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Variational analysis for Dirichlet impulsive differential equations with oscillatory nonlinearity

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Abstract. By using variational methods and critical point theory, we establish the existence of infinitely many solutions for second-order impulsive differential equations with Dirichlet boundary conditions, depending on two real parameters.

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1. Introduction

Many dynamical systems describing models in applied sciences have an *impulsive* dynamical behaviour due to abrupt changes at certain instants during the evolution process. The rigorous mathematical description of these phenomena leads to *impulsive differential equations*; they characterize various processes of the real world described by models that are subject to *sudden changes* in their states. Essentially, impulsive differential equations correspond to a *smooth evolution* that may *change instantaneously* or even *abruptly*, as happens in various applications that describe mechanical or natural phenomena. These changes correspond to impulses in the smooth system, such as for example in the model of a mechanical clock. Impulsive differential equations also study models in physics, population dynamics, ecology, industrial robotics, biotechnology, economics, optimal control, chaos theory. Associated with this development, a theory of impulsive differential equations has been given extensive attention. For an introduction of the basic theory of impulsive differential equations in $\mathbb{R}^n$ we refer to [3], [14], and [22]. Some classical tools have been used to study such problems in

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the literature, such as the coincidence degree theory of Mawhin, the method of upper and lower solutions with the monotone iterative technique, and some fixed point theorems in cones (see [10], [15], [20]). Recently, the existence and multiplicity of solutions for impulsive boundary value problems by using variational methods and critical point theory has been considered in [16], [23], [24], [25], [26].

For a general second order differential equation $\mathscr{F}(t,u,u') = 0$, one can consider impulses in the *position* u and the *velocity* u'. However, as argued in [16], it is natural to consider in the motion of spacecraft only instantaneous impulses depending on the position that result in jump discontinuities, but with no change in position. The impulses only on the velocity occurs also in impulsive mechanics, see [17].

The purpose of this paper is to show the variational structure underlying of a class of nonlinear impulsive differential equations. We take as a model a Dirichlet problem with impulses. For an excellent overview of the most significant mathematical methods employed in this paper we refer to Ciarlet [11].

2. Statement of the problem

The aim of this paper is to study the following nonlinear Dirichlet boundary-value problem

$$\begin{cases} -\big(p(t)u'(t)\big)' + q(t)u(t) = \lambda f\big(t,u(t)\big) + \mu g\big(t,u(t)\big), & t \in [0,T],\ t \neq t_j, \\ u(0) = u(T) = 0, \\ \Delta u'(t_j) = I_j\big(u(t_j)\big), & j = 1,2,\dots,m, \end{cases} \tag{1}$$

where $T > 0$, $p \in C^1([0,T],[0,+\infty[)$, $q \in L^\infty([0,T])$ with $\operatorname{ess\,inf}_{t\in[0,T]} q(t) \geq 0$, $\lambda \in \,]0,+\infty[$, $\mu \in [0,+\infty[$, $f,g : [0,T]\times\mathbb{R} \to \mathbb{R}$ are L^1-Carathéodory functions, $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T$, $\Delta u'(t_j) = u'(t_j^+) - u'(t_j^-) = \lim_{t\to t_j^+} u'(t) - \lim_{t\to t_j^-} u'(t)$, and $I_j : \mathbb{R} \to \mathbb{R}$ are continuous for every $j = 1,2,\dots,m$.

We are motivated by the paper of Bonanno et al. [4] in which, using two critical point theorems, the authors ensured the existence of at least three classical solutions for the nonlinear Dirichlet boundary-value problem

$$\begin{cases} -u''(t) + a(t)u'(t) + b(t)u(t) = \lambda g\big(t,u(t)\big), & t \in [0,T],\ t \neq t_j, \\ u(0) = u(T) = 0, \\ \Delta u'(t_j) = u'(t_j^+) - u'(t_j^-) = \mu I_j\big(u(t_j)\big), & j = 1,2,\dots,n, \end{cases}$$

where $\lambda \in \,]0,+\infty[$, $\mu \in \,]0,+\infty[$, $g : [0,T]\times\mathbb{R} \to \mathbb{R}$, $a,b \in L^\infty([0,T])$ satisfy the conditions $\operatorname{ess\,inf}_{t\in[0,T]} a(t) \geq 0$, $\operatorname{ess\,inf}_{t\in[0,T]} b(t) \geq 0$, $0 = t_0 < t_1 < t_2 < \cdots <$

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$t_n < t_{n+1} = T$, $\Delta u'(t_j) = u'(t_j^+) - u'(t_j^-) = \lim_{t \to t_j^+} u'(t) - \lim_{t \to t_j^-} u'(t)$, and $I_j : \mathbb{R} \to \mathbb{R}$ are continuous for every $j = 1, 2, \dots, n$, by choosing μ in a suitable way and for every λ lying in a precise interval.

Our goal in this paper is to obtain some sufficient conditions to guarantee that problem (1) has infinitely many classical solutions. To this end, we require that the primitive F of f satisfies a suitable oscillatory behavior either at infinity (for obtaining unbounded solutions) or at the origin (for finding arbitrarily small solutions), while G, the primitive of g, has an appropriate growth (see Theorems 4.1 and 5.4). Our analysis is mainly based on a general critical point theorem (see Lemma 3.1 below) contained in [5]; see also [21].

We also refer the interested reader to the papers [1], [2], [6], [7], [8], [9], [12], [13] and references therein, in which Ricceri's variational principle and its variants have been successfully used to obtain the existence of infinitely many solutions for boundary value problems.

We end this preliminary section with the following theorem, which is a direct consequence of our main result.

Theorem 2.1. *Let $h : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous and non-zero function. Define $H(t) = \int_0^t h(\xi)\, d\xi$ for all $t \in \mathbb{R}$ and assume that*

$$\liminf_{\xi \to +\infty} \frac{H(\xi)}{\xi^2} < \frac{6(e^{3T/4} - e^{T/4})}{e^T(e^T - 1)} \limsup_{\xi \to +\infty} \frac{H(\xi)}{\xi^2}.$$

Then, for each

$$\lambda \in \left] \frac{e^T(12 + T^2)}{3T(e^{3T/4} - e^{T/4}) \limsup_{\xi \to +\infty} \frac{H(\xi)}{\xi^2}}, \frac{2}{T(e^T - 1) \liminf_{\xi \to +\infty} \frac{H(\xi)}{\xi^2}} \right[,$$

for every arbitrary nonnegative continuous function $p : \mathbb{R} \to \mathbb{R}$, whose potential $P(t) := \int_0^t p(\xi)\, d\xi$ for all $t \in \mathbb{R}$, satisfies the condition

$$\limsup_{\xi \to +\infty} \frac{P(\xi)}{\xi^2} < +\infty,$$

and for every

$$\mu \in \left[0, \frac{2}{T(e^T - 1) \limsup_{\xi \to +\infty} \frac{P(\xi)}{\xi^2}} \left(1 - \lambda \frac{T(e^T - 1) \liminf_{\xi \to +\infty} \frac{H(\xi)}{\xi^2}}{2} \right) \right[,$$

the nonlinear problem

$$\begin{cases} -u''(t) + u'(t) + u(t) = \lambda h\big(u(t)\big) + \mu p\big(u(t)\big), & t \in [0, T],\ t \neq t_j, \\ u(0) = u(T) = 0, \\ \Delta u'(t_j) = I_j\big(u(t_j)\big), \quad j = 1, 2, \ldots, m, \end{cases}$$

has a sequence of classical solutions which is unbounded in $H_0^1(0, T)$.

3. Auxiliary results

We shall prove our results applying the following smooth version of Theorem 2.1 of [5], which is a more precise version of Ricceri's variational principle [21], Theorem 2.5. We point out that Ricceri's variational principle generalizes the celebrated three critical point theorem of Pucci and Serrin [18], [19] and is an useful result that gives alternatives for the multiplicity of critical points of certain functions depending on a parameter.

Lemma 3.1. *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous and coercive, and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, let*

$$\varphi(r) := \inf_{u \in \Phi^{-1}(-\infty, r)} \frac{\Big(\sup_{v \in \Phi^{-1}(-\infty, r)} \Psi(v)\Big) - \Psi(u)}{r - \Phi(u)},$$

$$\gamma := \liminf_{r \to +\infty} \varphi(r), \qquad \text{and} \qquad \delta := \liminf_{r \to (\inf_X \Phi)^+} \varphi(r).$$

Then the following properties hold:

(a) *For every $r > \inf_X \Phi$ and every $\lambda \in \big(0, 1/\varphi(r)\big)$, the restriction of the functional*

$$I_\lambda := \Phi - \lambda \Psi$$

to $\Phi^{-1}(-\infty, r)$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.

(b) *If $\gamma < +\infty$, then for each $\lambda \in (0, 1/\gamma)$, the following alternative holds: either*
  (b_1) *I_λ possesses a global minimum, or*
  (b_2) *there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that*

$$\lim_{n \to +\infty} \Phi(u_n) = +\infty.$$

(c) *If* $\delta < +\infty$, *then for each* $\lambda \in (0, 1/\delta)$, *the following alternative holds: either*
   (c_1) *there is a global minimum of* Φ *which is a local minimum of* I_λ, *or*
   (c_2) *there is a sequence* $\{u_n\}$ *of pairwise distinct critical points* (*local minima*) *of* I_λ *that converges weakly to a global minimum of* Φ.

In the Sobolev space $H_0^1(0, T)$, consider the inner product

$$(u, v) := \int_0^T p(t)u'(t)v'(t)\,dt + \int_0^T q(t)u(t)v(t)\,dt,$$

which induces the norm

$$\|u\| := \left(\int_0^T p(t)\big(u'(t)\big)^2\,dt + \int_0^T q(t)\big(u(t)\big)^2\,dt\right)^{1/2}.$$

Then the following Poincaré-type inequality holds:

$$\left[\int_0^T u^2(t)\,dt\right]^{1/2} \le \frac{T}{\pi}\left[\int_0^T (u')^2(t)\,dt\right]^{1/2}. \tag{2}$$

Proposition 3.2 ([4], Proposition 2.1). *Let* $u \in H_0^1(0, T)$. *Then*

$$\|u\|_\infty \le \frac{1}{2}\sqrt{\frac{T}{p^*}}\|u\|, \tag{3}$$

where $p^* := \min_{t\in[0,T]} p(t)$.

Let $f, g : [0, T] \times \mathbb{R} \to \mathbb{R}$ be two L^1-Carathéodory functions. We recall that $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function if

(a) the mapping $t \mapsto f(t, x)$ is measurable for every $x \in \mathbb{R}$;

(b) the mapping $x \mapsto f(t, x)$ is continuous for almost every $t \in [0, T]$;

(c) for every $\rho > 0$ there exists a function $l_\rho \in L^1([0, T])$ such that

$$\sup_{|x|\le\rho} |f(t, x)| \le l_\rho(t)$$

for almost every $t \in [0, T]$;

Corresponding to f, g we introduce the functions $F, G : [0, T] \times \mathbb{R} \to \mathbb{R}$ as follows

$$F(t, x) := \int_0^x f(t, \xi)\,d\xi, \qquad G(t, x) := \int_0^x g(t, \xi)\,d\xi,$$

for all $(t, x) \in [0, T] \times \mathbb{R}$.

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By a *classical solution* of problem (1), we mean a function

$$u \in \{w \in C([0,T] : w|_{[t_j, t_{j+1}]} \in H^2([t_j, t_{j+1}])\}$$

that satisfies the equation in (1) a.e. on $[0,T] \setminus \{t_1, \dots, t_m\}$, the limits $u'(t_j^+)$, $u'(t_j^-)$, $j = 1, \dots, m$, exist, satisfy the impulsive conditions $\Delta u'(t_j) = I_j(u(t_j))$ and the boundary condition $u(0) = u(T) = 0$.

We say that a function $u \in H_0^1(0,T)$ is a *weak solution* of problem (1), if u satisfies

$$\int_0^T p(t)u'(t)v'(t)\,dt + \int_0^T q(t)u(t)v(t)\,dt - \lambda \int_0^T f(t, u(t))v(t)\,dt \\ - \mu \int_0^T g(t, u(t))v(t)\,dt + \sum_{j=1}^m p(t_j) I_j(u(t_j))v(t_j) = 0,$$

for any $v \in H_0^1(0,T)$.

Lemma 3.3 ([4], Lemma 2.3). *The function $u \in H_0^1(0,T)$ is a weak solution of problem* (1) *if and only if u is a classical solution of* (1).

Lemma 3.4 ([4], Lemma 3.1). *Assume that*

(A1) *there exist constants $\alpha, \beta > 0$ and $\sigma \in [0,1[$ such that*

$$|I_j(x)| \le \alpha + \beta|x|^\sigma \qquad \textit{for all } x \in \mathbb{R},\ j = 1, 2, \dots, m.$$

Then, for any $u \in H_0^1(0,T)$, we have

$$\left|\sum_{j=1}^m p(t_j) \int_0^{u(t_j)} I_j(x)\,dx\right| \le \sum_{j=1}^m p(t_j)\left(\alpha\|u\|_\infty + \frac{\beta}{\sigma+1}\|u\|_\infty^{\sigma+1}\right). \tag{4}$$

Finally, put

$$\tilde{p} := \sum_{j=1}^m p(t_j), \qquad k := \frac{6p^*}{12\|p\|_\infty + T^2\|q\|_\infty}, \qquad \Gamma_c := \frac{\alpha}{c} + \left(\frac{\beta}{\sigma+1}\right)c^{\sigma-1},$$

where α, β, σ are given by (A1) and c is a positive constant.

4. Existence of infinitely many solutions

In this section we establish the main abstract result of this paper. Let

$$A := \liminf_{\xi \to +\infty} \frac{\int_0^T \max_{|x| \le \xi} F(t,x)\,dt}{\xi^2},$$

$$B := \limsup_{\xi \to +\infty} \frac{\int_{T/4}^{3T/4} F(t,\xi)\,dt}{\xi^2},$$

and

$$\lambda_1 := \frac{2p^*}{kTB}, \qquad \lambda_2 := \frac{2p^*}{TA}.$$

With the above notations we establish the following multiplicity property.

Theorem 4.1. *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function. Assume that* (A1) *holds and, moreover,*

(A2) $F(t,\xi) \ge 0$ *for all* $(t,\xi) \in \left(\left[0, \frac{T}{4}\right] \cup \left[\frac{3T}{4}, T\right]\right) \times \mathbb{R}$*;*

(A3) $A < kB$*.*

Then, for every $\lambda \in (\lambda_1, \lambda_2)$ and for every arbitrary L^1-Carathéodory function $g : [0,T] \times \mathbb{R} \to \mathbb{R}$, whose potential $G(t,x) := \int_0^x g(x,\xi)\,d\xi$ for all $(t,x) \in [0,T] \times \mathbb{R}$, is a nonnegative function satisfying the condition

$$G_\infty := \limsup_{\xi \to +\infty} \frac{\int_0^T \max_{|x| \le \xi} G(t,x)\,dt}{\xi^2} < +\infty, \tag{5}$$

if we put

$$\mu_{G,\lambda} := \frac{2p^*}{TG_\infty}\left(1 - \lambda \frac{TA}{2p^*}\right),$$

where $\mu_{G,\lambda} = +\infty$ when $G_\infty = 0$, problem (1) *has an unbounded sequence of classical solutions for every $\mu \in [0, \mu_{G,\lambda})$ in $H_0^1(0,T)$.*

Proof. Our aim is to apply Lemma 3.1(b) to problem (1). To this end, fix $\bar{\lambda} \in (\lambda_1, \lambda_2)$ and g satisfying our assumptions. Since $\bar{\lambda} < \lambda_2$, we have

$$\mu_{G,\bar{\lambda}} := \frac{2p^*}{TG_\infty}\left(1 - \bar{\lambda} \frac{TA}{2p^*}\right) > 0.$$

Now fix $\bar{\mu} \in (0, \mu_{G,\bar{\lambda}})$ and set

$$J(t,x) := F(t,x) + \frac{\bar{\mu}}{\bar{\lambda}} G(t,x)$$

for all $(t,x) \in [0,T] \times \mathbb{R}$. Take $X = H_0^1(0,T)$ and for each $u \in X$, let the functionals $\Phi, \Psi : X \to \mathbb{R}$ be defined by

$$\Phi(u) := \frac{1}{2}\|u\|^2,$$

$$\Psi(u) := \int_0^T J\big(t,u(t)\big)\,dt - \frac{1}{\bar{\lambda}}\sum_{j=1}^m p(t_j)\int_0^{u(t_j)} I_j(x)\,dx,$$

and put

$$E_{\bar{\lambda},\bar{\mu}}(u) := \Phi(u) - \bar{\lambda}\Psi(u), \qquad u \in X.$$

Using the property of f, g and the continuity of I_j, $j = 1,2,\dots,m$, we obtain that $\Phi, \Psi \in C^1(X,\mathbb{R})$ and for any $v \in X$, we have

$$\Phi'(u)(v) = \int_0^T p(t)u'(t)v'(t)\,dt + \int_0^T q(t)u(t)v(t)\,dt$$

and

$$\Psi'(u)(v) = \int_0^T f\big(t,u(t)\big)v(t)\,dt + \frac{\bar{\mu}}{\bar{\lambda}}\int_0^T g\big(t,u(t)\big)v(t)\,dt - \frac{1}{\bar{\lambda}}\sum_{j=1}^m p(t_j)I_j\big(u(t_j)\big)v(t_j).$$

So, with standard arguments, we deduce that the critical points of the functional $E_{\bar{\lambda},\bar{\mu}}$ are the weak solutions of problem (1) and so they are classical. We first observe that the functionals Φ and Ψ satisfy the regularity assumptions of Lemma 3.1.

First of all, we show that $\bar{\lambda} < 1/\gamma$. Hence, let $\{\xi_n\}$ be a sequence of positive numbers such that $\lim_{n\to+\infty} \xi_n = +\infty$ and

$$\lim_{n\to+\infty} \frac{\int_0^T \max_{|x|\le\xi_n} F(t,x)\,dx}{\xi_n^2} = A.$$

Put $r_n := \frac{2p^*}{T}\xi_n^2$ for all $n \in \mathbb{N}$. Then, for all $v \in X$ with $\Phi(v) < r_n$, taking (3) into account, one has $\|v\|_\infty < \xi_n$. Note that $\Phi(0) = \Psi(0) = 0$. Then, for all $n \in \mathbb{N}$,

$$\begin{aligned}\varphi(r_n) &= \inf_{u\in\Phi^{-1}(-\infty,r_n)} \frac{\Big(\sup_{v\in\Phi^{-1}(-\infty,r_n)} \Psi(v)\Big) - \Psi(u)}{r_n - \Phi(u)} \\ &\le \frac{\sup_{v\in\Phi^{-1}(-\infty,r_n)} \Psi(v)}{r_n}\end{aligned}$$

$$\leq \frac{\int_0^T \max_{|x| \leq \xi_n} J(t,x)\,dt + \frac{1}{\bar{\lambda}}\tilde{p}\left(\alpha\xi_n + \frac{\beta}{\sigma+1}\xi_n^{\sigma+1}\right)}{\frac{2p^*}{T}\xi_n^2}$$

$$\leq \frac{T}{2p^*}\left[\frac{\int_0^T \max_{|x| \leq \xi_n} F(t,x)\,dt}{\xi_n^2} + \frac{\bar{\mu}}{\bar{\lambda}}\frac{\int_0^T \max_{|x| \leq \xi_n} G(t,x)\,dt}{\xi_n^2} + \frac{1}{\bar{\lambda}}\tilde{p}\Gamma_{\xi_n}\right].$$

Since $\lim_{n\to+\infty} \Gamma_{\xi_n} = 0$, from the assumption (A3) and the condition (5), we have

$$\gamma \leq \liminf_{n\to+\infty} \varphi(r_n) \leq \frac{T}{2p^*}\left(A + \frac{\bar{\mu}}{\bar{\lambda}}G_\infty\right) < +\infty. \tag{6}$$

The assumption $\bar{\mu} \in (0, \mu_{G,\bar{\lambda}})$ immediately yields

$$\gamma \leq \frac{T}{2p^*}\left(A + \frac{\bar{\mu}}{\bar{\lambda}}G_\infty\right) < \frac{T}{2p^*}A + \frac{1 - \frac{T}{2p^*}\bar{\lambda}A}{\bar{\lambda}}.$$

Hence,

$$\bar{\lambda} = \frac{1}{\frac{T}{2p^*}A + \left(1 - \frac{T}{2p^*}\bar{\lambda}A\right)/\bar{\lambda}} < \frac{1}{\gamma}.$$

Let $\bar{\lambda}$ be fixed. We claim that the functional $E_{\bar{\lambda},\bar{\mu}}$ is unbounded from below. Since

$$\frac{1}{\bar{\lambda}} < \frac{kT}{2p^*}B,$$

there exist a sequence $\{\eta_n\}$ of positive numbers and $\tau > 0$ such that $\lim_{n\to+\infty} \eta_n = +\infty$ and

$$\frac{1}{\bar{\lambda}} < \tau < \frac{kT}{2p^*}\frac{\int_{T/4}^{3T/4} F(t,\eta_n)\,dt}{\eta_n^2} \tag{7}$$

for each $n \in \mathbb{N}$ large enough. For all $n \in \mathbb{N}$ define $w_n \in X$ by

$$w_n(t) := \begin{cases} \frac{4\eta_n}{T}t, & t \in [0, T/4], \\ \eta_n, & t \in]T/4, 3T/4], \\ \frac{4\eta_n}{T}(T-t), & t \in]3T/4, T]. \end{cases} \tag{8}$$

For any fixed $n \in \mathbb{N}$, one has

$$\Phi(w_n) \le \frac{(12\|p\|_\infty + T^2\|q\|_\infty)}{3T}\eta_n^2 = \frac{2p^*}{kT}\eta_n^2. \tag{9}$$

On the other hand, by (A2) and since G is nonnegative, from the definition of Ψ, we infer

$$\Psi(w_n) \ge \int_{T/4}^{3T/4} F(t,\eta_n)\,dt - \frac{1}{\bar\lambda}\frac{\tilde p}{k}\eta_n^2\Gamma_{(\eta_n/\sqrt{k})}. \tag{10}$$

By (7), (9) and (10), we see that

$$\begin{aligned} E_{\bar\lambda,\bar\mu}(w_n) &\le \frac{2p^*}{kT}\eta_n^2 - \bar\lambda\int_{T/4}^{3T/4} F(t,\eta_n)\,dt + \frac{\tilde p}{k}\eta_n^2\Gamma_{(\eta_n/\sqrt{k})} \\ &< \frac{2p^*}{kT}\eta_n^2(1-\bar\lambda\tau) + \frac{\tilde p}{k}\eta_n^2\Gamma_{(\eta_n/\sqrt{k})} \end{aligned} \tag{11}$$

for every $n \in \mathbb{N}$ large enough. Since $\sigma < 1$, $\bar\lambda\tau > 1$ and $\lim_{n\to+\infty}\eta_n = +\infty$, we have

$$\lim_{n\to+\infty} E_{\bar\lambda,\bar\mu}(w_n) = -\infty.$$

Then, the functional $E_{\bar\lambda,\bar\mu}$ is unbounded from below, and it follows that $E_{\bar\lambda,\bar\mu}$ has no global minimum. Therefore, by Lemma 3.1(b), there exists a sequence $\{u_n\}$ of critical points of $E_{\bar\lambda,\bar\mu}$ such that

$$\lim_{n\to+\infty}\|u_n\| = +\infty,$$

and the conclusion is achieved. □

Remark 4.2. Under the conditions $A = 0$ and $B = +\infty$, from Theorem 4.1 we see that for every $\lambda > 0$ and for each $\mu \in \left[0, \frac{2p^*}{TG_\infty}\right)$, problem (1) admits a sequence of classical solutions which is unbounded in X. Moreover, if $G_\infty = 0$, the result holds for every $\lambda > 0$ and $\mu \ge 0$.

5. Further results and particular cases

In this section we establish several useful consequences and particular cases of Theorem 4.1.

Corollary 5.1. *Let $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function. Suppose that hypotheses* (A1), (A2) *are fulfilled and*

$$A < \frac{2p^*}{T}, \qquad B > \frac{2p^*}{kT}.$$

Then, for every arbitrary L^1-Carathéodory function $g : [0, T] \times \mathbb{R} \to \mathbb{R}$, whose potential $G(t, x) := \int_0^x g(t, \xi)\, d\xi$ for all $(t, x) \in [0, T] \times \mathbb{R}$, is a nonnegative function satisfying the condition (5), *if we put*

$$\mu_{G,\lambda} := \frac{2p^*}{TG_\infty}\left(1 - \frac{TA}{2p^*}\right),$$

where $\mu_{G,\lambda} = +\infty$ when $G_\infty = 0$, the problem

$$\begin{cases} -\big(p(t)u'(t)\big)' + q(t)u(t) = f\big(t, u(t)\big) + \mu g\big(t, u(t)\big), & t \in [0, T],\ t \neq t_j, \\ u(0) = u(T) = 0, \\ \Delta u'(t_j) = I_j\big(u(t_j)\big), & j = 1, 2, \dots, m, \end{cases}$$

has an unbounded sequence of classical solutions for every $\mu \in [0, \mu_{G,\lambda})$ in $H_0^1(0, T)$.

The following result is a special case of Theorem 4.1.

Corollary 5.2. *Let* (A1) *holds and let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function. Put $F(\xi) := \int_0^\xi f(t)\, dt$ for all $\xi \in \mathbb{R}$ and assume that*

$$\liminf_{\xi \to +\infty} \frac{F(\xi)}{\xi^2} = 0, \qquad \limsup_{\xi \to +\infty} \frac{F(\xi)}{\xi^2} = +\infty.$$

Then, for every nonnegative continuous function $g : \mathbb{R} \to \mathbb{R}$ satisfying the condition

$$g_\infty := \lim_{\xi \to +\infty} \frac{\int_0^\xi g(x)\, dx}{\xi^2} < +\infty,$$

and for every $\mu \in \left[0, \frac{2p^*}{T^2 g_\infty}\right[$, the problem

$$\begin{cases} -\big(p(t)u'(t)\big)' + q(t)u(t) = f\big(u(t)\big) + \mu g\big(u(t)\big), & t \in [0, T],\ t \neq t_j, \\ u(0) = u(T) = 0, \\ \Delta u'(t_j) = I_j\big(u(t_j)\big), & j = 1, 2, \dots, m, \end{cases}$$

admits infinitely many distinct pairwise classical solutions.

Now, we point out a special situation of our main result when $\mu = 0$ and the nonlinear term has separated variables. To be precise, let $k \in L^1([0, T])$ such that $k(t) \geq 0$ a.e. $t \in [0, T]$, $k \not\equiv 0$, and let $l : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function.

Consider the following nonlinear Dirichlet boundary-value problem

$$\begin{cases} -u''(t) = \lambda k(t) l\big(u(t)\big), & t \in [0, T],\ t \neq t_j, \\ u(0) = u(T) = 0, \\ \Delta u'(t_j) = I_j\big(u(t_j)\big), & j = 1, 2, \dots, m. \end{cases} \tag{12}$$

Put $L(\xi) := \int_0^\xi l(x)\,dx$ for all $\xi \in \mathbb{R}$, and set $\|k\|_1 := \int_0^T k(t)\,dt$ and $k_0 := \int_{T/4}^{3T/4} k(t)\,dt$.

Corollary 5.3. *Let* (A1) *holds. Moreover, suppose that*

$$\liminf_{\xi \to +\infty} \frac{L(\xi)}{\xi^2} < \frac{k_0}{2\|k\|_1} \limsup_{\xi \to +\infty} \frac{L(\xi)}{\xi^2}.$$

Then, for each

$$\lambda \in \left] \frac{4}{(Tk_0) \limsup\limits_{\xi \to +\infty} \frac{L(\xi)}{\xi^2}}, \frac{2}{(T\|k\|_1) \liminf\limits_{\xi \to +\infty} \frac{L(\xi)}{\xi^2}} \right[,$$

problem (12) *has an unbounded sequence of classical solutions.*

Now put

$$\Im_c := \sum_{j=1}^m \min_{|\xi| \leq c} \int_0^\xi I_j(x)\,dx, \qquad \text{for all } c > 0,$$

$$A' := \liminf_{\xi \to 0^+} \frac{\int_0^T \max\limits_{|x| \leq \xi} F(t, x)\,dt}{\xi^2},$$

$$B' := \limsup_{\xi \to 0^+} \frac{\int_{T/4}^{3T/4} F(t, \xi)\,dt}{\xi^2},$$

and

$$\lambda_1' := \frac{2p^*}{kTB'}, \qquad \lambda_2' := \frac{2p^*}{TA'}.$$

Using Lemma 3.1(c) and arguing as in the proof of Theorem 4.1, we can obtain the following multiplicity result.

Theorem 5.4. *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function and $I_j(x) \le 0$ for all $x \in \mathbb{R}$, $j = 1, \dots, m$. Moreover, assume that* (A2) *and*

(A4) $A' < kB'$.

are satisfied. Then, for every $\lambda \in (\lambda_1', \lambda_2')$ and for every arbitrary L^1-Carathéodory function $g : [0,T] \times \mathbb{R} \to \mathbb{R}$, whose potential $G(t,x) := \int_0^x g(x,\xi)\, d\xi$ for all $(t,x) \in [0,T] \times \mathbb{R}$, is a nonnegative function satisfying the condition

$$G_0 := \limsup_{\xi \to 0^+} \frac{\int_0^T \max_{|x| \le \xi} G(t,x)\, dt}{\xi^2} < +\infty, \tag{13}$$

if we put

$$\mu'_{G,\lambda} := \frac{2p^*}{TG_0}\left(1 - \lambda \frac{TA'}{2p^*}\right),$$

where $\mu'_{G,\lambda} = +\infty$ when $G_0 = 0$, for every $\mu \in [0, \mu'_{G,\lambda})$ problem (1) *has a sequence of classical solutions, which strongly converges to zero in $H_0^1(0,T)$.*

Proof. Fix $\bar{\lambda} \in (\lambda_1', \lambda_2')$ and let g be a function that satisfies the condition (13). Since $\bar{\lambda} < \lambda_2'$, we obtain

$$\mu'_{G,\bar{\lambda}} := \frac{2p^*}{TG_0}\left(1 - \bar{\lambda} \frac{TA'}{2p^*}\right) > 0.$$

Now fix $\bar{\mu} \in (0, \mu'_{G,\bar{\lambda}})$ and set

$$J(t,x) := F(t,x) + \frac{\bar{\mu}}{\bar{\lambda}} G(t,x),$$

for all $(t,x) \in [0,T] \times \mathbb{R}$. We take Φ, Ψ and $E_{\bar{\lambda},\bar{\mu}}$ as in the proof of Theorem 4.1. Now, as it has been pointed out before, the functionals Φ and Ψ satisfy the regularity assumptions required in Lemma 3.1. As first step, we will prove that $\bar{\lambda} < 1/\delta$. Then, let $\{\xi_n\}$ be a sequence of positive numbers such that $\lim_{n\to+\infty} \xi_n = 0$ and

$$\lim_{n\to+\infty} \frac{\int_0^T \max_{|x| \le \xi_n} F(t,x)\, dx}{\xi_n^2} = A'.$$

By the fact that $\inf_X \Phi = 0$ and the definition of δ, we have $\delta = \liminf_{r\to 0^+} \varphi(r)$. Put $r_n := \frac{2p^*}{T}\xi_n^2$ for all $n \in \mathbb{N}$. Then, for all $v \in X$ with $\Phi(v) < r_n$, taking (3) into account, one has $\|v\|_\infty < \xi_n$. Thus, for all $n \in \mathbb{N}$,

$$\begin{aligned}
\varphi(r_n) &\le \frac{\sup_{v \in \Phi^{-1}(-\infty, r_n)} \Psi(v)}{r_n} \\
&\le \frac{\int_0^T \max_{|x| \le \xi_n} J(t,x)\,dt - \frac{1}{\bar{\lambda}}\|p\|_\infty \Im_{\xi_n}}{\frac{2p^*}{T}\xi_n^2} \\
&\le \frac{T}{2p^*}\left[\frac{\int_0^T \max_{|x|\le\xi_n} F(t,x)\,dt}{\xi_n^2} + \frac{\bar{\mu}}{\bar{\lambda}}\frac{\int_0^T \max_{|x|\le\xi_n} G(t,x)\,dt}{\xi_n^2} - \frac{1}{\bar{\lambda}}\|p\|_\infty \frac{\Im_{\xi_n}}{\xi_n^2}\right].
\end{aligned}$$

Since $\lim_{n\to+\infty} \frac{\Im_{\xi_n}}{\xi_n^2} = 0$, from the assumption (A4) and the condition (13), we have

$$\delta \le \liminf_{n\to+\infty} \varphi(r_n) \le \frac{T}{2p^*}\left(A' + \frac{\bar{\mu}}{\bar{\lambda}} G_0\right) < +\infty.$$

From $\bar{\mu} \in (0, \mu'_{G,\bar{\lambda}})$, the following inequalities hold

$$\delta \le \frac{T}{2p^*}\left(A' + \frac{\bar{\mu}}{\bar{\lambda}} G_0\right) < \frac{T}{2p^*}A' + \frac{1 - \frac{T}{2p^*}\bar{\lambda}A'}{\bar{\lambda}}.$$

Therefore

$$\bar{\lambda} = \frac{1}{\frac{T}{2p^*}A' + \left(1 - \frac{T}{2p^*}\bar{\lambda}A'\right)/\bar{\lambda}} < \frac{1}{\delta}.$$

Let $\bar{\lambda}$ be fixed. We claim that the functional $E_{\bar{\lambda},\bar{\mu}}$ does not have a local minimum at zero. Since

$$\frac{1}{\bar{\lambda}} < \frac{kT}{2p^*}B',$$

there exists a sequence $\{\eta_n\}$ of positive numbers and $\tau > 0$ such that $\lim_{n\to+\infty} \eta_n = 0$ and

$$\frac{1}{\bar{\lambda}} < \tau < \frac{kT}{2p^*}\frac{\int_{T/4}^{3T/4} F(t,\eta_n)\,dt}{\eta_n^2}$$

for each $n \in \mathbb{N}$ large enough. For all $n \in \mathbb{N}$, let $w_n \in X$ defined by (8) with the above η_n. Note that $\bar{\lambda}\tau > 1$. Then, since $I_j(x) \le 0$ for all $x \in \mathbb{R}$, $j = 1, \dots, m$, we obtain

$$
\begin{aligned}
E_{\bar{\lambda},\bar{\mu}}(w_n) &\leq \frac{2p^*}{kT}\eta_n^2 - \bar{\lambda}\int_{T/4}^{3T/4} F(t,\eta_n)\,dt + \sum_{j=1}^{m} p(t_j)\int_0^{w_n(t_j)} I_j(x)\,dx \\
&< \frac{2p^*}{kT}\eta_n^2(1-\bar{\lambda}\tau) < 0,
\end{aligned}
$$

for every $n \in \mathbb{N}$ large enough. Then, since

$$
\lim_{n\to+\infty} E_{\bar{\lambda},\bar{\mu}}(w_n) = E_{\bar{\lambda},\bar{\mu}}(0) = 0,
$$

we see that zero is not a local minimum of $E_{\bar{\lambda},\bar{\mu}}$. This, together with the fact that zero is the only global minimum of Φ, we deduce that the energy functional $E_{\bar{\lambda},\bar{\mu}}$ does not have a local minimum at the unique global minimum of Φ. Therefore, by Lemma 3.1(c), there exists a sequence $\{u_n\}$ of critical points of $E_{\bar{\lambda},\bar{\mu}}$ which converges weakly to zero. In view of the fact that the embedding $H_0^1(0,T) \hookrightarrow C^0([0,T])$ is compact, we know that the critical points converge strongly to zero, and the proof is complete. □

Remark 5.5. Applying Theorem 5.4, results similar to Corollaries 5.1, 5.2 and 5.3 can be obtained. We omit the discussions here.

Now, consider the nonlinear Dirichlet boundary value problem

$$
\begin{cases}
-u''(t) + a(t)u'(t) + b(t)u(t) = \lambda h\big(t,u(t)\big) + \mu p\big(t,u(t)\big), & t \in [0,T],\ t \neq t_j, \\
u(0) = u(T) = 0, \\
\Delta u'(t_j) = I_j\big(u(t_j)\big), \quad j = 1,2,\ldots,m,
\end{cases}
\tag{14}
$$

where $h, p : [0,T] \times \mathbb{R} \to \mathbb{R}$ are L^1-Carathéodory functions, $a, b \in L^\infty([0,T])$ satisfy the conditions $\operatorname{ess\,inf}_{t\in[0,T]} a(t) \geq 0$, $\operatorname{ess\,inf}_{t\in[0,T]} b(t) \geq 0$, and $I_j : \mathbb{R} \to \mathbb{R}$ are continuous for every $j = 1,2,\ldots,m$.

It is easy to see that the solutions of problem (1) are solutions of (14) if

$$
p(t) = e^{-\int_0^t a(\tau)\,d\tau}, \qquad q(t) = b(t)e^{-\int_0^t a(\tau)\,d\tau},
$$
$$
f(t,u) = h(t,u)e^{-\int_0^t a(\tau)\,d\tau}, \qquad g(t,u) = p(t,u)e^{-\int_0^t a(\tau)\,d\tau}.
$$

Let $A(t)$ be a primitive of $a(t)$ and put

$$
H(t,\xi) := \int_0^\xi h(t,x)\,dx, \qquad P(t,\xi) := \int_0^\xi p(t,x)\,dx,
$$
$$
\tilde{k} := \frac{6e^{-A(T)}}{12 + T^2\|be^{-A}\|_\infty}.
$$

By Theorem 4.1, we obtain the following multiplicity property for problem (14).

Theorem 5.6. *Let $h : [0, T] \times \mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function. Assume that* (A1) *holds and, moreover,*

(A5) $H(t, \xi) \geq 0$ *for all* $(t, \xi) \in \left(\left[0, \frac{T}{4}\right] \cup \left[\frac{3T}{4}, T\right]\right) \times \mathbb{R}$*;*

(A6) $A'' < \tilde{k}B''$*, where*

$$A'' := \liminf_{\xi \to +\infty} \frac{\int_0^T e^{-A(t)} \max_{|x| \leq \xi} H(t, x)\, dt}{\xi^2},$$
$$B'' := \limsup_{\xi \to +\infty} \frac{\int_{T/4}^{3T/4} e^{-A(t)} H(t, \xi)\, dt}{\xi^2},$$

Then, for every

$$\lambda \in \left] \frac{2}{\tilde{k} T e^{\|a\|_1} B''}, \frac{2}{T e^{\|a\|_1} A''} \right[$$

and for every arbitrary L^1-Carathéodory function $p : [0, T] \times \mathbb{R} \to \mathbb{R}$, whose potential $P(t, x) := \int_0^x p(x, \xi)\, d\xi$ for all $(t, x) \in [0, T] \times \mathbb{R}$, is a nonnegative function satisfying the condition

$$P_\infty := \limsup_{\xi \to +\infty} \frac{\int_0^T e^{-A(t)} \max_{|x| \leq \xi} P(t, x)\, dt}{\xi^2} < +\infty,$$

if we put

$$\mu_{P,\lambda} := \frac{2}{T e^{\|a\|_1} P_\infty} \left(1 - \lambda \frac{T e^{\|a\|_1} A''}{2}\right),$$

where $\mu_{P,\lambda} = +\infty$ when $P_\infty = 0$, problem (14) *has an unbounded sequence of classical solutions for every $\mu \in [0, \mu_{P,\lambda})$ in $H_0^1(0, T)$.*

Remark 5.7. Theorem 2.1 follows immediately from Theorem 5.6, setting $a(t) = b(t) \equiv 1$ for all $t \in \mathbb{R}$.

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