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A singular controllability problem with vanishing viscosity

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Objectives: The aim of this paper is to answer the question: Do the controls of a vanishing viscosity approximation of the one dimensional linear wave equation converge to a control of the conservative limit equation? The characteristic of our viscous term is that it contains the fractional power α of the Dirichlet Laplace operator. Through the parameter α we may increase or decrease the strength of the high frequencies damping which allows us to cover a large class of dissipative mechanisms. The viscous term, being multiplied by a small parameter ε devoted to tend to zero, vanishes in the limit. Our analysis enables us to evaluate the magnitude of the controls needed for each eigenmode and to show their uniform boundedness with respect to ε, under the assumption that $\alpha \in [0,1) \setminus \left\{\frac{1}{2}\right\}$. It follows that, under this assumption, our starting question has a positive answer.

Summary: For $T > 0$ and $\varepsilon \in (0,1)$ we consider the following one dimensional perturbed wave equation

$$
(1) \quad \begin{cases} u_{tt}(t,x) - \partial^2_{xx} u(t,x) + 2\varepsilon(-\partial^2_{xx})^{\alpha} u_t(t,x) + \varepsilon^2(-\partial^2_{xx})^{2\alpha} u(t,x) = v_\varepsilon(t) f(x) & (t,x) \in (0,T) \times (0,\pi) \\ u(t,0) = u(t,\pi) = 0 & t \in (0,T) \\ u(0,x) = u^0(x),\ u_t(0,x) = u^1(x) & x \in (0,\pi). \end{cases}
$$

We intend to control the solution of (1) by using a control $v_\varepsilon(t)$, depending only on time and acting on the system through a given shape function in space $f(x)$. Such types of controls are often used and sometimes called "lumped" or "bilinear" (see, for instance [1]). The profile where our control acts belong to $L^2(0,\pi)$ and verifies $\widehat{f}_n \neq 0$ for any $n \in \mathbb{N}^*$. In order to justify the damping mechanism introduced in (1), which involves the fractional power α of the Laplace operator, let us point out that sometimes it may be useful to control the amount of dissipation introduced in the system not only by means of the vanishing parameter ε but also by an adequate choice of the differential operator. In (1) this is achieved through the parameter α. Note that, if $\alpha \in \left[0, \frac{1}{2}\right)$, the imaginary parts of the eigenvalues λ_n dominate the real ones and (1) has the same hyperbolic character as in the limit case $\varepsilon = 0$. On the contrary, if $\alpha \in \left(\frac{1}{2}, 1\right)$, (1) has a parabolic type. In this case we are dealing with a truly singular control problem and the pass to the limit is sensibly more difficult. Finally, let us remark that $\alpha = \frac{1}{2}$ is a singular case in which the basic controllability properties (such as spectral controllability) do not hold and the case $\alpha = 1$ has been studied in [4] for a slightly different problem.

In this paper we reduce the controllability problem to a *moment problem.* The advantage of this method consists in the fact that the solutions of the moment problem are given in terms of an explicit *biorthogonal sequence* to a family Λ of exponential functions. Now, the main task is to show that there exists a biorthogonal sequence and to evaluate its L^2-norm. In order to do that, we define a family $(\Psi_m(z))_{m\in\mathbb{Z}^*}$ of entire functions of exponential type independent of ε (see, for instance, [6]) such that $\Psi_m(i\overline{\lambda}_n) = \delta_{mn}$. The inverse Fourier transform of $(\Psi_m)_{m\in\mathbb{Z}^*}$ will give us the biorthogonal sequence $(\theta_m)_{m\in\mathbb{Z}^*}$ that we are looking for. Such a method was used for the first time by Paley and Wiener [5] and, in the context of control problems, by Fattorini and Russell in the pioneering articles [2, 3] to prove the controllability of the one dimensional heat equation.

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