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Boualem Alleche, Vicențiu D. Rădulescu & Madjid Sebaoui

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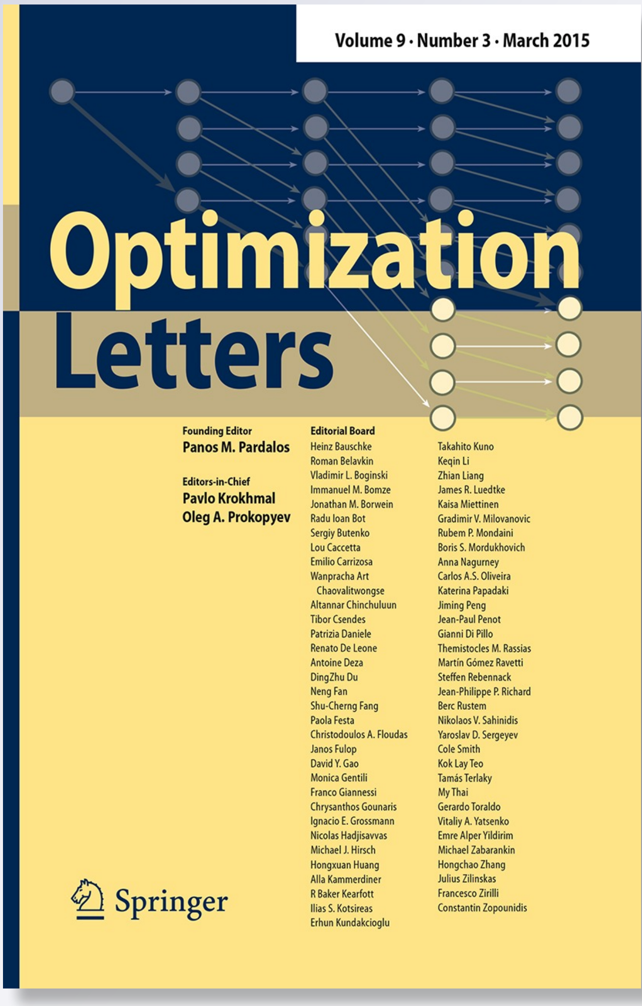

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ORIGINAL PAPER

The Tikhonov regularization for equilibrium problems and applications to quasi-hemivariational inequalities

Boualem Alleche · Vicenţiu D. Rădulescu · Madjid Sebaoui

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Abstract In this paper, we deal with the Tikhonov regularization method for pseudo-monotone equilibrium problems. Under mild conditions of semicontinuity and convexity, we show that strictly pseudo-monotone bifunctions can be also used as regularization bifunctions as well as strongly monotone bifunctions. We extend Berge's maximum theorem and establish the relationship between quasi-hemivariational inequalities and equilibrium problems. Applications of the Tikhonov regularization method to quasi-hemivariational inequalities are also given.

Keywords Equilibrium problem · Quasi-hemivariational inequality · Regularization method · Multivalued mapping · Semicontinuity · Pseudo-monotonicity

1 Introduction

Let C be a nonempty, closed and convex subset of a real Banach space E and let $\Phi : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying $\Phi(x, x) = 0$, for every $x \in C$. Such a bifunction Φ is called an *equilibrium bifunction*.

B. Alleche · M. Sebaoui
Laboratoire de Mécanique, Physique et Modélisation Mathématique,
Université de Médéa, 26000 Médéa, Algeria
e-mail: alleche.boualem@univ-medea.dz; alleche.boualem@gmail.com

M. Sebaoui
e-mail: msebaoui@gmail.com

V. D. Rădulescu (✉)
Department of Mathematics, Faculty of Science, King Abdulaziz University,
Jeddah, Saudi Arabia
e-mail: vicentiu.radulescu@math.cnrs.fr

Recall that an *equilibrium problem* in the sense of Blum, Muu and Oettli (see [9,27]) is a problem of the form:

$$\text{find } x^* \in C \text{ such that } \Phi(x^*, y) \geq 0 \quad \forall y \in C, \tag{EP}$$

where its set of solutions is denoted by $SEP(C, \Phi)$.

Equilibrium problems encompass several problems including variational inequalities, mathematical programming, Nash equilibrium, Kakutani fixed points, optimization and many other problems arising in nonlinear analysis. Recently, some practical models of interest in engineering and economics have been formulated as an equilibrium problem of the form (EP), see for example [18,20,22] and the references therein.

Equilibrium problems also encompass quasi-hemivariational inequalities. Recall that if E is a real Banach space which is continuously embedded in $L^p(\Omega; \mathbb{R}^n)$, for some $1 < p < +\infty$ and $n \geq 1$, where Ω is a bounded domain in $\mathbb{R}^m$, $m \geq 1$, then a *quasi-hemivariational inequality* is a problem of the form:

$$\begin{aligned}&\text{find } u \in E \text{ and } z \in \mathrm{A}(u) \text{ such that}\\&\langle z, v\rangle + h(u)\, J^0(iu; iv) - \langle Fu, v\rangle \geq 0 \quad \forall v \in E,\end{aligned}$$

where i is the canonical injection of E into $L^p(\Omega; \mathbb{R}^n)$, $A : E \rightrightarrows E^*$ is a nonlinear multivalued mapping, $F : E \to E^*$ is a nonlinear operator, $J : L^p(\Omega; \mathbb{R}^n) \to \mathbb{R}$ is a locally Lipschitz functional and $h : E \to \mathbb{R}$ is a given nonnegative functional. We denote by E^* the dual space of E and by $\langle ., .\rangle$ the duality pairing between E^* and E.

For technical reasons, we will consider the following quasi-hemivariational inequality:

$$\begin{aligned}&\text{find } u \in C \quad \text{and} \quad z \in A(u) \text{ such that}\\&\langle z, v-u\rangle + h(u)J^0(iu; iv - iu) - \langle Fu, v-u\rangle \geq 0 \quad \forall v \in C,\end{aligned} \tag{QHVI}$$

where its set of solutions is denoted by $\mathrm{SQHVI}(C, A)$. Note that in the special case when C is the whole space E, the above two formulations of quasi-hemivariational inequalities are one and the same.

Studies about inequality problems captured special attention in the last decades where one of the most recent and general type of inequalities is the *hemivariational inequalities* introduced in [29,30] as a variational formulation for several classes of mechanical problems with nonsmooth and nonconvex energy super-potentials. The theory of hemivariational inequalities has produced an abundance of important results both in pure and applied mathematics as well as in other domains such as mechanics and engineering sciences as it allowed mathematical formulations for new classes of interesting problems, see [4,12,28–30,33,35] and the references therein.

When h is equal to zero in the quasi-hemivariational inequality (QHVI) corresponding to convex super-potentials, we obtain the standard case of *variational inequalities*, which were studied earlier by many authors, see [21,24]. The setting corresponding to h equal to 1 describes the hemivariational inequalities. These inequality problems appear as a generalization of variational inequalities, but they are much more general than these ones, in the sense that they are not equivalent to minimum problems but give rise to substationarity problems. The general case when h is nonconstant corresponds to *quasi-hemivariational inequalities*, which were first studied in [28, Section 4.5], in relationship with relevant models in mechanics and engineering. We refer to [12,33,35] for recent contributions to the qualitative analysis of hemivariational and quasi-hemivariational inequalities. On can also consult [1,2,4] where some techniques on continuity of functions are introduced and new results in the field are obtained.

On the other hand, *regularization methods* that are widely used in convex optimization and variational inequalities have been also considered for equilibrium problems. The *proximal point method* as well as the *Tikhonov regularization method*, which are a fundamental regularization technique for handling ill-posed problems, have been recently applied to equilibrium problems, see [13,14,19,25,26] and the references therein.

In this paper, we deal first with the Tikhonov regularization method for *pseudo-monotone equilibrium problems*. Under weakened conditions of upper semicontinuity of bifunctions in their first variable on a subset and of convexity, we extend some results of [14,19] and prove that strictly pseudo-monotone bifunctions can be also used as regularization bifunctions as well as strongly monotone bifunctions. We extend Berge's maximum theorem and develop some results in the qualitative analysis of quasi-hemivariational inequalities to establish the relationship between quasi-hemivariational inequality problems and equilibrium problems. We also give examples and apply the Tikhonov regularization method to quasi-hemivariational inequalities.

2 Notations and preliminary results

Let X be Hausdorff topological space, $x \in X$ and $f : X \longrightarrow \mathbb{R}$ be a function. Recall that f is said to be

1. *upper semicontinuous* at x if for every $\epsilon > 0$, there exists an open neighborhood U of x such that

$$f(y) \leq f(x) + \epsilon \quad \forall y \in U;$$

2. *lower semicontinuous* at x if for every $\epsilon > 0$, there exists an open neighborhood U of x such that

$$f(y) \geq f(x) - \epsilon \quad \forall y \in U.$$

It is well-known that if X is a metric space (or more generally, a Fréchet-Urysohn space, see [3]), then f is upper (*resp*. lower) semicontinuous at $x \in X$ if and only if for every sequence $(x_n)_n$ in X converging to x, we have

$$f(x) \geq \limsup_{n\to+\infty} f(x_n) \quad (\textit{resp}.\ f(x) \leq \liminf_{n\to+\infty} f(x_n)),$$

where $\limsup_{n\to+\infty} f(x_n) = \inf_n \sup_{k\geq n} f(x_k)$ and $\liminf_{n\to+\infty} f(x_n) = \sup_n \inf_{k\geq n} f(x_k)$.

We say that f is upper (*resp*. lower) semicontinuous on a subset S of X if it is upper (*resp*. lower) semicontinuous at every point of S.

The notion of upper (*resp*. lower) semicontinuous function on a subset generalizes the notion of upper (*resp*. lower) semicontinuous function at a point.

If X is a metric space, these notions of upper and lower semicontinuous functions on a subset coincide respectively with those of sequentially upper and sequentially lower semicontinuous functions on a subset considered in [4].

Clearly, if S is an open subset of X, then $f : X \longrightarrow \mathbb{R}$ is upper (*resp*. lower) semicontinuous on S if and only if $f_{|S}$ is upper (*resp*. lower) semicontinuous on S, where $f_{|S}$ denotes the restriction of f on S. It is also not difficult to prove the following result which shows how easy is to construct upper (*resp*. lower) semicontinuous functions on subsets.

Proposition 2.1 *Let $f : X \longrightarrow \mathbb{R}$ be a function and let S be a subset of X. If the restriction $f_{|U}$ of f on an open subset U containing S is upper* (resp. *lower) semicontinuous on S, then any extension of $f_{|U}$ to the whole space X is upper (resp. lower) semicontinuous on S.*

The following result provides us with some properties of upper (*resp*. lower) semicontinuous functions.

Proposition 2.2 *Let $f : X \longrightarrow \mathbb{R}$ be a function, S a subset of X and $a \in \mathbb{R}$.*

1. *f is upper semicontinuous on S if and only if*

$$\overline{\{x \in X \mid f(x) \geq a\}} \cap S = \{x \in S \mid f(x) \geq a\}.$$

In particular, if f is upper semicontinuous on S, then the trace on S of any upper level set of f is closed in S.

2. *f is lower semicontinuous at S if and only if*

$$\overline{\{x \in X \mid f(x) \leq a\}} \cap S = \{x \in S \mid f(x) \leq a\}.$$

In particular, if f is lower semicontinuous on S, then the trace on S of any lower level set of f is closed in S.

Proof The second statement being similar to the first one, we prove only the case of upper semicontinuity. Let

$$x^* \in \overline{\{x \in X \mid f(x) \geq a\}} \cap S.$$

Clearly, $x^* \in S$. To prove that $f(x^*) \geq a$, we argue by contradiction and assume that $f(x^*) < a$. Take $\epsilon > 0$ such that $f(x^*) + \epsilon < a$. By upper semicontinuity of f at x^*, let U be an open neighborhood of x^* such that $f(y) \leq f(x^*) + \epsilon$, for every $y \in U$. It follows that

$$U \cap \{x \in X \mid f(x) \geq a\} = \emptyset,$$

which is a contradiction.

Conversely, let $x^* \in S$, $\epsilon > 0$ and put $a = f(x^*) + \epsilon$. We have $f(x^*) < a$ and then

$$x^* \notin \overline{\{x \in X \mid f(x) \geq a\}}.$$

Let U be an open neighborhood of x^* such that $\{x \in X \mid f(x) \geq a\} \cap U = \emptyset$. It follows that

$$f(y) < a = f\left(x^*\right) + \epsilon \quad \forall y \in U.$$

Finally, we have

$$\{x \in X \mid f(x) \geq a\} \cap S = \{x \in S \mid f(x) \geq a\},$$

which yields that the trace on S of any upper level set of f is closed in S. □

In the sequel, for $y \in C$, we define the following sets:

$$\Phi^+(y) = \{x \in C \mid \Phi(x, y) \geq 0\} \quad \text{and} \quad \Phi^-(y) = \{x \in C \mid \Phi(y, x) \leq 0\}.$$

Clearly, $x^* \in C$ is a solution of the equilibrium problem (EP) if and only if $x^* \in \bigcap_{y \in C} \Phi^+(y)$.

The following result of [4] is a generalization of the well-known Ky Fan's minimax inequality theorem (see [16,20]). We sketch the proof for the convenience of the reader.

Theorem 2.3 *Let $\Phi : C \times C \longrightarrow \mathbb{R}$ be an equilibrium bifunction and suppose the following assumptions hold:*

1. *Φ is quasiconvex in its second variable on C;*
2. *there exists a compact subset K of C and $y_0 \in K$ such that*

$$\Phi(x, y_0) < 0 \quad \forall x \in C \backslash K;$$

3. *Φ is upper semicontinuous in its first variable on K.*

Then, the equilibrium problem (EP) has a solution and its set of solutions $SEP(C, \Phi)$ is a nonempty compact set.

Proof Since Φ is an equilibrium bifunction, then $\overline{\Phi^+(y)}$ is nonempty and closed, for every $y \in C$.

By quasiconvexity of Φ in its second variable, the mapping $y \mapsto \Phi^+(y)$ is a KKM mapping (see for example, [2,4,7,15,16,20]), and since $\Phi^+(y_0)$ is contained in the compact subset K, then by Ky Fan's lemma, we have

$$\bigcap_{y \in C} \overline{\Phi^+(y)} \neq \emptyset.$$

On the other hand, we have

$$\bigcap_{v\in C}\overline{\Phi^{+}(y)}=\bigcap_{y\in C}\left(\overline{\Phi^{+}(y)}\cap K\right).$$

By Proposition 2.2, we have

$$\overline{\Phi^{+}(y)}\cap K=\Phi^{+}(y)\cap K\quad\forall y\in C.$$

Thus,

$$\bigcap_{y\in C}\Phi^{+}(y)=\bigcap_{y\in C}\overline{\Phi^{+}(y)}\neq\emptyset.$$

The compactness of the set of solutions is obvious. □

Remark 1 The set K in the condition 2 of Theorem 2.3 is known in the literature under the name of the *set of coerciveness*.

The Minty lemma for equilibrium problems deals in particular with properties such as compactness and convexity of the set of solutions of equilibrium problems (see for example, [22]). For more properties of the set of solutions of equilibrium problems, we need some additional concepts of monotonicity for bifunctions.

A bifunction $\Phi : C\times C\longrightarrow\mathbb{R}$ is called

1. *strongly monotone* on C with modulus β if

$$\Phi(x,y)+\Phi(y,x)\le-\beta\|x-y\|^{2},\quad\forall x,y\in C;$$

2. *monotone* on C if

$$\Phi(x,y)+\Phi(y,x)\le 0,\quad\forall x,y\in C;$$

3. *strictly pseudo-monotone* on C if

$$\Phi(x,y)\ge 0\Longrightarrow\Phi(y,x)<0,\quad\forall x,y\in C,x\neq y;$$

4. *pseudo-monotone* on C if

$$\Phi(x,y)\ge 0\Longrightarrow\Phi(y,x)\le 0,\quad\forall x,y\in C.$$

Every strongly monotone bifunction is both monotone and strictly pseudo-monotone and every strictly pseudo-monotone bifunction Φ is pseudo-monotone provided it is an equilibrium bifunction, that is, $\Phi(x,x)=0,\forall x\in C$.

The following result extends [7, Theorem 4.2] for equilibrium problems defined on non necessarily convex sets and its proof is elementary. We call such a problem, a *nonconvex equilibrium problem*.

Proposition 2.4 *Let $\Phi : C \times C \longrightarrow \mathbb{R}$ be a strictly pseudo-monotone bifunction. Then for every subset A of C, the following nonconvex equilibrium problem*

$$\text{find } x^* \in A \text{ such that } \Phi\left(x^*, y\right) \geq 0 \quad \forall y \in A$$

has at most one solution.

We also need the following notions about convexity of functions. A function $f : C \longrightarrow \mathbb{R}$ is said to be

1. *semistrictly quasiconvex* on C if, for every $x_1, x_2 \in C$ such that $f(x_1) \neq f(x_2)$, we have

$$f(\lambda x_1 + (1-\lambda)x_2) < \max\{f(x_1), f(x_2)\} \quad \forall \lambda \in]0, 1[;$$

2. *explicitly quasiconvex* on C if it is quasiconvex and semistrictly quasiconvex (see [6,17,23]).

Note that there is not any inclusion relationship between the class of semistrictly quasiconvex functions and that of quasiconvex functions. However, if f is a lower semicontinuous and semistrictly quasiconvex function, then f is explicitly quasiconvex, see [10].

Here, we obtain some additional properties of the set of solutions of equilibrium problems.

Theorem 2.5 *Under assumptions of Theorem 2.3 and suppose the following conditions hold:*

1. *Φ is pseudo-monotone;*
2. *Φ is explicitly quasiconvex in its second variable on C.*

Then the equilibrium problem (EP) has a solution and its set of solutions $SEP(C, \Phi)$ is nonempty compact set. If in addition, K is convex, then $SEP(C, \Phi)$ is convex.

Proof The first part of this theorem being proved above, we prove the second part. By pseudo-monotonicity, we have $\Phi^+(y) \subset \Phi^-(y)$, for every $y \in C$. Since $\bigcap_{y \in C} \Phi^+(y) \subset K$, then

$$\bigcap_{y \in C} \Phi^+(y) \subset \left(\bigcap_{y \in C} \Phi^-(y)\right) \cap K.$$

Now, by explicit quasiconvexity (see [2, Proposition 1.3]), we obtain

$$\left(\bigcap_{y \in C} \Phi^-(y)\right) \cap K \subset \bigcap_{y \in C} \Phi^+(y).$$

It follows that

$$\bigcap_{y\in C} \Phi^{+}(y) = \left(\bigcap_{y\in C} \Phi^{-}(y)\right) \cap K.$$

By quasiconvexity, the set $\Phi^{-}(y)$ is convex, for every y. Thus, the set of solutions $\mathrm{SEP}(C, \theta)$ is convex whenever K is convex. □

Note that Theorem 2.5 also holds if we replace upper semicontinuity of Φ in the first variable by upper hemicontinuity in the first variable and lower semicontinuity in the second variable. Recall that upper hemicontinuity is upper semicontinuity on line segments. The notion of upper hemicontinuity on a subset has been used in [2,4].

3 The Tikhonov regularization method for equilibrium problems

The *Tikhonov regularization method* (or *ridge regression* in statistics) (see [34]) is a powerful tool in convex optimization to handle discrete or continuous ill-posed problems. In the framework of monotone variational inequalities, the basic idea of this method is to perturb the problem with a strongly monotone operator depending on a regularization parameter to the monotone cost operator to obtain a strongly monotone variational inequality. The optimal regularization parameter is usually unknown and usually in practical problems it is determined by various methods, such as the discrepancy principle, cross-validation, L-curve method, Bayesian interpretation, restricted maximum likelihood, and unbiased predictive risk estimator. The resulting regularized inequality problem has a unique solution that depends on the regularization parameter. Next, passing to the limit as the parameter goes to a suitable value, the unique solution of the regularized problem tends to a solution of the original problem. We point out that if the cost operator is pseudo-monotone rather than monotone, then the monotonicity of the regularized problem may fail.

3.1 Main result

In this section, we define a regularized equilibrium problem for the equilibrium problem (EP). Let $\theta : C \times C \longrightarrow \mathbb{R}$ be an equilibrium bifunction that we call the *regularization equilibrium bifunction*. Then, for every $\epsilon > 0$, we define the equilibrium bifunction $\Phi_\epsilon : C \times C \longrightarrow \mathbb{R}$ by

$$\Phi_\epsilon(x, y) = \Phi(x, y) + \epsilon\theta(x, y)$$

and we associate with the equilibrium problem (EP), the regularized equilibrium problem defined as follows:

$$\text{find } x_\epsilon^* \in C \text{ such that } \Phi_\epsilon\left(x_\epsilon^*, y\right) \geq 0 \quad \forall y \in C, \tag{REP}$$

where its set of solutions is denoted by $\mathrm{SREP}(C, \Phi_\epsilon)$.

Note that when Φ or θ is pseudo-monotone, the regularized equilibrium bifunction Φ_ϵ does not inherit any monotonicity property from Φ and θ in general. Also, while the sum of two convex function is convex, this fact does not remain true for quasiconvex functions. The sum of two quasiconvex functions need not be quasiconvex even if one of the functions involved is linear.

The following result is an extension of [14, Theorem 2.9] (see also [19, Theorem 3.2]) in which a largest family of bifunctions including strictly pseudo-monotone bifunctions can be used. We avoid the lower semicontinuity of Φ and θ in their second variable on C, the convexity of Φ and θ is weakened to the quasiconvexity of the regularized bifunction and the upper semicontinuity of Φ and θ in their first variable is weakened to the set of uniform coerciveness. In this result we will never need the quasiconvexity of Φ or θ in their second variable but the quasiconvexity of the regularized equilibrium bifunctions Φ_{ϵ_n} in their second variable.

Theorem 3.1 *Let $(\epsilon_n)_n$ be a sequence of positive numbers such that $\lim_{n\to+\infty} \epsilon_n = 0$ and suppose the following conditions hold:*

1. *Φ and θ are pseudo-monotone on C;*
2. *$\Phi + \epsilon_n\theta$ is quasiconvex in the second variable on C, for every n;*
3. *there exist a compact subset K of C and $y_0 \in C$ such that $\Phi(x, y_0) < 0$, for every $x \in C \setminus K$;*
4. *Φ and θ are upper semicontinuous in the first variable on K.*

Then any cluster point $x^* \in C$ of a sequence $(x_n)_n$ with $x_n \in SREP\left(C, \Phi_{\epsilon_n}\right) \cap K$ for every n, is a solution to the nonconvex equilibrium problem:

$$\textit{find } x^* \in SEP(C, \Phi) \textit{ such that } \theta\left(x^*, y\right) \geq 0 \quad \forall y \in SEP(C, \Phi). \tag{NC-EP}$$

Assume in addition that the following hypotheses hold:

1. *θ is strictly pseudo-monotone on C;*
2. *there exists $A \subset K$ such that $\theta(x, y_0) < 0$, for every $x \in C\backslash A$.*

Then, the regularized equilibrium problem (REP) is solvable, for every n, and any sequence $(x_n)_n$ with $x_n \in SREP\left(C, \Phi_{\epsilon_n}\right)$ for every n, converges to the unique solution of the nonconvex equilibrium problem (NC-EP).

Proof Let $(x_n)_n$ be a sequence with $x_n \in \text{SREP}\left(C, \Phi_{\epsilon_n}\right) \cap K$ for every n, and admitting $x^* \in C$ as a cluster point. We have $x^* \in K$ and without loss of generality, we may assume that $(x_n)_n$ converges to x^*.

First we will prove that $x^* \in \text{SEP}(C, \Phi)$ and therefore $\text{SEP}(C, \Phi)$ is not empty. We know that for every n,

$$\Phi(x_n, y) + \epsilon_n\theta(x_n, y) \geq 0 \quad \forall y \in C.$$

By upper semicontinuity of Φ and θ in their first variable on K and the properties of the upper limits, we have

$$\Phi\left(x^*, y\right) \geq \limsup_{n\to+\infty} \Phi\left(x_n, y\right) + \limsup_{n\to+\infty} \epsilon_n\theta\left(x_n, y\right)$$
$$\geq \limsup_{n\to+\infty} \left(\Phi\left(x_n, y\right) + \epsilon_n\theta\left(x_n, y\right)\right) \geq 0 \quad \forall y \in C.$$

It results that $x^* \in \mathrm{SEP}\,(C, \Phi)$. Now, let $z \in \mathrm{SEP}\,(C, \Phi)$. By pseudo-monotonicity of Φ, we have $\Phi\left(x_n, z\right) \leq 0$, for every n. Then

$$\epsilon_n\theta\left(x_n, z\right) \geq -\Phi\left(x_n, z\right) \geq 0 \quad \forall n,$$

which implies that $\theta\left(x_n, z\right) \geq 0$. Letting n go to $+\infty$, we obtain by upper semicontinuity of θ in its first variable on K that $\theta\left(x^*, z\right) \geq 0$. Thus,

$$\theta\left(x^*, z\right) \geq 0 \quad \forall z \in \mathrm{SEP}\,(C, \Phi)$$

which completes the proof of the first part.

To prove the second part of the theorem, note that for every n,

$$\Phi\left(x_n, y_0\right) + \epsilon_n\theta\left(x_n, y_0\right) < 0 \quad \forall x \in C \setminus K.$$

By Theorem 2.3, the regularized equilibrium problem $(\mathrm{REP}\left(C, \Phi_{\epsilon_n}\right))$ is solvable and its set of solutions $\mathrm{SREP}\left(C, \Phi_{\epsilon_n}\right)$ is contained in K. Let $(x_n)_n$ be a sequence such that $x_n \in \mathrm{SREP}\left(C, \Phi_{\epsilon_n}\right)$, for every n. Then the sequence $(x_n)_n$ has a cluster point $x^* \in K$ and by the first part of the theorem, x^* is a solution to the nonconvex equilibrium problem (NC-EP). Since θ is strictly pseudo-monotone, then by Proposition 2.4, the above nonconvex equilibrium problem (NC-EP) has a unique solution. It follows that every subsequence of the sequence $(x_n)_n$ admits this unique solution of the nonconvex equilibrium problem (NC-EP) as a cluster point. Thus, the sequence $(x_n)_n$ converges to the unique solution of the nonconvex equilibrium problem (NC-EP). □

Remark 2
1. Note that in the case of a finite dimensional real Banach space E, if θ is strongly monotone on C, and θ and Φ are convex and lower semicontinuous in the second variable on C, then the sequence $(\Phi + \epsilon_n\theta)_n$ is uniformly coercive whenever Φ has a set of coerciveness, see [14, Corollary 2.6]. This means that even if we consider a strongly monotone bifunction θ as a regularization bifunction, Theorem 3.1 can also be seen as a generalization of [14, Theorem 2.9] since the upper semicontinuity on the first variable is weakened. We choose in this case θ such that both θ and Φ are upper semicontinuous in their first variable on the subset of the uniform coerciveness.
2. We point out that Theorem 3.1 provides us with a tool to use the Tikhonov regularization method in the case of equilibrium problems involving non upper semicontinuous bifunctions on their first variable.
3. Finally, even if strongly monotone bifunctions seem to be more widely used in the Tikhonov regularization method, our Theorem 3.1 presents a generalization in

several directions of [14, Theorem 2.9] and provides us with a largest family of bifunctions to use in the Tikhonov regularization method.

3.2 Examples of suitable bifunctions and discussion

First, we construct in what follows two bifunctions Φ and θ satisfying all the conditions of Theorem 3.1 without being upper semicontinuous in their first variable on the whole space C. The bifunction Φ is pseudo-monotone non strictly pseudo-montone and θ is strictly pseudo-monotone non strongly monotone on C. This example is obtained by modification of some equilibrium bifunctions constructed in the literature, see [1,5,7,8].

Example 3.2 Let $E = C = \mathbb{R}$, $K = [-1, +1]$ and $y_0 = 0$.

(I) First consider the bifunction $\Phi : C \times C \longrightarrow \mathbb{R}$ is defined by

$$\Phi(x, y) = \begin{cases} (x+2)(y-x) & \text{if } \ x \in]-\infty, -2[, \\ (x+1)(y-x) & \text{if } \ x \in [-2, -1[, \\ \max(x, 0)(y-x) & \text{otherwise.} \end{cases}$$

1. Clearly, $\Phi(x, x) = 0$, for every $x \in C$ and $\Phi(x, 0) < 0$, for every $x \notin [-1, +1]$.
2. To verify that Φ is pseudo-monotone on C, let $x, y \in C$ such that $\Phi(x, y) \geq 0$.
   (a) If $x \in]-\infty, -2[$, then $\Phi(x, y) = (x+2)(y-x)$. It follows that $y - x \leq 0$ and then, $y < -2$. Thus $\Phi(y, x) = (y+2)(x-y) \leq 0$.
   (b) If $x \in [-2, -1[$, then $y - x \leq 0$ and then, $y < -1$. If $y \in [-2, -1[$, then $\Phi(y, x) = (y+1)(x-y) \leq 0$, and if $y \in]-\infty, -2[$, then $\Phi(y, x) = (y+2)(x-y) \leq 0$.
   (c) If $x \geq -1$, then $y \geq x$. It follows that $y \geq -1$ and then $\Phi(y, x) = \max(y, 0)(x-y) \leq 0$.
3. Clearly, Φ is convex in its second variable on C and upper semicontinuous in its first variable on $[-1, +1]$.
4. To see that Φ is not upper semicontinuous in its first variable on C, consider $y > -2$ and take a sequence $(x_n)_n$ in $]-\infty, -2[$ converging to -2. We have

$$\Phi(-2, y) = -(y+2) < 0 = \limsup_{n\to+\infty}(x_n+2)(y+2) = \limsup_{n\to+\infty}\Phi(x_n, y).$$

5. Note that in addition, Φ is not lower semicontinuous in its first variable on C. To see this fact, consider $y < -2$ and take a sequence $(x_n)_n$ in $]-\infty, -2[$ converging to -2. We have

$$\Phi(-2, y) = -(y+2) > 0 = \liminf_{n\to+\infty}(x_n+2)(y+2) = \liminf_{n\to+\infty}\Phi(x_n, y).$$

6. Finally, let us point out that Φ is not strictly pseudo-monotone on C since $\Phi(x, y) = \Phi(y, x) = 0$ whenever $x, y \in [-1, 0]$.

(II) Now, consider the bifunction $\theta : C \times C \longrightarrow \mathbb{R}$ is defined by

$$\theta(x, y) = \begin{cases} \frac{y^4 - x^4}{65} & \text{if} \quad x = 2, \\ y^4 - x^4 & \text{otherwise.} \end{cases}$$

1. Clearly $\theta(x, x) = 0$, for every $x \in C$ and $\theta(x, 0) < 0$, for every $x \notin K = [-1, +1]$. It is also easy to see that θ is strictly pseudo-monotone and not strongly monotone on C.
2. To see that θ is convex in its second variable, let $x \in C$ be fixed.
   (a) if $x = 2$, then $\theta(2, y) = \frac{y^4 - 16}{65}$, for every $y \in C$. The function $y \mapsto \frac{y^4 - 16}{65}$ is convex on C.
   (b) if $x \neq 2$, then $\theta(x, y) = y^4 - x^4$, for every $y \in C$. The function $y \mapsto y^4 - x^4$ is convex on C.
3. To see that θ is upper semicontinuous in its first variable on $[-1, +1]$, let $y \in C$ be fixed and denote by $f : C \longrightarrow \mathbb{R}$ the function defined by

$$f(x) = \theta(x, y).$$

   The restriction $f_{|U}$ of f on the open set $U =]-\infty, 2[$ containing $[-1, +1]$ is defined by $f_{|U}(x) = y^4 - x^4$ which is continuous on U and then by Proposition 2.1, f is upper semicontinuous on $[-1, +1]$.
4. Finally, the bifunction θ is not upper semicontinuous in its first variable on C. Indeed, consider $y = 3$ for example. Let $(x_n)_n$ be a converging sequence to 2 such that $x_n \neq 2$, for every n. We have

$$\theta(2, 3) = 1 < 65 = \limsup_{n \to +\infty} \theta(x_n, 3).$$

Convexity and generalized convexity are important fields in many areas of mathematics and more particularly, in Optimization since convex and concave functions entail several useful properties. Moreover, quasiconvexity and by analogy, quasiconcavity, reveal properties of special interest in Economics Theory. For classical and recent investigations of the subject, we refer to [6,10] where the role of convexity and concavity is stressed.

As mentioned before, the sum of two quasiconvex functions need not be quasiconvex even if one of the functions involved is linear. This means that the sum of two non necessarily quasiconvex functions may be quasiconvex. Also, any quasiconvex function could be split into a sum of two functions and it seems that in general, nothing can justifies that these functions must be quasiconvex. In other words, this subject is very rich and for this reason, studies about convexity and generalized convexity abound in the literature. Characterizations by means of various notions including the notion of differentiability and different sufficient conditions to obtain quasiconvexity as well as other stronger notions such as convexity, strict convexity and strict quasiconvexity are deeply developed and many examples are constructed by several authors.

However, we recall here the following basic properties which will inspire us in the construction of our next examples:

1. If f is a quasiconvex function, then for every $\alpha \geq 0$, αf is quasiconvex.
2. Every monotone function of one real variable is quasiconvex.
3. The sum of two monotone functions of one real variable with the same sense of monotonicity is monotone, and therefore quasiconvex.

Now, we modify the bifunction θ of Example 3.2 in such a way that all its above properties are conserved but it is quasiconvex non convex bifunction in the second variable. Note that it could be more easy to construct further examples if we relax the condition on the semicontinuity of the bifunctions to the whole space rather than only on the set of coerciveness.

Example 3.3 Let $E = C = \mathbb{R}$, $K = [-1, +1]$ and $y_0 = 0$.

The bifunction $\theta : C \times C \longrightarrow \mathbb{R}$ is now defined by

$$\theta(x, y) = \begin{cases} \frac{y^4 - x^4}{65} & \text{if} \quad (x, y) \in \{2\} \times]-\infty, 3], \\ y^4 - x^4 & \text{otherwise.} \end{cases}$$

1. To see that θ is quasiconvex in its second variable, treat only the case of $x = 2$. In this case, we have

$$\theta(2, y) = \begin{cases} \frac{y^4 - 16}{65} & \text{if} \quad y \in]-\infty, 3], \\ y^4 - 16 & \text{otherwise.} \end{cases}$$

   We have that $\theta\,(2, 3) \leq \theta\,(2, y)$, for every $y \in]3, +\infty[$. A combination with the other properties of the bifunction θ yields easily that the function $y \mapsto \theta\,(2, y)$ is quasiconvex on C.
2. To see that the function $y \mapsto \theta\,(2, y)$ is not convex, choose $y_1 = 3$ and $y_2 = 4$ for example. Take the point

$$y = \frac{1}{2} y_1 + \left(1 - \frac{1}{2}\right) y_2 = \frac{7}{2}$$

   in the line segment between y_1 and y_2. We have

$$\theta\,(2, y) = \theta\left(2, \frac{7}{2}\right) = \frac{7^4}{16} - 16.$$

   In the other hand, we have

$$\frac{1}{2}\theta\,(2, y_1) + \left(1 - \frac{1}{2}\right)\theta\,(2, y_2) = \frac{1}{2} + \frac{1}{2}\left(4^4 - 16\right) = \frac{241}{2} < \theta\,(2, y)\,.$$

Now, we show that the regularized bifunction constructed from Φ and θ as in Theorem 3.1 is quasiconvex in its second variable.

Example 3.4 Let ϵ be a positive number and consider the regularized bifunction $\Phi_\epsilon = \Phi + \epsilon\theta$ as in Theorem 3.1. Then, the bifunction Φ_ϵ is defined on $C \times C$ by

$$\Phi_\epsilon(x, y) = \begin{cases} (x+2)(y-x) + \epsilon\left(y^4 - x^4\right) & \text{if } x \in]-\infty, -2[, \\ (x+1)(y-x) + \epsilon\left(y^4 - x^4\right) & \text{if } x \in [-2, -1[, \\ \max(x, 0)(y-x) + \epsilon\frac{y^4-x^4}{65} & \text{if } (x, y) \in \{2\} \times]-\infty, 3], \\ \max(x, 0)(y-x) + \epsilon\left(y^4 - x^4\right) & \text{otherwise.} \end{cases}$$

As above, only the quasiconvexity of the function $y \mapsto \Phi_\epsilon(2, y)$ is important to verify, the other cases come readily from the definition. In this case, we have

$$\Phi_\epsilon(2, y) = \begin{cases} 2(y-2) + \epsilon\frac{y^4-16}{65} & \text{if } y \in]-\infty, 3], \\ 2(y-2) + \epsilon\left(y^4 - 16\right) & \text{otherwise.} \end{cases}$$

By the same argument as above, remark that $\Phi_\epsilon(2, 3) \leq \Phi_\epsilon(2, y)$, for every $y \in]3, +\infty[$, and this completes the proof.

4 Applications to quasi-hemivariational inequalities

In this section we give some results on the relationship between equilibrium problems and quasi-hemivariational inequalities. We develop results in the qualitative analysis of quasi-hemivariational inequalities and give a generalization to Berge's maximum theorem in order to apply the Tikhonov regularization for quasi-hemivariational inequalities.

Recall that a function $\phi : E \to \mathbb{R}$ is called *locally Lipschitzian* if for every $u \in E$, there exists a neighborhood U of u and a constant $L_u > 0$ such that

$$|\phi(w) - \phi(v)| \leq L_u \|w - v\|_X \quad \forall w \in U, \forall v \in U.$$

If $\phi : E \to \mathbb{R}$ is locally Lipschitzian near $u \in E$, then the *Clarke generalized directional derivative* of ϕ at u in the direction of $v \in E$, denoted by $\Phi^0(u, v)$, is defined by

$$\phi^0(u, v) = \limsup_{\substack{w \to u \\ \lambda \downarrow 0}} \frac{\phi(w + \lambda v) - \phi(w)}{\lambda}.$$

Among several important properties of the generalized directional derivative of locally Lipschitzian functions, we will make use in the present paper of the following properties (for proofs and related properties, we refer to [11, Proposition 2.1.1]).

Suppose that $\phi : E \to \mathbb{R}$ is locally Lipschitzian near $u \in E$. Then,

1. the function $v \longmapsto \phi^0(u, v)$ is finite, positively homogeneous and subadditive;
2. the function $(u, v) \longmapsto \phi^0(u, v)$ is upper semicontinuous.

Remark 3 To avoid any confusion in the definition of semicontinuity on subsets, from now on and in all what follows, the functions h and F, and the multivalued mapping A will be considered from C rather than from E.

It is easily seen that any solution of the quasi-hemivariational inequality (QHVI) is a solution of the equilibrium problem (EP) where the equilibrium bifunction $\Phi : C \times C \to \mathbb{R}$ is defined by

$$\Phi(u, v) = \sup_{z \in A(u)} \langle z, v - u \rangle + h(u) J^0 (iu; iv - iu) - \langle Fu, v - u \rangle \quad \forall u, v \in C.$$

The converse needs some additional conditions on the multivalued mapping A and holds by a classical approach.

Theorem 4.1 *If A has nonempty, convex and weak* compact values, then any solution of the equilibrium problem (EP) is a solution of the quasi-hemivariational inequality problem (QHVI).*

Proof Let $u^* \in C$ be such that $\Phi(u^*, v) \geq 0$, for every $v \in C$, and assume that there does not exist $z \in A(u^*)$ satisfying

$$\langle z, v - u^* \rangle + h(u^*) J^0 \left(iu^*; iv - iu^*\right) - \langle Fu^*, v - u^* \rangle \geq 0 \quad \forall v \in C.$$

Clearly, for every $z \in A(u^*)$, there exist $v_z \in C$ and $\epsilon_z > 0$ such that

$$\langle z, v_z - u^* \rangle + h(u^*) J^0 \left(iu^*; iv_z - iu^*\right) - \langle Fu^*, v_z - u^* \rangle < -\epsilon_z.$$

Since, for every $v \in C$, the mapping defined on E^* by

$$z \mapsto \langle z, v - u^* \rangle + h(u^*) J^0 \left(iu^*; iv - iu^*\right) - \langle Fu^*, v - u^* \rangle$$

is weak* continue, then for every $z \in A(u^*)$, we choose a weak* open subset O_z of E^* such that

$$\langle z', v_z - u^* \rangle + h(u^*) J^0 \left(iu^*; iv_z - iu^*\right) - \langle Fu^*, v_z - u^* \rangle < -\epsilon_z \quad \forall z' \in O_z.$$

For every $z \in A(u^*)$, we have $z \in O_z$ and then, $\{O_z \mid z \in A(u^*)\}$ is a weak* open cover of $A(u^*)$. Since $A(u^*)$ is weak* compact, there exist $z_j \in C$, $j = 1, \ldots, n$ such that $\left\{O_{z_j} \mid j = 1, \ldots, n\right\}$ is a finite subcover of $A(u^*)$. Put $v_j = v_{z_j}$, $j = 1, \ldots, n$ and $\epsilon = \min\left\{\epsilon_{z_j} \mid j = 1, \ldots, n\right\}$. Clearly for all $z \in A(u^*)$, we have

$$\min_{j=1,\ldots,n} \Big(\langle z, v_j - u^* \rangle + h(u^*) J^0 \left(iu^*; iv_j - iu^*\right) - \langle Fu^*, v_j - u^* \rangle\Big) < -\epsilon.$$

The Clarke generalized directional derivative being finite, then for every $j = 1, \ldots, n$, the functions

$$z \mapsto \langle z, v_j - u^* \rangle + h(u^*) J^0 \left(iu^*; iv_j - iu^*\right) - \langle Fu^*, v_j - u^* \rangle,$$

defined on the convex set $A(u^*)$ are concave and proper with domain containing $A(u^*)$, and therefore by a standard result of convex analysis (see [32, Theorem 21.1]), there exist $\mu_j \geq 0$, $j = 1, \ldots, n$, with $\sum_{j=1}^{n} = 1$ such that for all $z \in A(u^*)$

$$\sum_{j=1}^{n} \mu_j \left(\langle z, v_j - u^* \rangle + h(u^*) J^0 \left(iu^*; iv_j - iu^* \right) - \langle Fu^*, v_j - u^* \rangle \right) < -\epsilon.$$

Set $v^* = \sum_{j=1}^{n} \mu_j v_j$. Then $v^* \in C$ and by the positive homogeneity and the subadditivity of the Clarke generalized directional derivative in its second variable, we have

$$\langle z, v^* - u^* \rangle + h(u^*) J^0 \left(iu^*; iv^* - iu^* \right) - \langle Fu^*, v^* - u^* \rangle < -\epsilon \quad \forall z \in A(u^*)$$

which implies that $\Phi\,(u^*, v^*) < 0$, a contradiction. □

We turn now into studying the properties inherited by the equilibrium bifunctions defined from quasi-hemivariational inequalities.

Theorem 4.2 *The bifunction Φ is lower semicontinuous and convex in its second variable on C.*

Proof From the positive homogeneity and the subadditivity of the Clarke generalized directional derivative in its second variable, the function

$$v \mapsto \langle z, v - u \rangle + h\,(u)\, J^0\,(iu; iv - iu) - \langle Fu, v - u \rangle$$

is convex, for every $u \in C$ and every $z \in A(u^*)$. It is also lower semicontinuous since the Clarke generalized directional derivative is lower semicontinuous. The bifunction Φ being the superior envelope of a family of convex and lower semicontinuous functions, it is then convex and lower semicontinuous in its second variable on C. □

The properties inherited by Φ in its first variable are more complicated and need additional conditions on the functions and multivalued mappings involved in the quasi-hemivariational inequalities.

Recall that a multivalued mapping T from a topological space X with values in the set of subsets of a topological space Y is called *upper semicontinuous* at a point $x \in X$ if whenever V an open subset containing $T\,(x)$, there exist an open neighborhood U of x such that $T\left(x'\right) \subset V$, for every $x' \in U$. We say that T is upper semicontinuous on a subset S of X if T is upper semicontinuous at every point of S.

The following result is a generalization of the well-known Berge's maximum theorem, see [31, Theorem 6.1.18].

Theorem 4.3 *Let X and Y be two Hausdorff topological spaces, S a nonempty subset of X, U an open subset containing S, $T : X \rightrightarrows Y$ a multivalued mapping and $\psi : Y \times X \to \mathbb{R} \cup \{+\infty\}$ a function. Suppose that ψ is upper semicontinuous on $Y \times U$ and T is upper semicontinuous on S with nonempty compact values on U. Then the value function $f : X \to \mathbb{R} \cup \{+\infty\}$ defined by*

$$f(x) = \sup_{y \in T(x)} \psi(y, x)$$

is upper semicontinuous on S.

Proof By Proposition 2.1, it suffices to prove that the restriction $g = f_{|U}$ of f on U is upper semicontiunuous on S. Let $a \in \mathbb{R}$ and by Proposition 2.2, we have to prove that

$$\overline{\{x \in U \mid g(x) \geq a\}} \cap S = \{x \in S \mid g(x) \geq a\},$$

where the closure is taken with respect to U. Let $x^* \in \overline{\{x \in U \mid g(x) \geq a\}} \cap S$ and choose a net $(x_\alpha)_{\alpha \in \Lambda}$ in $\{x \in U \mid g(x) \geq a\}$ converging in U to x^*. Since $x_\alpha \in U$, then the restriction of the function ψ on $Y \times \{x_\alpha\}$ is upper semicontinuous and therefore, by the Weierstrass theorem, it attains its maximum on the compact set $T(x_\alpha)$, for every $\alpha \in \Lambda$. Let $y_\alpha \in T(x_\alpha)$ be such that $g(x_\alpha) = \psi(y_\alpha, x_\alpha)$, for every $\alpha \in \Lambda$.

The net $(y_\alpha)_{\alpha \in \Lambda}$ has a cluster point in $T(x^*)$. Indeed, suppose the contrary holds. Then the compactness of $T(x^*)$ yields the existence of an open set V containing $T(x^*)$ and $\alpha_0 \in \Lambda$ such that $y_\alpha \notin V$, for every $\alpha \geq \alpha_0$. It follows by upper semicontinuity of T at x^* the existence of an open neighborhood W of x^* such that $T(x) \subset V$, for every $x \in W$. Let $\alpha_1 \in \Lambda$ be such that $x_\alpha \in W$, for every $\alpha \geq \alpha_1$. Thus $y_\alpha \in V$, for every $\alpha \geq \alpha_1$. Contradiction.

Take now $y^* \in T(x)$ and $(y_\alpha)_{\alpha \in \Gamma}$ a subnet of $(y_\alpha)_{\alpha \in \Lambda}$ converging to y^*. The net $((y_\alpha, x_\alpha))_{\alpha \in \Gamma}$ is in $Y \times U$, converging to (y^*, x^*) and satisfies

$$\psi(y_\alpha, x_\alpha) \geq a \quad \forall \alpha \in \Gamma.$$

By upper semicontinuity of ψ on $Y \times U$, it follows that $g(x^*) \geq \psi(y^*, x^*) \geq a$, which completes the proof.

We give in what follows a sufficient condition for the upper semicontinuity in its first variable of the equilibrium bifunction Φ.

Corollary 4.4 *Let K be a subset of C, U be an open subset containing K and suppose the following conditions hold:*

1. *the nonlinear multivalued mapping A is upper semicontinuous on K with respect to the strong topology of E^* and has nonempty compact values on U;*
2. *for every $v \in C$, the mapping $u \in C \mapsto h(u) J^0(iu; iv - iu)$ is upper semicontinuous on U;*
3. *for every $v \in C$, the mapping $u \in C \mapsto \langle F(u), v - u\rangle$ is lower semicontinuous on U.*

Then Φ is upper semicontinuous in its first variable on K.

Proof Let $v \in C$ be fixed and define the function $\psi : E^* \times C \to \mathbb{R}$ by

$$\psi(z, u) = \langle z, v - u\rangle + h(u) J^0(iu; iv - iu) - \langle Fu, v - u\rangle.$$

The function ψ being a sum of upper semicontinuous functions on $E^* \times U$, it is upper semicontinuous on $E^* \times U$, where E^* is equipped with the strong topology. It follows by Theorem 4.3 that the value function $u \mapsto \Phi(u, v)$ is upper semicontinuous on K.

Corollary 4.5 *Let K be a subset of C, U be an open subset containing K and suppose the following conditions hold:*

1. *the nonlinear multivalued mapping A is upper semicontinuous on K with respect to the weak* topology of E^* and has nonempty weak* compact values on U;*
2. *for every $v \in C$, the mapping*

$$(z, u) \in E^* \times U \mapsto \langle z, v - u \rangle + h(u) J^0(iu; iv - iu) - \langle Fu, v - u \rangle$$

is upper semicontinuous on $E^* \times U$.

Then Φ is upper semicontinuous in its first variable on K.

Recall that a multivalued mapping $T : C \to 2^{E^*}$ is said to be:

1. *pseudo-monotone* on C if

$$\langle z, u - v \rangle \leq 0 \Longrightarrow \langle t, v - u \rangle \geq 0 \quad \forall u, v \in C, \forall z \in A(u), \forall t \in A(v);$$

2. *strictly pseudo-monotone* if

$$\langle z, u - v \rangle \leq 0 \Longrightarrow \langle t, v - u \rangle > 0 \quad \forall u, v \in C, \forall z \in A(u), \forall t \in A(v).$$

It is well-known that if T has weak* compact values, then T is pseudo-monotone (*resp*. strictly pseudo-monotone) if and only if the equilibrium bifunction θ is pseudo-monotone (*resp*. strictly pseudo-monotone) where $\theta : C \times C \to \mathbb{R}$ is defined by

$$\theta(u, v) = \sup_{z \in T(u)} \langle z, v - u \rangle.$$

This follows from the fact that for every $u, v \in C$, by the weak* compactness of the values of T, there exist $z \in T(u)$ and $t \in T(v)$ such that

$$\theta(u, v) = \langle z, v - u \rangle \quad \text{and} \quad \theta(v, u) = \langle t, u - v \rangle.$$

Now, to apply the Tikhonov regularization for quasi-hemivariational inequalities, first we take a multivalued function $G : C \to 2^{E^*}$ and $\epsilon > 0$, and define the multivalued function $A_\epsilon : C \to 2^{E^*}$ by

$$A_\epsilon(x) = A(x) + \epsilon G(x).$$

The regularized quasi-hemivariational inequality has the following form:

$$\text{Find } u \in C \text{ and } z \in A_\epsilon(u) \text{ such that}$$

$$\langle z, v-u\rangle + h(u)\, J^0(iu; iv-iu) - \langle Fu, v-u\rangle \geq 0 \quad \forall v \in C. \qquad \text{(RQHVI)}$$

As previously, we denote its set of solutions by SRQHVI(C, A_ϵ).

We say that a quasi-hemivariational inequality (QHVI) is pseudo-monotone on C if the associated equilibrium bifunction Φ is pseudo-monotone on C.

Theorem 4.6 *Let K be a compact subset of C, U an open subset containing K and $(\epsilon_n)_n$ is a sequence of positive numbers such that $\lim_{n\to+\infty} \epsilon_n = 0$. Suppose that the following assumptions hold:*

1. *G is pseudo-monotone on C, upper semicontinuous on K with respect to the strong topology of E^* and has nonempty, convex and compact values on C;*
2. *the quasi-hemivariational inequality (QHVI) is pseudo-monotone on C;*
3. *A is upper semicontinuous on K with respect to the strong topology of E^* and has nonempty, convex and compact values on C;*
4. *for every $v \in C$, the mapping $u \in C \mapsto h(u)\, J^0(iu; iv-iu)$ is upper semicontinuous on U;*
5. *for every $v \in C$, the mapping $u \in C \mapsto \langle F(u), v-u\rangle$ is lower semicontinuous on U.*
6. *there exists $v_0 \in C$ such that*

$$\langle z, v_0-u\rangle + h(u)\, J^0(iu; iv_0-iu) - \langle Fu, v_0-u\rangle < 0 \quad \forall u \in C \setminus K, \forall z \in A(u).$$

Then any cluster point $x^* \in C$ of a sequence $(x_n)_n$ with $x_n \in SRQHVI(C, A_{\epsilon_n}) \cap K$ for every n, is a solution to the multivalued variational inequality:

$$\begin{aligned} &\textit{Find } u \in SQHVI(C, A) \ \textit{and} z \in G(u) \ \textit{such that} \\ &\langle z, v-u\rangle \geq 0 \quad \forall v \in SQHVI(C, A). \end{aligned}$$

Assume in addition, that the following conditions hold:

1. *G is strictly pseudo-monotone on C;*
2. *there exists $K' \subset K$ such that $\langle z, v_0-u\rangle < 0$, for every $u \in C \backslash K'$ and every $z \in G(u)$.*

Then the regularized quasi-hemivariational inequality (RQHVI(C, F_{ϵ_n})) is solvable, for every n, and any sequence $(x_n)_n$ with $x_n \in SRQHVI(C, F_{\epsilon_n})$ for every n, converges to the unique solution of the multivalued variational inequality problem:

$$\begin{aligned} &\textit{Find } u \in SQHVI(C, A) \ \textit{and } z \in G(u) \ \textit{such that} \\ &\langle z, v-u\rangle \geq 0 \quad \forall v \in SQHVI(C, A). \end{aligned}$$

Proof Note that

$$\sup_{z \in A_\epsilon(x)} \langle z, y-x\rangle = \sup_{z \in A(x)} \langle z, y-x\rangle + \epsilon \sup_{z \in G(x)} \langle z, y-x\rangle \quad \forall x, y \in C.$$

The result holds now easily from the results developed above and by applying Theorem 3.1. ◻

Remark 4 Under assumptions of Theorem 4.6, the set of solutions of the quasi-hemivariational inequality (QHVI) is nonempty and compact. It is also convex whenever K is convex.

5 Conclusions

In this work, we have proved that under weakened conditions of semicontinuity and convexity, the Tikhonov regularization method can be applied to pseudo-monotone equilibrium problems with strictly pseudo-monotone bifunctions as regularized equilibrium bifunctions as well as strongly monotone bifunctions. We have obtained a generalization of Berge's maximum theorem and developed new techniques in the qualitative analysis of quasi-hemivariational inequalities in order to establish the relationship between quasi-hemivariational inequalities and equilibrium problems. We have also applied the Tikhonov regularization method to quasi-hemivariational inequality problems.

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