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Multiple Solutions of Neumann Problems: An Orlicz–Sobolev Space Setting

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Abstract In the present paper, we establish the range of two parameters for which a non-homogeneous boundary value problem admits at least three weak solutions. The proof of the main results relies on recent variational principles due to Ricceri.

Keywords Three solutions · Non-homogeneous differential operator · Orlicz–Sobolev space

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1 Introduction

In this paper, we are concerned with the qualitative analysis of weak solutions for a large class of non-linear elliptic equations with Neumann boundary condition. The main features of this paper are the following:

(i) the presence of a non-homogeneous differential operator and the treatment in a suitable Orlicz–Sobolev function space;
(ii) the use of the Ricceri three-critical point theorem, which is a powerful analytic tool for multiplicity results in non-linear problems with a variational structure.

Orlicz–Sobolev spaces have been used in the last decades to model various phenomena. These function spaces play a significant role in many fields of mathematics, such as approximation theory, partial differential equations, calculus of variations, non-linear potential theory, the theory of quasi-conformal mappings, non-Newtonian fluids, image processing, differential geometry, geometric function theory, and probability theory. These spaces consist of functions that have weak derivatives and satisfy certain integrability conditions.

We refer to Chen, Levine and Rao [1], who proposed a framework for image restoration based on a Laplace operator with variable exponent. A second major application of non-homogeneous differential operators with variable exponent is the modeling of some materials with inhomogeneities, for instance, electrorheological (non-Newtonian) fluids (sometimes referred to as 'smart fluids'), cf. [2–7]. Materials requiring such more advanced theory have been studied experimentally since the middle of the last century. The first major discovery in electrorheological fluids is due to Willis Winslow in 1949. These fluids have the interesting property that their viscosity depends on the electric field in the fluid. Winslow noticed that in such fluids (for instance, lithium polymethacrylate) viscosity in an electrical field is inversely proportional to the strength of the field. The field induces string-like formations in the fluid, which are parallel to the field. They can raise the viscosity by as much as five orders of magnitude. This phenomenon is known as the *Winslow effect*. For a general account of the underlying physics we refer to Halsey [8] and Pfeiffer et al. [9]. An overview of Orlicz–Sobolev spaces is given in the monographs by Rao and Ren [10] and Rădulescu and Repovš [6]. On the other hand, the study of differential equations and variational problems with non-linear boundary condition have been studied recently in many papers, see [11–16].

The classical three-critical point theorem of Pucci and Serrin [17,18] asserts that if $f : X \to \mathbb{R}$ is of class C^1 (X is a Banach space), satisfies the Palais–Smale condition, and has two local minima, then f has a third critical point. The general variational principle of Ricceri [19,20] extends the Pucci–Serrin theorem and provides alternatives for the multiplicity of critical points of certain functionals depending on a parameter. We refer to [21] for several applications of the Ricceri variational principles.

In this work, we are concerned with the following problem involving non-homogeneous differential operators:

$$\begin{cases} -\operatorname{div}\big(\alpha(x,|\nabla u(x)|)\nabla u(x)\big) + \alpha\big(x,|u(x)|\big)u(x) = \lambda f\big(x,u(x)\big) & \text{for } x \in \Omega, \\ \alpha\big(x,|\nabla u(x)|\big)\dfrac{\partial u}{\partial \nu}(x) = \mu g\Big(\gamma\big(u(x)\big)\Big) & \text{for } x \in \partial\Omega, \end{cases} \qquad \big(N^{f,g}_{\lambda,\mu}\big)$$

where Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 3$) with smooth boundary $\partial\Omega$, $\frac{\partial u}{\partial \nu}$ is the outer unit normal derivative, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, $g : \mathbb{R} \to \mathbb{R}$ is a continuous function, λ is a positive parameter, μ is a non-negative parameter, and the functions $\alpha(x, t) : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ and γ will be specified later.

In this paper, motivated by the above facts and the recent papers [22–26], we establish some new sufficient conditions under which the problem $(N^{f,g}_{\lambda,\mu})$ possesses three weak solutions in the Orlicz–Sobolev space. At first, we present two three solutions existence results under algebraic conditions on f (see Theorems 3.1, 3.2). Next, assuming that the growth of f is sublinear at infinity, we establish the existence of three weak solutions for the problem $(N^{f,g}_{\lambda,\mu})$ (see Theorem 3.3).

This paper is organized as follows. In Sect. 2, some preliminaries and the abstract Orlicz–Sobolev spaces setting are presented. In Sect. 3, we discuss the existence of three weak solutions for the problem $(N^{f,g}_{\lambda,\mu})$. We also point out special cases of the results, and we illustrate the results by presenting an example.

2 Functional Setting

We now state two critical point theorems which are the main tools for the proofs of our results. The first results of this type have been obtained in [19]. In the first result, the coercivity of the functional $J - \lambda I$ is required; in the second one, a suitable sign hypothesis is assumed. Theorem 2.1 is a particular case of Theorem 1 of Ricceri [20], while for Theorem 2.2, we refer to Bonanno and Candito [27, Corollary 3.1].

Theorem 2.1 *Let X be a reflexive real Banach space; $J : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive, and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*, $I : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, such that*

$$J(0) = I(0) = 0\,.$$

Assume that there exist $r > 0$ and $\bar{x} \in X$, with $r < J(\bar{x})$ such that

(i) $\sup_{J(x)\leq r} I(x) < rI(\bar{x})/J(\bar{x})$,
(ii) *for each λ in*

$$\Lambda_r := \left]\frac{J(\bar{x})}{I(\bar{x})}, \frac{r}{\sup_{J(x)\leq r} I(x)}\right[,$$

the functional $J - \lambda I$ is coercive.

Then, for each $\lambda \in \Lambda_r$ the functional $J - \lambda I$ has at least three distinct critical points in X.

Theorem 2.2 *Let X be a reflexive real Banach space; $J : X \to \mathbb{R}$ be a convex, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*, $I : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, such that*

$$\inf_X J = J(0) = I(0) = 0\,.$$

Assume that there exist two positive constants $r_1, r_2 > 0$ *and* $\bar{x} \in X$, *with* $2r_1 < J(\bar{x}) < \frac{r_2}{2}$, *such that*

(j) $\dfrac{\sup_{J(x)\le r_1} I(x)}{r_1} < \dfrac{2}{3}\dfrac{I(\bar{x})}{J(\bar{x})}$,
(jj) $\dfrac{\sup_{J(x)\le r_2} I(x)}{r_2} < \dfrac{1}{3}\dfrac{I(\bar{x})}{J(\bar{x})}$,
(jjj) *for each* λ *in*

$$\Lambda^*_{r_1,r_2} := \left]\frac{3}{2}\frac{J(\bar{x})}{I(\bar{x})}, \min\left\{\frac{r_1}{\sup_{J(x)\le r_1} I(x)}, \frac{r_2}{2\sup_{J(x)\le r_2} I(x)}\right\}\right[$$

and for every $x_1, x_2 \in X$, *which are local minima for the functional* $J - \lambda I$, *and such that* $I(x_1) \ge 0$ *and* $I(x_2) \ge 0$, *one has* $\inf_{t\in[0,1]} I\big(tx_1 + (1-t)x_2\big) \ge 0$.

Then, for each $\lambda \in \Lambda^*_{r_1,r_2}$ *the functional* $J - \lambda I$ *has at least three distinct critical points which lie in* $J^{-1}(]-\infty, r_2[)$.

In order to study the problem $(N^{f,g}_{\lambda,\mu})$, let us introduce the functional spaces where it will be discussed. We will give just a brief review of some basic concepts and facts of the theory of Orlicz–Sobolev spaces, useful for what follows, for more details we refer the readers to Adams [28], Diening [29], Musielak [30], Rao and Ren [10], Rădulescu [5], Rădulescu and Repovš [6].

We now recall some facts on the theory of Orlicz–Sobolev spaces that will be used in the present paper. Suppose that the function $\alpha(x,t) : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ is such that the mapping $\varphi(x,t) : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$, defined by

$$\varphi(x,t) = \begin{cases} \alpha\big(x, |t|\big)t & \text{for } t \neq 0, \\ 0 & \text{for } t = 0 \end{cases}$$

satisfies the condition (φ) for all $x \in \Omega$, $\varphi(x,\cdot) : \mathbb{R} \to \mathbb{R}$ is an odd, increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$, and

$$\Phi(x,t) = \int_0^t \varphi(x,s)\,\mathrm{d}s, \quad \forall x \in \overline{\Omega},\ \ t \ge 0$$

belongs to class Φ (see [30], p. 33), i.e., the function Φ satisfies the following conditions:

(Φ_1) for all $x \in \Omega$, $\varphi(x,\cdot) : [0,+\infty) \to \mathbb{R}$ is a non-decreasing continuous function, with $\Phi(x,0) = 0$ and $\Phi(x,t) > 0$ whenever $t > 0$, $\lim_{t\to\infty} \Phi(x,t) = \infty$,
(Φ_2) for every $t \ge 0$, $\Phi(\cdot,t) : \Omega \to \mathbb{R}$ is a measurable function.

Since $\varphi(x,\cdot)$ satisfies condition (φ), we deduce that $\Phi(x,\cdot)$ is convex and increasing from $\mathbb{R}^+$ to $\mathbb{R}^+$.

For the function Φ, we define the *generalized Orlicz class*,

$$K_\Phi(\Omega) = \left\{ u : \Omega \to \mathbb{R},\ \text{measurable};\ \int_\Omega \Phi(x, |u(x)|)\, \mathrm{d}x < \infty \right\}$$

and the *generalized Orlicz space*,

$$L^\Phi(\Omega) = \left\{ u : \Omega \to \mathbb{R},\ \text{measurable};\ \lim_{\lambda \to 0^+} \int_\Omega \Phi(x, \lambda |u(x)|)\, \mathrm{d}x = 0 \right\}.$$

The space $L^\Phi(\Omega)$ is a Banach space endowed with the *Luxemburg norm*

$$|u|_\Phi = \inf \left\{ \mu > 0;\ \int_\Omega \Phi\left(x, \frac{|u(x)|}{\mu}\right) \mathrm{d}x \le 1 \right\}$$

or the equivalent norm (the *Orlicz norm*)

$$|u|_{(\Phi)} = \sup \left\{ \left| \int_\Omega uv\, \mathrm{d}x \right|;\ v \in L^{\overline{\Phi}}(\Omega),\ \int_\Omega \overline{\Phi}(x, |v(x)|)\, \mathrm{d}x \le 1 \right\},$$

where $\overline{\Phi}$ denotes the *conjugate Young* function of Φ, that is,

$$\overline{\Phi}(x, t) = \sup_{s>0} \left\{ ts - \Phi(x, s);\ s \in \mathbb{R} \right\}, \qquad \forall\, x \in \overline{\Omega},\ t \ge 0.$$

Furthermore, for Φ and $\overline{\Phi}$ conjugate Young functions, the Hölder type inequality holds true

$$\left| \int_\Omega uv\, \mathrm{d}x \right| \le B \times |u|_\Phi \times |v|_{\overline{\Phi}}, \qquad \forall\, u \in L^\Phi(\Omega),\ v \in L^{\overline{\Phi}}(\Omega), \tag{2.1}$$

where B is a positive constant (see [30], Theorem 13.13). In this paper we assume that there exist two positive constants φ_0 and φ^0 such that

$$1 < \varphi_0 \le \frac{t\varphi(x, t)}{\Phi(x, t)} \le \varphi^0 < \infty, \qquad \forall\, x \in \overline{\Omega},\ t \ge 0. \tag{2.2}$$

The above relation implies that Φ satisfies the Δ_2-*condition*, i.e.

$$\Phi(x, 2t) \le K \times \Phi(x, t), \qquad \forall\, x \in \overline{\Omega},\ t \ge 0, \tag{2.3}$$

where K is a positive constant (see [31, Proposition 2.3]). Relation (2.3) and Theorem 8.13 in [30] imply that $L^\Phi(\Omega) = K_\Phi(\Omega)$. Furthermore, we assume that Φ satisfies the following condition:

$$\text{for each } x \in \overline{\Omega},\ \text{the function } [0, \infty) \ni t \to \Phi(x, \sqrt{t}) \text{ is convex}. \tag{2.4}$$

Relation (2.4) assures that $L^{\Phi}(\Omega)$ is an uniformly convex space and thus, a reflexive space (see [31, Proposition 2.2]).

On the other hand, we point out that assuming that Φ and Ψ belong to class Φ and

$$\Psi(x,t) \leq K_1 \times \Phi(x, K_2 \times t) + \eta(x), \qquad \forall\, x \in \overline{\Omega},\ t \geq 0\,, \tag{2.5}$$

where $\eta \in L^1(\Omega)$, $\eta(x) \geq 0$ a.e. $x \in \Omega$ and K_1, K_2 are positive constants, then by Theorem 8.5 in [30] there exists the continuous embedding $L^{\Phi}(\Omega) \subset L^{\Psi}(\Omega)$. Next, we define the *generalized Orlicz–Sobolev space*

$$W^{1,\Phi}(\Omega) = \left\{ u \in L^{\Phi}(\Omega);\ \frac{\partial u}{\partial x_i} \in L^{\Phi}(\Omega), \quad i = 1, \ldots, N \right\}.$$

On $W^{1,\Phi}(\Omega)$ we define the equivalent norms

$$\begin{aligned}
\|u\|_{1,\Phi} &= |\,|\nabla u|\,|_{\Phi} + |u|_{\Phi},\\
\|u\|_{2,\Phi} &= \max\left\{|\,|\nabla u|\,|_{\Phi}, |u|_{\Phi}\right\},\\
\|u\| &= \inf\left\{\mu > 0;\ \int_{\Omega}\left[\Phi\left(x, \frac{|u(x)|}{\mu}\right) + \Phi\left(x, \frac{|\nabla u(x)|}{\mu}\right)\right] \mathrm{d}x \leq 1\right\}.
\end{aligned}$$

More precisely, for every $u \in W^{1,\Phi}(\Omega)$, we have

$$\|u\| \leq 2\|u\|_{2,\Phi} \leq 2\|u\|_{1,\Phi} \leq 4\|u\| \tag{2.6}$$

(see [31, Proposition2.4]). The generalized Orlicz–Sobolev space $W^{1,\Phi}(\Omega)$ endowed with one of the above norms is a reflexive Banach space.

In the following, we will use the norm $\|\cdot\|$ on $E := W^{1,\Phi}(\Omega)$ and we suppose that $\gamma : E \to L^{\Phi}(\Omega)$ is the trace operator.

The following lemma is useful in the proof of our results.

Lemma 2.3 *Let $u \in E$. Then*

$$\int_{\Omega}\Big(\Phi\big(x, |\nabla u(x)|\big) + \Phi\big(x, |u(x)|\big)\Big)\,\mathrm{d}x \geq \|u\|^{\varphi_0} \qquad \text{if } \|u\| > 1; \tag{2.7}$$

$$\int_{\Omega}\Big(\Phi\big(x, |\nabla u(x)|\big) + \Phi\big(x, |u(x)|\big)\Big)\,\mathrm{d}x \geq \|u\|^{\varphi^0} \qquad \text{if } \|u\| < 1. \tag{2.8}$$

For the proof of the previous result see, for instance, Lemma 2.3 of [32].

We point out that assuming that Φ and Ψ belong to class Φ, satisfying relation (2.5) and $\inf_{x\in\Omega}\Phi(x,1) > 0$, $\inf_{x\in\Omega}\Psi(x,1) > 0$ then there exists the continuous embedding $W^{1,\Phi}(\Omega) \hookrightarrow W^{1,\Psi}(\Omega)$.

In this paper we study the problem $(N^{f,g}_{\lambda,\mu})$ in the particular case when Φ satisfies

$$M \times |t|^{p(x)} \leq \Phi(x,t), \qquad \forall\, x \in \overline{\Omega},\ t \geq 0\,, \tag{2.9}$$

where $p(x) \in C(\overline{\Omega})$ with $p^- := \inf_{x\in\Omega} p(x) > N$ for all $x \in \overline{\Omega}$, and $M > 0$ is a constant.

By the relation (2.9) we deduce that E is continuously embedded in $W^{1,p(x)}(\Omega)$ (see relation (2.5) with $\Psi(x,t) = |t|^{p(x)}$).

Moreover, as pointed out in [33] and [34], $W^{1,p(x)}(\Omega)$ is continuously embedded in $W^{1,p^-}(\Omega)$ and since $p^- > N$, we deduce that $W^{1,p^-}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$. Thus, E is compactly embedded in $C^0(\overline{\Omega})$, and there exists a constant $m > 0$ such that

$$\|u\|_\infty \leq m\, \|u\|, \quad \forall\, u \in E, \tag{2.10}$$

where $\|u\|_\infty := \sup_{x\in\overline{\Omega}} |u(x)|$.

Example 2.4 Define

$$\varphi(x,t) = p(x)\frac{|t|^{p(x)-2}t}{\log\left(1+|t|\right)} \qquad \text{for} \quad t \neq 0, \quad \text{and} \quad \varphi(x,0) = 0,$$

where $p(x) \in C(\overline{\Omega})$ satisfying $N < p(x) < +\infty$ for all $x \in \overline{\Omega}$. Some simple computations imply

$$\Phi(x,t) = \frac{|t|^{p(x)}}{\log\left(1+|t|\right)} + \int_0^{|t|} \frac{s^{p(x)}}{(1+s)(\log\left(1+s\right))^2}\, \mathrm{d}s,$$

and the relations (φ), (Φ_1), and (Φ_2) are verified. For each $x \in \overline{\Omega}$ fixed, by Example 3 on p. 243 in [14], we have

$$p(x) - 1 \leq \frac{t\varphi(x,t)}{\Phi(x,t)} \leq p(x), \qquad \forall\, t \geq 0\,.$$

Thus, the relation (2.2) holds true with $\varphi_0 = p^- - 1$ and $\varphi^0 = p^+ := \sup_{x\in\Omega} p(x)$. Next, Φ satisfies the condition (2.9) since

$$\Phi(x,t) \geq t^{p(x)-1}, \qquad \forall\, x \in \overline{\Omega},\ t \geq 0\,.$$

Finally, we point out that trivial computations imply that $\frac{d^2\left(\Phi(x,\sqrt{t})\right)}{\mathrm{d}t^2} \geq 0$ for all $x \in \overline{\Omega}$ and $t \geq 0$. Thus, the relation (2.4) is satisfied.

We say that $u \in E$ is a *weak solution* of the problem $(N^{f,g}_{\lambda,\mu})$ if

$$\begin{aligned}\int_\Omega \alpha\left(x, |\nabla u(x)|\right)\nabla u(x)\, \times\, \nabla v(x)\, \mathrm{d}x + \int_\Omega \alpha\left(x, |u(x)|\right)u(x)v(x)\, \mathrm{d}x \\ = \lambda \int_\Omega f\left(x, u(x)\right)v(x)\, \mathrm{d}x + \mu \int_{\partial\Omega} g\Big(\gamma\left(u(x)\right)\Big)\gamma\left(v(x)\right)\mathrm{d}\sigma\end{aligned}$$

for every $v \in E$.

3 Main Results

We recall that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and $F(x,t) := \int_0^t f(x,s)\,\mathrm{d}s$.

Fix $d \neq 0$ and $c \geq m$ [see relation (2.10)] such that

$$\frac{\int_\Omega \max_{|t|\leq c} F(x,t)\,\mathrm{d}x}{\left(\frac{c}{m}\right)^{\varphi_0}} < \frac{\int_\Omega F(x,d)\,\mathrm{d}x}{\int_\Omega \Phi(x,|d|)\,\mathrm{d}x}.$$

Pick

$$\lambda \in \Lambda_1 := \left]\frac{\int_\Omega \Phi(x,|d|)\,\mathrm{d}x}{\int_\Omega F(x,d)\,\mathrm{d}x}, \frac{\left(\frac{c}{m}\right)^{\varphi_0}}{\int_\Omega \max_{|t|\leq c} F(x,t)\,\mathrm{d}x}\right[, \tag{3.1}$$

and set

$$\delta_1 := \min\left\{\frac{c^{\varphi_0} - \lambda m^{\varphi_0}\int_\Omega \max_{|t|\leq c} F(x,t)\,\mathrm{d}x}{m^{\varphi_0} a(\partial\Omega)\max_{|t|\leq c} G(t)}, \left|\frac{\int_\Omega \Phi(x,|d|)\,\mathrm{d}x - \lambda\int_\Omega F(x,d)\,\mathrm{d}x}{a(\partial\Omega)G(d)}\right|\right\} \tag{3.2}$$

and

$$\overline{\delta}_1 := \min\left\{\delta_1, \frac{1}{\max\left\{0, 2m^{\varphi_0}a(\partial\Omega)\limsup_{|\xi|\to+\infty}\frac{G(\xi)}{|\xi|^{\varphi_0}}\right\}}\right\}, \tag{3.3}$$

where $a(\partial\Omega) := \int_{\partial\Omega} d\sigma$, and we read $\frac{1}{0} = +\infty$ whenever this case occurs.

With the above notations we establish the following result.

Theorem 3.1 *Suppose that there exist $d \neq 0$ and $c \geq m$, with*

$$\int_\Omega \Phi(x,|d|)\,\mathrm{d}x > \left(\frac{c}{m}\right)^{\varphi_0},$$

such that

(A1)

$$\frac{\int_\Omega \max_{|t|\leq c} F(x,t)\,\mathrm{d}x}{\left(\frac{c}{m}\right)^{\varphi_0}} < \frac{\int_\Omega F(x,d)\,\mathrm{d}x}{\int_\Omega \Phi(x,|d|)\,\mathrm{d}x};$$

(A2)

$$\limsup_{|\xi|\to+\infty}\frac{\sup_{x\in\Omega} F(x,\xi)}{|\xi|^{\varphi_0}} < \frac{\int_\Omega \max_{|t|\leq c} F(x,t)\,\mathrm{d}x}{2c^{\varphi_0}|\Omega|}.$$

Then, for every $\lambda \in \Lambda_1$, where Λ_1 is given by (3.1), and for every continuous function $g : \mathbb{R} \to \mathbb{R}$ such that

$$\limsup_{|\xi|\to+\infty} \frac{G(\xi)}{|\xi|^{\varphi_0}} < +\infty,$$

there exists $\overline{\delta}_1 > 0$ *given by* (3.3) *such that, for each* $\mu \in [0, \overline{\delta}_1[$, *the problem* $(N^{f,g}_{\lambda,\mu})$ *admits at least three weak solutions in* E.

Proof Fix λ, μ and g as in the conclusion. Our aim is to apply Theorem 2.1. For each $u \in E$, let the functionals $J, I : E \to \mathbb{R}$ be defined by

$$J(u) := \int_\Omega \Big(\Phi(x, |\nabla u(x)|) + \Phi(x, |u(x)|)\Big)\, \mathrm{d}x,$$
$$I(u) := \int_\Omega F(x, u(x))\, \mathrm{d}x + \frac{\mu}{\lambda}\int_{\partial\Omega} G(\gamma(u(x)))\, \mathrm{d}\sigma,$$

and put

$$T_{\lambda,\mu}(u) := J(u) - \lambda I(u).$$

Similar arguments as those used in [31, Lemma 4.2] imply that $J \in C^1(E, \mathbb{R})$ with the derivative given by

$$\big\langle J'(u), v\big\rangle = \int_\Omega \alpha\big(x, |\nabla u(x)|\big)\nabla u(x) \times \nabla v(x)\, \mathrm{d}x + \int_\Omega \alpha\big(x, |u(x)|\big)u(x)\, v(x)\, \mathrm{d}x$$

for every $v \in E$. Also J is bounded from below. Moreover, $I \in C^1(E, \mathbb{R})$ and

$$\big\langle I'(u), v\big\rangle = \int_\Omega f\big(x, u(x)\big)v(x)\, \mathrm{d}x + \frac{\mu}{\lambda}\int_{\partial\Omega} g\Big(\gamma\big(u(x)\big)\Big)\gamma\big(v(x)\big)\, \mathrm{d}\sigma$$

for every $v \in E$.

So, with standard arguments, we deduce that the critical points of the functional $T_{\lambda,\mu}$ are the weak solutions of problem $(N^{f,g}_{\lambda,\mu})$. We will verify (i) and (ii) of Theorem 2.1. Let w be the function defined by $w(x) := d$ for all $x \in \overline{\Omega}$ and put

$$r := \Big(\frac{c}{m}\Big)^{\varphi_0}.$$

Clearly, $w \in E$ and from the condition $\int_\Omega \Phi\big(x, |d|\big)\, \mathrm{d}x > (\frac{c}{m})^{\varphi_0}$, one has

$$J(w) = \int_\Omega \Phi\big(x, |d|\big)\, \mathrm{d}x > r.$$

Also, we have

$$I(w) = \int_\Omega F(x, d)\, \mathrm{d}x + \frac{\mu}{\lambda} a(\partial\Omega)G(d).$$

By Lemma 2.3 and the fact $\max\left\{r^{1/\varphi_0}, r^{1/\varphi^0}\right\} = r^{1/\varphi_0}$, we deduce

$$\left\{u \in E : J(u) < r\right\} \subseteq \left\{u \in E : \|u\| < r^{1/\varphi_0}\right\} = \left\{u \in E : \|u\| < \frac{c}{m}\right\}.$$

Moreover, due to (2.10), we have

$$|u(x)| \leq \|u\|_\infty \leq m\|u\| \leq c, \qquad \forall x \in \overline{\Omega}.$$

Hence,

$$\left\{u \in E : \|u\| < \frac{c}{m}\right\} \subseteq \left\{u \in E : \|u\|_\infty \leq c\right\}.$$

Therefore,

$$\frac{\sup_{u\in J^{-1}\left(]-\infty,r[\right)} I(u)}{r} \leq \left(\frac{m}{c}\right)^{\varphi_0} \int_\Omega \max_{|t|\leq c} F(x,t)\,\mathrm{d}x + \frac{\mu}{\lambda}\left(\frac{m}{c}\right)^{\varphi_0} a(\partial\Omega) \max_{|t|\leq c} G(t).$$

If $\max_{|t|\leq c} G(t) = 0$, it is clear that we get

$$\frac{\sup_{u\in J^{-1}(]-\infty,r[)} I(u)}{r} < \frac{1}{\lambda}, \tag{3.4}$$

while, if $\max_{|t|\leq c} G(t) > 0$, it turns out to be true bearing in mind that

$$\mu < \frac{c^{\varphi_0} - \lambda m^{\varphi_0} \int_\Omega \max_{|t|\leq c} F(x,t)\,\mathrm{d}x}{m^{\varphi_0} a(\partial\Omega) \max_{|t|\leq c} G(t)}.$$

On the other hand, taking into account that

$$0 < J(w) = \int_\Omega \Phi(x, |d|)\,\mathrm{d}x,$$

we have

$$\begin{aligned}\frac{I(w)}{J(w)} &= \frac{\displaystyle\int_\Omega F(x,d)\,\mathrm{d}x + \frac{\mu}{\lambda} a(\partial\Omega) G(d)}{\displaystyle\int_\Omega \Phi(x,|d|)\,\mathrm{d}x} \\ &= \frac{\displaystyle\int_\Omega F(x,d)\,\mathrm{d}x}{\displaystyle\int_\Omega \Phi\big(x,|d|\big)\,\mathrm{d}x} + \frac{\mu}{\lambda}\frac{a(\partial\Omega)G(d)}{\displaystyle\int_\Omega \Phi\big(x,|d|\big)\,\mathrm{d}x}.\end{aligned}$$

Hence, if $G(d) \geq 0$, one has

$$\frac{I(w)}{J(w)} > \frac{1}{\lambda}, \tag{3.5}$$

while, if $G(d) < 0$, it holds since

$$\mu < \frac{\int_\Omega \Phi\big(x, |d|\big)\,\mathrm{d}x - \lambda \int_\Omega F(x,d)\,\mathrm{d}x}{a(\partial\Omega)G(d)}.$$

Therefore, from (3.4) and (3.5), condition (i) of Theorem 2.1 is fulfilled. Now, to prove the coercivity of the functional $T_{\lambda,\mu}$, first, we assume that

$$\limsup_{|\xi|\to+\infty} \frac{\sup_{x\in\Omega} F(x,\xi)}{|\xi|^{\varphi_0}} > 0.$$

So, we can fix $\varepsilon > 0$ satisfying

$$\limsup_{|\xi|\to+\infty} \frac{\sup_{x\in\Omega} F(x,\xi)}{|\xi|^{\varphi_0}} < \varepsilon < \frac{\int_\Omega \max_{|t|\le c} F(x,t)\,\mathrm{d}x}{2c^{\varphi_0}|\Omega|},$$

from (A2) there exists a positive constant h_ε such that

$$F(x,\xi) \le \varepsilon|\xi|^{\varphi_0} + h_\varepsilon \quad \forall x \in \Omega, \quad \forall \xi \in \mathbb{R}.$$

Taking into account (2.10) and since

$$\lambda < \frac{\left(\frac{c}{m}\right)^{\varphi_0}}{\int_\Omega \max_{|t|\le c} F(x,t)\,\mathrm{d}x},$$

it follows that

$$\begin{aligned} \lambda \int_\Omega F\big(x, u(x)\big)\,\mathrm{d}x &\le \lambda\varepsilon \int_\Omega \big(u(x)\big)^{\varphi_0}\,\mathrm{d}x + \lambda h_\varepsilon|\Omega| \\ &< \frac{\left(\frac{c}{m}\right)^{\varphi_0}}{\int_\Omega \max_{|t|\le c} F(x,t)\,\mathrm{d}x}\left(\varepsilon \int_\Omega \big(u(x)\big)^{\varphi_0}\mathrm{d}x + h_\varepsilon|\Omega|\right) \\ &\le \frac{\left(\frac{c}{m}\right)^{\varphi_0}|\Omega|}{\int_\Omega \max_{|t|\le c} F(x,t)\,\mathrm{d}x}\big(\varepsilon m^{\varphi_0}\|u\|^{\varphi_0} + h_\varepsilon\big) \end{aligned} \tag{3.6}$$

for all $u \in E$. Moreover, since $\mu < \overline{\delta}$ one has

$$\limsup_{|\xi|\to+\infty} \frac{G(\xi)}{|\xi|^{\varphi_0}} < \frac{1}{2\mu m^{\varphi_0} a(\partial\Omega)},$$

then, there is a positive constant h_μ such that

$$G(\xi) \le \frac{1}{2\mu m^{\varphi_0} a(\partial\Omega)}|\xi|^{\varphi_0} + h_\mu$$

for each $\xi \in \mathbb{R}$. Thus, taking again (2.10) into account, it follows that

$$\begin{aligned}\int_{\partial\Omega} G\Big(\gamma\big(u(x)\big)\Big)\,\mathrm{d}\sigma &\leq \frac{1}{2\mu m^{\varphi_0} a(\partial\Omega)} \int_{\partial\Omega} \big(u(x)\big)^{\varphi_0}\,\mathrm{d}\sigma + h_\mu a(\partial\Omega)\\ &\leq \frac{1}{2\mu}\|u\|^{\varphi_0} + h_\mu a(\partial\Omega)\end{aligned} \tag{3.7}$$

for all $u \in E$. Finally, if $\|u\| \geq 1$, putting together (3.6) and (3.7) we have

$$\begin{aligned}T_{\lambda,\mu}(u) &= J(u) - \lambda I(u)\\ &\geq \|u\|^{\varphi_0} - \frac{\left(\frac{c}{m}\right)^{\varphi_0}|\Omega|}{\int_\Omega \max_{|t|\leq c} F(x,t)\,\mathrm{d}x}\big(\varepsilon m^{\varphi_0}\|u\|^{\varphi_0} + h_\varepsilon\big) - \frac{1}{2}\|u\|^{\varphi_0} - \mu h_\mu a(\partial\Omega)\\ &= \left(\frac{1}{2} - \frac{m^{\varphi_0}|\Omega|}{\int_\Omega \max_{|t|\leq c} F(x,t)\,\mathrm{d}x}\varepsilon\right)\|u\|^{\varphi_0} - \frac{\left(\frac{c}{m}\right)^{\varphi_0}|\Omega| h_\varepsilon}{\int_\Omega \max_{|t|\leq c} F(x,t)\,\mathrm{d}x} - \mu h_\mu a(\partial\Omega).\end{aligned}$$

On the other hand, if

$$\limsup_{|\xi|\to+\infty} \frac{\sup_{x\in\Omega} F(x,\xi)}{|\xi|^{\varphi_0}} \leq 0,$$

there exists $h_\varepsilon > 0$ such that $F(x,\xi) \leq h_\varepsilon$ for each $\xi \in \mathbb{R}$ and $x \in \Omega$, and arguing as before we obtain

$$T_{\lambda,\mu}(u) \geq \frac{1}{2}\|u\|^{\varphi_0} - \frac{\left(\frac{c}{m}\right)^{\varphi_0}|\Omega| h_\varepsilon}{\int_\Omega \max_{|t|\leq c} F(x,t)\,\mathrm{d}x} - \mu h_\mu a(\partial\Omega).$$

Both cases lead to the coercivity of $T_{\lambda,\mu}$ and condition (ii) of Theorem 2.1 is verified. Since, from (3.4) and (3.5),

$$\lambda \in \Lambda_1 \subseteq \left]\frac{J(w)}{I(w)}, \frac{r}{\sup_{J(u)\leq r} I(u)}\right[,$$

Theorem 2.1 ensures the existence of at least three-critical points for the functional $T_{\lambda,\mu}$ in E, which are the weak solutions of the problem $(N^{f,g}_{\lambda,\mu})$. This completes the proof. □

Now, we state a variant of Theorem 3.1 in which no asymptotic condition on g is requested; on the other, the functions f and g are supposed to be non-negative.

Fixing $d \neq 0$ and $c_1, c_2 > 0$ such that

$$\frac{3}{2}\frac{\int_\Omega \Phi\big(x,|d|\big)\,\mathrm{d}x}{\int_\Omega F(x,d)\,\mathrm{d}x} < \min\left\{\frac{\left(\frac{c_1}{m}\right)^{\varphi_0}}{\int_\Omega F(x,c_1)\,\mathrm{d}x}, \frac{\left(\frac{c_2}{m}\right)^{\varphi_0}}{2\int_\Omega F(x,c_2)\,\mathrm{d}x}\right\},$$

and picking

$$\lambda \in \Lambda_2 := \left]\frac{3}{2}\frac{\int_\Omega \Phi(x,|d|)\,\mathrm{d}x}{\int_\Omega F(x,d)\,\mathrm{d}x}, \min\left\{\frac{\left(\frac{c_1}{m}\right)^{\varphi_0}}{\int_\Omega F(x,c_1)\,\mathrm{d}x}, \frac{\left(\frac{c_2}{m}\right)^{\varphi_0}}{2\int_\Omega F(x,c_2)\,\mathrm{d}x}\right\}\right[, \tag{3.8}$$

put

$$\delta_2 := \min\left\{\frac{\left(\frac{c_1}{m}\right)^{\varphi_0} - \lambda\int_\Omega F(x,c_1)\,\mathrm{d}x}{a(\partial\Omega)G(c_1)}, \frac{\left(\frac{c_2}{m}\right)^{\varphi_0} - 2\lambda\int_\Omega F(x,c_2)\,\mathrm{d}x}{2a(\partial\Omega)G(c_2)}\right\}. \tag{3.9}$$

With the above notations we have the following multiplicity result.

Theorem 3.2 *Suppose that there exist $d \neq 0$ and two constants c_1, c_2 with $\min\{c_1, c_2\} \geq m$ and*

$$2\left(\frac{c_1}{m}\right)^{\varphi_0} < \int_\Omega \Phi\big(x,|d|\big)\,\mathrm{d}x < \frac{1}{2}\left(\frac{c_2}{m}\right)^{\varphi_0},$$

such that

(B1) $f(x,\xi) \geq 0$ *for all* $(x,\xi) \in \Omega \times \mathbb{R}$*;*
(B2)

$$\max\left\{\frac{\int_\Omega F(x,c_1)\,\mathrm{d}x}{\left(\frac{c_1}{m}\right)^{\varphi_0}}, \frac{2\int_\Omega F(x,c_2)\,\mathrm{d}x}{\left(\frac{c_2}{m}\right)^{\varphi_0}}\right\} < \frac{2}{3}\frac{\int_\Omega F(x,d)\,\mathrm{d}x}{\int_\Omega \Phi(x,|d|)\,\mathrm{d}x},$$

Then, for every $\lambda \in \Lambda_2$ is given by (3.8), and for every non-negative continuous function $g : \mathbb{R} \to \mathbb{R}$, there exists $\delta_2 > 0$ given by (3.9) such that, for each $\mu \in [0, \delta_2[$, the problem $(N^{f,g}_{\lambda,\mu})$ admits at least three distinct weak solutions u_i, $i = 1, 2, 3$, such that

$$0 \leq u_i(x) \leq c_2, \qquad \forall x \in \Omega,\ i = 1, 2, 3.$$

Proof Fix λ, μ, and g as in the conclusion and take E, J, I, and $T_{\lambda,\mu}$ as in the proof of Theorem 3.1. We observe that the regularity assumptions of Theorem 2.2 on J and I are satisfied. Then, our aim is to verify (j) and (jj).

Put $w(x) := d$ for all $x \in \overline{\Omega}$, $r_1 := \left(\frac{c_1}{m}\right)^{\varphi_0}$ and $r_2 := \left(\frac{c_2}{m}\right)^{\varphi_0}$. Therefore, by using the conditions

$$2\left(\frac{c_1}{m}\right)^{\varphi_0} < \int_\Omega \Phi(x,|d|)\,\mathrm{d}x < \frac{1}{2}\left(\frac{c_2}{m}\right)^{\varphi_0},$$

one has $2r_1 < J(w) < \frac{r_2}{2}$. Since $\mu < \delta_2$ and $G(d) \geq 0$, one has

$$\begin{aligned}\frac{1}{r_1}\sup_{J(u)<r_1} I(u) &= \frac{1}{r_1}\sup_{J(u)<r_1}\left[\int_\Omega F\big(x,u(x)\big)\,\mathrm{d}x + \frac{\mu}{\lambda}\int_{\partial\Omega} G\Big(\gamma\big(u(x)\big)\Big)\,\mathrm{d}\sigma\right]\\ &\le \frac{\displaystyle\int_\Omega F(x,c_1)\,\mathrm{d}x + \frac{\mu}{\lambda}a(\partial\Omega)G(c_1)}{\left(\frac{c_1}{m}\right)^{\varphi_0}}\\ &< \frac{1}{\lambda} < \frac{2}{3}\frac{\displaystyle\int_\Omega F(x,d)\,\mathrm{d}x + \frac{\mu}{\lambda}a(\partial\Omega)G(d)}{\displaystyle\int_\Omega \Phi(x,|d|)\,\mathrm{d}x}\\ &\le \frac{2}{3}\frac{I(w)}{J(w)},\end{aligned}$$

and

$$\begin{aligned}\frac{2}{r_2}\sup_{J(u)<r_2} I(u) &= \frac{2}{r_2}\sup_{J(u)<r_2}\left[\int_\Omega F\big(x,u(x)\big)\,\mathrm{d}x + \frac{\mu}{\lambda}\int_{\partial\Omega} G\Big(\gamma\big(u(x)\big)\Big)\,\mathrm{d}\sigma,\right]\\ &\le \frac{\displaystyle 2\int_\Omega F(x,c_2)\,\mathrm{d}x + \frac{2\mu}{\lambda}a(\partial\Omega)G(c_2)}{\left(\frac{c_2}{m}\right)^{\varphi_0}}\\ &< \frac{1}{\lambda} < \frac{2}{3}\frac{\displaystyle\int_\Omega F(x,d)\,\mathrm{d}x + \frac{\mu}{\lambda}a(\partial\Omega)G(d)}{\displaystyle\int_\Omega \Phi(x,|d|)\,\mathrm{d}x}\\ &\le \frac{2}{3}\frac{I(w)}{J(w)}.\end{aligned}$$

Therefore, conditions (j) and (jj) of Theorem 2.2 are satisfied. Finally, we verify that $T_{\lambda,\mu}$ satisfies the assumption (jjj) of Theorem 2.2. Let u_1 and u_2 be two local minima for $T_{\lambda,\mu}$. Then, u_1 and u_2 are critical points for $T_{\lambda,\mu}$, and so, they are weak solutions for the problem $(N^{f,g}_{\lambda,\mu})$. We claim that the weak solutions obtained are non-negative. Indeed, let $v_0 \in E$ be one (non-trivial) weak solution of the problem $(N^{f,g}_{\lambda,\mu})$. Arguing by contradiction, if we assume that v_0 is negative at a point of Ω the set

$$\Omega^- := \big\{x \in \Omega : v_0(x) < 0\big\},$$

is non-empty and open. Moreover, let us consider $v_0^\star := \min\{v_0, 0\}$, one has $v_0^\star \in E$. So, taking into account that v_0 is a weak solution and by choosing $v = v_0^\star$, from our sign assumptions on the data, we have

$$\begin{aligned}&\int_{\Omega^-}\alpha\big(x,|\nabla v_0(x)|\big)|\nabla v_0(x)|^2\,\mathrm{d}x + \int_{\Omega^-}\alpha\big(x,|v_0(x)|\big)|v_0(x)|^2\,\mathrm{d}x\\ &\quad= \lambda\int_{\Omega^-} f\big(x,v_0(x)\big)v_0(x)\,\mathrm{d}x + \mu\int_{\partial\Omega} g\Big(\gamma\big(v_0(x)\big)\Big)\gamma\big(v_0(x)\big)\,\mathrm{d}\sigma \le 0.\end{aligned}$$

Therefore,

$$\int_{\Omega^-} \alpha\big(x, |\nabla v_0(x)|\big)|\nabla v_0(x)|^2 \,\mathrm{d}x + \int_{\Omega^-} \alpha\big(x, |v_0(x)|\big)|v_0(x)|^2 \,\mathrm{d}x = 0,$$

which means

$$\int_{\Omega^-} \varphi\big(x, |\nabla v_0(x)|\big)|\nabla v_0(x)| \,\mathrm{d}x + \int_{\Omega^-} \varphi\big(x, |v_0(x)|\big)|v_0(x)| \,\mathrm{d}x = 0.$$

Now, from the previous relation and bearing in mind that $t\varphi(x,t) \geq \Phi(x,t)$ for every $x \in \overline{\Omega}$ and $t \geq 0$, we find that

$$\int_{\Omega^-} \Phi\big(x, |\nabla v_0(x)|\big) \,\mathrm{d}x + \int_{\Omega^-} \Phi\big(x, |v_0(x)|\big) \,\mathrm{d}x = 0.$$

Hence, by Lemma (2.3) we observe that $\|v_0\|_{W^{1,\Phi}(\Omega^-)} = 0$ which is absurd. Then, we obtain $u_1(x) \geq 0$ and $u_2(x) \geq 0$ for all $x \in \Omega$. So, one has $I\big(su_1 + (1-s)u_2\big) \geq 0$ for all $s \in [0,1]$. Therefore, also (jjj) holds. From Theorem 2.2 the functional $J - \lambda I$ has at least three distinct critical points which are weak solutions of $(N^{f,g}_{\lambda,\mu})$. This completes the proof. □

Now, we state a variant of Theorem 3.1 in which the growth of $f(x,\cdot)$ is $(\varphi_0 - 1)$-sublinear at infinity. Fixing $d' \neq 0$ and $c' < m$ such that

$$\frac{\int_\Omega \max_{|t|\leq c'} F(x,t)\,\mathrm{d}x}{\left(\frac{c'}{m}\right)^{\varphi^0}} < \frac{\int_\Omega F(x,d')\,\mathrm{d}x}{\int_\Omega \Phi(x,|d'|)\,\mathrm{d}x}$$

and picking

$$\lambda \in \Lambda_3 := \left] \frac{\int_\Omega \Phi\big(x,|d'|\big)\,\mathrm{d}x}{\int_\Omega F(x,d')\,\mathrm{d}x}, \frac{\left(\frac{c'}{m}\right)^{\varphi^0}}{\int_\Omega \max_{|t|\leq c'} F(x,t)\,\mathrm{d}x} \right[, \tag{3.10}$$

put

$$\delta_3 := \min\left\{ \frac{c'^{\varphi^0} - \lambda m^{\varphi^0} \int_\Omega \max_{|t|\leq c'} F(x,t)\,\mathrm{d}x}{m^{\varphi^0} a(\partial\Omega) \max_{|t|\leq c'} G(t)}, \frac{\int_\Omega \Phi\big(x,|d'|\big)\,\mathrm{d}x - \lambda \int_\Omega F(x,d')\,\mathrm{d}x}{a(\partial\Omega) G(d')} \right\} \tag{3.11}$$

and

$$\overline{\delta}_3 := \min\left\{ \delta_3, \frac{1}{\max\left\{0, 2m^{\varphi_0} a(\partial\Omega) \limsup_{|\xi|\to+\infty} \frac{G(\xi)}{|\xi|^{\varphi_0}}\right\}} \right\}, \tag{3.12}$$

where $a(\partial\Omega) := \int_{\partial\Omega} d\sigma$, and we read $\frac{1}{0} = +\infty$ whenever this case occurs.

Theorem 3.3 *Suppose that there exist $d' \neq 0$ and $c' < m$, with*

$$\int_\Omega \Phi\big(x, |d'|\big)\, \mathrm{d}x > \left(\frac{c'}{m}\right)^{\varphi^0},$$

such that

(C1)

$$\frac{\int_\Omega \max_{|t|\le c'} F(x,t)\, \mathrm{d}x}{\left(\frac{c'}{m}\right)^{\varphi^0}} < \frac{\int_\Omega F(x,d')\, \mathrm{d}x}{\int_\Omega \Phi(x,|d'|)\, \mathrm{d}x};$$

(C2) *There exist $c_0 > 0$ and $0 < s < \varphi_0 - 1$ such that $|f(x,t)| \le c_0(1+|t|^s)$ for every $(x,t) \in \Omega \times \mathbb{R}$.*

Then, for every $\lambda \in \Lambda_3$, where Λ_3 is given by (3.10), and for every continuous function $g : \mathbb{R} \to \mathbb{R}$ such that

$$\limsup_{|\xi|\to+\infty} \frac{G(\xi)}{|\xi|^{\varphi_0}} < +\infty,$$

there exists $\overline{\delta}_3 > 0$ given by (3.12) such that, for each $\mu \in [0, \overline{\delta}_3[$, the problem $(N^{f,g}_{\lambda,\mu})$ possesses at least three weak solutions in E.

Proof Our aim is to apply again Theorem 2.1 to $(N^{f,g}_{\lambda,\mu})$. Fix λ, μ, and g as in the conclusion and take E, J, I and w as in the proof of Theorem 3.1. We observe that the regularity assumptions of Theorem 2.1 on J and I are satisfied. Put $r' := \left(\frac{c'}{m}\right)^{\varphi^0}$ and $T_{\lambda,\mu}(u) = J(u) - \lambda I(u)$. By the same argument as given in the proof of Theorem 3.1, we can show the condition (i) of Theorem 2.1 is fulfilled. Now, we prove for every $\lambda > 0$ and $\mu \ge 0$, the functional $T_{\lambda,\mu}$ is coercive. Indeed, by Lemma 2.3 we deduce that for any $u \in E$ with $\|u\| > 1$ we have $J(u) \ge \|u\|^{\varphi_0}$. Hence J is coercive. On the other hand, by (C2), one has that there exists a positive constant θ such that

$$\int_\Omega F(x, u(x))d\, \mathrm{d}x \le \theta(\|u\|_\infty + \|u\|_\infty^{s+1}), \quad \forall u \in E.$$

Since E is compactly embedded into $C^0(\overline{\Omega})$, there exists $\theta_1 > 0$ such that

$$\int_\Omega F\big(x, u(x)\big)\, \mathrm{d}x \ge \theta_1(\|u\| + \|u\|^{s+1}) \tag{3.13}$$

for every $u \in E$. On the other hand, arguing as in the proof of Theorem 3.1, there exists a positive constant h_μ such that

$$\int_{\partial\Omega} G\Big(\gamma\big(u(x)\big)\Big)\, \mathrm{d}\sigma \le \frac{1}{2\mu}\|u\|^{\varphi_0} + h_\mu a(\partial\Omega) \tag{3.14}$$

for each $u \in E$. Taking (3.13) and (3.14) into account and since $J(u) \geq \|u\|^{\varphi_0}$ for any $u \in E$ with $\|u\| > 1$, we obtain

$$T_{\lambda,\mu}(u) \geq \frac{1}{2}\|u\|^{\varphi_0} - \lambda\theta_1(\|u\| + \|u\|^{s+1}) - \mu h_\mu a(\partial\Omega)$$

for any $u \in E$ with $\|u\| > 1$. Since $1 < s + 1 < \varphi_0$ it follows that

$$\lim_{\|u\|\to+\infty} T_{\lambda,\mu}(u) = +\infty$$

for every $\lambda > 0$ and $\mu \geq 0$. Hence, $T_{\lambda,\mu}$ is a coercive functional. Then, also condition (ii) holds. Since all the assumptions of Theorem 2.1 are satisfied, for each $\lambda \in \Lambda_3$ is given by (3.10) there exists $\overline{\delta}_3$ given by (3.12) such that, for each $\mu \in [0, \overline{\delta}_3[$ the functional $T_{\lambda,\mu}$ has at least three distinct critical points in E, which are the weak solutions of the problem $(N^{f,g}_{\lambda,\mu})$. The proof is complete. □

A particular case of Theorem (3.3) is established in the following result.

Theorem 3.4 *Let $b : \Omega \to \mathbb{R}$ be a bounded measurable and positive function and $f : \mathbb{R} \to \mathbb{R}$ be a continuous and non-negative function. Set $F(\xi) := \int_0^\xi f(t)\,\mathrm{d}t$. Further, suppose that there exist $d'' \neq 0$ and $c'' < m$, with*

$$\int_\Omega \Phi(x, |d''|)\,\mathrm{d}x > \left(\frac{c''}{m}\right)^{\varphi^0},$$

such that

(D1)

$$\frac{F(c'')}{\left(\frac{c''}{m}\right)^{\varphi^0}} < \frac{F(d'')}{\int_\Omega \Phi\big(x, |d''|\big)\,\mathrm{d}x};$$

(D2) *There exist $c_0 > 0$ and $0 < s < \varphi_0 - 1$ such that $|f(t)| \leq c_0(1 + |t|^s)$ for every $t \in \mathbb{R}$.*

Then, for every λ belonging to

$$\left]\frac{\int_\Omega \Phi(x, |d''|)\,\mathrm{d}x}{\|b\|_{L^1(\Omega)} F(d'')}, \frac{\left(\frac{c''}{m}\right)^{\varphi^0}}{\|b\|_{L^1(\Omega)} F(c'')}\right[,$$

and for every continuous function $g : \mathbb{R} \to \mathbb{R}$ such that

$$\limsup_{|\xi|\to+\infty} \frac{G(\xi)}{|\xi|^{\varphi_0}} < +\infty,$$

there exists $\overline{\delta}_4 > 0$ *defined by*

$$\overline{\delta}_4 := \min\left\{\delta_4, \frac{1}{\max\left\{0, 2m^{\varphi_0}a(\partial\Omega)\limsup_{|\xi|\to+\infty}\frac{G(\xi)}{|\xi|^{\varphi_0}}\right\}}\right\},$$

where

$$\delta_4 := \min\left\{\frac{c''^{\varphi^0} - \lambda m^{\varphi^0}\|b\|_{L^1(\Omega)}F(c'')}{m^{\varphi^0}a(\partial\Omega)\max_{|t|\le c''}G(t)}, \left|\frac{\int_\Omega \Phi\big(x, |d''|\big)\,\mathrm{d}x - \lambda\|b\|_{L^1(\Omega)}F(d'')}{a(\partial\Omega)G(d'')}\right|\right\},$$

for each $\mu \in [0, \overline{\delta}_4[$, *the problem*

$$\begin{cases} -\mathrm{div}\big(\alpha(x, |\nabla u(x)|)\nabla u(x)\big) + \alpha\big(x, |u(x)|\big)u(x) = \lambda b(x) f\big(u(x)\big) & \text{for } x \in \Omega, \\ \alpha(x, |\nabla u(x)|)\dfrac{\partial u}{\partial \nu}(x) = \mu g\big(\gamma\big(u(x)\big)\big) & \text{for } x \in \partial\Omega \end{cases} \qquad (N^{bf,g}_{\lambda,\mu})$$

possesses at least three weak solutions in E.

Remark 3.5 The same conclusion of Theorem 3.4 holds under the assumption that $b : \Omega \to \mathbb{R}$ is a bounded measurable function with $\operatorname{ess\,inf}_{x\in\Omega} b(x) \ge 0$ and $\int_\Omega b(x)\,\mathrm{d}x > 0$.

A direct consequence of the previous result reads as follows.

Corollary 3.6 *Let* $b : \Omega \to \mathbb{R}$ *be a bounded measurable and positive function. Moreover, let* $f : \mathbb{R} \to \mathbb{R}$ *be a non-negative (not identically zero) and continuous function such that*

$$\lim_{t\to 0^+}\frac{f(t)}{t^{\varphi^0 - 1}} = 0. \qquad (l_0)$$

Further, assume that condition (C2) *holds. Then, for each*

$$\lambda > \frac{1}{\|b\|_{L^1(\Omega)}}\inf_{\rho\in S}\frac{\int_\Omega \Phi(x, |\rho|)\,\mathrm{d}x}{F(\rho)},$$

where $S := \{\rho > 0 : F(\rho) > 0\}$, *the following problem*

$$\begin{cases} -\mathrm{div}\big(\alpha(x, |\nabla u(x)|)\nabla u(x)\big) + \alpha(x, |u(x)|)u(x) = \lambda b(x) f(u(x)) & \text{for } x \in \Omega, \\ \dfrac{\partial u}{\partial \nu}(x) = 0 & \text{for } x \in \partial\Omega \end{cases} \qquad \big(N^{bf}_{\lambda}\big)$$

possesses at least three weak solutions in E.

Proof Fix

$$\lambda > \frac{1}{\|b\|_{L^1(\Omega)}}\inf_{\rho\in S}\frac{\int_\Omega \Phi(x, |\rho|)\,\mathrm{d}x}{F(\rho)}.$$

Then there exists $\overline{\rho}$ such that $F(\overline{\rho}) > 0$ and

$$\lambda > \frac{1}{\|b\|_{L^1(\Omega)}} \frac{\int_\Omega \Phi(x, |\overline{\rho}|)\, dx}{F(\overline{\rho})}.$$

By using condition (l_0) one has

$$\lim_{\xi \to 0^+} \frac{F(\xi)}{|\xi|^{\varphi^0}} = 0.$$

Therefore, we can find a positive constant $\overline{c}$ such that

$$\overline{c} < m \min\left\{1, \left(\int_\Omega \Phi(x, |\overline{\rho}|)\, dx\right)^{1/\varphi^0}\right\},$$

and

$$\frac{F(\overline{c})}{\overline{c}^{\varphi^0}} < \frac{1}{m^{\varphi^0}} \min\left\{\frac{F(\overline{\rho})}{\int_\Omega \Phi(x, |\overline{\rho}|)\, dx}, \frac{1}{\lambda \|b\|_{L^1(\Omega)}}\right\}.$$

Hence,

$$\lambda \in \left]\frac{\int_\Omega \Phi(x, |\overline{\rho}|)\, dx}{\|b\|_{L^1(\Omega)} F(\overline{\rho})}, \frac{\left(\frac{\overline{c}}{m}\right)^{\varphi^0}}{\|b\|_{L^1(\Omega)} F(\overline{c})}\right[.$$

All the hypotheses of Theorem 3.4 are satisfied, and the problem N_λ^{bf} admits at least three distinct weak solutions. The proof is complete. □

Example 3.7 Let Ω be a non-empty bounded open subset of the Euclidean space $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$. Define $f : \mathbb{R} \to \mathbb{R}$ as follows:

$$f(t) = \begin{cases} 0 & \text{if} \quad t < 0, \\ t^{\varphi^0} & \text{if} \quad 0 \leq t \leq 1, \\ t^s & \text{if} \quad t > 1, \end{cases}$$

where $s \in]0, \varphi_0 - 1[$. Further, let $b : \Omega \to \mathbb{R}$ be a bounded measurable and positive function. From Corollary 3.6, for each parameter

$$\lambda > \frac{1}{\|b\|_{L^1(\Omega)}} \inf_{\rho > 0} \frac{\int_\Omega \Phi(x, |\rho|)\, dx}{F(\rho)},$$

$$\begin{cases} -\operatorname{div}\left(p(x)\dfrac{|\nabla u|^{p(x)-2}\nabla u}{\log(1+|\nabla u|)}\right)+p(x)\dfrac{|u|^{p(x)-2}u}{\log(1+|u|)}=\lambda b(x)f(u(x)) & \text{for } x\in\Omega, \\ \dfrac{\partial u}{\partial \nu}(x)=0 & \text{for } x\in\partial\Omega \end{cases}$$

admits at least three non-negative weak solutions.

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