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IV.1 Critical Point Theory

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The solutions of many problems are found to be stationary points of some associated "energy" functionals. Often such a functional is unbounded from above and below, so that it has no maximum or minimum. This forces one to look for saddle points, which are obtained by mini-max arguments. One specifies a functional I on a Banach space X, and two points 0 and e of X, where 0 is the origin. For each smooth path $g(t)$ which connects 0 with e, one defines the functional $\max_t I(g(t))$. One then tries to minimize this functional with respect to the choice of $g(t)$ in the collection G of all such paths. Thus, it is natural to define

$$b = \inf_{g(t)\in G} \sup_{u\in g(t)} I(u) \tag{1}$$

and to prove, under various hypotheses, that b is a critical value of I. Indeed, it seems intuitively obvious that b defined in relation (1) is a critical value of I. However, this is not true in general, as showed by the following example in the plane: let $I(x,y) = x^2-(x-1)^3y^2$. Because I has a proper local minimum at the origin, then $b>0$ whenever $e\neq 0$. Moreover, choosing $e=(2,2)$, we observe that $I=1$ along the line $x=1$, which separates 0 from e, so that $b\geq 1$, hence it is not a critical value. In fact, $b=1$, but there is no path $g(t)$ in G such that $I(g(t))\leq 1$.

One of the most important mini-max properties is the so-called *mountain pass theorem*, which is a deep result in modern nonlinear analysis. It marks the beginning of a new approach to critical point theory. The mountain pass theorem was established by Ambrosetti and Rabinowitz in [1]. Their original proof relies on some deep deformation techniques developed by Palais and Smale [8, 9], who put the main ideas of the Morse theory into the framework of differential topology on infinite-dimensional manifolds.

In the statement of the mountain pass theorem, the strict inequality in the geometric condition

$$\inf_{S(0,R)} I > \max\{I(0), I(e)\}\,, \tag{2}$$

which means that the mountain ridge separating the points 0 and e has an altitude which is strictly higher than those of 0 and e, plays an essential role in the proof.

In paper [ii] below P. Pucci and J. Serrin studied what happens when the equality holds in relation (2). In such a case, Pucci and Serrin located a critical point of level b on the sphere $S(0,R)$. Their argument is based on the observation that there is a critical point in the closure of an open ring A around the sphere such that the distance between the boundary of A and $S(0, R)$ can be taken arbitrarily small. However, in the infinite-dimensional case, they established that the mountain pass theorem still holds true when the strict inequality in (2) is replaced a non-strict inequality. Their result in this case is the following.

Theorem (P. Pucci & J. Serrin [ii]). *Let X be a real Banach space and let $I : X \to \mathbb{R}$ be a C^1-functional satisfying the following conditions:*

(i) *there exist numbers a, r, R such that $0 < r < R$ and $I(u) \geq a$ for all $u \in A := \{u \in X : \ r < \|u\| < R\}$;*

(ii) *$I(0) \leq a$ and $I(e) \leq a$ for some $e \in X$ with $\|e\| \geq R$.*

Then I has a critical point x_0 in X, different from 0 and e, with critical value $b \geq a$; in addition, $x_0 \in A$ when $b = a$.

We point out that the restriction of the mountain ridge to an annulus centered at 0 is not necessary and the above result remains true when the annulus is replaced by a topological annulus.

This theorem implies the existence of an infinite number of critical points in the ring A, since the preceding theorem applies in any of its sub-rings.

We point out that another extension of the mountain pass theorem is due to Ghoussoub and Preiss [4]. We refer to the monographs [5] and [6] for applications to nonlinear PDEs and to the survey [10] for an historical development of the mountain pass theory.

P. Pucci and J. Serrin [i] have also established two interesting corollaries of their mountain pass theorem in the limiting case. The first one is the *three critical point* theorem, which asserts that a C^1 functional with the Palais-Smale property that has two local minimum points has a third critical point. We refer to Ricceri [11] and Bonanno [2] for results concerning the *stability*, resp. the *location* of the critical points in the three critical point theorem. The second relevant application of the limiting case of the mountain pass theorem found in [i,ii] states that a v-periodic C^1 functional with a local minimum e has a critical point $x_0 \neq e + kv$, $k = 0, \pm 1, \pm 2, \ldots$. This critical point yields a second independent solution of the forced pendulum equation studied by Mawhin and Willem [7].

Brezis and Nirenberg [3] also proved a version of the mountain pass theorem that includes the limiting case corresponding to mountains of zero altitude. Their proof combines a pseudo-gradient lemma, an original perturbation argument and Ekeland's variational principle.

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