Recovering the initial state of dynamical systems using observers

Ghislain Haine
Department of Mathematics, Computer Science, Control

Introduction

In many areas of science, we need to recover the initial data of a physical system from partial observation over some finite time interval. In oceanography and meteorology, this problem is known as ***data assimilation***. It also arises in medical imaging, for instance in thermoacoustic tomography where the problem is to recover an initial data for a 3D wave type equation from surface measurement [8].

In the last decade, new algorithms based on time reversal (see Fink [4, 5]) appeared to answer this question. We can mention, for instance, the Back and Forth Nudging proposed by Auroux and Blum [1], the Time Reversal Focusing by Phung and Zhang [10], the algorithm proposed by Ito, Ramdani and Tucsnak [7] and finally, the one we will consider here, the forward–backward observers based algorithm proposed by Ramdani, Tucsnak and Weiss [11].

Under some assumptions, we prove that this algorithm allows to reconstruct the ***observable part*** of the initial state, *i.e.* the part which contributes to the measurement, and give necessary and sufficient conditions to get exponential decay.

The dynamical system

Let X and Y be two Hilbert spaces, $A : \mathcal{D}(A) \to X$ a skew-adjoint operator (possibly unbounded), and $C \in \mathcal{L}(X, Y)$. We consider the system

$$\begin{cases} \dot{z}(t) = Az(t), & \forall\, t \geq 0, \\ z(0) = z_0 \in X. \end{cases} \tag{1}$$

Such a system is often used to model vibrating system (acoustic or elastic waves) or quantum system (Schrödinger equation).

We observe system (1) *via* the operator C during a finite time interval $(0, \tau)$ with $\tau > 0$, leading to the measurement

$$y(t) = Cz(t), \quad \forall\, t \in (0, \tau). \tag{2}$$

The inverse problem

Considering systems (1)–(2), a natural question arises

Is it possible to recover z_0 of (1) from the measurement y given by (2) ?

If we denote $\Psi_\tau : X \to L^2_{\ell oc}([0, \infty), Y)$ the continuous linear operator which associates z_0 to the measurement y, *i.e.* $y = \Psi_\tau z_0$, it is clear that the problem is well-posed if Ψ_τ is left-invertible. In other words, it is well-posed if there exists a constant $k_\tau > 0$ such that

$$\|\Psi_\tau x\| \geq k_\tau \|x\|, \quad \forall\, x \in X. \tag{3}$$

If the above inequality (3) holds, we say that (1)–(2) (or (A, C)) is exactly observable.

The iterative algorithm

Under some assumptions, Ramdani, Tucsnak and Weiss [11] proposed the iterative use of the following back and forth ***observers*** (4)–(5) to reconstruct z_0 from y. In our case, these assumptions are equivalent to the exact observability assumption (3) (see [11, Proposition 3.7]).
Let $A^+ = A - \gamma C^*C$ and $A^- = -A - \gamma C^*C$ be the generator of the two exponentially stable C_0-semigroups (denoted $\mathbb{T}^+$ and $\mathbb{T}^-$ respectively) for all $\gamma > 0$ (see Liu [9] or [11, Proposition 3.7]), and let $z_0^+ \in X$ be the initial guess (usually $z_0^+ = 0$). Then the algorithm reads: for all $n \in \mathbb{N}^*$

$$\begin{cases} \dot{z}_n^+(t) = A^+ z_n^+(t) + \gamma C^* y(t), & \forall\, t \in (0, \tau), \\ z_1^+(0) = z_0^+, \\ z_n^+(0) = z_{n-1}^-(0), & \forall\, n \geq 2, \end{cases} \tag{4}$$

$$\begin{cases} \dot{z}_n^-(t) = -A^- z_n^-(t) - \gamma C^* y(t), & \forall\, t \in (0, \tau), \\ z_n^-(\tau) = z_n^+(\tau), & \forall\, n \geq 1. \end{cases} \tag{5}$$

Convergence of the algorithm

Denoting $e_n^+ = z_n^+ - z$ and $e_n^- = z_n^- - z$, one can easily show that e_n^+ and e_n^- are the respective solutions of

$$\begin{cases} \dot{e}_n^+(t) = A^+ e_n^+(t), & \forall\, t \in (0, \tau), \\ e_1^+(0) = z_0^+ - z_0, \\ e_n^+(0) = z_{n-1}^-(0) - z_0, & \forall\, n \geq 2, \end{cases}$$

$$\begin{cases} \dot{e}_n^-(t) = -A^- e_n^-(t), & \forall\, t \in (0, \tau), \\ e_n^-(\tau) = z_n^+(\tau) - z(\tau), & \forall\, n \geq 1. \end{cases}$$

Then for all $n \in \mathbb{N}^*$

$$\left\|e_n^-(0)\right\| = \left\|\left(\mathbb{T}_\tau^- \mathbb{T}_\tau^+\right)^n e_1^+(0)\right\| \leq \left\|\mathbb{T}_\tau^- \mathbb{T}_\tau^+\right\|^n \left\|e_1^+(0)\right\|,$$

which can be rewritten

$$\left\|z_n^-(0) - z_0\right\| \leq \left\|\mathbb{T}_\tau^- \mathbb{T}_\tau^+\right\|^n \left\|z_0^+ - z_0\right\|, \quad \forall\, n \in \mathbb{N}^*.$$

By the exponential stability of $\mathbb{T}^+$ and $\mathbb{T}^-$, there exists a $\tau > 0$ such that

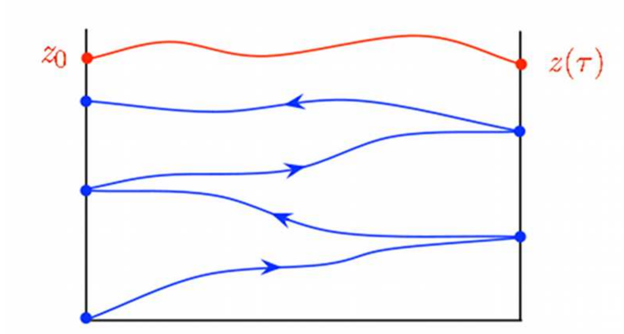
$$\left\|\mathbb{T}_\tau^- \mathbb{T}_\tau^+\right\| < 1. \tag{6}$$

Ito, Ramdani and Tucsnak [7, Lemma 2.2] show that every time of observability (*i.e.* every time $\tau > 0$ such that (3) holds) leads to (6).

In other words, $z_n^-(0)$ converges exponentially to z_0: there exists a constant $\alpha \in (0, 1)$ such that

$$\left\|z_n^-(0) - z_0\right\| \leq \alpha^n \left\|z_0^+ - z_0\right\|, \quad \forall\, n \leq 1.$$

The convergence of the algorithm by back and forth observers can be summarize by this illustration:



Partial reconstruction of z_0

Note that systems (4)–(5) are always well defined without the exact observability assumption. Then two questions arise naturally:

1. **Given arbitrary C and $\tau > 0$, does the algorithm converge ?**
2. **If it does, what is the limit of $z_n^-(0)$, and how is it related to z_0 ?**

Using the operator Ψ_τ, we have the orthogonal decomposition

$$X = \operatorname{Ker} \Psi_\tau \oplus \overline{\operatorname{Ran}(\Psi_\tau)^*}. \tag{7}$$

Theorem . *With the previous definitions and notation, and denoting by Π the orthogonal projector from X onto $\overline{\operatorname{Ran}(\Psi_\tau)^*}$, the following statements hold true:*

1. *We have for all $z_0, z_0^+ \in X$*
$$\left\|(I - \Pi)\left(z_n^-(0) - z_0\right)\right\| = \left\|(I - \Pi)\left(z_0^+ - z_0\right)\right\|, \quad \forall\, n \geq 1.$$
2. *The sequence $\left(\left\|\Pi\left(z_n^-(0) - z_0\right)\right\|\right)_{n \geq 1}$ is strictly decreasing and verifies*
$$\left\|\Pi\left(z_n^-(0) - z_0\right)\right\| \longrightarrow 0, \quad n \to \infty.$$
3. *The rate of convergence is exponential, i.e. there exists a constant $\alpha \in (0, 1)$, independent of z_0 and z_0^+, such that*
$$\left\|\Pi\left(z_n^-(0) - z_0\right)\right\| \leq \alpha^n \left\|\Pi\left(z_0^+ - z_0\right)\right\|, \quad \forall\, n \geq 1,$$
if and only if $\operatorname{Ran}(\Psi_\tau)^*$ *is closed in X.*

In general, it is difficult to characterise the projector Π. However, if the initial guess z_0^+ belongs to $\overline{\operatorname{Ran}(\Psi_\tau)^*}$ (for instance, $z_0^+ = 0$), then all iterative approximations $z_n^-(0)$ belong to $\overline{\operatorname{Ran}(\Psi_\tau)^*}$.

Corollary . *Under the assumptions of the previous Theorem, if $z_0^+ \in \overline{\operatorname{Ran}(\Psi_\tau)^*}$, then*

$$\left\|z_n^-(0) - \Pi z_0\right\| \longrightarrow 0, \quad n \to \infty.$$

Furthermore, the decay rate is exponential if and only if $\operatorname{Ran}(\Psi_\tau)^*$ *is closed in X.*

Generalization and example

Using the framework of well-posed linear systems, we can use a result of Curtain and Weiss [3] to handle the case of (some) unbounded observation operators and derive a result similar to Theorem **??** (formally, we also take $A^\pm = \pm A - \gamma C^*C$, with a suitably chosen $\gamma > 0$).

Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, with smooth boundary $\partial\Omega = \overline{\Gamma_0} \cup \overline{\Gamma_1}$, $\Gamma_0 \cap \Gamma_1 = \emptyset$ and Γ_0 and Γ_1 being relatively open in $\partial\Omega$. Denote by ν the unit normal vector of Γ_1 pointing towards the exterior of Ω. Consider the following wave system

$$\begin{cases} \ddot{w}(x,t) - \Delta w(x,t) = 0, & \forall x \in \Omega, t > 0, \\ w(x,t) = 0, & \forall x \in \Gamma_0, t > 0, \\ w(x,t) = u(x,t), & \forall x \in \Gamma_1, t > 0, \\ w(x,0) = w_0(x), \ \dot{w}(x,0) = w_1(x), & \forall x \in \Omega, \end{cases} \tag{8}$$

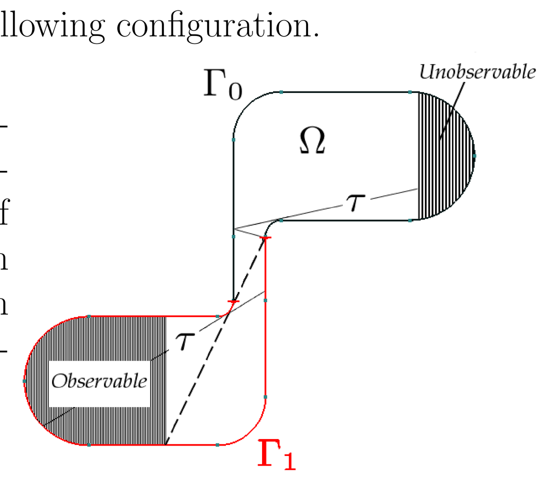
with u the input function (the control), and (w_0, w_1) the initial state. We observe this system on Γ_1, leading to

$$y(x,t) = -\frac{\partial(-\Delta)^{-1}\dot{w}(x,t)}{\partial\nu}, \quad \forall x \in \Gamma_1, t > 0. \tag{9}$$

Using a result of Guo and Zhang [6], we can show that the system (8)–(9) fits into the framework of well-posed linear systems and we can thus apply the generalization of the main Theorem to recover the observable part of the initial data (w_0, w_1).

For instance, let us consider the following configuration.

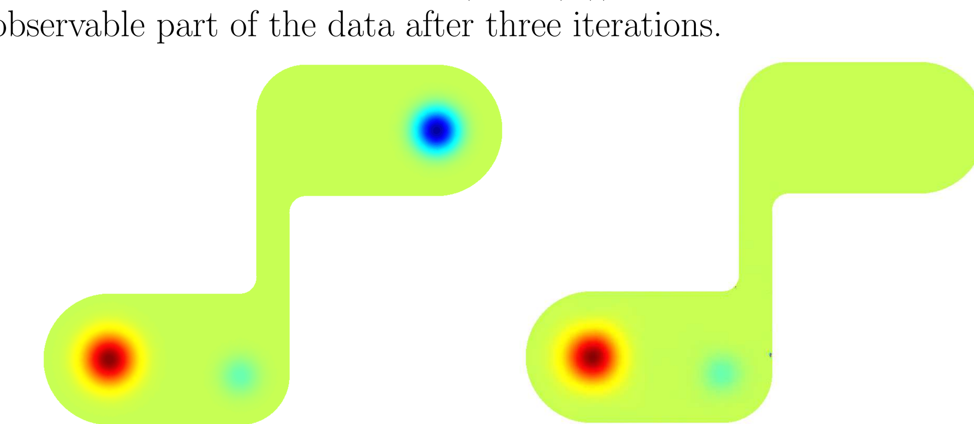
We can easily obtain two subdomains of Ω (the striped ones), using the geometric optic condition of Bardos, Lebeau and Rauch [2], such that all initial data with support in the left (resp. right) one are observable (resp. unobservable).



Simulations

We used Gmsh[a] to mesh our domain, and GetDP[b] to discretize (4)–(5) by finite elements in space and a Crank-Nicolson scheme in time.

We choose a suitable initial data to bring out these inclusions (in particular $w_1 \equiv 0$). We perform some simulations (using GMSH and GetDP) and obtain 6% of relative error (in $L^2(\Omega)$) on the reconstruction of the observable part of the data after three iterations.



The initial position and its reconstruction after 3 iterations

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[a] http://geuz.org/gmsh/
[b] http://www.geuz.org/getdp/