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ON A NONHOMOGENEOUS QUASILINEAR EIGENVALUE PROBLEM IN SOBOLEV SPACES WITH VARIABLE EXPONENT

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ABSTRACT. We consider the nonlinear eigenvalue problem

$$-\mathrm{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)=\lambda|u|^{q(x)-2}u$$

in Ω, $u=0$ on $\partial\Omega$, where Ω is a bounded open set in $\mathbb{R}^N$ with smooth boundary and p, q are continuous functions on $\overline{\Omega}$ such that $1<\inf_\Omega q<\inf_\Omega p<\sup_\Omega q$, $\sup_\Omega p<N$, and $q(x)<Np(x)/(N-p(x))$ for all $x\in\overline{\Omega}$. The main result of this paper establishes that any $\lambda>0$ sufficiently small is an eigenvalue of the above nonhomogeneous quasilinear problem. The proof relies on simple variational arguments based on Ekeland's variational principle.

1. Introduction and preliminary results

A basic result in the elementary theory of linear partial differential equations asserts that the spectrum of the Laplace operator in $H_0^1(\Omega)$ is discrete, where Ω is a bounded open set in $\mathbb{R}^N$ with smooth boundary. More precisely, the problem

$$\begin{cases} -\Delta u=\lambda u & \text{in} \quad \Omega,\\ u=0 & \text{on} \quad \partial\Omega \end{cases}$$

has an unbounded sequence of eigenvalues $0<\lambda_1<\lambda_2\le\ldots\le\lambda_n\le\ldots$. This celebrated result goes back to the Riesz-Fredholm theory of self-adjoint and compact operators on Hilbert spaces. The anisotropic case

$$\begin{cases} -\Delta u=\lambda a(x)u & \text{in} \quad \Omega,\\ u=0 & \text{on} \quad \partial\Omega \end{cases}$$

was considered by Bocher [5], Hess and Kato [16], Minakshisundaram and Pleijel [20, 22]. For instance, Minakshisundaram and Pleijel proved that the above eigenvalue problem has an unbounded sequence of positive eigenvalues if $a\in L^\infty(\Omega)$, $a\ge 0$ in Ω, and $a>0$ in $\Omega_0\subset\Omega$, where $|\Omega_0|>0$. Eigenvalue problems for homogeneous quasilinear problems have been intensively studied in the last decades (see, e.g., Anane [4] and Lindqvist [18]).

This paper is motivated by recent advances in elastic mechanics and electrorheological fluids (sometimes referred to as "smart fluids"), where some processes are modeled by nonhomogeneous quasilinear operators (see Acerbi and Mingione [1],

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Diening [7], Halsey [15], Ruzicka [23], Zhikov [26, 27], and the references therein). We refer mainly to the $p(x)$–Laplace operator $\Delta_{p(x)}u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$, where p is a continuous nonconstant function. This differential operator is a natural generalization of the p-Laplace operator $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, where $p>1$ is a real constant. However, the $p(x)$-Laplace operator possesses more complicated nonlinearities than the p-Laplace operator, due to the fact that $\Delta_{p(x)}$ is not homogeneous. Recent qualitative properties of solutions to quasilinear problems in Sobolev spaces with variable exponent have been obtained by Alves and Souto [2], Chabrowski and Fu [6], and Mihăilescu and Rădulescu [19]. We also refer to El Hamidi [12], where it is proved a multiplicity result for a class of symmetric systems involving $(p(x), q(x))$-Laplace operators.

In this paper we are concerned with the nonhomogeneous eigenvalue problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda |u|^{q(x)-2}u, & \text{for} \quad x\in\Omega,\\ u=0, & \text{for} \quad x\in\partial\Omega\,, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^N$ ($N\geq 3$) is a bounded domain with smooth boundary, $\lambda>0$ is a real number, and p, q are continuous on $\overline{\Omega}$.

The case $p(x)=q(x)$ was considered by Fan, Zhang and Zhao in [14] who, using the Ljusternik-Schnirelmann critical point theory, established the existence of a sequence of eigenvalues. Denoting by Λ the set of all nonnegative eigenvalues, Fan, Zhang and Zhao showed that $\sup\Lambda=+\infty$, and they pointed out that only under additional assumptions we have $\inf\Lambda>0$. We remark that for the p-Laplace operator (corresponding to $p(x)\equiv p$) we always have $\inf\Lambda>0$.

In this paper we study problem (1.1) under the basic assumption

$$1<\min_{x\in\overline{\Omega}} q(x) < \min_{x\in\overline{\Omega}} p(x) < \max_{x\in\overline{\Omega}} q(x). \tag{1.2}$$

Our main result establishes the existence of a continuous family of eigenvalues for problem (1.1) in a neighborhood of the origin. More precisely, we show that there exists $\lambda^\star>0$ such that *any* $\lambda\in(0,\lambda^\star)$ is an eigenvalue for problem (1.1).

We start with some preliminary basic results on the theory of Lebesgue–Sobolev spaces with variable exponent. For more details we refer to the book by Musielak [21] and the papers by Edmunds et al. [8, 9, 10], Kovacik and Rákosník [17], and Samko and Vakulov [24].

Assume that $p\in C(\overline{\Omega})$ and $p(x)>1$, for all $x\in\overline{\Omega}$.

Set

$$C_+(\overline{\Omega}) = \{h;\ h\in C(\overline{\Omega}),\ h(x)>1 \text{ for all } x\in\overline{\Omega}\}.$$

For any $h\in C_+(\overline{\Omega})$ we define

$$h^+ = \sup_{x\in\Omega} h(x) \qquad \text{and} \qquad h^- = \inf_{x\in\Omega} h(x).$$

For any $p(x)\in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \{u;\ u \text{ is a measurable real-valued function and } \int_\Omega |u(x)|^{p(x)}\,dx<\infty\}.$$

We define a norm, the so-called *Luxemburg norm*, on this space by the formula

$$|u|_{p(x)} = \inf\left\{\mu>0;\ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)}\,dx\leq 1\right\}.$$

We remember that the variable exponent Lebesgue spaces are separable and reflexive Banach spaces. If $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponent, so that $p_1(x) \le p_2(x)$ almost everywhere in Ω, then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$.

We denote by $L^{p^{'}(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $1/p(x)+1/p^{'}(x) = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p^{'}(x)}(\Omega)$ the Hölder type inequality

$$\left|\int_\Omega uv\ dx\right| \le \left(\frac{1}{p^-} + \frac{1}{p^{'-}}\right) |u|_{p(x)}|v|_{p^{'}(x)} \tag{1.3}$$

holds true.

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the *modular* of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)}\ dx.$$

If (u_n), $u \in L^{p(x)}(\Omega)$, then the following relations hold true:

$$|u|_{p(x)} > 1 \quad \Rightarrow \quad |u|^{p^-}_{p(x)} \le \rho_{p(x)}(u) \le |u|^{p^+}_{p(x)}, \tag{1.4}$$

$$|u|_{p(x)} < 1 \quad \Rightarrow \quad |u|^{p^+}_{p(x)} \le \rho_{p(x)}(u) \le |u|^{p^-}_{p(x)}, \tag{1.5}$$

$$|u_n - u|_{p(x)} \to 0 \quad \Leftrightarrow \quad \rho_{p(x)}(u_n - u) \to 0. \tag{1.6}$$

Next, we define $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\| = |\nabla u|_{p(x)}.$$

The space $(W_0^{1,p(x)}(\Omega), \|\cdot\|)$ is a separable and reflexive Banach space. We note that if $s(x) \in C_+(\overline{\Omega})$ and $s(x) < p^\star(x)$ for all $x \in \overline{\Omega}$, then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{s(x)}(\Omega)$ is compact and continuous, where $p^\star(x) = \frac{Np(x)}{N-p(x)}$ if $p(x) < N$ or $p^\star(x) = +\infty$ if $p(x) \ge N$.

We refer to Kováčik and Rákosník [17] for more properties of Lebesgue and Sobolev spaces with variable exponent.

2. The main result

We say that $\lambda \in \mathbb{R}$ is an eigenvalue of problem (1.1) if there exists $u \in W_0^{1,p(x)}(\Omega) \setminus \{0\}$ such that

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v\ dx - \lambda \int_\Omega |u|^{q(x)-2}uv\ dx = 0,$$

for all $v \in W_0^{1,p(x)}(\Omega)$. We point out that if λ is an eigenvalue of problem (1.1), then the corresponding $u \in W_0^{1,p(x)}(\Omega) \setminus \{0\}$ is a weak solution of (1.1).

Our main result is given by the following theorem.

Theorem 2.1. *Assume that condition* (1.2) *is fulfilled,* $\max_{x\in\overline{\Omega}} p(x) < N$ *and* $q(x) < p^\star(x)$ *for all* $x \in \overline{\Omega}$. *Then there exists* $\lambda^\star > 0$ *such that any* $\lambda \in (0, \lambda^\star)$ *is an eigenvalue for problem* (1.1).

The above result implies

$$\inf_{u\in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^{p(x)}\,dx}{\displaystyle\int_\Omega |u|^{q(x)}\,dx} = 0.$$

Thus, for any positive constant C, there exists $u_0 \in W_0^{1,p(x)}(\Omega)$ such that

$$C\int_\Omega |u_0|^{q(x)}\,dx \geq \int_\Omega |\nabla u_0|^{p(x)}\,dx.$$

Let E denote the generalized Sobolev space $W_0^{1,p(x)}(\Omega)$.

For any $\lambda > 0$ the energy functional corresponding to problem (1.1) is defined as $J_\lambda : E \to \mathbb{R}$,

$$J_\lambda(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx - \lambda\int_\Omega \frac{1}{q(x)}|u|^{q(x)}\,dx.$$

Standard arguments imply that $J_\lambda \in C^1(E,\mathbb{R})$ and

$$\langle J_\lambda^{'}(u), v\rangle = \int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v\,dx - \lambda\int_\Omega |u|^{q(x)-2}uv\,dx,$$

for all $u,\ v\in E$. Thus the weak solutions of (1.1) coincide with the critical points of J_λ. If such an weak solution exists and is nontrivial, then the corresponding λ is an eigenvalue of problem (1.1).

Lemma 2.2. *There exists $\lambda^\star > 0$ such that for any $\lambda\in(0,\lambda^\star)$ there exist ρ, $a > 0$ such that $J_\lambda(u) \geq a > 0$ for any $u\in E$ with $\|u\| = \rho$.*

Proof. Since $q(x) < p^\star(x)$ for all $x\in\overline{\Omega}$ it follows that E is continuously embedded in $L^{q(x)}(\Omega)$. So, there exists a positive constant c_1 such that

$$|u|_{q(x)} \leq c_1\|u\|, \quad \forall\, u\in E. \tag{2.1}$$

We fix $\rho\in(0,1)$ such that $\rho < 1/c_1$. Then relation (2.1) implies

$$|u|_{q(x)} < 1, \quad \forall\, u\in E, \text{ with } \|u\| = \rho.$$

Furthermore, relation (1.5) yields

$$\int_\Omega |u|^{q(x)}\,dx \leq |u|_{q(x)}^{q^-}, \quad \forall\, u\in E, \text{ with } \|u\| = \rho. \tag{2.2}$$

Relations (2.1) and (2.2) imply

$$\int_\Omega |u|^{q(x)}\,dx \leq c_1^{q^-}\|u\|^{q^-}, \quad \forall\, u\in E, \text{ with } \|u\| = \rho. \tag{2.3}$$

Taking into account relations (1.5) and (2.3) we deduce that for any $u \in E$ with $\|u\| = \rho$ the following inequalities hold true:

$$\begin{aligned} J_\lambda(u) &\geq \frac{1}{p^+}\int_\Omega |\nabla u|^{p(x)}\,dx - \frac{\lambda}{q^-}\int_\Omega |u|^{q(x)}\,dx \\ &\geq \frac{1}{p^+}\|u\|^{p^+} - \frac{\lambda}{q^-}c_1^{q^-}\|u\|^{q^-} \\ &= \frac{1}{p^+}\rho^{p^+} - \frac{\lambda}{q^-}c_1^{q^-}\rho^{q^-} \\ &= \rho^{q^-}\left(\frac{1}{p^+}\rho^{p^+-q^-} - \frac{\lambda}{q^-}c_1^{q^-}\right). \end{aligned}$$

By the above inequality we remark that if we define

$$\lambda^\star = \frac{\rho^{p^+-q^-}}{2p^+}\cdot\frac{q^-}{c_1^{q^-}}, \tag{2.4}$$

then for any $\lambda \in (0, \lambda^\star)$ and any $u \in E$ with $\|u\| = \rho$ there exists $a = \frac{\rho^{p^+}}{2p^+} > 0$ such that

$$J_\lambda(u) \geq a > 0.$$

The proof of Lemma 2.2 is complete. □

Lemma 2.3. *There exists $\phi \in E$ such that $\phi \geq 0$, $\phi \neq 0$ and $J_\lambda(t\phi) < 0$, for $t > 0$ small enough.*

Proof. Assumption (1.2) implies that $q^- < p^-$. Let $\epsilon_0 > 0$ be such that $q^- + \epsilon_0 < p^-$. On the other hand, since $q \in C(\overline{\Omega})$ it follows that there exists an open set $\Omega_0 \subset \Omega$ such that $|q(x) - q^-| < \epsilon_0$ for all $x \in \Omega_0$. Thus, we conclude that $q(x) \leq q^- + \epsilon_0 < p^-$ for all $x \in \Omega_0$.

Let $\phi \in C_0^\infty(\Omega)$ be such that $\mathrm{supp}(\phi) \supset \overline{\Omega}_0$, $\phi(x) = 1$ for all $x \in \overline{\Omega}_0$ and $0 \leq \phi \leq 1$ in Ω. Then using the above information for any $t \in (0,1)$ we have

$$\begin{aligned} J_\lambda(t\phi) &= \int_\Omega \frac{t^{p(x)}}{p(x)}|\nabla\phi|^{p(x)}\,dx - \lambda\int_\Omega \frac{t^{q(x)}}{q(x)}|\phi|^{q(x)}\,dx \\ &\leq \frac{t^{p^-}}{p^-}\int_\Omega |\nabla\phi|^{p(x)}\,dx - \frac{\lambda}{q^+}\int_\Omega t^{q(x)}|\phi|^{q(x)}\,dx \\ &\leq \frac{t^{p^-}}{p^-}\int_\Omega |\nabla\phi|^{p(x)}\,dx - \frac{\lambda}{q^+}\int_{\Omega_0} t^{q(x)}|\phi|^{q(x)}\,dx \\ &\leq \frac{t^{p^-}}{p^-}\int_\Omega |\nabla\phi|^{p(x)}\,dx - \frac{\lambda\cdot t^{q^-+\epsilon_0}}{q^+}\int_{\Omega_0} |\phi|^{q(x)}\,dx. \end{aligned}$$

Therefore

$$J_\lambda(t\phi) < 0$$

for $t < \delta^{1/(p^- - q^- - \epsilon_0)}$ with

$$0 < \delta < \min\left\{1, \frac{\frac{\lambda\cdot p^-}{q^+}\int_{\Omega_0}|\phi|^{q(x)}\,dx}{\int_\Omega |\nabla\phi|^{p(x)}\,dx}\right\}.$$

Finally, we point out that $\int_\Omega |\nabla\phi|^{p(x)}\,dx > 0$. Indeed, it is clear that

$$\int_{\Omega_0} |\phi|^{q(x)}\,dx \leq \int_\Omega |\phi|^{q(x)}\,dx \leq \int_\Omega |\phi|^{q^-}\,dx.$$

On the other hand, $W_0^{1,p(x)}(\Omega)$ is continuously embedded in $L^{q^-}(\Omega)$, and thus there exists a positive constant c_2 such that

$$|\phi|_{q^-} \leq c_2\|\phi\|.$$

The last two inequalities imply that

$$\|\phi\| > 0,$$

and combining that fact with relations (1.4) or (1.5) we deduce that

$$\int_\Omega |\nabla\phi|^{p(x)}\,dx > 0.$$

The proof of Lemma 2.3 is complete. $\square$

Proof of Theorem 2.1. Let $\lambda^\star > 0$ be defined as in (2.4) and $\lambda \in (0, \lambda^\star)$. By Lemma 2.2 it follows that on the boundary of the ball centered at the origin and of radius ρ in E, denoted by $B_\rho(0)$, we have

$$\inf_{\partial B_\rho(0)} J_\lambda > 0. \tag{2.5}$$

On the other hand, by Lemma 2.3, there exists $\phi \in E$ such that $J_\lambda(t\phi) < 0$ for all $t > 0$ small enough. Moreover, relations (2.3) and (1.5) imply that for any $u \in B_\rho(0)$ we have

$$J_\lambda(u) \geq \frac{1}{p^+}\|u\|^{p^+} - \frac{\lambda}{q^-}c_1^{q^-}\|u\|^{q^-}.$$

It follows that

$$-\infty < \underline{c} := \inf_{\overline{B_\rho(0)}} J_\lambda < 0.$$

We now let $0 < \epsilon < \inf_{\partial B_\rho(0)} J_\lambda - \inf_{B_\rho(0)} J_\lambda$. Applying Ekeland's variational principle to the functional $J_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$, we find $u_\epsilon \in \overline{B_\rho(0)}$ such that

$$\begin{aligned} J_\lambda(u_\epsilon) &< \inf_{\overline{B_\rho(0)}} J_\lambda + \epsilon, \\ J_\lambda(u_\epsilon) &< J_\lambda(u) + \epsilon \cdot \|u - u_\epsilon\|, \quad u \neq u_\epsilon. \end{aligned}$$

Since

$$J_\lambda(u_\epsilon) \leq \inf_{\overline{B_\rho(0)}} J_\lambda + \epsilon \leq \inf_{B_\rho(0)} J_\lambda + \epsilon < \inf_{\partial B_\rho(0)} J_\lambda\,,$$

we deduce that $u_\epsilon \in B_\rho(0)$. Now, we define $I_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$ by $I_\lambda(u) = J_\lambda(u) + \epsilon \cdot \|u - u_\epsilon\|$. It is clear that u_ϵ is a minimum point of I_λ and thus

$$\frac{I_\lambda(u_\epsilon + t\cdot v) - I_\lambda(u_\epsilon)}{t} \geq 0$$

for small $t > 0$ and any $v \in B_1(0)$. The above relation yields

$$\frac{J_\lambda(u_\epsilon + t\cdot v) - J_\lambda(u_\epsilon)}{t} + \epsilon\cdot\|v\| \geq 0.$$

Letting $t \to 0$ it follows that $\langle J_\lambda^{'}(u_\epsilon), v\rangle + \epsilon\cdot\|v\| > 0$, and we infer that $\|J_\lambda^{'}(u_\epsilon)\| \leq \epsilon$.

We deduce that there exists a sequence $\{w_n\} \subset B_\rho(0)$ such that

$$J_\lambda(w_n) \to \underline{c} \quad \text{and} \quad J_\lambda^{'}(w_n) \to 0. \tag{2.6}$$

It is clear that $\{w_n\}$ is bounded in E. Thus, there exists $w \in E$ such that, up to a subsequence, $\{w_n\}$ converges weakly to w in E. Since $q(x) < p^\star(x)$ for all $x \in \overline{\Omega}$ we deduce that E is compactly embedded in $L^{q(x)}(\Omega)$, hence $\{w_n\}$ converges strongly to w in $L^{q(x)}(\Omega)$. So, by relations (1.6) and (1.3),

$$\lim_{n\to\infty} \int_\Omega |w_n|^{q(x)-2} w_n (w_n - w)\, dx = 0.$$

On the other hand, relation (2.6) yields

$$\lim_{n\to\infty} \langle J_\lambda^{'}(w_n), w_n - w \rangle = 0.$$

Using the above information we find

$$\lim_{n\to\infty} \int_\Omega |\nabla w_n|^{p(x)-2} \nabla w_n \nabla (w_n - w)\, dx = 0. \tag{2.7}$$

Relation (2.7) and the fact that $\{w_n\}$ converges weakly to w in E enable us to apply Theorem 3.1 in Fan and Zhang [13] in order to obtain that $\{w_n\}$ converges strongly to w in E. So, by (2.6),

$$J_\lambda(w) = \underline{c} < 0 \quad \text{and} \quad J_\lambda^{'}(w) = 0. \tag{2.8}$$

We conclude that w is a nontrivial weak solution for problem (1.1) and thus any $\lambda \in (0, \lambda^\star)$ is an eigenvalue of problem (1.1).

The proof of Theorem (2.1) is complete. $\square$

Let us now assume that the hypotheses of Theorem (2.1) are fulfilled and, furthermore,

$$\max_{\overline{\Omega}} p(x) < \max_{\overline{\Omega}} q(x).$$

Then, using similar arguments as in the proof of Lemma 2.3, we find some $\psi \in E$ such that

$$\lim_{t\to\infty} J_\lambda(t\psi) = -\infty.$$

That fact combined with Lemma 2.2 and the mountain pass theorem (see [3]) implies that there exists a sequence $\{u_n\}$ in E such that

$$J_\lambda(u_n) \to \overline{c} > 0 \quad \text{and} \quad J_\lambda^{'}(u_n) \to 0 \text{ in } E^\star. \tag{2.9}$$

However, relation (2.9) is not useful because we cannot show that the sequence $\{u_n\}$ is bounded in E since the functional J_λ does not satisfy a relation of the Ambrosetti-Rabinowitz type. This enables us to affirm that we cannot obtain a critical point for J_λ by using this method.

On the other hand, we point out that we will fail in trying to show that the functional J_λ is coercive since by relation (1.2) we have $q^+ > p^-$. Thus, we cannot apply (as in the homogeneous case) a result as Theorem 1.2 in Struwe [25] in order to obtain a critical point of the functional J_λ.

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