B(y,r);\ V(x) \leqslant M\} \to 0 \text{ as } |y| \to +\infty \quad \forall M > 0,$$

where m is the Lebesgue measure in $\mathbb{R}^N$;

(S_2) $a\colon \mathbb{R}^N \to \mathbb{R}$ is a continuous function and $a \in L^{2/(1-p)}(\mathbb{R}^N)$, $0 < p < 1$.

If V and a both change sign on $\mathbb{R}^N$, various difficulties arise. To the authors' knowledge, few results are known in this case. On this subject, Costa and Tehrani [18] have proved the existence of at least one non-trivial solution for the equation

$$-\Delta u + V(x)u = \lambda u + g(x,u)$$

under the following conditions:

(VC_1) $V \in C^{\beta}(\mathbb{R}^N)$ $(0 < \beta < 1)$ and $\lim\limits_{|x|\to+\infty} V(x) = 0$;

(VC_2) $\displaystyle\int_{\mathbb{R}^N} (|\nabla\varphi|^2 + V(x)\varphi^2)\,\mathrm{d}x < 0$ for some $\varphi \in C_0^1(\mathbb{R}^N)$;

(GC_1) $|g(x,s)| \leqslant b_1(x)|s|^{\alpha} + b_2(x)$ for some $0 < \alpha < 1$ and a class of integrable functions b_1 and b_2;

(GC_2) $\lambda < 0$ is an eigenvalue of the Schrödinger operator $L_V = -\Delta + V(x)$ in $\mathbb{R}^N$.

(GC_2) $$\lim_{\substack{\|u_0\|\to+\infty,\\ u_0\in\mathrm{Ker}(-\Delta+V-\lambda)}} \frac{1}{\|u_0\|^{2\alpha}} \int_{\mathbb{R}^N} G(x,u_0(x))\,\mathrm{d}x = \pm\infty.$$

Tehrani [39] studied the perturbed equation

$$-\Delta u + V(x)u = a(x)g(u) + f, \tag{2.3}$$

where a and V change sign on $\mathbb{R}^N$, $f \in L^2(\mathbb{R}^N)$ and g is a sublinear function. With further assumptions on a, V, f and g, he proved the existence of at least one non-trivial solution.

Costa and Chabrowski [16] considered the p-Laplacian equation

$$-\Delta_p u - \lambda V(x)|u|^{p-2}u = a(x)|u|^{q-2}u, \quad x \in \mathbb{R}^N, \tag{2.4}$$

where $\lambda \in \mathbb{R}$ is a parameter, $1 < q < p < p^* = Np/(N-p)$, $V \in L^{N/p}(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N)$, $a \in L^{\infty}(\mathbb{R}^N)$ and $\lim_{|x|\to+\infty} a(x) = a_{\infty} < 0$. With further assumptions on a and V, they proved the existence of $\lambda_1 > 0$ and $\lambda_{-1} < 0$ such that (2.4) admits at least one positive solution for $\lambda_{-1} < \lambda < \lambda_1$ and two positive solutions for $\lambda > \lambda_1$ and $\lambda < \lambda_{-1}$.

Benrhouma [9] proved the existence of at least three solutions for (2.3) with $g(u) = |u|^p \operatorname{sgn}(u)$, $0 < p < 1$, V changing sign and $a < 0$.

In all works cited above, where a and V change sign the authors proved the existence of at most three solutions. In this paper, we prove the existence of infinitely many solutions of (2.1) with a and V changing sign, under various assumptions on these potential functions.

Denote by s the best Sobolev constant,

$$s = \inf\left\{\|\nabla u\|_2^2,\ u \in W^{1,2}(\mathbb{R}^N),\ \int_{\mathbb{R}^N} |u(x)|^{2N/(N-2)}\,\mathrm{d}x = 1\right\}, \quad N \geqslant 3.$$

We suppose the following hypotheses on g:

(G_1) $g \in C(\mathbb{R}, \mathbb{R})$, g is odd and there exist $c > 0$, $q \in]0, 1[$ such that

$$|g(x)| \leqslant c|x|^q \quad \text{for all } x \in \mathbb{R};$$

(G_2) $\displaystyle\lim_{x\to 0} \frac{G(x)}{|x|^2} = +\infty, \quad \text{where } G(x) = \int_0^x g(t)\,\mathrm{d}t \quad \forall x \in \mathbb{R};$

(G_3) G is positive on $\mathbb{R} \setminus \{0\}$.

We give three theorems on the existence of infinitely many solutions to the nonlinear problem (2.1).

Theorem 2.1. *Assume that $g(x) = |x|^{q-1}x$, $0 < q < 1$, and that V satisfies:*

(V_1) *$V \in L^\infty(\mathbb{R}^N)$, $\lim_{|x|\to+\infty} V(x) = v_\infty > 0$ and*

$$\|V^-\|_{N/2} < s,$$

where $u^\mp(x) = \max\{\mp u(x), 0\}$ for all $x \in \mathbb{R}^N$ and for all $u \in E$.

Assume also that a satisfies:

(A_1) *$a \in L^\infty(\mathbb{R}^N)$, $\lim_{|x|\to+\infty} a(x) = a_\infty < 0$ and there exist $y = (y_1, \dots, y_N) \in \mathbb{R}^N$, $R_0 > 0$ such that*

$$a(x) > 0 \quad \textit{for all } x \in B(y, R_0).$$

Then (2.1) possesses a sequence of non-trivial solutions converging to 0.

In the next two theorems we change the assumption of boundedness of a by the integrability condition. The last assumption was supported to make the energy functional associated with (2.1) well defined and to guarantee that the functional $F(u) = \int_{\mathbb{R}^N} a(x)G(u(x))\,\mathrm{d}x$ has a compact gradient. This compactness property in turn was used to prove the required Palais–Smale condition, which is essential in the application of the critical point theory. We then have the following two multiplicity properties.

Theorem 2.2. *Suppose that g satisfies (G_1)–(G_3) and the potentials V and a satisfy the following hypotheses:*

(V_2) *$V \in L^{N/2}(\mathbb{R}^N)$ and*

$$\|V^-\|_{N/2} < s;$$

(A_2) *$a \in L^{2^*/(2^*-(q+1))}(\mathbb{R}^N)$ and there exist $y \in \mathbb{R}^N$ and $R_0 > 0$ such that*

$$a(x) > 0 \quad \forall x \in B(y, R_0).$$

Then (2.1) possesses a bounded sequence of non-trivial solutions.

THEOREM 2.3. *Assume that g satisfies (G_1)–(G_3), V satisfies (V_1) and a satisfies:*

(A_3) *$a \in L^{2/(1-q)}(\mathbb{R}^N)$ and there exist $y = (y_1, \ldots, y_N) \in \mathbb{R}^N$ and $R_0 > 0$ such that*

$$a(x) > 0 \quad \forall x \in B(y, R_0).$$

Then (2.1) possesses a bounded sequence of non-trivial solutions.

This paper is organized as follows. In § 2 we give some notation, we present the variational framework and we recall some definitions and standard results. Then §§ 3–5 are dedicated to the proof of theorems 2.1, 2.2 and 2.3.

3. Notation and preliminary results

In this section we present some notation and preliminaries that will be useful in the following. We make the following definitions:

- $\|u\|_m = \left(\int_{\mathbb{R}^N} |u(x)|^m \, \mathrm{d}x \right)^{1/m}, \quad 1 \leqslant m < +\infty$;
- $2^* = \dfrac{2N}{N-2}$ if $N \geqslant 3$ and $2^* = +\infty$ if $n \in \{1, 2\}$;
- B_R denotes the ball centred at the origin of radius $R > 0$ in $\mathbb{R}^N$ and $B_R^c = \mathbb{R}^N \backslash B_R$;
- $F'(u)$ is the Fréchet derivative of F at u.

Let F_1, F_2 be Banach spaces and let $T\colon F_1 \to F_2$. T is said to be a sequentially compact operator if, given any bounded sequence (x_n) in F_1, $(T(x_n))$ has a convergent subsequence in F_2.

Let $E = H^1(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N)$ $(0 < q < 1)$ be the reflexive Banach space endowed with the norm

$$\|u\| = \|\nabla u\|_2 + \|u\|_{q+1}.$$

Let $X = D^{1,2}(\mathbb{R}^N) = \{u \in L^{2^*}(\mathbb{R}^N);\ \nabla u \in (L^2(\mathbb{R}^N))^N\}$, endowed with the norm

$$\|u\|_X = \left(\int_{\mathbb{R}^N} |\nabla u(x)|^2 \, \mathrm{d}x \right)^{1/2},$$

be a Hilbert space. Moreover, the embedding $X \subset L^{2^*}(\mathbb{R}^N)$ is continuous, which implies that

$$S := \inf \left\{ \int_{\mathbb{R}^N} |\nabla u(x)|^2 \, \mathrm{d}x;\ u \in X,\ \int_{\mathbb{R}^N} |u(x)|^{2^*} \, \mathrm{d}x = 1 \right\} > 0.$$

We refer the reader to [40, pp. 8 and 9] for more details.

Let

$$Y = \left\{ u \in H^1(\mathbb{R}^N);\ \int_{\mathbb{R}^N} V^+(x) u^2(x) \, \mathrm{d}x < +\infty \right\},$$

under the hypotheses $V \in L^\infty(\mathbb{R}^N)$ and $\operatorname{esslim}_{x\to+\infty} V(x) > 0$. We endow Y with the inner product

$$\langle u, v\rangle = \int_{\mathbb{R}^N} \nabla u \nabla v + \int_{\mathbb{R}^N} V^+(x) uv \, \mathrm{d}x$$

and the associated norm $\|\cdot\|_Y$, which is equivalent to the usual norm

$$\|u\|_{H^1} = \|\nabla u\|_2 + \|u\|_2.$$

Consider the functionals

$$\begin{aligned}
I(u) &= \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2(x) + V(x)u^2(x)) \, \mathrm{d}x - \int_{\mathbb{R}^N} a(x)G(u(x)) \, \mathrm{d}x, \\
\varphi(u) &= \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2(x) \, \mathrm{d}x - \int_{\mathbb{R}^N} a(x)G(u(x)) \, \mathrm{d}x, \\
\psi(u) &= -\frac{1}{2}\int_{\mathbb{R}^N} V^-(x)u^2(x) \, \mathrm{d}x - \int_{\mathbb{R}^N} a(x)G(u(x)) \, \mathrm{d}x.
\end{aligned}$$

Under suitable assumptions on a, G and V (to be fixed later), I, φ and ψ are well defined and of class C^1 on X, Y or E. A critical point of I is a weak solution of (2.1).

Next, let us recall that a Palais–Smale (PS) sequence for the functional I is a sequence (u_n) such that

$$I(u_n) \text{ is bounded and } \|I'(u_n)\| \to 0.$$

The functional I is said to satisfy the PS condition if any PS sequence possesses a convergent subsequence.

A first main difficulty that appears in the study of (2.1) is the loss of compactness. In order to overcome this difficulty, we use the Lions compactness principle [31]. A second main difficulty is to satisfy the geometric conditions required by the Ambrosetti–Rabinowitz theorem [4]. We use a geometrical construction of subsets to overcome this difficulty. Let us give a definition and recall the mountain pass theorem of Ambrosetti and Rabinowitz.

Definition 3.1. Let E be a Banach space. A subset A of E is said to be symmetric if $u \in E$ implies $-u \in E$. For a closed symmetric set A that does not contain the origin, we define the genus $\gamma(A)$ of A by the smallest integer k such that there exists an odd continuous mapping from A to $\mathbb{R}^k\setminus\{0\}$. If there does not exist such a k, we define $\gamma(A) = +\infty$. We set $\gamma(\emptyset) = 0$. Let Γ_k denote the family of closed symmetric subsets A of E such that $0 \notin A$ and $\gamma(A) \geqslant k$.

Theorem 3.2 (Ambrosetti and Rabinowitz [4]). *Let E be an infinite-dimensional Banach space and let $I \in C^1(E, \mathbb{R})$ satisfy the following conditions.*

(1) *I is even, bounded from below, $I(0) = 0$ and I satisfies the PS condition.*

(2) *For each $k \in \mathbb{N}$ there exists $A_k \in \Gamma_k$ such that*

$$\sup_{u\in A_k} I(u) < 0.$$

Under assumptions (1) and (2) we define c_k by

$$c_k = \inf_{A \in \Gamma_k} \sup_{u \in A} I(u).$$

Then each c_k is a critical value of I, $c_k \leqslant c_{k+1} < 0$ for $k \in \mathbb{N}$ and (c_k) converges to zero. Moreover, if $c_k = c_{k+1} = \cdots = c_{k+p} = c$, then $\gamma(K_c) \geqslant p+1$. The critical set K_c is defined by

$$K_c = \{u \in E;\ I'(u) = 0, I(u) = c\}.$$

4. Proof of theorem 2.1

In this section we consider the case in which a is bounded and we define I on the function space $E = H^1(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N)$.

LEMMA 4.1. *Assume that* (A$_1$) *and* (V$_1$) *hold. Then any PS sequence of I is bounded in E.*

Proof. By standard arguments, I is well defined and of class C^1 on E.

Let (u_n) be a PS sequence of I. There then exists $\alpha > 0$ such that $I(u_n) \leqslant \alpha$. Applying Hölder's inequality and conditions (A$_1$) and (V$_1$), we have

$$\begin{aligned}
\alpha \geqslant I(u_n) &= \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u_n|^2(x) + V(x)u_n(x)^2)\,\mathrm{d}x - \frac{1}{q+1}\int_{\mathbb{R}^N} a(x)|u_n(x)|^{q+1}\,\mathrm{d}x \\
&\geqslant \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_n(x)|^2\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} V^-(x)u_n(x)^2\,\mathrm{d}x \\
&\quad - \frac{1}{q+1}\int_{\mathbb{R}^N} a^+(x)|u_n|^{q+1}(x)\,\mathrm{d}x + \frac{1}{q+1}\int_{\mathbb{R}^N} a^-(x)|u_n|^{q+1}(x)\,\mathrm{d}x \\
&\geqslant \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_n|^2(x)\,\mathrm{d}x - \frac{\|V^-\|_{N/2}}{2s}\|\nabla u_n\|_2^2 \\
&\quad - \frac{1}{q+1}\int_{\mathbb{R}^N} a^+(x)|u_n|^{q+1}(x)\,\mathrm{d}x.
\end{aligned} \tag{4.1}$$

By (A$_1$), there exists $R > 0$ such that

$$-\|a\|_\infty \leqslant a(x) \leqslant \frac{a_\infty}{2} < 0 \quad \forall |x| \geqslant R \quad \text{and} \quad a^+ \in L^m(\mathbb{R}^N) \quad \forall 1 \leqslant m \leqslant +\infty. \tag{4.2}$$

Combining (4.1) and (4.2), we infer that

$$\begin{aligned}
\alpha \geqslant I(u_n) &\geqslant \frac{1}{2}\|\nabla u_n\|_2^2 - \frac{\|V^-\|_{N/2}}{2s}\|\nabla u_n\|_2^2 - s^{(-q-1)/2}\|a^+\|_{2^*/(2^*-(q+1))}\|\nabla u_n\|_2^{q+1} \\
&\geqslant \left(\frac{1}{2} - \frac{\|V^-\|_{N/2}}{2s}\right)\|\nabla u_n\|_2^2 - s^{(-q-1)/2}\|a^+\|_{2^*/(2^*-(q+1))}\|\nabla u_n\|_2^{q+1},
\end{aligned}$$

and hence there exists $\beta > 0$ such that

$$\|\nabla u_n\|_2 \leqslant \beta \quad \forall n \in \mathbb{N}. \tag{4.3}$$

On the other hand, there exists $c > 0$ such that

$$
\begin{aligned}
c + \frac{\|u_n\|}{2} &\geqslant -\frac{1}{2}\langle I'(u_n), u_n\rangle + I(u_n) \\
&= \left(\frac{1}{2} - \frac{1}{q+1}\right)\int_{\mathbb{R}^N} a(x)|u_n|^{q+1}(x)\,\mathrm{d}x \\
&= \left(\frac{1}{q+1} - \frac{1}{2}\right)\int_{\mathbb{R}^N} a^-(x)|u_n|^{q+1}\,\mathrm{d}x \\
&\quad - \left(\frac{1}{q+1} - \frac{1}{2}\right)\int_{\mathbb{R}^N} a^+(x)|u_n|^{q+1}\,\mathrm{d}x \\
&= \left(\frac{1}{q+1} - \frac{1}{2}\right)\int_{\mathbb{R}^N} (a^-(x) + \chi_{B_R}(x))|u_n|^{q+1}\,\mathrm{d}x \\
&\quad - \left(\frac{1}{q+1} - \frac{1}{2}\right)\int_{\mathbb{R}^N} (a^+(x) + \chi_{B_R}(x))|u_n|^{q+1}\,\mathrm{d}x \\
&\geqslant \left(\frac{1}{q+1} - \frac{1}{2}\right)\min\left\{\frac{-a_\infty}{2}, 1\right\}\int_{\mathbb{R}^N} |u_n|^{q+1}(x)\,\mathrm{d}x \\
&\quad - s^{(-q-1)/2}\left(\frac{1}{q+1} - \frac{1}{2}\right)\|a^+ + \chi_{B_R}\|_{2^*/(2^*-(q+1))}\|\nabla u_n\|_2^{q+1}.
\end{aligned}
$$

Thus, there is a constant $c > 0$ such that

$$
\int_{\mathbb{R}^N} |u_n|^{q+1}\,\mathrm{d}x \leqslant c(\|\nabla u_n\|_2 + \|u_n\|_{q+1} + \|a^+ + \chi_R\|_{2^*/(2^*-(q+1))}\|\nabla u_n\|_2^{q+1}).
$$

Relation (4.3) yields

$$
\|u_n\|_{q+1}^{q+1} \leqslant c + c\|u_n\|_{q+1} \quad \text{for all } n \in \mathbb{N}. \tag{4.4}
$$

Combining (4.3) and (4.4), we get

$$
\|u_n\| \leqslant c \quad \forall n \in \mathbb{N}.
$$

The proof is complete. □

We need the following lemma to prove that the PS condition is satisfied for I on E.

LEMMA 4.2. *There exists a constant $c > 0$ such that for all real numbers x, y,*

$$
||x + y|^{q+1} - |x|^{q+1} - |y|^{q+1}| \leqslant c|x|^q|y|. \tag{4.5}
$$

Proof. If $x = 0$, the inequality (4.5) is trivial.

Suppose that $x \neq 0$. We consider the continuous function f defined on $\mathbb{R}\backslash\{0\}$ by

$$
f(t) = \frac{|1 + t|^{q+1} - |t|^{q+1} - 1}{|t|}.
$$

Then $\lim_{|t|\to+\infty} f(t) = 0$ and $\lim_{t\to 0\pm} f(t) = \pm(q+1)$. Thus, there exists a constant $c > 0$ such that $|f(t)| \leqslant c$ for all $t \in \mathbb{R}\backslash\{0\}$. In particular, $|f(y/x)| \leqslant c$, so

$$\left|\left|1+\frac{y}{x}\right|^{q+1} - \left|\frac{y}{x}\right|^{q+1} - 1\right| \leqslant c\left|\frac{y}{x}\right|.$$

Multiplying by $|x|^{q+1}$, we obtain the desired result. □

LEMMA 4.3. *Assume* (A_1) *and* (V_1) *hold. Then* I *satisfies the PS condition on* E.

Proof. Let (u_n) be a PS sequence. By lemma 4.1, (u_n) is bounded in E. There then exists a subsequence $u_n \rightharpoonup u$ in E, $u_n \to u$ in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ for all $1 \leqslant p \leqslant 2^*$ and $u_n \to u$ almost everywhere (a.e.) in $\mathbb{R}^N$.

Fix $\varphi \in D(\mathbb{R}^N)$. By the weak convergence of (u_n) to u, we obtain

$$\int_{\mathbb{R}^N} \nabla u_n \nabla\varphi(x) + V(x)u_n\varphi(x)\,\mathrm{d}x \to \int_{\mathbb{R}^N} \nabla u\nabla\varphi + V(x)u\varphi(x)\,\mathrm{d}x. \tag{4.6}$$

By compactness Sobolev embedding, $u_n \to u$ in $L^{q+1}(\mathrm{supp}(\varphi))$, and hence there exists a function $h \in L^{q+1}(\mathbb{R}^N)$ such that

$$a(x)|u_n|^{q-1}u_n\varphi \to a(x)|u|^{q-1}u\varphi \quad \text{a.e. in } \mathbb{R}^N$$

and

$$|a|\,|u_n|^q|\varphi| \leqslant \|a\|_\infty|h|\,|\varphi| \quad \text{in } \mathbb{R}^N.$$

Using the Lebesgue dominated convergence theorem, we deduce that

$$\int_{\mathbb{R}^N} a(x)|u_n|^{q-1}u_n\varphi(x)\,\mathrm{d}x \to \int_{\mathbb{R}^N} a(x)|u|^{q-1}u\varphi(x)\,\mathrm{d}x. \tag{4.7}$$

Combining (4.6) and (4.7), we obtain

$$0 = \lim_{n\to+\infty}\langle I'(u_n),\varphi\rangle = \langle I'(u),\varphi\rangle \quad \forall\varphi \in D(\mathbb{R}^N).$$

Then,

$$\langle I'(u), u\rangle = 0. \tag{4.8}$$

Since $u_n \rightharpoonup u$ in E, we have $\|u\| \leqslant \liminf_{n\to+\infty}\|u_n\| = \lim_{n\to+\infty}\|u_n\|$. We distinguish two cases.

CASE 1 (compactness). $\|u\| = \lim_{n\to+\infty}\|u_n\|$, so

$$\limsup_{n\to+\infty}\|u_n\|_{q+1} \leqslant \|u\|_{q+1} + \|\nabla u\|_2 - \liminf_{n\to+\infty}\|\nabla u_n\|_2.$$

Since

$$\|\nabla u\|_2 \leqslant \liminf_{n\to+\infty}\|\nabla u_n\|_2, \qquad \|u\|_{q+1} \leqslant \liminf_{n\to+\infty}\|u_n\|_{q+1},$$

we obtain

$$\|u\|_{q+1} \leqslant \liminf_{n\to+\infty}\|u_n\|_{q+1} \leqslant \limsup_{n\to+\infty}\|u_n\|_{q+1} \leqslant \|u\|_{q+1},$$

and thus

$$u_n \to u \quad \text{a.e. in } \mathbb{R}^N,$$
$$\|u_n\|_{q+1} \to \|u\|_{q+1}.$$

By the Brezis–Lied lemma [13], we infer that

$$u_n \to u \quad \text{in } L^{q+1}(\mathbb{R}^N). \tag{4.9}$$

Therefore, $\|\nabla u_n\|_2 \to \|\nabla u\|_2$. On the other hand,

$$\int_{\mathbb{R}^N} |\nabla u_n - \nabla u|^2 \, \mathrm{d}x = \int_{\mathbb{R}^N} |\nabla u_n|^2 \, \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla u|^2 \, \mathrm{d}x - 2\int_{\mathbb{R}^N} \nabla u_n \nabla u \, \mathrm{d}x,$$

and hence

$$\int_{\mathbb{R}^N} \nabla u_n \nabla u \, \mathrm{d}x \to \int_{\mathbb{R}^N} |\nabla u|^2 \, \mathrm{d}x.$$

Therefore,

$$\|\nabla u_n - \nabla u\|_2 \to 0. \tag{4.10}$$

Combining (4.9) and (4.10), we deduce that $u_n \to u$ in E and the PS condition for I is satisfied.

CASE 2 (dichotomy). $\|u\| < \lim_{n\to+\infty} \|u_n\|$. We prove that this case cannot occur. Set $v_n = u_n - u$.

STEP 1 (There exists $(y_n) \subset \mathbb{R}^N$ such that $v_n(\cdot + y_n) \rightharpoonup v \neq 0$ in E). If not, for all $(y_n) \subset \mathbb{R}^N$, $v_n(\cdot + y_n) \rightharpoonup 0$ in E. Then

$$\forall R > 0 \quad \sup_{y\in\mathbb{R}^N} \int_{B(y,R)} |v_n|^{q+1}(x) \, \mathrm{d}x \to 0.$$

By [31, lemma I.1, p. 231],

$$v_n \to 0 \text{ in } L^p(\mathbb{R}^N) \quad \forall q+1 < p < 2^*. \tag{4.11}$$

On the other hand,

$$\begin{aligned}
&\langle I'(u_n), u_n\rangle \\
&\quad = \int_{\mathbb{R}^N} (|\nabla u_n|^2 + V(x)u_n^2) \, \mathrm{d}x - \int_{\mathbb{R}^N} a(x)|u_n|^{q+1} \, \mathrm{d}x \\
&\quad = \int_{\mathbb{R}^N} (|\nabla v_n|^2 + V(x)v_n^2) \, \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla u|^2 \, \mathrm{d}x + \int_{\mathbb{R}^N} (V(x)u^2 + 2\nabla v_n \nabla u) \, \mathrm{d}x \\
&\qquad + \int_{\mathbb{R}^N} 2V(x)v_n u \, \mathrm{d}x - \int_{\mathbb{R}^N} a(x)(|u_n|^{q+1} - |u|^{q+1}) \, \mathrm{d}x - \int_{\mathbb{R}^N} a(x)|u|^{q+1} \, \mathrm{d}x.
\end{aligned} \tag{4.12}$$

By (4.5) in lemma 4.2, we obtain

$$\begin{aligned}
|a(x)| \, ||u_n|^{q+1} - |u|^{q+1} - |v_n|^{q+1}| &= |a(x)| \, ||v_n + u|^{q+1} - |u|^{q+1} - |v_n|^{q+1}| \\
&\leqslant c|a(x)| \, |u|^q v_n.
\end{aligned}$$

Since $v_n \rightharpoonup 0$ in E, we deduce that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N} a(x)(|u_n|^{q+1}-|u|^{q+1})\,\mathrm{d}x = \lim_{n\to+\infty}\int_{\mathbb{R}^N} a(x)|v_n|^{q+1}(x)\,\mathrm{d}x. \tag{4.13}$$

Using Hölder's inequality in combination with (4.2) and (4.11), we obtain

$$\int_{\mathbb{R}^N}(a^+(x)+\chi_{B_R}(x))|v_n|^{q+1}\,\mathrm{d}x \leqslant \|a^+ + \chi_{B_R}\|_{2/(1-q)}\|v_n\|^{q+1}_{L^2(B(0,R))} \to 0. \tag{4.14}$$

Passing to the limit in (4.12) and using (4.2), (4.11), (4.13) and (4.14), we obtain

$$\begin{aligned}
0 = \lim_{n\to+\infty}\langle I'(u_n),u_n\rangle &= \langle I'(u),u\rangle + \lim_{n\to+\infty}\biggl(\int_{\mathbb{R}^N}(|\nabla v_n|^2 - a(x)|v_n|^{q+1})\,\mathrm{d}x\biggr)\\
&= \langle I'(u),u\rangle + \lim_{n\to+\infty}\biggl(\int_{\mathbb{R}^N}|\nabla v_n|^2 + (a^-(x)+\chi_{B_R})|v_n|^{q+1}\biggr)\\
&\qquad - \lim_{n\to+\infty}\int_{\mathbb{R}^N}(a^+(x)+\chi_{B_R})|v_n|^{q+1}\,\mathrm{d}x\\
&= \lim_{n\to+\infty}\int_{\mathbb{R}^N}|\nabla v_n|^2 + (a^-(x)+\chi_{B_R})|v_n|^{q+1}\,\mathrm{d}x.\\
&\geqslant \lim_{n\to+\infty}\int_{\mathbb{R}^N}(|\nabla v_n|^2 + \min(\tfrac{1}{2}-a_\infty,1)|v_n|^{q+1})\,\mathrm{d}x\\
&\geqslant \lim_{n\to+\infty}\min(1,\min(\tfrac{1}{2}-a_\infty,1))\int_{\mathbb{R}^N}(|\nabla v_n|^2+|v_n|^{q+1})\,\mathrm{d}x.
\end{aligned}$$

Then $v_n \to 0$ in E, which yields a contradiction.

STEP 2. (y_n) is not bounded. Indeed, suppose that (y_n) is bounded and there exists a subsequence of (y_n), also denoted by (y_n), such that $y_n \to y_0$. Then, for all $\varphi \in D(\mathbb{R}^N)$,

$$0 = \lim_{n\to+\infty}\int_{\mathbb{R}^N}\varphi(x-y_n)v_n\,\mathrm{d}x = \lim_{n\to+\infty}\int_{\mathbb{R}^N}\varphi(x)v_n(x+y_n)\,\mathrm{d}x = \int_{\mathbb{R}^N}\varphi(x)v(x)\,\mathrm{d}x.$$

Hence, $v=0$ a.e. in $\mathbb{R}^N$, a contradiction.

STEP 3. We show that v is a solution of the following problem:

$$\left.\begin{aligned} -\Delta u + v_\infty u = a_\infty |u|^{q-1}u \quad \text{in } \mathbb{R}^N,\\ u\in E.\end{aligned}\right\} \tag{P_∞}$$

We first prove that (P_∞) admits only the trivial solution. Thus, since v solves (P_∞), we will obtain a contradiction.

Since (y_n) is not bounded, $u_n(\cdot + y_n) \rightharpoonup v$ is in E. In fact, $u(\cdot + y_n) \rightharpoonup \psi \in E$, and hence

$$0 = \lim_{n\to+\infty}\int_{\mathbb{R}^N} u(x+y_n)\varphi(x)\,\mathrm{d}x = \int_{\mathbb{R}^N}\psi(x)\varphi(x)\,\mathrm{d}x \quad \forall\varphi\in D(\mathbb{R}^N).$$

It follows that $\psi = 0$ a.e. Therefore,

$$u_n(\cdot + y_n) \rightharpoonup v \quad \text{in } E. \tag{4.15}$$

Let $\varphi \in D(\mathbb{R}^N)$. We have

$$\begin{aligned}\langle I'(u_n), \varphi(\cdot - y_n)\rangle &= \int_{\mathbb{R}^N} (\nabla u_n \nabla \varphi(x-y_n) + V(x) u_n \varphi(x-y_n))\,\mathrm{d}x \\ &\quad - \int_{\mathbb{R}^N} a(x)|u_n|^{q-1} u_n \varphi(x-y_n)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^N} \nabla u_n(x+y_n)\nabla\varphi(x) + V(x+y_n)u_n(x+y_n)\varphi(x)\,\mathrm{d}x \\ &\quad - \int_{\mathbb{R}N} a(x+y_n)|u_n|^{q-1}(x+y_n)u_n(x+y_n)\varphi(x)\,\mathrm{d}x.\end{aligned}$$

Relation (4.15) yields

$$\int_{\mathbb{R}^N} \nabla u_n(x+y_n)\nabla\varphi(x)\,\mathrm{d}x \to \int_{\mathbb{R}^N} \nabla v(x)\nabla\varphi(x)\,\mathrm{d}x. \tag{4.16}$$

Since $(u_n(\cdot + y_n))$ is bounded in E, $u_n(\cdot + y_n) \to v$ in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ for all $1 \leqslant p \leqslant 2^*$ (up to a subsequence), $u_n(x+y_n) \to v$ a.e. in $\mathbb{R}^N$ and there exists $K \in L^p(\mathbb{R}^N)$ such that $\varphi|u_n(\cdot + y_n)| \leqslant |K|$ in $\mathbb{R}^N$, $1 \leqslant p \leqslant 2^*$. Then, by ($\mathrm{V}_1$), we obtain

$$\begin{gathered}V(x+y_n)u_n(x+y_n)\varphi \to v_\infty v\varphi \quad \text{a.e. in } \mathbb{R}^N, \\ |V(x+y_n)u_n(x+y_n)\varphi| \leqslant \|V\|_\infty |K|\,|\varphi| \in L^1(\mathbb{R}^N).\end{gathered}$$

Applying Lebesgue's dominated convergence theorem, we deduce that

$$\int_{\mathbb{R}^N} V(x+y_n)u_n(x+y_n)\varphi(x)\,\mathrm{d}x \to v_\infty \int_{\mathbb{R}^N} v(x)\varphi(x)\,\mathrm{d}x. \tag{4.17}$$

From hypothesis (A_1), we find

$$\begin{gathered}a(x+y_n)|u_n(x+y_n)|^{q-1}u_n(x+y_n)\varphi \to a_\infty |v|^{q-1}v\varphi \quad \text{a.e. in } \mathbb{R}^N, \\ |a(x+y_n)|\,|u_n(x+y_n)|^q|\varphi| \leqslant \|a\|_\infty |K|^q|\varphi| \in L^1(\mathbb{R}^N).\end{gathered}$$

Next, by Lebesgue's dominated convergence theorem, we obtain

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} a(x+y_n)|u_n|^{q-1}(x+y_n)u_n(x+y_n)\,\mathrm{d}x = a_\infty \int_{\mathbb{R}^N} |v|^{q-1}v\varphi(x)\,\mathrm{d}x. \tag{4.18}$$

Combining (4.16)–(4.18), we deduce that for all $\varphi \in D(\mathbb{R}^N)$,

$$\begin{aligned}0 &= \lim_{n\to+\infty} \langle I'(u_n), \varphi(\cdot - y_n)\rangle \\ &= \int_{\mathbb{R}^N} (\nabla v(x)\nabla\varphi(x) + v_\infty v\varphi)\,\mathrm{d}x - a_\infty \int_{\mathbb{R}^N} |v|^{q-1}v\varphi(x)\,\mathrm{d}x.\end{aligned}$$

Thus, v is a weak solution of (P_∞), and hence $v = 0$, which yields a contradiction. From steps 1, 2, and 3, we conclude that the dichotomy does not occur. The proof is complete. □

Lemma 4.4. *Assume that* (A_1) *and* (V_1) *are fulfilled. Then, for each* $k \in \mathbb{N}$, *there exists* $A_k \in \Gamma_k$ *such that* $\sup_{u\in A_k} I(u) < 0$.

Proof. We use some ideas developed in [27].

Let R_0 and y_0 be fixed by assumption (A_1) and consider the cube

$$D(R_0) = \{(x_1, \dots, x_N) \in \mathbb{R}^N \colon |x_i - y_i| < R_0,\ 1 \leqslant i \leqslant N\}.$$

Fix $k \in \mathbb{N}$ arbitrarily. Let $n \in \mathbb{N}$ be the smallest integer such that $n^N \geqslant k$. We divide $D(R_0)$ equally into n^N small cubes (denote them by D_i with $1 \leqslant i \leqslant n^N$) with planes parallel to each face of $D(R_0)$. The edge of D_i has the length of $a = R_0/n$. We construct new cubes E_i in D_i such that E_i has the same centre as that of D_i. The faces of E_i and D_i are parallel and the edge of E_i has the length $\frac{1}{2}a$. Thus, we can construct a function ψ_i, $1 \leqslant i \leqslant k$, such that

$$\begin{aligned} &\sup(\psi_i) \subset D_i, \qquad \operatorname{supp}(\psi_i) \cap \operatorname{supp}(\psi_j) = \emptyset \quad (i \neq j), \\ &\psi_i(x) = 1 \quad \text{for } x \in E_i, \qquad 0 \leqslant \psi_i(x) \leqslant 1 \quad \forall x \in \mathbb{R}^N. \end{aligned}$$

We define

$$S^{k-1} = \Big\{(t_1, \dots, t_k) \in \mathbb{R}^k \colon \max_{1 \leqslant i \leqslant k} |t_i| = 1\Big\}, \tag{4.19}$$

$$W_k = \Big\{ \sum_{i=1}^{k} t_i \psi_i(x) \colon (t_1, \dots, t_k) \in S^{k-1} \Big\} \subset E. \tag{4.20}$$

Since the mapping $(t_1, \dots, t_k) \to \sum_{i=1}^k t_i \psi_i$ from S^{k-1} to W_k is odd and homeomorphic, we have $\gamma(W_k) = \gamma(S^{k-1}) = k$. But W_k is compact in E, and thus there is a constant $\alpha_k > 0$ such that

$$\|u\|^2 \leqslant \alpha_k \quad \text{for all } u \in W_k.$$

We recall the inequality

$$\|u\|_2 \leqslant c \|\nabla u\|_2^r \|u\|_{q+1}^{1-r} \leqslant c\|u\| \tag{4.21}$$

with $r = 2^*(q-1)/2(2^* - q - 1)$. Then there is a constant $c_k > 0$ such that

$$\|u\|_2^2 \leqslant c_k \quad \text{for all } u \in W_k.$$

Let $z > 0$ and $u = \sum_{i=1}^k t_i \psi_i(x) \in W_k$. We have

$$I(zu) \leqslant \frac{z^2}{2}\alpha_k + z^2 \frac{\|V\|_\infty}{2} c_k - \frac{1}{q+1} \sum_{i=1}^{k} \int_{D_i} a(x) |z t_i \psi_i|^{q+1} \,\mathrm{d}x. \tag{4.22}$$

By (4.19), there exists $j \in [1, k]$ such that $|t_j| = 1$ and $|t_i| \leqslant 1$ for $i \neq j$. Then

$$\begin{aligned} \sum_{i=1}^{k} \int_{D_i} a(x) |z t_i \psi_i|^{q+1} \,\mathrm{d}x = \int_{E_j} a(x) |z t_j \psi_j|^{q+1} \,\mathrm{d}x + \int_{D_j \setminus E_j} a(x) |z t_j \psi_j(x)|^{q+1} \,\mathrm{d}x \\ + \sum_{i \neq j} \int_{D_i} a(x) |z t_i \psi_i|^{q+1} \,\mathrm{d}x. \end{aligned} \tag{4.23}$$

Since $\psi_j(x) = 1$ for $x \in E_j$ and $|t_j| = 1$, we have

$$\int_{E_j} a(x) |z t_j \psi_j|^{q+1} \,\mathrm{d}x = |z|^{q+1} \int_{E_j} a(x) \,\mathrm{d}x. \tag{4.24}$$

On the other hand, by (A_1),

$$\int_{D_j\setminus E_j} a(x)|zt_j\psi_j|^{q+1}\,\mathrm{d}x + \sum_{i\neq j}\int_{D_i} a(x)|zt_i\psi_i|^{q+1}\,\mathrm{d}x \geqslant 0. \tag{4.25}$$

Relations (4.22)–(4.25) yield

$$\frac{I(zu)}{z^2} \leqslant \frac{\alpha_k}{2} + \frac{\|V\|_\infty}{2}c_k - \frac{|z|^{q+1}}{z^2}\inf_{1\leqslant i\leqslant k}\left(\int_{E_i} a(x)\,\mathrm{d}x\right). \tag{4.26}$$

By (4.26), we conclude that

$$\lim_{z\to 0}\sup_{u\in W_k}\frac{I(zu)}{z^2} = -\infty.$$

We fix z small enough such that

$$\sup\{I(u), u\in A_k\} < 0, \quad \text{where } A_k = zW_k \in \Gamma_k.$$

This concludes the proof. □

Lemma 4.5. *Assume that* (A_1) *and* (V_1) *hold. Then* I *is bounded from below.*

Proof. By (A_1), we obtain

$$a^+ \in L^p(\mathbb{R}^N) \quad \text{for all } 1\leqslant p\leqslant +\infty. \tag{4.27}$$

Then

$$\begin{aligned} I(u) &= \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + V(x)u^2)\,\mathrm{d}x - \frac{1}{q+1}\int_{\mathbb{R}^N} a(x)|u|^{q+1}\,\mathrm{d}x \\ &\geqslant \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 - V^-(x)u^2)\,\mathrm{d}x - \frac{1}{q+1}\int_{\mathbb{R}^N} a^+(x)|u|^{q+1}\,\mathrm{d}x \\ &\geqslant \left(\frac{1}{2} - \frac{\|V^-\|_{N/2}}{2s}\right)\|\nabla u\|_2^2 - \frac{\|a^+\|_{2^*/(2^*-q-1)}}{s^{(q+1)/2}}\|\nabla u\|_2^{q+1}. \end{aligned}$$

In view of (V_1), we conclude the proof. □

Proof of theorem 2.1 concluded. We have $I(0)=0$ and I is even. Combining lemmas 4.3, 4.4 and 4.5, we deduce that theorem 3.2(1) and (2) are satisfied. Thus, there exists a sequence $(u_n)\subset E$ such that $I(u_n)<0$, $I'(u_n)=0$ and $I(u_n)\to 0$ for all $n\geqslant 0$, and hence u_n is a weak solution of (2.1).

By (V_1), we deduce that

$$\begin{aligned} \frac{1}{q+1}\langle I'(u_n), u_n\rangle - I(u_n) &= \left(\frac{1}{q+1} - \frac{1}{2}\right)\int_{\mathbb{R}^N}(|\nabla u_n|^2 + V(x)u_n^2)\,\mathrm{d}x \\ &\geqslant \left(\frac{1}{q+1} - \frac{1}{2}\right)\left(\frac{1}{2} - \frac{1}{2s}\|V^-\|_{N/2}\right)\|\nabla u_n\|_2^2. \end{aligned}$$

It follows that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}|\nabla u_n|^2\,\mathrm{d}x = 0. \tag{4.28}$$

On the other hand, by Hölder's inequality, (4.2) and (4.28), we have

$$
\begin{aligned}
0 &= \lim_{n\to+\infty}(I(u_n) - \tfrac{1}{2}\langle I'(u_n), u_n\rangle) \\
&= \lim_{n\to+\infty}\left(\frac{1}{2} - \frac{1}{q+1}\right)\int_{\mathbb{R}^N} a(x)|u_n|^{q+1} \\
&= \left(\frac{1}{q+1} - \frac{1}{2}\right)\lim_{n\to+\infty}\left(\int_{\mathbb{R}^N}(a^-(x) + \chi_{B_R})|u_n|^{q+1}\,\mathrm{d}x \right.\\
&\qquad\qquad\left. - \int_{\mathbb{R}^N}(a^+(x) + \chi_{B_R})|u_n|^{q+1}\right) \\
&\geqslant \left(\frac{1}{q+1} - \frac{1}{2}\right)\lim_{n\to+\infty}\left(\int_{\mathbb{R}^N}(a^-(x) + \chi_{B_R})|u_n|^{q+1}\,\mathrm{d}x \right.\\
&\qquad\qquad\left. - \frac{\|a^+ + \chi_{B_R}\|^{2^*/(2^*-q-1)}}{S^{(q+1)/2}}\|\nabla u_n\|_2^{q+1}\right) \\
&= \left(\frac{1}{q+1} - \frac{1}{2}\right)\lim_{n\to+\infty}\int_{\mathbb{R}^N}(a^-(x) + \chi_{B_R})|u_n|^{q+1}\,\mathrm{d}x \\
&\geqslant \left(\frac{1}{q+1} - \frac{1}{2}\right)\min\left(\frac{-a_\infty}{2}, 1\right)\lim_{n\to+\infty}\|u_n\|_{q+1}^{q+1}.
\end{aligned}
$$

This shows that

$$
\lim_{n\to+\infty}\int_{\mathbb{R}^N}|u_n|^{q+1}\,\mathrm{d}x = 0, \tag{4.29}
$$

and hence $\lim_{n\to+\infty} u_n = 0$ in E. This concludes the proof. □

5. Proof of theorem 2.2

In this section we define I and φ on X. We use standard arguments based on the fact that I' is a sequentially compact operator in order to prove that I satisfies the PS condition. We then deduce that (2.1) admits infinitely many non-trivial solutions in X.

To prove theorem 2.2, we need the following auxiliary results.

Lemma 5.1. *Assume that (A_2), (V_2) and (G_1) are satisfied. Then φ' is a sequentially compact operator on X.*

Proof. By standard arguments, the functionals I and φ are well defined and of class C^1 on X.

Let $(u_n) \subset X$ be a bounded sequence. Then, for all $h \in X$,

$$
\langle\varphi'(u_n) - \varphi'(u), h\rangle = \int_{\mathbb{R}^N}[V(x)(u_n - u) - a(x)(g(u_n) - g(u))]h(x)\,\mathrm{d}x.
$$

Let $R > 0$ and $h \in X$ be such that $\|h\| = 1$. We have

$$
\langle\varphi'(u_n) - \varphi'(u), h\rangle = J_1(n, h, R) + J_2(n, h, R),
$$

where

$$J_1(n,h,R)=\int_{B_R}[V(x)(u_n-u)-a(x)(g(u_n)-g(u))]h(x)\,\mathrm{d}x,$$
$$J_2(n,h,R)=\int_{B_R^c}[V(x)(u_n-u)-a(x)(g(u_n)-g(u))]h(x)\,\mathrm{d}x.$$

By Hölder's inequality, (V_2), (A_2) and (G_1), we obtain

$$\begin{aligned}
&|J_2(n,h,R)|\\
&\quad\leqslant \int_{B_R^c}|V(x)(u_n-u)h(x)-a(x)(g(u_n)-g(u))h(x)|\,\mathrm{d}x\\
&\quad\leqslant \biggl(\int_{B_R^c}|V(x)|^{N/2}\,\mathrm{d}x\biggr)^{2/N}\biggl(\int_{B_R^c}|u_n-u|^{2^*}\,\mathrm{d}x\biggr)^{1/2^*}\biggl(\int_{B_R^c}|h(x)|^{2^*}\,\mathrm{d}x\biggr)^{1/2^*}\\
&\qquad + c\biggl(\int_{B_R^c}|a(x)|^{2^*/(2^*-(q+1))}\,\mathrm{d}x\biggr)^{(2^*-(q+1))/2^*}\biggl(\int_{B_R^c}(|u_n(x)+u(x)|)^{2^*}\,\mathrm{d}x\biggr)^{q/2^*}\\
&\qquad\qquad\times\biggl(\int_{B_R^c}|h(x)|^{2^*}\,\mathrm{d}x\biggr)^{1/2^*}\\
&\quad\leqslant \biggl(\int_{B_R^c}|V(x)|^{N/2}\,\mathrm{d}x\biggr)^{2/N}+\biggl(\int_{B_R^c}|a(x)|^{2^*/(2^*-(q+1))}\,\mathrm{d}x\biggr)^{(2^*-(q+1))/2^*}.
\end{aligned}$$

The last expression can be made arbitrarily small by taking $R>0$ large enough.

For J_1, since $V\in L^{N/2}(\mathbb{R}^N)$ and $a\in L^{2^*/(2^*-(q+1))}(\mathbb{R}^N)$, we deduce that for all $\varepsilon>0$ there exists $\eta>0$ such that

$$\biggl(\int_K|a(x)|^{2^*/(2^*-(q+1))}\,\mathrm{d}x\biggr)^{(2^*-(q+1))/2^*}+\biggl(\int_K|V(x)|^{N/2}\,\mathrm{d}x\biggr)^{2/N}<\varepsilon$$

for all $K\subset B_R$ with $m(K)<\eta$ (see [20]). Moreover,

$$\begin{aligned}
&\int_K|V(x)(u_n-u)-a(x)(g(u_n)-g(u))|\,|h(x)|\,\mathrm{d}x\\
&\qquad\leqslant c\biggl(\int_K|V(x)|^{N/2}\,\mathrm{d}x\biggr)^{2/N}+c\biggl(\int_K|a(x)|^{2^*/(2^*-(q+1))}\,\mathrm{d}x\biggr)^{(2^*-(q+1))/2^*}\\
&\qquad\leqslant c\varepsilon,
\end{aligned}$$

where c is independent of n and h. By using the Vitali convergence theorem, we deduce that $J_1(n,h,R)\to 0$ as $n\to+\infty$ uniformly for $\|h\|=1$. We conclude that $\varphi'(u_n)\to\varphi'(u)$ strongly in X'. The proof is complete. □

LEMMA 5.2. *Assume that* (V_2), (A_2) *and* (G_1) *are satisfied. Then any PS sequence of* I *is bounded in* X.

Proof. Let $(u_n) \subset X$ be a PS sequence. There then exists $\alpha > 0$ such that $I(u_n) \leqslant \alpha$. By Hölder's inequality and conditions (A_2), (V_2) and (G_1), we have

$$\begin{aligned} \alpha \geqslant I(u_n) &\geqslant \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u_n|^2(x) - V^-(x)u_n(x)^2)\,\mathrm{d}x - \int_{\mathbb{R}^N} a(x)G(u_n(x))\,\mathrm{d}x \\ &\geqslant \left(\frac{1}{2} - \frac{1}{2s}\|V^-\|_{N/2}\right)\|u_n\|_X^2 - s^{(-q-1)/2}\|a\|_{2^*/(2^*-(q+1))}\|u_n\|_X^{q+1}. \end{aligned}$$

Since $0 < q < 1$, the last inequality shows that (u_n) is bounded in X. The proof is complete. □

As a consequence, we obtain the following result.

Lemma 5.3. *Assume that (V_2), (A_2) and (G_1) are satisfied. Then I satisfies the PS condition in X.*

Proof. Set

$$\begin{gathered} F\colon D^{1,2}(\mathbb{R}^N) \to (D^{1,2}(\mathbb{R}^N))', \\ u \mapsto F(u), \\ \langle F(u), v\rangle = \int_{\mathbb{R}^N} \nabla u \nabla v\,\mathrm{d}x \quad \forall v \in D^{1,2}(\mathbb{R}^N). \end{gathered}$$

Then F is an isomorphism. Let (u_n) be a PS sequence of I; hence,

$$u_n = F^{-1}(\varphi'(u_n)) + o(1). \tag{5.1}$$

By lemma 5.2, (u_n) is bounded in X. Since φ' is a compact operator and using (5.1), we deduce that (u_n) is strongly convergent in X (up to a subsequence). □

Lemma 5.4. *Assume that (G_1), (V_2) and (A_2) are satisfied. Then I is bounded from below.*

Proof. By (G_1), (V_2) and (A_2), we have

$$\begin{aligned} I(u) &= \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u(x)|^2 + V(x)u^2(x))\,\mathrm{d}x - \int_{\mathbb{R}^N} a(x)G(u(x))\,\mathrm{d}x \\ &\geqslant \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u(x)|^2\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} V^-(x)u^2(x)\,\mathrm{d}x - \int_{\mathbb{R}^N} a(x)G(u(x))\,\mathrm{d}x \\ &\geqslant \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u(x)|^2\,\mathrm{d}x - \frac{1}{2s}\|V^-\|_{N/2}\|u\|_X^2 - s^{(-q-1)/2}\|a\|_{2^*/(2^*-(q+1))}\|u\|_X^{q+1} \\ &\geqslant \left(\frac{1}{2} - \frac{1}{2s}\|V^-\|_{N/2}\right)\|u\|_X^2 - s^{(-q-1)/2}\|a\|_{2^*/(2^*-(q+1))}\|u\|_X^{q+1}. \end{aligned}$$

Since $1 < q + 1 < 2$, we deduce that I is bounded from below. The proof is complete. □

Next, we prove the geometric condition required by theorem 3.2.

Lemma 5.5. *Assume that (A_2), (V_2), (G_1), (G_2) and (G_3) are satisfied. Then, for each $k \in \mathbb{N}$ there exists an $A_k \in \Gamma_k$ such that $\sup_{u\in A_k} I(u) < 0$.*

Proof. By using conditions (G_2) and (G_3), the proof is similar to that of lemma 4.4. □

Proof of theorem 2.2 concluded. The energy functional I is even and $I(0) = 0$. By lemmas 5.3 and 5.4, theorem 3.2(1) is satisfied. In view of lemma 5.5, theorem 3.2(2) is also satisfied. Thus, there exists a sequence (u_k) such that $c_k = I(u_k)$ is a critical value of I, $c_k < 0$, $c_k \to 0$ for all $k \geqslant 0$. This means that (u_k) are weak solutions of (2.1) and (u_k) is a PS sequence of I. Then, by lemma 5.2, (u_k) is bounded. □

REMARK 5.6. If $g(x) = |x|^{q-1}x$, $0 < q < 1$, then $u_n \to 0$ in X. In fact, by (V_2), we have

$$0 = \frac{1}{q+1}\langle I'(u_n), u_n\rangle - I(u_n) = \left(\frac{1}{q+1} - \frac{1}{2}\right)\int_{\mathbb{R}^N} (|\nabla u_n|^2 + V(x)|u_n|^2)\,\mathrm{d}x$$
$$\geqslant \left(\frac{1}{q+1} - \frac{1}{2}\right)\left(1 - \frac{\|V^-\|_{N/2}}{s}\right)\int_{\mathbb{R}^N} |\nabla u_n|^2(x)\,\mathrm{d}x.$$

Since $I'(u_n) = 0$ and $\lim_{n\to+\infty} I(u_n) = 0$, we deduce that $u_n \to 0$ in X.

6. Proof of theorem 2.3

In this section we change the condition (V_2) to (V_1) and we suppose that a satisfies (A_3). Under the last conditions, if the functional I is not well defined on either X or E, then we define it on the space Y. We first establish that $(Y, \langle\cdot\rangle)$ is a Hilbert space and it is embedded into $L^p(\mathbb{R}^N)$ for $2 \leqslant p \leqslant 2^*$. The proof of the following result relies on standard arguments and we will omit it.

LEMMA 6.1. *Assume that (V_1) holds. Then*

$$u \to \left(\int_{\mathbb{R}^N} (|\nabla u(x)|^2 + V^+(x)u^2(x))\,\mathrm{d}x\right)^{1/2}$$

defines a norm on Y, which is equivalent to the usual norm in $H^1(\mathbb{R}^N)$,

$$\|u\|_{H^1} = \|\nabla u\|_2 + \|u\|_2.$$

By using lemma 6.1, the proof of theorem 2.3, with slight modifications, is similar to that of theorem 2.2.

REMARK 6.2. If $g(x) = |x|^{q-1}x$, $0 < q < 1$, then $u_n \to 0$ in Y.

REMARK 6.3. In theorems 2.1, 2.2 and 2.3 we can suppose that u_0 is a non-negative solution of (2.1), since

$$I(u_0) = I(|u_0|) = c_0.$$

In such a case, u_0 is called a ground state for I.

Acknowledgements

The authors are grateful to the anonymous referee and to the handling editor for the careful reading of the paper and for their constructive remarks. V.R. acknowledges support from grant no. CNCS PCE-47/2011.

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(*Issued 5 June 2015*)