MINIMIZATION PROBLEMS AND CORRESPONDING RENORMALIZED ENERGIES

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1. Introduction and theoretical setting

Let G be a smooth bounded simply connected domain in $\mathbf{R}^2$. Let $a = (a_1, ..., a_k)$ be a configuration of distinct points in G and $\overline{d} = (d_1, ..., d_k) \in \mathbf{Z}^k$. We also consider a smooth boundary data $g : \partial G \to S^1$ whose topological degree is $d = d_1 + ... + d_k$. Let also $\rho > 0$ be sufficiently small and denote

$$\Omega_\rho = G \setminus \bigcup_{i=1}^{k} \overline{B(a_i, \rho)} \ , \ \Omega = G \setminus \{a_1, ...a_k\} \ .$$

As in [BBH4] we consider the classes of functions

(1) $$\mathcal{E}_{\rho,g} = \{v \in H^1(\Omega_\rho; S^1);\ v = g \text{ on } \partial G \text{ and } \deg(v, \partial B(a_i, \rho)) = d_i,\ \text{for } i = 1, ..., k\}$$

(2) $$\mathcal{F}_\rho = \{v \in H^1(\Omega_\rho; S^1);\ \deg(v, \partial G) = d \text{ and } \deg(v, \partial B(a_i, \rho)) = d_i,\ \text{for } i = 1, ..., k\}$$

(3) $$\mathcal{F}_{\rho,A} = \{v \in \mathcal{F}_\rho;\ \int_{\partial G} \left| \frac{\partial v}{\partial \tau} \right|^2 \leq A\}$$

and the minimization problems

(4) $$E_{\rho,g} = \inf_{v \in \mathcal{E}_{\rho,g}} \int_\Omega | \nabla v |^2$$

(5) $$F_\rho = \inf_{v \in \mathcal{F}_\rho} \int_\Omega | \nabla v |^2 \ .$$

$$F_{\rho,A} = \inf_{v \in \mathcal{F}_{\rho,A}} \int_\Omega |\nabla v|^2 \quad . \tag{6}$$

F. Bethuel, H. Brezis and F. Hélein proved in [BBH4] that the minimization problems (4) and (5) have unique solutions, say u_ρ respectively v_ρ. By analysing the behaviour of u_ρ as $\rho \to 0$ they obtained the renormalized energy $W(a, \overline{d}, g)$ by the following asymptotic expansion:

$$\frac{1}{2}\int_{\Omega_\rho} |\nabla u_\rho|^2 = \pi \left(\sum_{i=1}^k d_i^2 \right) \log \frac{1}{\rho} + W(a, \overline{d}, g) + O(\rho) \ , \quad \text{as } \rho \to 0. \tag{7}$$

We shall omit $\overline{d}$ in $W(a, \overline{d}, g)$ when $k = d$ and each d_j equals 1.

By considering the behavior of v_ρ as $\rho \to 0$ we obtain in the first part of this paper a notion of renormalized energy $\widetilde{W}(a, \overline{d})$ when only singularities and degrees are prescribed. This will appear in a similar asymptotic expansion:

$$\frac{1}{2}\int_{\Omega_\rho} |\nabla v_\rho|^2 = \pi \left(\sum_{i=1}^k d_i^2 \right) \log \frac{1}{\rho} + \widetilde{W}(a, \overline{d}) + O(\rho) \ , \quad \text{as } \rho \to 0. \tag{8}$$

The connection between the two energies is given by

$$\widetilde{W}(a, \overline{d}) = \inf_{\substack{g:\partial G \to S^1 \\ \deg(g, \partial G) = d}} W(a, \overline{d}, g). \tag{9}$$

Moreover the infimum in (9) is atteint. We give thereafter an explicit formula for $\widetilde{W}(a, \overline{d})$.

We recall that in [BBH4] the study of the minimization problems (4) and (5) is related to the unique solutions Φ_ρ respectively $\hat{\Phi}_\rho$ of the following linear problems:

$$\begin{cases} \Delta \Phi_\rho = 0 \quad \text{in } \Omega_\rho \\ \Phi_\rho = C_i = \text{Const.} \quad \text{on each } \ \partial\omega_i \text{ with } \ \omega_i = B(a_i, \rho) \\ \displaystyle\int_{\partial\omega_i} \frac{\partial \Phi_\rho}{\partial \nu} = 2\pi d_i \quad i = 1, ...k \\ \displaystyle\frac{\partial \Phi_\rho}{\partial \nu} = g \wedge g_\tau \quad \text{on } \partial G \\ \displaystyle\int_{\partial G} \Phi_\rho = 0 \end{cases} \tag{10}$$

and

$$\begin{cases} \Delta \hat{\Phi}_\rho = 0 \quad \text{in } \Omega \\ \hat{\Phi}_\rho = C_i = \text{Const.} \quad \text{on } \ \partial\omega_i \ i = 1, ..., k \\ \hat{\Phi}_\rho = 0 \quad \text{on } \partial G \\ \displaystyle\int_{\partial\omega_i} \frac{\partial \hat{\Phi}_\rho}{\partial \nu} = 2\pi d_i \quad i = 1, ..., k \ . \end{cases} \tag{11}$$

We also recall that Φ_ρ converges uniformly as $\rho \to 0$ to Φ_0, which is the unique solution of

$$(12)\qquad \begin{cases} \Delta\Phi_0 = 2\pi \sum_{j=1}^{k} d_j \delta_{a_j} & \text{in } G \\ \dfrac{\partial \Phi_0}{\partial \nu} = g \wedge g_\tau & \text{on } \partial G \\ \displaystyle\int_{\partial G} \Phi_0 = 0 \ . \end{cases}$$

The explicit formula for $W(a, \overline{d}, g)$ found in [BBH4] is

$$(13)\qquad W(a,\overline{d},g) = -\pi \sum_{i \neq j} d_i d_j \log \mid a_i - a_j \mid + \frac{1}{2} \int_{\partial G} \Phi_0 (g \wedge g_\tau) - \pi \sum_{i=1}^{k} d_i R_0(a_i) \ ,$$

where

$$R_0(x) = \Phi_0(x) - \sum_{j=1}^{k} d_j \log \mid x - a_j \mid \ .$$

The expression we obtain for $\widetilde{W}(a, \overline{d})$ is lied to $\hat{\Phi}_0$, which is the local uniform limit of $\hat{\Phi}_\rho$ as $\rho \to 0$ and is the unique solution of the problem

$$(14)\qquad \begin{cases} \Delta\hat{\Phi}_0 = 2\pi \sum_{j=1}^{k} d_j \delta_{a_j} & \text{in } G \\ \hat{\Phi}_0 = 0 & \text{on } \partial G \ . \end{cases}$$

In the second part of this section, considering the minimization problem (6) we find a variant of the formula (8), but for $\widetilde{W}$ replaced by $\widetilde{W}_A$, which is a corresponding notion of renormalized energy that satisfies

$$\widetilde{W}_A(a, \overline{d}) = \inf \{ W(a, \overline{d}, g);\ \deg\,(g; \partial G) = d \ \text{ and } \ \int_{\partial G} \mid \frac{\partial g}{\partial \tau} \mid^2 \leq A \} \ .$$

In Section 3 we calculate explicitly W and $\widetilde{W}$ in a particular case and deduce auxiliary results.

In the last section we minimize the Ginzburg-Landau energy

$$E_\varepsilon(u) = \frac{1}{2} \int_G \mid \nabla u \mid^2 + \frac{1}{4\varepsilon^2} \int_G (1 - \mid u \mid^2)^2$$

in the class

$$\mathcal{H}_{d,A} = \{u \in H^1(G;\mathbf{R}^2);\ \mid u \mid = 1 \ \text{ on } \ \partial G,\ \deg(u,\partial G) = d \ \text{ and } \ \int_{\partial G} \mid \frac{\partial u}{\partial \tau} \mid^2 \leq A\} \ .$$

We prove that $\mathcal{H}_{d,A}$ is non-empty if A is big enough and the infimum of E_ε is atteint. If u_ε is a minimizer, we prove the convergence as $\varepsilon \to 0$ of u_ε to $u_\star$, which is a canonical harmonic map with values in S^1 and d singularities, say $a_1, \cdots, a_d$. Moreover, the configuration $a = (a_1, \cdots, a_d)$ minimizes the renormalized energy $\widetilde{W}_A$.

We recall here (see [BBH4]) that v is a canonical harmonic map with values in S^1 and boundary data g if it is harmonic and satisfies

$$\begin{cases} v \wedge \dfrac{\partial v}{\partial x_1} = -\dfrac{\partial \Phi_0}{\partial x_2} & \text{in } \ \Omega \\ v \wedge \dfrac{\partial v}{\partial x_2} = \dfrac{\partial \Phi_0}{\partial x_1} & \text{in } \ \Omega \ , \end{cases}$$

or, equivalently,

$$\frac{\partial}{\partial x_1}\left(v \wedge \frac{\partial v}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(v \wedge \frac{\partial v}{\partial x_2}\right) = 0 \quad \text{in } \ \mathcal{D}'(G) \ .$$

If v is canonical and has singularities $a_1, \cdots, a_k \in G$ with topological degrees $d_1, \cdots, d_k$ then v has the form

$$v(z) = \left(\frac{z - a_1}{\mid z - a_1 \mid}\right)^{d_1} \cdots \left(\frac{z - a_k}{\mid z - a_k \mid}\right)^{d_k} e^{i\varphi(z)} \ ,$$

where φ is a smooth harmonic function in G.

2. The renormalized energy for prescribed singularities and degrees

We know from Chapter I in [BBH4] that

$$\begin{cases} v_\rho \wedge \dfrac{\partial v_\rho}{\partial x_1} = -\dfrac{\partial \hat{\Phi}_\rho}{\partial x_2} & \text{in } \Omega_\rho \\ v_\rho \wedge \dfrac{\partial v_\rho}{\partial x_2} = \dfrac{\partial \hat{\Phi}_\rho}{\partial x_1} & \text{in } \Omega_\rho \ . \end{cases} \tag{15}$$

So

$$\mid \nabla v_\rho \mid = \mid \nabla \hat{\Phi}_\rho \mid \quad \text{in } \Omega_\rho \ . \tag{16}$$

Lemma 1. *$\hat{\Phi}_\rho$ converges to $\hat{\Phi}_0$ in $L^\infty(\Omega_\rho)$ as $\rho \to 0$. More precisely, there exists $C > 0$ such that*

$$\|\hat{\Phi}_\rho - \hat{\Phi}_0\|_{L^\infty(\Omega_\rho)} \leq C\rho. \tag{17}$$

Lemma 2. *Let v be a solution of*

$$\begin{cases} \Delta v = 0 \quad \text{in } \Omega_\rho \\ v = 0 \quad \text{on } \partial G \\ \displaystyle\int_{\partial \omega_j} \frac{\partial v}{\partial \nu} = 0 \quad \text{for each } j \ . \end{cases} \tag{18}$$

Then

$$\sup_{\Omega_\rho} v - \inf_{\Omega_\rho} v \leq \sum_{j=1}^{k} (\sup_{\omega_j} v - \inf_{\omega_j} v) \ .$$

Proof of Lemma 2. We shall adapt the proof of Lemma I.3 in [BBH4]. Let

$$\alpha_j = \inf_{\partial \omega_j} v \ , \quad \beta_j = \sup_{\partial \omega_j} v \ \text{ and } \quad I_j = [\alpha_j, \beta_j] \ .$$

We shall prove for the instant that

$$\bigcup_{j=1}^{k} I_j \ \text{ is connected} \ . \tag{19}$$

Suppose, by contradiction, it is not true. Then, there exist $t_0 \in \mathbf{R}\,, \delta > 0$ and $1 \leq i \leq k$ such that

$$\beta_j \leq t_0 - \delta \quad \text{ if } 1 \leq j \leq i$$

$$\alpha_j \geq t_0 + \delta \quad \text{ if } i+1 \leq j \leq k.$$

We may suppose, without loss of generality that $t_0 \neq 0$, say $t_0 > 0$. We may also suppose that $t_0 - \delta \geq 0$. Choose $\theta \in C^\infty(\mathbf{R}, [0,1])$ such that

$$\theta(t) = \begin{cases} 0 & \text{if } t \leq t_0 - \delta \\ 1 & \text{if } t \geq t_0 + \delta \ . \end{cases}$$

We multiply $\Delta v = 0$ with $\theta(v)$ and then integrate on Ω_ρ. Observing that $\theta(v) = 0$ on ∂G we deduce

$$0 = \int_{\Omega_\rho} \theta'(v) \mid \nabla v \mid^2 - \int_{\partial G} \frac{\partial v}{\partial \nu} \theta(v) + \sum_{j=1}^{k} \int_{\partial \omega_j} \frac{\partial v}{\partial \nu} \theta(v) = \int_{\Omega_\rho} \theta'(v) \mid \nabla v \mid^2 \ .$$

So $\nabla v = 0$ on $B = \{x \in \Omega_\rho; t_0 - \delta < v(x) < t_0 + \delta\}$ which is a contradiction.

We distinguish two cases:

Case 1. $\inf_{\Omega_\rho} v < 0$ and $\sup_{\Omega_\rho} v > 0$.

In this case, from the connectedness of $\bigcup_{j=1}^{k} I_j$, $v = 0$ on ∂G and the maximum principle, our conclusion follows obviously.

Case 2. $\inf_{\Omega_\rho} v = 0$ or $\sup_{\Omega_\rho} v = 0$.

We shall treat only the first case. Suppose $v \not\equiv 0$ on Ω_ρ (otherwise the conclusion is obvious). By the Hopf maximum principle, $\frac{\partial v}{\partial \nu} < 0$ on ∂G, which contradicts $\int_{\partial G} \frac{\partial v}{\partial \nu} = 0$.

□

Proof of Lemma 1. We apply Lemma 2 to the function $v = \hat{\Phi}_\rho - \hat{\Phi}_0$. Since $\hat{\Phi}_\rho =$ Const. on each $\partial B(a_j, \rho)$, it follows that

$$\sup_{\Omega_\rho} (\hat{\Phi}_\rho - \hat{\Phi}_0) - \inf_{\Omega_\rho} (\hat{\Phi}_\rho - \hat{\Phi}_0) \leq \sum_{j=1}^{k} \left(\sup_{\partial B(a_j,\rho)} \hat{\Phi}_0 - \inf_{\partial B(a_j,\rho)} \hat{\Phi}_0 \right) \leq C\rho \ .$$

Using now the fact that $\hat{\Phi}_\rho - \hat{\Phi}_0 = 0$ on ∂G we obtain

$$\|\hat{\Phi}_\rho - \hat{\Phi}_0\|_{L^\infty(\Omega_\rho)} \leq C\rho \ . \tag{20}$$

□

Remark. By Lemma 1 and standard elliptic estimates it follows that $\hat{\Phi}_\rho$ converges in $C^k_{\mathrm{loc}}(\Omega \cup \partial G)$ as $\rho \to 0$, for each $k \geq 0$.

Theorem 1. *As $\rho \to 0$ then (up to a subsequence) v_ρ converges in $C^k_{\mathrm{loc}}(\Omega \cup \partial G)$ to v_0, which is a canonical harmonic map.*

Moreover, the limits of two such sequences differ by a multiplicative constant of modulus 1.

Proof. We may write, locally on $\Omega_\rho \cup \partial G$, $v_\rho = e^{i\varphi_\rho}$ with $0 \leq \varphi_\rho \leq 2\pi$. Thus, by (15),

$$\begin{cases} \dfrac{\partial \varphi_\rho}{\partial x_1} = -\dfrac{\partial \hat{\Phi}_\rho}{\partial x_2} & \text{in } \Omega_\rho \\ \dfrac{\partial \varphi_\rho}{\partial x_2} = \dfrac{\partial \hat{\Phi}_\rho}{\partial x_1} & \text{in } \Omega_\rho \ . \end{cases} \tag{21}$$

Hence, up to a subsequence, φ_ρ converges in $C^k_{\text{loc}}(\Omega \cup \partial G)$. This means that v_ρ converges (up to a subsequence) in $C^k_{\text{loc}}(\Omega \cup \partial G)$ to some v_0. Denote by $g_\rho = v_{\rho|\partial G}$. It is clear that g_ρ converges to some g_0 and v_0 satisfies

$$\begin{cases} v_0 \wedge \dfrac{\partial v_0}{\partial x_1} = -\dfrac{\partial \hat{\Phi}_0}{\partial x_2} & \text{in } \Omega \\ v_0 \wedge \dfrac{\partial v_0}{\partial x_2} = \dfrac{\partial \hat{\Phi}_0}{\partial x_1} & \text{in } \Omega \\ v_0 = g_0 & \text{on } \partial G\ , \end{cases} \tag{22}$$

which means that v_0 is a canonical harmonic map.

We now consider two sequences $v_{\rho n}$ and $v_{\nu n}$ which converge to v_1 and v_2. Locally,

$$\varphi_{\rho_n} \to \varphi_1 \qquad \text{and} \qquad \varphi_{\nu_n} \to \varphi_2\ .$$

Thus, $\nabla\varphi_1 = \nabla\varphi_2$, so φ_1 and φ_2 differ locally by an additive constant, which means that v_1 and v_2 differ locally by a multiplicative constant of modulus 1. By the connectedness of Ω, this constant is global. □

Let

$$\hat{R}_0(x) = \hat{\Phi}_0(x) - \sum_{j=1}^{k} d_j \log |\, x - a_j \,|\ .$$

We observe that $\hat{R}_0$ is a smooth harmonic function in G.

Theorem 2. *We have the following asymptotic estimate:*

$$\frac{1}{2}\int_{\Omega_\rho} |\, \nabla v_\rho \,|^2 = \pi \left(\sum_{j=1}^{k} d_j^2 \right) \log\frac{1}{\rho} + \widetilde{W}(a, \overline{d}) + O(\rho)\,, \quad \text{as } \rho \to 0\ , \tag{23}$$

where

$$\widetilde{W}(a, \overline{d}) = -\pi \sum_{i \neq j} d_i d_j \, \log |\, a_i - a_j \,| - \pi \sum_{j=1}^{k} d_j \hat{R}_0(a_j)\ . \tag{24}$$

Proof. We follow the ideas of the proof of Theorem I.7 in [BBH4].

Since $\hat{\Phi}_\rho$ is harmonic in Ω_ρ and $\hat{\Phi}_\rho = 0$ on ∂G we may write

$$\frac{1}{2}\int_{\Omega_\rho} |\, \nabla v_\rho \,|^2 = \frac{1}{2}\int_{\Omega_\rho} |\, \nabla \hat{\Phi}_\rho \,|^2 = -\frac{1}{2}\sum_{j=1}^{k} \int_{\partial B(a_j,\rho)} \frac{\partial \hat{\Phi}_\rho}{\partial \nu}\, \hat{\Phi}_\rho = -\pi \sum_{j=1}^{k} d_j\, \hat{\Phi}_\rho \Big(\partial B(a_j, \rho) \Big)\ .$$

By Lemma 1 and the expression of $\hat{R}_0$ we easily deduce (23). □

Theorem 3. *The following equality holds:*

$$\widetilde{W}(a,\overline{d}) = \inf_{\deg(g;\partial G)=d} W(a,\overline{d},g) \tag{25}$$

and the infimum is atteint.

Proof. *Step 1.* $\widetilde{W}(a,\overline{d}) \leq \inf_{\deg(g;\partial G)=d} W(a,\overline{d},g)$.

Suppose not, then there exist $\varepsilon > 0$ and $g : \partial G \to S^1$ with $\deg(g;\partial G) = d$ such that

$$W(a,\overline{d},g) + \varepsilon \leq \widetilde{W}(a,\overline{d}) \ . \tag{26}$$

Thus, if u_ρ is a solution of (4), then

$$\frac{1}{2}\int_{\Omega_\rho} \mid \nabla u_\rho \mid^2 = \pi\left(\sum_{j=1}^k d_j^2\right) \log\frac{1}{\rho} + W(a,\overline{d},g) + O(\rho) \geq \tag{27}$$

$$\geq \frac{1}{2}\int_{\Omega_\rho} \mid \nabla v_\rho \mid^2 = \pi\left(\sum_{j=1}^k d_j^2\right) \log\frac{1}{\rho} + \widetilde{W}(a,\overline{d}) + O(\rho) \ , \quad \text{as } \rho \to 0 \ .$$

We obtain a contradiction by (26) and (27).

Step 2. If g_ρ and g_0 are as in the proof of Theorem 1, then

$$\widetilde{W}(a,\overline{d}) = W(a,\overline{d},g_0) \ .$$

For $r > 0$ let $u_{\rho,r}$ be a solution of the minimization problem

$$\min_{u\in\mathcal{E}_{r,g_\rho}} \int_{\Omega_r} \mid \nabla u \mid^2 \ . \tag{28}$$

Denote $u_{\rho,\rho} = u_\rho$ and $\Phi_{\rho,r}$ the solution of the associated linear problem (see (10)). Let $\Phi_{\rho,0}$ be the solution of (12) for g replaced by g_ρ.

We recall (see Theorem I.6 in [BBH4]) that

$$\Phi_{\rho,r} \to \Phi_{\rho,0} \qquad \text{in } C^k_{\text{loc}}(\Omega \cup \partial G) \ \text{ as } r \to 0 \tag{29}$$

and

$$\mid \frac{1}{2}\int_{\Omega_r} \mid \nabla u_{\rho,r} \mid^2 - \pi\left(\sum_{j=1}^k d_j^2\right) \log\frac{1}{\rho} - W(a,\overline{d},g_\rho) \mid \leq C_{g_\rho} r \ , \tag{30}$$

where $C_g = C(g) > 0$ is a constant which depends on the boundary data g.

Our aim is to prove that C_{g_ρ} is uniformly bounded for $\rho > 0$. Indeed, analysing the proof of Theorem I.7 in [BBH4] we observe that C_{g_ρ} depends on $\widetilde{C}_{g_\rho}$, which appears in

$$\|\Phi_{\rho,r} - \Phi_{\rho,0}\|_{L^\infty(\Omega_r)} \leq \sum_{j=1}^{k} \left[\sup_{\partial B(a_j,r)} \Phi_{\rho,0} - \inf_{\partial B(a_j,r)} \Phi_{\rho,0} \right] \leq \widetilde{C}_{g_\rho} r \ . \tag{31}$$

It is clear at this stage, by the convergence of g_ρ and elliptic estimates, that $\widetilde{C}_{g_\rho}$ is uniformly bounded.

Observe now that the map $C^1(\partial G; S^1) \ni g \longmapsto W(a, \overline{d}, g)$ is continuous. We have

$$| W(a,\overline{d},g_0) - \widetilde{W}(a,\overline{d}) | \leq | \frac{1}{2} \int_{\Omega_\rho} | \nabla v_\rho |^2 - \pi \left(\sum_{j=1}^{k} d_j^2 \right) \log \frac{1}{\rho} - \widetilde{W}(a,\overline{d}) | +$$

$$+ | \frac{1}{2} \int_{\Omega_\rho} | \nabla v_\rho |^2 - \pi \left(\sum_{j=1}^{k} d_j^2 \right) \log \frac{1}{\rho} - W(a,\overline{d},g_\rho) | + | W(a,\overline{d},g_\rho) - W(a,\overline{d},g_0) | \leq$$

$$\leq O(\rho) + C\rho + | W(a,\overline{d},g_\rho) - W(a,\overline{d},g_0) | \to 0 \qquad \text{as } \rho \to 0 \ .$$

Thus

$$\widetilde{W}(a,\overline{d}) = W(a,\overline{d},g_0) \ ,$$

which concludes the proof of this step. □

Theorem 4. *For fixed A, if w_ρ is a solution of the minimization problem (6) then the following holds:*

$$\frac{1}{2} \int_{\Omega_\rho} | \nabla w_\rho |^2 = \pi \left(\sum_{j=1}^{k} d_j^2 \right) \log \frac{1}{\rho} + \widetilde{W}_A(a,\overline{d}) + o(1) \ , \quad \text{as } \rho \to 0 \ , \tag{32}$$

where

$$\widetilde{W}_A(a,\overline{d}) = \inf \{ W(a,\overline{d},g);\ \deg(g;\partial G) = d \ \ \text{and} \ \int_{\partial G} | \frac{\partial g}{\partial \tau} |^2 \leq A \} \ , \tag{33}$$

and the infimum is atteint.

Moreover, w_ρ converges in $C^{0,\alpha}(\Omega \cup \partial G)$ to the canonical harmonic map associated to g_0, a, $\overline{d}$.

Proof. The existence of w_ρ is obvious. Let $g_\rho = w_\rho |_{\partial G}$. It follows from Chapter I in [BBH4] that

$$\frac{1}{2} \int_{\Omega_\rho} | \nabla w_\rho |^2 = \pi \left(\sum_{j=1}^{k} d_j^2 \right) \log \frac{1}{\rho} + W(a,\overline{d},g_\rho) + O_{g_\rho}(\rho) \ , \quad \text{as } \ \rho \to 0 \ , \tag{34}$$

where $O_g(\eta)$ stands for a quantity X such that $| X |\leq C_g\eta$ and C_g depends only on g, a and $\overline{d}$.

By the boundedness of g_ρ in $H^1(\partial G)$ we may suppose that (up to a subsequence)

$$g_\rho \rightharpoonup g_0 \qquad \text{weakly in} \quad H^1(\partial G), \quad \text{as} \quad \rho \to 0 \ .$$

As in the proof of Theorem 3 (see (31)) we deduce that C_{g_ρ} is uniformly bounded.

We now prove that the map $g \longmapsto W(a, \overline{d}, g)$ is continuous in the weak topology of $H^1(\partial G)$. Taking into account the weak convergence of g_ρ to g_0 and the Sobolev embedding Theorem we obtain

$$g_\rho \wedge \frac{\partial g_\rho}{\partial \tau} \rightharpoonup g_0 \wedge \frac{\partial g_0}{\partial \tau} \qquad \text{weakly in} \quad L^2(\partial G), \quad \text{as} \quad \rho \to 0 \ .$$

Using (12), it follows that

$$\Phi_{\rho,0} \rightharpoonup \Phi_0 \qquad \text{weakly in} \quad H^1(G), \ \text{as} \ \rho \to 0 \ .$$

So, by the Rellich Theorem,

$$\Phi_{\rho,0} \to \Phi_0 \qquad \text{strongly in} \quad L^2(G), \quad \text{as} \quad \rho \to 0 \ .$$

Therefore,

$$\int_{\partial G} \Phi_{\rho,0} \left(g_\rho \wedge \frac{\partial g_\rho}{\partial \tau} \right) \to \int_{\partial G} \Phi_0 \left(g_0 \wedge \frac{\partial g_0}{\partial \tau} \right) \quad \text{as } \rho \to 0 \ .$$

We also deduce, using elliptic estimates, that for each i,

$$R_{\rho,0}(a_i) \to R_0(a_i) \qquad \text{as} \quad \rho \to 0 \ .$$

Thus, by (13), we obtain the continuity of the map $g \longmapsto W(a, \overline{d}, g)$. Hence, by (34), we easily deduce (32).

The fact that the infimum in (33) is atteint may be deduced with similar arguments as in the proof of Theorem 3.

The convergence of w_ρ to a canonical harmonic map follows easily from the convergence of g_ρ. □

3. Renormalized energies in a particular case and related properties

We shall calculate in the first part of this section the expressions of $\widetilde{W}(a,\overline{d},g)$ when $G = B(0;1)$ and $g(\theta) = e^{id\theta}$, for an arbitrary configuration $a = (a_1, ..., a_k)$.

Proposition 1. *The expression of the renormalized energy $\widetilde{W}(a,\overline{d})$ is given by*

$$\widetilde{W}(a,\overline{d}) = -\pi \sum_{i\neq j} d_i d_j \, \log \mid a_i - a_j \mid + \pi \sum_{i\neq j} d_i d_j \, \log \mid 1 - a_i \overline{a}_j \mid + \pi \sum_{j=1}^{k} d_j^2 \, \log(1- \mid a_j \mid^2) \ .$$

Proof. If $\hat{R}_0$ is that defined in the preceding section, then

$$\begin{cases} \Delta \hat{R}_0 = 0 \quad \text{in } B_1 \\ \hat{R}_0(x) = -\sum_{j=1}^{k} d_j \, \log \mid x - a_j \mid \quad \text{if } x \in \partial B_1 \ . \end{cases}$$

It follows from the linearity of this problem that it is sufficient to calculate $\hat{R}_0$ when the configuration of points consists of one point, say a. Hence, by the Poisson formula, for each $x \in B_1$,

$$\hat{R}_0(x) = -\frac{d}{2\pi}\,(1- \mid x \mid^2) \int_{\partial B_1} \frac{\log \mid z - a \mid}{\mid z - x \mid^2} dz \ . \tag{35}$$

We first observe that

$$\hat{R}_0(x) = 0 \qquad \text{if} \ \ a = 0 \ . \tag{36}$$

If $a \neq 0$ and $a^\star = \dfrac{a}{\mid a \mid^2}$, then

$$\hat{R}_0(x) = -\frac{d}{2\pi}\,(1- \mid x \mid^2) \int_{\partial B_1} \frac{\log \mid z - a^\star \mid + \log \mid a \mid}{\mid z - x \mid^2} dz = \tag{37}$$

$$= -d \log \mid x - a^\star \mid - d \log \mid a \mid \ .$$

Hence, by (36) and (37)

$$\hat{R}_0(x) = \begin{cases} 0 \qquad \text{if } a = 0 \\ -\, d \log \mid x - a^\star \mid - d \log \mid a \mid \quad \text{if } a \neq 0 \ . \end{cases} \tag{38}$$

In the case of a general configuration $a = (a_1, ..., a_k)$ one has

$$\hat{R}_0(x) = -\sum_{j=1}^{k} d_j \log \mid x - a_j^\star \mid - \sum_{j=1}^{k} d_j \log \mid a_j \mid \ . \tag{39}$$

Applying now Theorem 2 we obtain

$$\widetilde{W}(a,\overline{d}) = -\pi \sum_{i\neq j} d_i d_j \ \log \mid a_i - a_j \mid + \pi \sum_{i\neq j} d_i d_j \ \log \mid 1 - a_i \overline{a}_j \mid + \pi \sum_{j=1}^{k} d_j^2 \ \log(1 - \mid a_j \mid^2) \ .$$

□

Proposition 2. *The expression of $W(a,\overline{d},g)$ in the particular case considered above is given by*

$$W(a,\overline{d},g) = \tag{40}$$

$$= -\pi \sum_{i\neq j} d_i d_j \ \log \mid a_i - a_j \mid - \pi \sum_{i\neq j} d_i d_j \ \log \mid 1 - a_i \overline{a}_j \mid - \pi \sum_{j=1}^{k} d_j^2 \ \log(1 - \mid a_j \mid^2) \ .$$

Proof. We shall use the expression (13) for the renormalized energy $W(a,\overline{d},g)$. As above, we observe that it suffices to calculate R_0 for one point, say a.

We define on $B(0;1) \setminus \{a\}$ the function $\mathcal{G}$ by

$$\mathcal{G}(x) = \begin{cases} \dfrac{d}{2\pi} \ \log \mid x - a \mid + \dfrac{d}{2\pi} \ \log \mid x - a^\star \mid - \dfrac{d}{4\pi} \ \mid x \mid^2 + \mathcal{C} & \text{if } a \neq 0 \\ \dfrac{d}{2\pi} \ \log \mid x \mid - \dfrac{d}{4\pi} \ \mid x \mid^2 + \mathcal{C} & \text{if } a = 0 \end{cases} \tag{41}$$

and we choose the constant $\mathcal{C}$ such that

$$\int_{\partial B_1} \mathcal{G} = 0 \ .$$

It follows that, for every $a \in B_1$,

$$\mathcal{C} = \frac{d}{4\pi} + \frac{d}{2\pi} \ \log \mid a \mid \ . \tag{42}$$

The function $\mathcal{G}$ satisfies

$$\begin{cases} \Delta \mathcal{G} = d\delta_a - \dfrac{d}{\pi} & \text{in } B_1 \\ \dfrac{\partial \mathcal{G}}{\partial \nu} = 0 & \text{on } \partial B_1 \\ \displaystyle\int_{\partial B_1} \mathcal{G} = 0 \ . \end{cases} \tag{43}$$

It follows now from (12) that

$$\begin{cases} \Delta\left(\dfrac{\Phi_0}{2\pi}\right) = d\delta_a & \text{in } B_1 \\ \dfrac{\partial}{\partial\nu}\left(\dfrac{\Phi_0}{2\pi}\right) = \dfrac{d}{2\pi} & \text{on } \partial B_1 \\ \displaystyle\int_{\partial B_1} \dfrac{\Phi_0}{2\pi} = 0 \ . \end{cases}$$

Thus the function $\Psi = \dfrac{\Phi_0}{2\pi} - \dfrac{d}{4\pi}(| x |^2 - 1)$ satisfies

$$\begin{cases} \Delta\Psi = d\delta_a - \dfrac{d}{\pi} & \text{in } B_1 \\ \dfrac{\partial\Psi}{\partial\nu} = 0 & \text{on } \partial B_1 \\ \displaystyle\int_{\partial B_1} \Psi = 0 \ . \end{cases} \tag{44}$$

By unicity arguments, it follows from (43) and (44) that

$$\frac{\Phi_0}{2\pi} - \frac{d}{4\pi}(| x |^2 - 1) = \frac{d}{2\pi}\log | x - a | + \frac{1}{2\pi}\log | x - a^\star | - \frac{d}{4\pi} | x |^2 + \mathcal{C} \ . \tag{45}$$

Taking into account the expression of $\mathcal{C}$ given in (42), as well as the link between Φ_0 and R_0 we obtain (40). □

Remark. It follows by Theorem 3 and Propositions 1 and 2 that

$$\sum_{i\neq j} d_i d_j \log | a_i - a_j | + \sum_{j=1}^{k} d_j^2 \log(1 - | a_j |^2) \leq 0 \ .$$

A very interesting problem is the study of configurations which minimize $W(a, \overline{d}, g)$ with $\overline{d}$ and g prescribed. This relies on the behaviour of minimizers of the Ginzburg-Landau energy (see [BBH4] for further details).

Proposition 3. *If $k = 2$ and $d_1 = d_2 = 1$, then the minimal configuration for W is unique (up to a rotation) and consists of two points which are symmetric with respect to the origin.*

Proof. Let a and b be two distinct points in B_1. Then

$$-\frac{W}{\pi} = \log(| a |^2 + | b |^2 - 2 | a | \cdot | b | \cdot \cos\varphi) + \log(1 + | a |^2 | b |^2 - 2 | a | \cdot | b | \cdot \cos\varphi) +$$

$$+\log(1-\mid a\mid^2)+\log(1-\mid b\mid^2)\ ,$$

where φ denotes the angle between the vectors $\overrightarrow{Oa}$ and $\overrightarrow{Ob}$. So, a necessary condition for the minimum of W is $\cos\varphi=-1$, that is the points a, O and b are colinear, with O between a and b. Hence one may suppose that the points a and b lie on the real axis and $-1<b<0<a<1$. Denote

$$f(a,b)=2\log(a-b)+2\log(1-ab)+\log(1-a^2)+\log(1-b^2)\ .$$

A straightforward calculation, based on the Jensen inequality and the symmetry of f, shows that $a=-b=5^{-1/4}$. □

4. The behavior of minimizers of the Ginzburg-Landau energy

We assume throughout this section that G is strictly starshaped about the origin.

In [BBH2] and [BBH4], F. Bethuel, H. Brezis and F. Hélein studied the behavior of minimizers of the Ginzburg-Landau energy E_ε in

$$H^1_g(G;\mathbf{R}^2)=\{u\in H^1(G;\mathbf{R}^2);\ u=g\ \text{ on }\ \partial G\}\ ,$$

for some smooth fixed $g:\partial G\to S^1$, $\deg(g;\partial G)=d>0$. Our aim is to study a similar problem, that is the behavior of minimizers u_ε of E_ε in the class

$$\mathcal{H}_{d,A}=\{u\in H^1(G;\mathbf{R}^2);\ \mid u\mid=1 \text{ on } \partial G,\ \deg(u,\partial G)=d \text{ and } \int_{\partial G}\mid\frac{\partial u}{\partial\tau}\mid^2\leq A\}\ . \tag{46}$$

It would have seemed more naturally to minimize E_ε in the class

$$\mathcal{H}_d=\{u\in H^1(G;\mathbf{R}^2);\ \mid u\mid=1 \text{ on } \partial G,\ \deg(u,\partial G)=d\}$$

but, as observed by F. Bethuel, H. Brezis and F. Hélein, the infimum of E_ε is not atteint. To show this, they considered the particular case when $G=B_1$, $d=1$ and $g(x)=x$. This is the reason which imposed us to take the infimum of E_ε on the class $\mathcal{H}_{d,A}$, that was also considered by F. Bethuel, H. Brezis and F. Hélein.

Theorem 5. *For each sequence $\varepsilon_n\to 0$, there is a subsequence (also denoted by ε_n) and exactly d points $a_1,\cdots,a_d$ in G such that*

$$u_{\varepsilon_n}\to u_\star\quad \textit{in } H^1_{\rm loc}(\overline{G}\setminus\{a_1,\cdots,a_d\};\mathbf{R}^2)\ ,$$

where $u_\star$ is a canonical harmonic map with values in S^1 and singularities $a_1,\cdots,a_d$ of degrees +1.

Moreover, the configuration $a = (a_1, \cdots, a_d)$ is a minimum point of

$$\widetilde{W}_A(a, \overline{d}) := \min \{W(a, \overline{d}, g);\ \deg(g; \partial G) = d \ \text{ and } \int_{\partial G} |\frac{\partial g}{\partial \tau}|^2 \leq A\}\ .$$

Proof. *Step 1.* The existence of u_ε.

For fixed ε, let u_ε^n be a minimizing sequence for E_ε in $\mathcal{H}_{d,A}$. It follows that (up to a subsequence)

$$u_\varepsilon^n \rightharpoonup u_\varepsilon \quad \text{weakly in } H^1$$

and, by the boundedness of $u_\varepsilon^n |_{\partial G}$ in $H^1(\partial G)$, we obtain that

$$u_{\varepsilon_n} |_{\partial G} \to u_\varepsilon |_{\partial G} \quad \text{strongly in } H^{\frac{1}{2}}(\partial G)\ .$$

This means that, if $g_\varepsilon = u_\varepsilon |_{\partial G}$, then

$$\deg(g_\varepsilon; \partial G) = d\ .$$

By the lower semi-continuity of E_ε, u_ε is a minimizer of E_ε. Moreover, this u_ε satisfies the Ginzburg-Landau energy

$$-\Delta u_\varepsilon = \frac{1}{\varepsilon^2} u_\varepsilon (1 - |u_\varepsilon|^2) \quad \text{in } G\ . \tag{47}$$

Step 2. A fundamental estimate.

As in the proof of Theorem III.2 in [BBH4], multiplying (47) by $x \cdot \nabla u_\varepsilon$ and integrating on G, we find

$$\begin{aligned} \frac{1}{2} \int_{\partial G} (x \cdot \nu) \left(\frac{\partial u_\varepsilon}{\partial \nu} \right)^2 + \frac{1}{2\varepsilon^2} \int_G (1 - |u_\varepsilon|^2)^2 = \\ = \frac{1}{2} \int_{\partial G} (x \cdot \nu) \left(\frac{\partial g_\varepsilon}{\partial \tau} \right)^2 - \int_{\partial G} (x \cdot \tau) \frac{\partial u_\varepsilon}{\partial \nu} \frac{\partial g_\varepsilon}{\partial \tau}\ . \end{aligned} \tag{48}$$

Using now the boundedness of g_ε in $H^1(\partial G)$ and the fact that G is strictly starshaped we easily obtain

$$\int_{\partial G} |\frac{\partial u_\varepsilon}{\partial \nu}|^2 + \frac{1}{\varepsilon^2} \int_G (1 - |u_\varepsilon|^2)^2 \leq C\ , \tag{49}$$

where C depends only on A and d.

Step 3. A fundamental Lemma.

The following result is an adapted version of Theorem III.3 in [BBH4] which is essential towards locating the singularities at the limit.

Lemma 3. *There exist positive constants λ_0 and μ_0 (which depend only on G, d and A) such that if*

$$\frac{1}{\varepsilon^2}\int_{G\cap B_{2\ell}}(1-\mid u_\varepsilon\mid^2)^2 \le \mu_0 \ ,$$

where $B_{2\ell}$ is some disk of radius 2ℓ in $\mathbf{R}^2$ with

$$\frac{\ell}{\varepsilon}\ge\lambda_0 \quad \textit{and} \quad \ell\le 1 \ ,$$

then

$$\mid u_\varepsilon(x)\mid\ge\frac{1}{2} \quad \textit{if } x\in G\cap B_\ell \ . \tag{50}$$

The proof of Lemma is essentially the same as of the cited theorem, after observing that

$$\|\nabla u_\varepsilon\|_{L^\infty(G)}\le\frac{C}{\varepsilon} \ ,$$

where C depends only on G, d and A.

Step 4. The convergence of u_ε.

Using Lemma 1 and the estimate (49), we may apply the methods developed in Chapters III-V in [BBH4] to determine the "bad" disks, as well as the fact that their number is uniformly bounded. The same techniques allow us to prove the weak convergence in $H^1_{\rm loc}(G\setminus\{a_1,\cdots,a_k\};\mathbf{R}^2)$ of a subsequence, also denoted by u_{ε_n}, to some $u_\star$.

As in [BBH4], Chapter X (see also [S]) one may prove that, for each $p<2$,

$$u_{\varepsilon_n}\to u_\star \quad \text{in } W^{1,p}(G) \ .$$

This allows us to pass at the limit in

$$\frac{\partial}{\partial x_1}\left(u_{\varepsilon_n}\wedge\frac{\partial u_{\varepsilon_n}}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(u_{\varepsilon_n}\wedge\frac{\partial u_{\varepsilon_n}}{\partial x_2}\right)=0 \quad \text{in } \ \mathcal{D}'(G)$$

and to deduce that $u_\star$ is a canonical harmonic map.

The strong convergence of (u_{ε_n}) in $H^1_{\rm loc}(\overline{G}\setminus\{a_1,\cdots,a_k\};\mathbf{R}^2)$ follows as in [BBH4], Theorem VI.1 with the techniques from [BBH3], Theorem 2, Step 1.

We then observe that for all j, $\deg(u_\star,a_j)\ne 0$. Indeed, if not, then as in Step 1 of Theorem 2 in [BBH3], the H^1-convergence is extended up to a_j, which becomes a "removable singularity". The fact that all these degrees equal $+1$ and the points $a_1,\cdots,a_d$ are not on the boundary may be deduced as in Theorem VI.2 [BBH4].

The following steps are devoted to characterize the limit configuration as a minimum point of the renormalized energy $\widetilde{W}_A$.

Step 5. An upper bound for $E_\varepsilon(u_\varepsilon)$.

For $R > 0$, let $I(R)$ be the infimum of E_ε on $H^1_g(G)$ with $G = B(0; \frac{\varepsilon}{R})$ and $g(x) = \dfrac{x}{| x |}$ on ∂G. Following the ideas of the proof of Lemma VIII.1 in [BBH4] one may show that if $b = (b_j)$ is an arbitrary configuration of d distinct points in G and g is such that $\deg(g, \partial G) = d$ and $\int_{\partial G} | \frac{\partial g}{\partial \tau} |^2 \leq A$, then there exists $\eta_0 > 0$ such that, for each $\eta < \eta_0$,

$$E_\varepsilon(u_\varepsilon) \leq dI\left(\frac{\varepsilon}{\eta}\right) + W(b, g) + \pi d \log \frac{1}{\eta} + O(\eta) \ , \quad \text{as } \eta \to 0 \tag{51}$$

for $\varepsilon > 0$ small enough. Here $O(\eta)$ stands for a quantity which is bounded by C_η, where C is a constant depending only on g.

Step 6. A lower bound for $E_{\varepsilon_n}(u_{\varepsilon_n})$.

With the same proof as of Step 2 of Theorem 1 in [LR] one may show that if $a_1, \cdots, a_d$ are the singularities of $u_\star$ and $\eta > 0$, then there is $N_0 = N_0(\eta) \in \mathbf{N}$ such that, for each $n \geq N_0$,

$$E_{\varepsilon_n}(u_{\varepsilon_n}) \geq dI\left(\frac{\varepsilon_n}{\eta(1+\eta)}\right) + \pi d \log \frac{1}{\eta} + W(a, g_0) + O(\eta) \ , \tag{52}$$

where $O(\eta)$ is a quantity bounded by $C\eta$, where C depends only on g_0.

Step 7. The limit configuration is a minimum point for $\widetilde{W}_A$.

Taking into account that (see [BBH4], Chapter III)

$$I(\varepsilon) = \pi \ | \log \varepsilon | + \gamma + O(\varepsilon) \ ,$$

we obtain by (51) and (52)

$$\begin{aligned} W(b, g) - \pi d \log \varepsilon_n + d\gamma + O\left(\frac{\varepsilon_n}{\eta}\right) &\geq \\ \geq W(a, g_0) - \pi d \log \varepsilon_n + d\gamma + O(\eta) \ . \end{aligned} \tag{53}$$

Adding $\pi d \log \varepsilon_n$ in (53) and passing to the limit firstly as $n \to \infty$ and then as $\eta \to 0$, we find

$$W(a, g_0) \leq W(b, g) \ . \tag{54}$$

As b and g are arbitrary chosen it follows that $a = (a_1, \cdots, a_d)$ is a global minimum point of

$$\widetilde{W}_A(b) = \min \{ W(b, g); \ \deg(g; \partial G) = d \ \text{ and } \int_{\partial G} | \frac{\partial g}{\partial \tau} |^2 \leq A \} \ . \tag{55}$$

□

Remark. The infimum in (55) is atteint because of the continuity of the mapping $\mathcal{H}_{d,A} \ni g \longmapsto W(b, g)$ with respect to the weak topology of $H^1(\partial G)$.

Acknowledgements. We are grateful to Prof. H. Brezis for his constant support during the preparation of this work. We also thank Th. Cazenave for useful discussions.

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