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Eigenvalue problems for anisotropic elliptic equations: An Orlicz–Sobolev space setting

Mihai Mihăilescu [a,b], Gheorghe Moroşanu [a], Vicenţiu Rădulescu [b,c,*]

[a] Department of Mathematics, Central European University, 1051 Budapest, Hungary
[b] Department of Mathematics, University of Craiova, 200585 Craiova, Romania
[c] Institute of Mathematics "Simion Stoilow" of the Romanian Academy, P.O. Box 1-764, 014700 Bucharest, Romania

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ABSTRACT

The paper studies a class of anisotropic eigenvalue problems involving an elliptic, nonhomogeneous differential operator on a bounded domain from $\mathbb{R}^N$ with a smooth boundary. Some results regarding the existence or non-existence of eigenvalues are obtained. In each case the competition between the growth rates of the anisotropic coefficients plays an essential role in the description of the set of eigenvalues.

1. Introduction

Eigenvalue problems involving nonhomogeneous elliptic operators have captured special attention in the last decade. Numerous papers have been devoted to the study of various phenomena which occur on the spectrum of such differential operators. We just refer to the recent advances in [1–13]. The present paper wishes to extend the above investigations by considering a new class of eigenvalue problems that will by described in the following.

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ be a bounded domain with smooth boundary $\partial\Omega$. Consider that, for each $i \in \{1, \dots, N\}$, φ_i are odd, increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$, λ is a positive real and $q : \overline{\Omega} \to (1, \infty)$ is a continuous function. The goal of this paper is to study the following anisotropic eigenvalue problem:

$$\begin{cases} -\sum_{i=1}^{N} \partial_i\left(\varphi_i(\partial_i u)\right) = \lambda |u|^{q(x)-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Since the operator in the divergence form is nonhomogeneous we introduce an Orlicz–Sobolev space setting for problems of this type. Actually, the fact that Eq. (1) is of anisotropic type means that a classical Orlicz–Sobolev space setting is not adequate. This leads us to seek weak solutions for problem (1) in a more general Orlicz–Sobolev type space, which will be

* Corresponding author at: Department of Mathematics, University of Craiova, 200585 Craiova, Romania. Tel.: +40 251412615.
E-mail addresses: mmihailes@yahoo.com (M. Mihăilescu), Morosanug@ceu.hu (G. Moroşanu), vicentiu.radulescu@imar.ro, vicentiu.radulescu@gmail.com (V. Rădulescu).

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introduced later in this paper. On the other hand, the term arising in the right-hand side of (1) is also nonhomogeneous and its particular form appeals to a suitable variable exponent Lebesgue space setting.

We first recall some basic facts about Orlicz spaces. For more details we refer to the books by Adams and Hedberg [14], Adams [15], Musielak [16] and Rao and Ren [17] and the papers by Clément et al. [18,19], García-Huidobro et al. [3] and Gossez [20].

Assume that $\varphi_i : \mathbb{R} \to \mathbb{R}$, $i \in \{1, \dots, N\}$, are odd, increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$. Define

$$\Phi_i(t) = \int_0^t \varphi_i(s)\mathrm{d}s, \qquad (\Phi_i)^\star(t) = \int_0^t (\varphi_i)^{-1}(s)\mathrm{d}s, \quad \text{for all } t \in \mathbb{R},\ i \in \{1, \dots, N\}.$$

We observe that Φ_i, $i \in \{1, \dots, N\}$, are *Young functions*; i.e., $\Phi_i(0) = 0$, Φ_i are convex, and $\lim_{x\to\infty} \Phi_i(x) = +\infty$. Furthermore, since $\Phi_i(x) = 0$ if and only if $x = 0$, $\lim_{x\to 0} \Phi_i(x)/x = 0$, and $\lim_{x\to\infty} \Phi_i(x)/x = +\infty$, then Φ_i are called *N-functions*. The functions $(\Phi_i)^\star$, $i \in \{1, \dots, N\}$, are called the *complementary* functions of Φ_i, $i \in \{1, \dots, N\}$, and they satisfy

$$(\Phi_i)^\star(t) = \sup\{st - \Phi_i(s); s \geq 0\}, \quad \text{for all } t \geq 0.$$

We also observe that $(\Phi_i)^\star$, $i \in \{1, \dots, N\}$, are also N-functions, and Young's inequality holds true:

$$st \leq \Phi_i(s) + (\Phi_i)^\star(t), \quad \text{for all } s, t \geq 0.$$

The Orlicz spaces $L_{\Phi_i}(\Omega)$, $i \in \{1, \dots, N\}$, defined by the N-functions Φ_i (see [14,15,18]) are the spaces of measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\|u\|_{L_{\Phi_i}} := \sup\left\{\int_\Omega uv\ \mathrm{d}x; \int_\Omega (\Phi_i)^\star(|g|)\mathrm{d}x \leq 1\right\} < \infty.$$

Then $(L_{\Phi_i}(\Omega), \|\cdot\|_{L_{\Phi_i}})$, $i \in \{1, \dots, N\}$, are Banach spaces whose norms are equivalent to the Luxemburg norms

$$\|u\|_{\Phi_i} := \inf\left\{k > 0; \int_\Omega \Phi_i\left(\frac{u(x)}{k}\right)\mathrm{d}x \leq 1\right\}.$$

For Orlicz spaces, Hölder's inequality reads as follows (see [17, Inequality 4, p. 79]):

$$\int_\Omega uv\mathrm{d}x \leq 2\,\|u\|_{L_{\Phi_i}}\|v\|_{L_{(\Phi_i)^\star}} \quad \text{for all } u \in L_{\Phi_i}(\Omega) \text{ and } v \in L_{(\Phi_i)^\star}(\Omega),\ i \in \{1, \dots, N\}.$$

We denote by $W^1L_{\Phi_i}(\Omega)$, $i \in \{1, \dots, N\}$, the Orlicz–Sobolev spaces defined by

$$W^1L_{\Phi_i}(\Omega) := \left\{u \in L_{\Phi_i}(\Omega); \frac{\partial u}{\partial x_i} \in L_{\Phi_i}(\Omega), i = 1, \dots, N\right\}.$$

These are Banach spaces with respect to the norms

$$\|u\|_{1,\Phi_i} := \|u\|_{\Phi_i} + \||\nabla u|\|_{\Phi_i}, \quad i \in \{1, \dots, N\}.$$

We also define the Orlicz–Sobolev spaces $W_0^1L_{\Phi_i}(\Omega)$, $i \in \{1, \dots, N\}$, as the closure of $C_0^1(\Omega)$ in $W^1L_{\Phi_i}(\Omega)$. By [20, Lemma 5.7], we obtain that on $W_0^1L_{\Phi_i}(\Omega)$, $i \in \{1, \dots, N\}$, we may consider the equivalent norm

$$\|u\|_i := \||\nabla u|\|_{\Phi_i}.$$

Moreover, it can be proved that the above norm is equivalent to the following norm:

$$\|u\|_{i,1} = \sum_{j=1}^N \|\partial_j u\|_{\Phi_i}$$

(see Proposition 1 in this paper).

For an easier manipulation of Orlicz–Sobolev spaces, we define

$$(p_i)_0 := \inf_{t>0} \frac{t\varphi_i(t)}{\Phi_i(t)} \quad \text{and} \quad (p_i)^0 := \sup_{t>0} \frac{t\varphi_i(t)}{\Phi_i(t)}, \quad i \in \{1, \dots, N\}.$$

In this paper we assume that for each $i \in \{1, \dots, N\}$ we have

$$1 < (p_i)_0 \leq \frac{t\varphi_i(t)}{\Phi_i(t)} \leq (p_i)^0 < \infty, \quad \forall\, t \geq 0. \tag{2}$$

The above relation implies that each Φ_i, $i \in \{1, \dots, N\}$, satisfies the Δ_2-condition; i.e.,

$$\Phi_i(2t) \leq K\Phi_i(t), \quad \forall\, t \geq 0, \tag{3}$$

where K is a positive constant (see [21, Proposition 2.3]).

Furthermore, in this paper we assume that for each $i \in \{1, \ldots, N\}$ the function Φ_i satisfies the following condition:

$$\text{the function } [0, \infty) \ni t \to \Phi_i(\sqrt{t}) \text{ is convex.} \tag{4}$$

Conditions (3) and (4) ensure that for each $i \in \{1, \ldots, N\}$ the Orlicz spaces $L_{\Phi_i}(\Omega)$ are uniformly convex spaces, and thus reflexive Banach spaces (see [21, Proposition 2.2]). That fact implies that the Orlicz–Sobolev spaces $W_0^1 L_{\Phi_i}(\Omega)$, $i \in \{1, \ldots, N\}$, are also reflexive Banach spaces.

Remark 1. We point out certain examples of functions $\varphi : \mathbb{R} \to \mathbb{R}$ which are odd, increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$ and satisfy conditions (2) and (4). For more details, the reader can consult [19, Examples 1–3, p. 243].

(1) Let

$$\varphi(t) = |t|^{p-2}t, \quad \forall\, t \in \mathbb{R},$$

with $p > 1$. For this function, it can be proved that

$$(\varphi)_0 = (\varphi)^0 = p.$$

(2) Consider

$$\varphi(t) = \log(1 + |t|^r)|t|^{p-2}t, \quad \forall\, t \in \mathbb{R},$$

with $p, r > 1$. In this case, it can be proved that

$$(\varphi)_0 = p, \qquad (\varphi)^0 = p + r.$$

(3) Let

$$\varphi(t) = \frac{|t|^{p-2}t}{\log(1 + |t|)}, \quad \text{if } t \neq 0,\ \varphi(0) = 0,$$

with $p > 2$. In this case we have

$$(\varphi)_0 = p - 1, \qquad (\varphi)^0 = p.$$

Finally, we introduce a natural generalization of the Orlicz–Sobolev spaces $W_0^1 L_{\Phi_i}(\Omega)$ that will enable us to study problem (1) with sufficient accuracy. For this purpose, let us denote by $\vec{\Phi} : \overline{\Omega} \to \mathbb{R}^N$ the vectorial function $\vec{\Phi} = (\Phi_1, \ldots, \Phi_N)$. We define $W_0^1 L_{\vec{\Phi}}(\Omega)$, the *anisotropic Orlicz–Sobolev space*, as the closure of $C_0^1(\Omega)$ with respect to the norm

$$\|u\|_{\vec{\Phi}} = \sum_{i=1}^{N} |\partial_i u|_{\Phi_i}.$$

It is natural to endow the space $W_0^1 L_{\vec{\Phi}}(\Omega)$ with the norm $\|\cdot\|_{\vec{\Phi}}$ since Proposition 1 below is valid. In the case when $\Phi_i(t) = |t|^{\theta_i}$, where θ_i are constants for any $i \in \{1, \ldots, N\}$ the resulting anisotropic Sobolev space is denoted by $W_0^{1,\vec{\theta}}(\Omega)$, where $\vec{\theta}$ is the constant vector $(\theta_1, \ldots, \theta_N)$. The theory of such spaces was developed in [22–27]. It was proved that $W_0^{1,\vec{\theta}}(\Omega)$ is a reflexive Banach space for any $\vec{\theta} \in \mathbb{R}^N$ with $\theta_i > 1$ for all $i \in \{1, \ldots, N\}$. This result can be easily extended to $W_0^1 L_{\vec{\Phi}}(\Omega)$. Indeed, denoting $X = L_{\Phi_1}(\Omega) \times \cdots \times L_{\Phi_N}(\Omega)$ and considering the operator $T : W_0^1 L_{\vec{\Phi}}(\Omega) \to X$, defined by $T(u) = \nabla u$, it is clear that $W_0^1 L_{\vec{\Phi}}(\Omega)$ and X are isometric by T, since $\|Tu\|_X = \sum_{i=1}^N |\partial_i u|_{\Phi_i} = \|u\|_{\vec{\Phi}}$. Thus, $T(W_0^1 L_{\vec{\Phi}}(\Omega))$ is a closed subspace of X, which is a reflexive Banach space. By [28, Proposition III.17], it follows that $T(W_0^1 L_{\vec{\Phi}}(\Omega))$ is reflexive, and consequently $W_0^1 L_{\vec{\Phi}}(\Omega)$ is also a reflexive Banach space.

On the other hand, in order to facilitate the manipulation of the space $W_0^1 L_{\vec{\Phi}}(\Omega)$, we introduce $\vec{P}^0, \vec{P}_0 \in \mathbb{R}^N$ as

$$\vec{P}^0 = ((p_1)^0, \ldots, (p_N)^0), \qquad \vec{P}_0 = ((p_1)_0, \ldots, (p_N)_0),$$

and $(P^0)_+, (P_0)_+, (P_0)_- \in \mathbb{R}^+$ as

$$(P^0)_+ = \max\{(p_1)^0, \ldots, (p_N)^0\}, \qquad (P_0)_+ = \max\{(p_1)_0, \ldots, (p_N)_0\}, \qquad (P_0)_- = \min\{(p_1)_0, \ldots, (p_N)_0\}.$$

Throughout this paper we assume that

$$\sum_{i=1}^{N} \frac{1}{(p_i)_0} > 1, \tag{5}$$

and define $P_0^\star \in \mathbb{R}^+$ and $P_{0,\infty} \in \mathbb{R}^+$ by

$$(P_0)^{\star} = \frac{N}{\sum_{i=1}^{N} 1/(p_i)_0 - 1}, \qquad P_{0,\infty} = \max\{(P_0)_+, (P_0)^{\star}\}.$$

Next, we recall some background facts concerning the variable exponent Lebesgue spaces. For more details, we refer to the book by Musielak [16] and the papers by Edmunds et al. [29–31], Kovacik and Rákosník [32], Mihăilescu and Rădulescu [33], and Samko and Vakulov [34].

Set

$$C_+(\overline{\Omega}) = \{h;\, h \in C(\overline{\Omega}),\, h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any $h \in C_+(\overline{\Omega})$, we define

$$h^+ = \sup_{x\in\Omega} h(x) \quad \text{and} \quad h^- = \inf_{x\in\Omega} h(x).$$

For any $q(x) \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space $L^{q(x)}(\Omega)$ (see [32]). On $L^{q(x)}(\Omega)$, we define the Luxemburg norm by the formula

$$|u|_{q(x)} = \inf\left\{\mu > 0;\, \int_\Omega \left|\frac{u(x)}{\mu}\right|^{q(x)} \mathrm{d}x \le 1\right\}.$$

We remember that $L^{q(x)}(\Omega)$ is a separable and reflexive Banach space. If $0 < |\Omega| < \infty$ and $q_1, q_2 \in C_+(\overline{\Omega})$ satisfy $q_1(x) \le q_2(x)$ almost everywhere in Ω, then there exists the continuous embedding $L^{q_2(x)}(\Omega) \hookrightarrow L^{q_1(x)}(\Omega)$.

Let $L^{p'(\cdot)}(\Omega)$ be the conjugate space of $L^{p(\cdot)}(\Omega)$, obtained by conjugating the exponent pointwise; i.e., $1/p(x) + 1/p'(x) = 1$, [32, Corollary 2.7]. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, the following Hölder-type inequality

$$\left|\int_\Omega uv\, \mathrm{d}x\right| \le \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) |u|_{p(\cdot)} |v|_{p'(\cdot)} \tag{6}$$

is valid.

If $(u_n), u \in L^{q(x)}(\Omega)$, then the following relations hold true:

$$|u|_{q(x)} > 1 \Rightarrow |u|_{q(x)}^{q^-} \le \int_\Omega |u|^{q(x)}\, \mathrm{d}x \le |u|_{q(x)}^{q^+} \tag{7}$$

$$|u|_{q(x)} < 1 \Rightarrow |u|_{q(x)}^{q^+} \le \int_\Omega |u|^{q(x)}\, \mathrm{d}x \le |u|_{q(x)}^{q^-} \tag{8}$$

$$|u_n - u|_{q(x)} \to 0 \Leftrightarrow \int_\Omega |u_n - u|^{q(x)}\, \mathrm{d}x \to 0. \tag{9}$$

2. Main results

In the following, for each $i \in \{1, \ldots, N\}$, we define $a_i : [0, \infty) \to \mathbb{R}$ by

$$a_i(t) = \begin{cases} \dfrac{\varphi_i(t)}{t}, & \text{for } t > 0 \\ 0, & \text{for } t = 0. \end{cases}$$

Since the φ_i are odd, we deduce that, actually, $\varphi_i(t) = a_i(|t|)t$ for each $t \in \mathbb{R}$ and each $i \in \{1, \ldots, N\}$.

We say that $\lambda \in \mathbb{R}$ is an *eigenvalue* of problem (1) if there exists $u \in W_0^1 L_{\vec{\Phi}}(\Omega) \setminus \{0\}$ such that

$$\int_\Omega \left\{\sum_{i=1}^{N} |a_i(|\partial_i u|)|\, \partial_i u \partial_i w - \lambda |u|^{q(x)-2} uw\right\} \mathrm{d}x = 0,$$

for all $w \in W_0^1 L_{\vec{\Phi}}(\Omega)$. For $\lambda \in \mathbb{R}$ an eigenvalue of problem (1), the function u from the above definition will be called a *weak solution* of problem (1) corresponding to the eigenvalue λ.

The main results of this paper are given by the following theorems.

Theorem 1. *Assume that the function $q \in C(\overline{\Omega})$ verifies the hypothesis*

$$(P^0)_+ < q^- \le q^+ < (P_0)^{\star}. \tag{10}$$

Then any $\lambda > 0$ is an eigenvalue of problem (1).

Theorem 2. *Assume that the function $q \in C(\overline{\Omega})$ satisfies the conditions*

$$1 < q^- < (P_0)_- \quad \text{and} \quad q^+ < P_{0,\infty}. \tag{11}$$

Then there exists $\lambda_\star > 0$ such that any $\lambda \in (0, \lambda_\star)$ is an eigenvalue of problem (1).

Theorem 3. *Assume that the function $q \in C(\overline{\Omega})$ satisfies the inequalities*

$$1 < q^- \le q^+ < (P_0)_-. \tag{12}$$

Then there exist two positive constants $\lambda_\star > 0$ and $\lambda^\star > 0$ such that any $\lambda \in (0, \lambda_\star) \cup (\lambda^\star, \infty)$ is an eigenvalue of problem (1).

Remark 2. By Theorem 3, it is not clear if $\lambda_\star < \lambda^\star$ or $\lambda_\star \ge \lambda^\star$. In the first case, an interesting question concerns the existence of eigenvalues of problem (1) in the interval $[\lambda_\star, \lambda^\star]$. We propose to the reader the study of these open problems.

In order to state the next result, we define

$$\lambda_1 = \inf_{u \in W_0^1 L_{\vec{\Phi}}(\Omega)\setminus\{0\}} \frac{\int_\Omega \sum_{i=1}^N \Phi_i(|\partial_i u|)\mathrm{d}x}{\int_\Omega \frac{1}{q(x)} |u|^{q(x)}\,\mathrm{d}x},$$

and

$$\lambda_0 = \inf_{u \in W_0^1 L_{\vec{\Phi}}(\Omega)\setminus\{0\}} \frac{\int_\Omega \sum_{i=1}^N a_i(|\partial_i u|)|\partial_i u|^2\,\mathrm{d}x}{\int_\Omega |u|^{q(x)}\,\mathrm{d}x}.$$

Theorem 4. *Assume that there exist $j_1, j_2, k \in \{1, \ldots, N\}$ such that*

$$(p_{j_1})_0 = q^- \quad \text{and} \quad (p_{j_2})^0 = q^+, \tag{13}$$

and

$$q^+ < \min\{(p_k)_0, (P_0)^\star\}. \tag{14}$$

Then $0 < \lambda_0 \le \lambda_1$, and every $\lambda \in (\lambda_1, \infty)$ is an eigenvalue of problem (1), while no $\lambda \in (0, \lambda_0)$ can be an eigenvalue of problem (1).

Remark 3. At this stage, we are not able to say whether $\lambda_0 = \lambda_1$ or $\lambda_0 < \lambda_1$. In the latter case, an interesting question concerns the existence of eigenvalues of problem (1) in the interval $[\lambda_0, \lambda_1]$. We propose to the reader the study of these open problems.

3. Variational setting and auxiliary results

From now on, E denotes the anisotropic Orlicz–Sobolev space $W_0^1 L_{\vec{\Phi}}(\Omega)$. Define the functionals $J, I, J_1, I_1 : E \to \mathbb{R}$ by

$$J(u) = \int_\Omega \sum_{i=1}^N \Phi_i(|\partial_i u|)\mathrm{d}x, \qquad I(u) = \int_\Omega \frac{1}{q(x)}|u|^{q(x)}\,\mathrm{d}x,$$

$$J_1(u) = \int_\Omega \sum_{i=1}^N a_i(|\partial_i u|)|\partial_i u|^2\,\mathrm{d}x, \qquad I_1(u) = \int_\Omega |u|^{q(x)}\,\mathrm{d}x.$$

Standard arguments imply that $J, I \in C^1(E, \mathbb{R})$ and their Fréchet derivatives are given by

$$\langle J'(u), v\rangle = \int_\Omega \sum_{i=1}^N a_i(|\partial_i u|)\partial_i u \partial_i v\,\mathrm{d}x,$$

$$\langle I'(u), v\rangle = \int_\Omega |u|^{q(x)-2} u v \mathrm{d}x,$$

for all $u, v \in E$.

Next, for each $\lambda \in \mathbb{R}$, we define the energetic functional associated with problem (1), $T_\lambda : E \to \mathbb{R}$, by

$$T_\lambda(u) = J(u) - \lambda I(u).$$

Clearly, $T_\lambda \in C^1(E, \mathbb{R})$ with

$$\langle T_\lambda'(u), v\rangle = \langle J'(u), v\rangle - \lambda\langle I'(u), v\rangle,$$

for all $u, v \in E$. Thus, λ is an eigenvalue of problem (1) if and only if there exists $u \in E \setminus \{0\}$ a critical point of T_λ. In other words, the main idea in proving Theorems 1–4 will be to look for nontrivial critical points of functional T_λ.

In order to do that, we begin by proving certain auxiliary results which will facilitate the proof of the main result.

Lemma 1. *Assume that $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with a smooth boundary. Assume that relation (5) is fulfilled. For any $q \in C(\overline{\Omega})$ verifying*

$$1 < q(x) < P_{0,\infty} \quad \text{for all } x \in \overline{\Omega}, \tag{15}$$

the embedding

$$W_0^1 L_{\vec{\Phi}}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$$

is compact.

Proof. First, we point out that $L_{\Phi_i}(\Omega)$ is continuously embedded in $L^{(p_i)_0}(\Omega)$ for any $i \in \{1, \ldots, N\}$. Indeed, by [15, Lemma 8.12(b)] it is enough to show that Φ_i dominates $\Psi_i := |t|^{(p_i)_0}$ near infinity; i.e., there exists $k > 0$ and $t_0 > 0$ such that

$$\Psi_i(t) \; (=|t|^{(p_i)_0}) \leq \Phi_i(k \cdot t), \quad \forall\, t \geq t_0.$$

That is a simple consequence of the definitions of $(p_i)_0$ combined with relation (15) (see, e.g., the proof of [10, Lemma 2] for more details).

Thus, for each $i \in \{1, \ldots, N\}$ there exists a positive constant $C_i > 0$ such that

$$|\varphi|_{(p_i)_0} \leq C_i \|\varphi\|_{\Phi_i} \quad \text{for all } \varphi \in L^{\Phi_i}(\Omega).$$

If $u \in W_0^1 L_{\vec{\Phi}}(\Omega)$ then $\partial_i u \in L_{\Phi_i}(\Omega)$ for each $i \in \{1, \ldots, N\}$. The above inequalities imply that

$$\|u\|_{\vec{P}_0} = \sum_{i=1}^N \left|\partial_{x_i} u\right|_{(p_i)_0} \leq C \sum_{i=1}^N \|\partial_i u\|_{\Phi_i} = C\|u\|_{\vec{P}_0},$$

where $C = \max\{C_1, \ldots, C_N\}$. Thus, we deduce that $W_0^1 L_{\vec{\Phi}}(\Omega)$ is continuously embedded in $W_0^1 L_{\vec{P}_0}(\Omega) = W_0^{1,\vec{P}_0}(\Omega)$. On the other hand, since relation (15) holds true, we infer that $q^+ < P_{0,\infty}$. This fact, combined with the result of [22, Theorem 1], implies that $W_0^{1,\vec{P}_0}(\Omega)$ is compactly embedded in $L^{q^+}(\Omega)$. Finally, since $q(x) \leq q^+$ for each $x \in \overline{\Omega}$, we deduce that $L^{q^+}(\Omega)$ is continuously embedded in $L^{q(\cdot)}(\Omega)$. The above piece of information yields the conclusion that $W_0^1 L_{\vec{\Phi}}(\Omega)$ is compactly embedded in $L^{q(\cdot)}(\Omega)$. The proof of Lemma 1 is complete. □

Lemma 2. *Assume that the hypothesis of Theorem 1 is fulfilled. Then there exist $\eta > 0$ and $\alpha > 0$ such that $T_\lambda(u) \geq \alpha > 0$ for any $u \in E$ with $\|u\|_{\vec{\Phi}} = \eta$.*

Proof. First, we point out that

$$|u(x)|^{q^-} + |u(x)|^{q^+} \geq |u(x)|^{q(x)} \quad \text{for all } x \in \overline{\Omega}. \tag{16}$$

Using the above inequality and the definition of T_λ, we find that

$$T_\lambda(u) \geq \sum_{i=1}^N \int_\Omega \Phi_i(|\partial_i u|)\mathrm{d}x - \frac{\lambda}{q^-}\left(|u|_{q^-}^{q^-} + |u|_{q^+}^{q^+}\right), \tag{17}$$

for any $u \in E$.

Since (10) holds, then by Lemma 1 it follows that E is continuously embedded both in $L^{q^-}(\Omega)$ and in $L^{q^+}(\Omega)$. We deduce there exist two positive constants B_1 and B_2 such that

$$B_1\|u\|_{\vec{\Phi}} \geq |u|_{q^+}, \qquad B_2\|u\|_{\vec{\Phi}} \geq |u|_{q^-} \quad \text{for all } u \in E. \tag{18}$$

Next, we focus our attention on the case when $u \in E$ and $\|u\|_{\vec{\Phi}} < 1$. For such an element u, we have $\|\partial_i u\|_{\Phi_i} < 1$ and, by a relation similar to the third inequality in [10, Lemma 1], we obtain

$$\frac{\|u\|_{\vec{\Phi}}^{(P^0)_+}}{N^{(P^0)_+-1}} = N\left(\sum_{i=1}^N \frac{1}{N}\|\partial_i u\|_{\Phi_i}\right)^{(P^0)_+} \leq \sum_{i=1}^N \|\partial_i u\|_{\Phi_i}^{(P^0)_+} \leq \sum_{i=1}^N \|\partial_i u\|_{\Phi_i}^{(p_i)^0} \leq \sum_{i=1}^N \int_\Omega \Phi_i\left(|\partial_i u|\right)\mathrm{d}x. \tag{19}$$

Relations (17)–(19) imply that

$$\begin{aligned}T_\lambda(u) &\geq \frac{\|u\|_{\vec{\Phi}}^{(P^0)_+}}{N^{(P^0)_+-1}} - \frac{\lambda}{q^-}\left[\left(B_1\|u\|_{\vec{\Phi}}\right)^{q^+} + \left(B_2\|u\|_{\vec{\Phi}}\right)^{q^-}\right] \\ &= \left(B_3 - B_4\|u\|_{\vec{\Phi}}^{q^+-(P^0)_+} - B_5\|u\|_{\vec{\Phi}}^{q^--(P^0)_+}\right)\|u\|_{\vec{\Phi}}^{(P^0)_+},\end{aligned}$$

for any $u \in E$ with $\|u\|_{\vec{\Phi}} < 1$, where B_3, B_4 and B_5 are positive constants.

Since the function $g : [0, 1] \to \mathbb{R}$ defined by

$$g(t) = B_3 - B_4 t^{q^+-(P^0)_+} - B_5 t^{q^--(P^0)_+}$$

is positive in a neighborhood of the origin, the conclusion of the lemma follows at once. $\square$

Lemma 3. *Assume that the hypothesis of Theorem 1 is fulfilled. Then there exists $e \in E$ with $\|e\|_{\vec{\Phi}} > \eta$ (where η is given in Lemma 2) such that $J_\lambda(e) < 0$.*

Proof. Let $\psi \in C_0^\infty(\Omega)$, $\psi \geq 0$ and $\psi \not\equiv 0$, be fixed and let $t > 1$. Using relation (11) in [10], we find that

$$\begin{aligned}T_\lambda(t\psi) &= \int_\Omega \left\{\sum_{i=1}^N \Phi_i(t|\partial_i\psi|) - \lambda\frac{t^{q(x)}}{q(x)}|\psi|^{q(x)}\right\}dx \\ &\leq \int_\Omega \left\{\sum_{i=1}^N t^{(p_i)^0}\Phi_i(|\partial_i\psi|) - \lambda\frac{t^{q(x)}}{q(x)}|\psi|^{q(x)}\right\}dx \\ &\leq t^{(P^0)_+}\sum_{i=1}^N \int_\Omega \Phi_i(|\partial_i\psi|)\,dx - \frac{\lambda t^{q^-}}{q^+}\int_\Omega |\psi|^{q(x)}\,dx.\end{aligned}$$

Since $q^- > (P^0)_+$ by (10), it is clear that $\lim_{t\to\infty} T_\lambda(t\psi) = -\infty$. Then, for $t > 1$ large enough, we can take $e = t\psi$ such that $\|e\|_{\vec{\Phi}} > \eta$ and $T_\lambda(e) < 0$. This completes the proof. $\square$

Lemma 4. *Assume that the hypotheses of Theorem 2 are fulfilled. Then there exists $\lambda_\star > 0$ such that for any $\lambda \in (0, \lambda_\star)$ there are $\rho, a > 0$ such that $T_\lambda(u) \geq a > 0$ for any $u \in E$ with $\|u\|_{\vec{\Phi}} = \rho$.*

Proof. Since (11) holds, by Lemma 1 it follows that E is continuously embedded in $L^{q(\cdot)}(\Omega)$. Thus, there exists a positive constant c_1 such that

$$|u|_{q(\cdot)} \leq c_1\|u\|_{\vec{\Phi}} \quad \text{for all } u \in E. \tag{20}$$

We fix $\rho \in (0, 1)$ such that $\rho < 1/c_1$. Then, relation (20) implies that

$$|u|_{q(\cdot)} < 1 \quad \text{for all } u \in E,\ \text{with } \|u\|_{\vec{\Phi}} = \rho.$$

Furthermore, relation (8) yields

$$\int_\Omega |u|^{q(x)}\,dx \leq |u|_{q(\cdot)}^{q^-} \quad \text{for all } u \in E,\ \text{with } \|u\|_{\vec{\Phi}} = \rho. \tag{21}$$

Relations (20) and (21) imply that

$$\int_\Omega |u|^{q(x)}\,dx \leq c_1^{q^-}\|u\|_{\vec{\Phi}}^{q^-} \quad \text{for all } u \in E,\ \text{with } \|u\|_{\vec{\Phi}} = \rho. \tag{22}$$

Taking into account relations (19) and (22), we deduce that for any $u \in E$ with $\|u\|_{\vec{\Phi}} = \rho$ the following inequalities hold true:

$$\begin{aligned}T_\lambda(u) &\geq \frac{1}{N^{(P^0)_+-1}}\|u\|_{\vec{\Phi}}^{(P^0)_+} - \frac{\lambda}{q^-}\int_\Omega |u|^{q(x)}\,dx \\ &\geq \frac{1}{N^{(P^0)_+-1}}\|u\|_{\vec{\Phi}}^{(P^0)_+} - \frac{\lambda c_1^{q^-}}{q^-}\|u\|_{\vec{\Phi}}^{q^-} \\ &= \rho^{q^-}\left(\frac{1}{N^{(P^0)_+-1}}\rho^{(P^0)_+-q^-} - \frac{\lambda c_1^{q^-}}{q^-}\right).\end{aligned}$$

Hence, if we define

$$\lambda_\star = \frac{q^-}{2c_1^{q^-} N^{(P^0)_+-1}} \rho^{(P^0)_+-q^-}, \tag{23}$$

then for any $\lambda \in (0, \lambda_\star)$ and $u \in E$ with $\|u\|_{\vec{\Phi}} = \rho$ the number $a = \rho^{(P^0)_+}/2N^{(P^0)_+-1}$ is such that

$$T_\lambda(u) \geq a > 0.$$

This completes the proof. $\square$

Lemma 5. *Assume that the hypothesis of Theorem 2 is fulfilled. Then there exists $\theta \in E$ such that $\theta \geq 0$, $\theta \not\equiv 0$ and $T_\lambda(t\theta) < 0$ for $t > 0$ small enough.*

Proof. Assumption (11) implies that $q^- < (P_0)_-$. Let $\epsilon_0 > 0$ be such that $q^- + \epsilon_0 < (P_0)_-$. On the other hand, since $q \in C(\overline{\Omega})$, it follows that there exists an open set $\Omega_2 \subset \Omega$ such that $|q(x) - q^-| < \epsilon_0$ for all $x \in \Omega_2$. Thus, we conclude that $q(x) \leq q^- + \epsilon_0 < (P_0)_-$ for all $x \in \Omega_2$.

Let $\theta \in C_0^\infty(\Omega)$ be such that $\text{supp}(\theta) \supset \overline{\Omega}_2$, $\theta(x) = 1$ for all $x \in \overline{\Omega}_2$ and $0 \leq \theta \leq 1$ in Ω. Then, using the above information and the definition of $(p_i)_0$, for any $t \in (0, 1)$, we have

$$\begin{aligned}
T_\lambda(t\theta) &= \int_\Omega \left\{ \sum_{i=1}^N \Phi_i\left(t\,|\partial_i\theta|\right) - \lambda \frac{t^{q(x)}}{q(x)} |\theta|^{q(x)} \right\} \mathrm{d}x \\
&\leq \sum_{i=1}^N t^{(p_i)_0} \int_\Omega \Phi_i\left(|\partial_i\theta|\right) \mathrm{d}x - \frac{\lambda}{q^+} \int_\Omega t^{q(x)} |\theta|^{q(x)}\, \mathrm{d}x \\
&\leq t^{(P_0)_-} \sum_{i=1}^N \int_\Omega \Phi_i\left(|\partial_i\theta|\right) \mathrm{d}x - \frac{\lambda}{q^+} \int_{\Omega_2} t^{q(x)} |\theta|^{q(x)}\, \mathrm{d}x \\
&\leq t^{(P_0)_-} \sum_{i=1}^N \int_\Omega \Phi_i\left(|\partial_i\theta|\right) \mathrm{d}x - \frac{\lambda\, t^{q^-+\epsilon_0}}{q^+} \int_{\Omega_2} |\theta|^{q(x)}\, \mathrm{d}x.
\end{aligned}$$

Therefore,

$$T_\lambda(t\theta) < 0,$$

for $t < \delta^{1/((P_0)_- - q^- - \epsilon_0)}$ with

$$0 < \delta < \min\left\{1, \frac{\lambda}{q^+} \int_{\Omega_2} |\theta|^{q(x)} \mathrm{d}x \Big/ \sum_{i=1}^N \int_\Omega \Phi_i\left(|\partial_i\theta|\right) \mathrm{d}x \right\}.$$

This is possible since we claim that $\sum_{i=1}^N \int_\Omega \Phi_i(|\partial_i\theta|)\mathrm{d}x > 0$. Indeed, it is clear that

$$\int_{\Omega_2} |\theta|^{q(x)}\, \mathrm{d}x \leq \int_\Omega |\theta|^{q(x)}\, \mathrm{d}x \leq \int_\Omega |\theta|^{q^-}\, \mathrm{d}x.$$

On the other hand, E is continuously embedded in $L^{q^-}(\Omega)$, and thus there exists a positive constant c_2 such that

$$|\theta|_{q^-} \leq c_2 \|\theta\|_{\vec{\Phi}}.$$

The last two inequalities imply that

$$\|\theta\|_{\vec{p}(\cdot)} > 0,$$

and combining this fact with relation (7) or relation (8), the claim follows at once. The proof of the lemma is now completed. $\square$

Lemma 6. *Assume that the hypotheses of Theorem 3 are fulfilled. Then the functional T_λ is coercive on E.*

Proof. By relations (17) and (18) we deduce that, for all $u \in E$,

$$T_\lambda(u) \geq \sum_{i=1}^N \int_\Omega \Phi_i\left(|\partial_i u|\right) \mathrm{d}x - \frac{\lambda}{q^-} \left[\left(B_1 \|u\|_{\vec{\Phi}}\right)^{q^+} + \left(B_2 \|u\|_{\vec{\Phi}}\right)^{q^-} \right]. \tag{24}$$

Now, we focus our attention on the elements $u \in E$ with $\|u\|_{\vec{\Phi}} > 1$. Using the same techniques as in the proof of (27) combined with relation (24), we find that

$$T_\lambda(u) \geq \frac{1}{N^{(P_0)_- - 1}} \|u\|_{\vec{\Phi}}^{(P_0)_-} - N - \frac{\lambda}{q^-}\left[\left(B_1\|u\|_{\vec{\Phi}}\right)^{q^+} + \left(B_2\|u\|_{\vec{\Phi}}\right)^{q^-}\right],$$

for any $u \in E$ with $\|u\|_{\vec{\Phi}} > 1$. Since by relation (12) we have $(P_0)_- > q^+ \geq q^-$, we infer that $T_\lambda(u) \to \infty$ as $\|u\|_{\vec{\Phi}} \to \infty$. In other words, T_λ is coercive in E, completing the proof. □

Lemma 7. *Assume that condition (13) in Theorem 4 is fulfilled. Then there exists a positive constant $D > 0$ such that*

$$\int_\Omega |u|^{q(x)}\,dx \leq D\left(\int_\Omega \Phi_{j_1}(|\partial_{j_1}u|) + \int_\Omega \Phi_{j_2}(|\partial_{j_2}u|)\right), \quad \forall\, u \in C_0^1(\Omega).$$

Proof. First, we point out that for any $x \in \Omega$ the following inequality holds true:

$$|u(x)|^{q(x)} \leq |u(x)|^{q^-} + |u(x)|^{q^-}, \quad \forall\, u \in C_0^1(\Omega).$$

Integrating the above inequality with respect to x over Ω, we get

$$\int_\Omega |u|^{q(x)}\,dx \leq \int_\Omega |u|^{q^+}\,dx + \int_\Omega |u|^{q^-}\,dx, \quad \forall\, u \in C_0^1(\Omega).$$

Combining the above inequality with inequality (11) in [22], we deduce that there exists a positive constant $C_1 > 0$ such that

$$\int_\Omega |u|^{q(x)}\,dx \leq C_1\left(\int_\Omega |\partial_{j_1}u|^{q^+}\,dx + \int_\Omega |\partial_{j_2}u|^{q^-}\,dx\right), \quad \forall\, u \in C_0^1(\Omega).$$

On the other hand, by a variant of [10, Lemma 3], we infer that there exists a positive constant $C_2 > 0$ such that

$$\int_\Omega \left(|\partial_{j_1}u|^{(p_{j_1})_0} + |\partial_{j_2}u|^{(p_{j_2})^0}\right)dx \leq C_2\int_\Omega \left(\Phi_{j_1}(|\partial_{j_1}u|) + \Phi_{j_2}(|\partial_{j_2}u|)\right)dx, \quad \forall\, u \in C_0^1(\Omega).$$

Combining the last two inequalities. we obtain the conclusion of the lemma. □

Lemma 8. *Let $\lambda > 0$ be fixed. Assume that the hypotheses of Theorem 4 are fulfilled. The following relation holds true:*

$$\lim_{\|u\|_{\vec{\Phi}}\to\infty} T_\lambda(u) = \infty.$$

Proof. First, we show that

$$\lim_{\|u\|_{\vec{\Phi}}\to\infty} \frac{J(u)}{I(u)} = \infty.$$

Assume by contradiction that the above relation does not hold true. Then there exists an $M > 0$ such that for each $n \in \mathbb{N}^\star$ there exists a $u_n \in E$ with $\|u_n\|_{\vec{\Phi}} > n$ and

$$\frac{J(u_n)}{I(u_n)} \leq M. \tag{25}$$

While $\|u_n\|_{\vec{\Phi}} = \sum_{i=1}^N \|\partial_i u_n\|_{\Phi_i} \to \infty$ as $n \to \infty$, the sequence $\{\|\partial_k u_n\|_{\Phi_k}\}_n$ (with k given in inequality (14)) is either bounded or unbounded.

On the other hand, it is not difficult to see that

$$\int_\Omega |u|^{q(x)} \leq \int_\Omega |u|^{q^-}\,dx + \int_\Omega |u|^{q^+}\,dx, \quad \forall\, u \in E.$$

Next, using relation (11) in [22], we find that there exists a positive constant c_1 such that

$$\int_\Omega |u|^{q^-}\,dx + \int_\Omega |u|^{q^+}\,dx \leq c_1\left(\int_\Omega |\partial_k u|^{q^-}\,dx + \int_\Omega |\partial_k u|^{q^+}\,dx\right), \quad \forall\, u \in E.$$

Since by inequality (14) we have $q^+ < (p_k)_0$, a similar proof to that of [10, Lemma 2] shows that $L_{\Phi_k}(\Omega)$ is continuously embedded in $L^{q^\pm}(\Omega)$. The above pieces of information lead to the existence of a positive constant c_2 such that

$$\int_\Omega |u|^{q(x)} \leq c_2[\|\partial_k u\|_{\Phi_k}^{q^+} + \|\partial_k u\|_{\Phi_k}^{q^-}], \quad \forall\, u \in E. \tag{26}$$

If $\{\|\partial_k u_n\|_{\Phi_k}\}_n$ is bounded, then by inequality (26) we have that $\{I(u_n)\}_n$ is also bounded. On the other hand, denoting

$$\alpha_{i,n} = \begin{cases} (P^0)_+, & \text{if } \|\partial_i u_n\|_{\Phi_i} < 1 \\ (P_0)_-, & \text{if } \|\partial_i u_n\|_{\Phi_i} > 1, \end{cases}$$

and using inequalities (C.9) and (C.10) in [19] (see also [10, Lemma 1]), we find that

$$\begin{aligned} J(u_n) &= \int_\Omega \sum_{i=1}^N \Phi_i(|\partial_i u_n|)\mathrm{d}x \\ &\geq \sum_{i=1}^N \|\partial_i u_n\|_{\Phi_i}^{\alpha_{i,n}} \\ &\geq \sum_{i=1}^N \|\partial_i u_n\|_{\Phi_i}^{(P_0)_-} - \sum_{\{i;\alpha_{i,n}=(P^0)_+\}} \left(\|\partial_i u_n\|_{\Phi_i}^{(P_0)_-} - \|\partial_i u_n\|_{\Phi_i}^{(P^0)_+}\right) \\ &\geq \frac{1}{N^{(P_0)_- -1}}\|u_n\|_{\vec{\Phi}}^{(P_0)_-} - N. \end{aligned} \tag{27}$$

Consequently, in this case we obtain that $\lim_{n\to\infty} \frac{J(u_n)}{I(u_n)} = \infty$, which contradicts (25).

Now, we assume that $\|\partial_k u_n\|_{\Phi_k} \to \infty$, as $n \to \infty$, on a subsequence of u_n denoted again u_n. We can assume that $\|\partial_k u_n\|_{\Phi_k} > 1$ for all n. Using inequality (C.10) in [19] and relation (26), we find that

$$\frac{J(u_n)}{I(u_n)} \geq \frac{c_5 \int_\Omega \Phi_k(|\partial_k u_n|)\mathrm{d}x}{c_2[\|\partial_k u_n\|_{\Phi_k}^{q^+} + \|\partial_k u_n\|_{\Phi_k}^{q^-}]} \geq \frac{c_5\|\partial_k u_n\|_{\Phi_k}^{(p_k)_0}}{c_2[\|\partial_k u_n\|_{\Phi_k}^{q^+} + \|\partial_k u_n\|_{\Phi_k}^{q^-}]} \quad \forall\, u \in E,\ n \in \mathbb{N}^\star,$$

where c_5 is a positive constant. Since by the hypothesis of Theorem 4 we have $(p_k)_0 > q^+$, the above inequalities show that $J(u_n)/I(u_n) \to \infty$, as $n \to \infty$, which again contradicts (25).

Next, we turn back to the proof of the relation given in Lemma 8. Assume by contradiction that the conclusion of Lemma 8 is not valid. Then there exists an $M_1 > 0$ such that for each $n \in \mathbb{N}^\star$ there exists a $v_n \in E$ with $\|v_n\|_{\vec{\Phi}} > n$ and

$$|T_\lambda(v_n)| = |J(v_n) - \lambda I(v_n)| \leq M_1.$$

Thus, it is clear that $\|v_n\|_{\vec{\Phi}} \to \infty$ as $n \to \infty$, and since we proved that

$$J(v_n) \geq \frac{1}{N^{(P_0)_- -1}}\|v_n\|_{\vec{\Phi}}^{(P_0)_-} - N,$$

it follows that $J(v_n) \to \infty$ as $n \to \infty$. Thus, we find that for each n large enough we have

$$\left|1 - \frac{I(v_n)}{J(v_n)}\right| \leq \frac{M_1}{J(v_n)}.$$

Then, passing to the limit as $n \to \infty$ in the above inequality and taking into account the facts that $J(v_n)/I(v_n) \to \infty$ (or, equivalently $I(v_n)/J(v_n) \to 0$) and $J(v_n) \to \infty$ as $n \to \infty$, we obtain a contradiction. Therefore, the conclusion of Lemma 8 is valid. $\square$

To end this section we prove the following proposition:

Proposition 1. *For each $i \in \{1, \ldots, N\}$ the norms $\|\cdot\|_i$ and $\|\cdot\|_{i,1}$ are equivalent.*

Proof. We fix $i \in \{1, \ldots, N\}$. First, we introduce a third norm on E, namely,

$$\|u\|_{i,2} = \max_{j\in\{1,\ldots,N\}} \{\|\partial_j u\|_{\Phi_i}\}.$$

Undoubtedly, we have

$$\|u\|_{i,2} \leq \|u\|_{i,1} \leq N\|u\|_{i,2}, \quad \forall\, u \in E.$$

Thus, the norms $\|\cdot\|_{i,1}$ and $\|\cdot\|_{i,2}$ are equivalent.

Next, we show that

$$\|u\|_i \leq N^{1/2}\|u\|_{i,2}, \quad \forall\, u \in E.$$

Indeed, since Φ_i satisfies condition (4), we have

$$\int_\Omega \Phi_i\left(\frac{|\nabla u(x)|}{N^{1/2}\|u\|_{i,2}}\right)\mathrm{d}x = \int_\Omega \Phi_i\left(\sqrt{\frac{\sum_{j=1}^N |\partial_j u|^2/\|u\|_{i,2}^2}{N}}\right)\mathrm{d}x \le \sum_{j=1}^N \frac{1}{N}\int_\Omega \Phi_i\left(\frac{|\partial_j u(x)|}{\|u\|_{i,2}}\right)\mathrm{d}x.$$

Next, by the definition of $\|\cdot\|_{\Phi_i}$ and $\|\cdot\|_{i,2}$ and the fact that Φ_i is an increasing function, we deduce that

$$\int_\Omega \Phi_i\left(\frac{|\partial_j u(x)|}{\|u\|_{i,2}}\right)\mathrm{d}x \le \int_\Omega \Phi_i\left(\frac{|\partial_j u(x)|}{\|\partial_j u\|_{\Phi_i}}\right)\mathrm{d}x \le 1, \quad \forall\, j\in\{1,\dots,N\}.$$

The last two inequalities imply that

$$\int_\Omega \Phi_i\left(\frac{|\nabla u(x)|}{N^{1/2}\|u\|_{i,2}}\right) \le 1;$$

i.e., $\|u\|_i \le N^{1/2}\|u\|_{i,2}$ for all $u\in E$.

Finally, we verify that

$$\|u\|_{i,1} \le N^2\|u\|_i, \quad \forall\, u\in E.$$

In order to prove that, first, we remember that using [19, Lemma C.4(ii)] we find that

$$N\Phi_i(t) \le \Phi_i(Nt), \quad \forall\, t\ge 0. \tag{28}$$

Using the fact that Φ_i is increasing, we deduce that

$$\int_\Omega \sum_{j=1}^N \Phi_i\left(\frac{|\partial_j u(x)|}{\|u\|_i}\right)\mathrm{d}x \le N\int_\Omega \Phi_i\left(\frac{|\nabla u(x)|}{\|u\|_i}\right)\mathrm{d}x \le N, \quad \forall\, j\in\{1,\dots,N\}.$$

Next, using the above inequality and (28), we obtain

$$\int_\Omega \sum_{j=1}^N \Phi_i\left(\frac{|\partial_j u(x)|}{N\|u\|_i}\right)\mathrm{d}x \le \frac{1}{N}\int_\Omega \sum_{j=1}^N \Phi_i\left(\frac{|\partial_j u(x)|}{\|u\|_i}\right)\mathrm{d}x \le 1, \quad \forall\, j\in\{1,\dots,N\}.$$

Thus, we have found that

$$\|\partial_j u\|_{\Phi_i} \le N\|u\|_i, \quad \forall\, j\in\{1,\dots,N\}.$$

Summing from $i=1$ to N, we get that $\|u\|_{i,1}\le N^2\|u\|_i$ for all $u\in E$.

The conclusion of the proposition is now clear. □

4. Proof of Theorem 1

By Lemmas 2 and 3 and the mountain pass theorem of Ambrosetti and Rabinowitz [35], we deduce the existence of a sequence $(u_n)\subset E$ such that

$$T_\lambda(u_n)\to \bar{c}>0 \quad \text{and} \quad T'_\lambda(u_n)\to 0\ (\text{in } E^\star) \quad \text{as } n\to\infty. \tag{29}$$

We prove that (u_n) is bounded in E. In order to do that, we assume by contradiction that passing eventually to a subsequence, still denoted by (u_n), we have $\|u_n\|_{\vec{\Phi}}\to\infty$ and that $\|u_n\|_{\vec{\Phi}}>1$ for all n.

Relation (29) and the above considerations imply that for n large enough we have

$$\begin{aligned}
1+\bar{c}+\|u_n\|_{\vec{\Phi}} &\ge T_\lambda(u_n) - \frac{1}{q^-}\langle T'_\lambda(u_n), u_n\rangle \\
&\ge \sum_{i=1}^N \int_\Omega \left(\Phi_i(|\partial_i u_n|) - \frac{1}{q^-}\varphi_i(|\partial_i u_n|)|\partial_i u_n|\right)\mathrm{d}x \\
&\ge \left(1-\frac{(P^0)_+}{q^-}\right)\sum_{i=1}^N \int_\Omega \Phi_i(|\partial_i u_n|)\mathrm{d}x.
\end{aligned}$$

Using similar arguments as in the proof of relation (27), we obtain

$$\begin{aligned} 1+\overline{c}+\|u_n\|_{\vec{\Phi}} &\geq \left(1-\frac{(P^0)_+}{q^-}\right)\sum_{i=1}^{N}\int_{\Omega}\Phi_i(|\partial_i u_n|)\mathrm{d}x \\ &\geq \left(1-\frac{(P^0)_+}{q^-}\right)\left(\frac{1}{N^{(P_0)_- -1}}\|u_n\|_{\vec{\Phi}}^{(P_0)_-}-N\right). \end{aligned} \tag{30}$$

Dividing by $\|u_n\|_{\vec{\Phi}}^{(P_0)_-}$ in the above inequality and passing to the limit as $n\to\infty$, we obtain a contradiction. It follows that (u_n) is bounded in E. This information, combined with the fact that E is reflexive, implies that there exist a subsequence, still denoted by (u_n), and $u_0\in E$ such that (u_n) converges weakly to u_0 in E. Since, by Lemma 1, the space E is compactly embedded in $L^{q(\cdot)}(\Omega)$, it follows that (u_n) converges strongly to u_0 in $L^{q(\cdot)}(\Omega)$. Then, by inequality (6), we deduce that

$$\lim_{n\to\infty}\int_{\Omega}|u_n|^{q(x)-2}u_n(u_n-u_0)\mathrm{d}x=0.$$

This fact and relation (29) yield

$$\lim_{n\to\infty}\langle T_\lambda'(u_n),u_n-u_0\rangle=0.$$

Thus, we deduce that

$$\lim_{n\to\infty}\sum_{i=1}^{N}\int_{\Omega}a_i\left(|\partial_i u_n|\right)\partial_i u_n\left(\partial_i u_n-\partial_i u_0\right)\mathrm{d}x=0. \tag{31}$$

Since the (u_n) converge weakly to u_0 in E, by relation (31), we find that

$$\lim_{n\to\infty}\sum_{i=1}^{N}\int_{\Omega}\left(a_i\left(|\partial_i u_n|\right)\partial_i u_n-a_i\left(|\partial_i u_0|\right)\partial_i u_0\right)\left(\partial_i u_n-\partial_i u_0\right)\mathrm{d}x=0. \tag{32}$$

Since, for each $i\in\{1,\dots,N\}$, Φ_i is convex, we have

$$\Phi_i(|\partial_i u(x)|)\leq\Phi_i\left(\left|\frac{\partial_i u(x)+\partial_i v(x)}{2}\right|\right)+a_i(|\partial_i u(x)|)\partial_i u(x)\cdot\frac{\partial_i u(x)-\partial_i v(x)}{2},$$

and

$$\Phi_i(|\partial_i v(x)|)\leq\Phi_i\left(\left|\frac{\partial_i u(x)+\partial_i v(x)}{2}\right|\right)+a_i(|\partial_i v(x)|)\partial_i v(x)\cdot\frac{\partial_i v(x)-\partial_i u(x)}{2},$$

for every $u,v\in E$, $x\in\Omega$ and $i\in\{1,\dots,N\}$. Adding the above two relations and integrating over Ω, we find that

$$\frac{1}{2}\int_{\Omega}(a_i(|\partial_i u|)\partial_i u-a_i(|\partial_i v|)\partial_i v)\cdot(\partial_i u-\partial_i v)\mathrm{d}x\geq\int_{\Omega}\Phi_i(|\partial_i u|)\mathrm{d}x+\int_{\Omega}\Phi_i(|\partial_i v|)\mathrm{d}x-2\int_{\Omega}\Phi_i\left(\left|\frac{\partial_i u+\partial_i v}{2}\right|\right)\mathrm{d}x, \tag{33}$$

for any $u,v\in E$ and each $i\in\{1,\dots,N\}$.

On the other hand, since for each $i\in\{1,\dots,N\}$ we know that $\Phi_i:[0,\infty)\to\mathbb{R}$ is an increasing, continuous function with $\Phi_i(0)=0$, and $t\mapsto\Phi_i(\sqrt{t})$ is convex, we deduce by Lamperti [36] that

$$\frac{1}{2}\left[\int_{\Omega}\Phi_i(|\partial_i u|)\mathrm{d}x+\int_{\Omega}\Phi_i(|\partial_i v|)\mathrm{d}x\right]\geq\int_{\Omega}\Phi_i\left(\left|\frac{\partial_i u+\partial_i v}{2}\right|\right)\mathrm{d}x+\int_{\Omega}\Phi_i\left(\left|\frac{\partial_i u-\partial_i v}{2}\right|\right)\mathrm{d}x, \tag{34}$$

for any $u,v\in E$ and each $i\in\{1,\dots,N\}$.

By (33) and (34), it follows that for each $i\in\{1,\dots,N\}$ we have

$$\int_{\Omega}(a_i(|\partial_i u|)\partial_i u-a_i(|\partial_i v|)\partial_i v)\cdot(\partial_i u-\partial_i v)\mathrm{d}x\geq 4\int_{\Omega}\Phi_i\left(\left|\frac{\partial_i u-\partial_i v}{2}\right|\right)\mathrm{d}x,\quad\forall\,u,v\in E. \tag{35}$$

Relations (32) and (35) show that actually (u_n) converges strongly to u_0 in E. Then, by relation (29), we have

$$T_\lambda(u_0)=\overline{c}>0\quad\text{and}\quad T_\lambda'(u_0)=0;$$

i.e., u_0 is a nontrivial weak solution of Eq. (1). $\square$

5. Proof of Theorem 2

Let $\lambda_\star > 0$ be defined as in (23) and $\lambda \in (0, \lambda_\star)$. By Lemma 4, it follows that on the boundary of the ball centered at the origin and of radius ρ in E, denoted by $B_\rho(0)$, we have

$$\inf_{\partial B_\rho(0)} T_\lambda > 0. \tag{36}$$

On the other hand, by Lemma 5, there exists $\theta \in E$ such that $T_\lambda(t\theta) < 0$ for all $t > 0$ small enough. Moreover, relations (19), (22) and (8) imply that for any $u \in B_\rho(0)$ we have

$$T_\lambda(u) \geq \frac{1}{N^{(P^0)_+-1}} \|u\|_{\vec{\Phi}}^{(P^0)_+} - \frac{\lambda c_1^{q^-}}{q^-} \|u\|_{\vec{\Phi}}^{q^-}.$$

It follows that

$$-\infty < \underline{c} := \inf_{\overline{B_\rho(0)}} T_\lambda < 0.$$

We let now $0 < \epsilon < \inf_{\partial B_\rho(0)} T_\lambda - \inf_{B_\rho(0)} T_\lambda$. Applying Ekeland's variational principle (see [37]) to the functional $T_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$, we find $u_\epsilon \in \overline{B_\rho(0)}$ such that

$$T_\lambda(u_\epsilon) < \inf_{\overline{B_\rho(0)}} T_\lambda + \epsilon$$

$$T_\lambda(u_\epsilon) < J_\lambda(u) + \epsilon \|u - u_\epsilon\|_{\vec{\Phi}}, \quad u \neq u_\epsilon.$$

Since

$$T_\lambda(u_\epsilon) \leq \inf_{\overline{B_\rho(0)}} T_\lambda + \epsilon \leq \inf_{B_\rho(0)} T_\lambda + \epsilon < \inf_{\partial B_\rho(0)} T_\lambda,$$

we deduce that $u_\epsilon \in B_\rho(0)$. Now, we define $H_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$ by $H_\lambda(u) = T_\lambda(u) + \epsilon \|u - u_\epsilon\|_{\vec{\Phi}}$. It is clear that u_ϵ is a minimum point of H_λ, and thus

$$\frac{H_\lambda(u_\epsilon + t\,v) - H_\lambda(u_\epsilon)}{t} \geq 0,$$

for small $t > 0$ and any $v \in B_1(0)$. The above relation yields

$$\frac{T_\lambda(u_\epsilon + t\,v) - T_\lambda(u_\epsilon)}{t} + \epsilon \|v\|_{\vec{\Phi}} \geq 0.$$

Letting $t \to 0$, it follows that $\langle T_\lambda'(u_\epsilon), v\rangle + \epsilon\|v\|_{\vec{\Phi}} > 0$, and we infer that $\|T_\lambda'(u_\epsilon)\| \leq \epsilon$.

We deduce that there exists a sequence $(w_n) \subset B_\rho(0)$ such that

$$T_\lambda(w_n) \to \underline{c} \quad \text{and} \quad T_\lambda'(w_n) \to 0. \tag{37}$$

It is clear that (w_n) is bounded in E. Thus, there exists $w \in E$ such that, up to a subsequence, (w_n) converges weakly to w in E. Actually, with similar arguments to those used at the end of Theorem 1 we can show that (w_n) converges strongly to w in E. Thus, by (37),

$$T_\lambda(w) = \underline{c} < 0 \quad \text{and} \quad T_\lambda'(w) = 0; \tag{38}$$

i.e., w is a nontrivial weak solution for problem (1). This completes the proof. $\square$

6. Proof of Theorem 3

The existence of a positive constant $\lambda_\star$ such that any $\lambda \in (0, \lambda_\star)$ is an eigenvalue of problem (1) is an immediate consequence of Theorem 2. In order to prove the second part of Theorem 3, we will show that for λ positive and large enough the functional T_λ possesses a nontrivial global minimum point in E.

Lemma 1 and some similar arguments as those used in the proof of [38, Theorem 2] show that T_λ is weakly lower semi-continuous. By Lemma 6, the functional T_λ is also coercive on E. These two facts enable us to apply [39, Theorem 1.2] in order to find that there exists $u_\lambda \in E$ a global minimizer of T_λ, and thus a weak solution of problem (1).

We show that u_λ is not trivial for λ large enough. Indeed, letting $t_0 > 1$ be a fixed real and Ω_1 be an open subset of Ω with $|\Omega_1| > 0$, we deduce that there exists $v_0 \in C_0^\infty(\Omega) \subset E$ such that $v_0(x) = t_0$ for any $x \in \overline{\Omega}_1$ and $0 \leq v_0(x) \leq t_0$ in $\Omega \setminus \Omega_1$. We have

$$\begin{aligned} T_\lambda(v_0) &= \int_\Omega \left\{ \sum_{i=1}^N \Phi_i(|\partial_i v_0|) - \frac{\lambda}{q(x)} |v_0|^{q(x)} \right\} dx \\ &\leq L - \frac{\lambda}{q^+} \int_{\Omega_1} |v_0|^{q(x)}\, dx \leq L - \frac{\lambda}{q^+} t_0^{q^-} |\Omega_1|, \end{aligned}$$

where L is a positive constant. Thus, there exists $\lambda^\star > 0$ such that $T_\lambda(u_0) < 0$ for any $\lambda \in [\lambda^\star, \infty)$. It follows that $T_\lambda(u_\lambda) < 0$ for any $\lambda \geq \lambda^\star$, and thus u_λ is a nontrivial weak solution of problem (1) for λ large enough. The proof of Theorem 3 is complete. □

7. Proof of Theorem 4

• First, we note that by Lemma 7 we can easily infer that

$$\lambda_1 = \inf_{u \in E \setminus \{0\}} \frac{J(u)}{I(u)} > 0.$$

On the other hand, by the definition of $(p_i)_0$, $i \in \{1, \ldots, N\}$, we have

$$a_i(t) \cdot t^2 = \varphi_i(t) \cdot t \geq (p_i)_0 \Phi_i(t), \quad \forall\, t > 0.$$

Combining that idea with the inequality given in Lemma 7, we conclude that

$$\lambda_0 = \inf_{u \in E \setminus \{0\}} \frac{J_1(u)}{I_1(u)} > 0.$$

• Second, we point out that no $\lambda \in (0, \lambda_0)$ can be an eigenvalue of problem (1).

Indeed, assuming by contradiction that there exists $\lambda \in (0, \lambda_0)$ an eigenvalue of problem (1), it follows that there exists a $w_\lambda \in E \setminus \{0\}$ such that

$$\langle J'(w_\lambda), v\rangle = \lambda \langle I'(w_\lambda), v\rangle, \quad \forall\, v \in E.$$

Thus, for $v = w_\lambda$ we find that

$$\langle J'(w_\lambda), w_\lambda\rangle = \lambda \langle I'(w_\lambda), w_\lambda\rangle;$$

i.e.,

$$J_1(w_\lambda) = \lambda I_1(w_\lambda).$$

The fact that $w_\lambda \in E \setminus \{0\}$ ensures that $I_1(w_\lambda) > 0$. Since $\lambda < \lambda_0$, the above information yields

$$J_1(w_\lambda) \geq \lambda_0 I_1(w_\lambda) > \lambda I_1(w_\lambda) = J_1(w_\lambda).$$

Clearly, the above inequalities lead to a contradiction. Consequently, no $\lambda \in (0, \lambda_0)$ can be an eigenvalue of problem (1).

• Third, we will prove that every $\lambda \in (\lambda_1, \infty)$ is an eigenvalue of problem (1).

Let $\lambda \in (\lambda_1, \infty)$ be arbitrary but fixed. By Lemma 8 we can obtain that T_λ is coercive; i.e., $\lim_{\|u\|_{\vec{\Phi}} \to \infty} T_\lambda(u) = \infty$. On the other hand, Lemma 1 and some similar arguments to those used in the proof of [38, Theorem 2] show that T_λ is weakly lower semi-continuous. These two facts enable us to apply [39, Theorem 1.2] in order to prove that there exists $u_\lambda \in E$ a global minimum point of T_λ, and thus a critical point of T_λ. In order to conclude that λ is an eigenvalue of problem (1), it is enough to show that u_λ is not trivial. Indeed, since $\lambda_1 = \inf_{u \in E \setminus \{0\}} \frac{J(u)}{I(u)}$ and $\lambda > \lambda_1$, it follows that there exists $v_\lambda \in E$ such that

$$J(v_\lambda) < \lambda I(v_\lambda),$$

or

$$T_\lambda(v_\lambda) < 0.$$

Thus,

$$\inf_E T_\lambda < 0,$$

and we conclude that u_λ is a nontrivial critical point of T_λ; i.e., λ is an eigenvalue of problem (1).

• Finally, we note that by the above arguments we can infer that $\lambda_0 \leq \lambda_1$.

The proof of Theorem 4 is complete. □

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