Robustness of Polynomial Stability of Semigroups

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1 Introduction

In this presentation we study the robustness of polynomial stability of a strongly continuous semigroup with respect to finite rank perturbations. The results have an application in the theory of robust output regulation for distributed parameter systems with infinite-dimensional exosystems.

We consider a semigroup $T(t)$ generated by $A : \mathcal{D}(A) \subset X \to X$ on a Hilbert space X. The semigroup is called *polynomially stable* if $T(t)$ is uniformly bounded, if $i\mathbb{R} \subset \rho(A)$, and if there exist constants $\alpha > 0$ and $M > 0$ such that

$$\|T(t)A^{-1}\| \leq \frac{M}{t^{1/\alpha}} \qquad \forall t > 0. \tag{1}$$

It should be noted that the assumption of uniform boundedness together with the estimate (1) imply that $T(t)$ is also strongly stable, i.e. $\|T(t)x\| \to 0$ as $t \to \infty$ for all $x \in X$.

2 Robustness of Polynomial Stability of Semigroups

In this section we show that the polynomial stability of $T(t)$ is robust with respect to certain classes of perturbations of its generator. The results are based on a recent characterization of polynomial stability on Hilbert spaces [1, 2].

We consider the preservation of the polynomial stability of $T(t)$ under finite rank perturbations of the form $A + BC$, where $B \in \mathcal{L}(\mathbb{C}^m, X)$ and $C \in \mathcal{L}(X, \mathbb{C}^m)$. We assume that for some $\beta, \gamma \in \mathbb{N}_0$ the operators B and C satisfy

$$\mathcal{R}(B) \subset \mathcal{D}((-A)^\beta) \qquad \text{and} \qquad \mathcal{R}(C^*) \subset \mathcal{D}((-A^*)^\gamma). \tag{2}$$

The above conditions immediately imply that $(-A)^\beta B$ and $(-A^*)^\gamma C^*$ are bounded operators. The main result of the presentation is stated in the following theorem.

Theorem 2.1. *If $\beta + \gamma \geq \alpha$, then there exists $\delta > 0$ such that for all B and C satisfying (2) and $\|(-A)^\beta B\| \cdot \|(-A^*)^\gamma C^*\| < \delta$ we have $\sigma(A+BC) \subset \mathbb{C}^-$, the semigroup $T_{A+BC}(t)$ generated by $A+BC$ is uniformly bounded, and there exists $M > 0$ such that*

$$\|T_{A+BC}(t)(A+BC)^{-1}\| \leq \frac{M}{t^{1/\alpha}}, \qquad \forall t > 0.$$

In particular, the perturbed semigroup is strongly and polynomially stable.

With slight modifications Theorem 2.1 extends to the situation where β and γ are not integers [4], as well as for perturbations that are not of finite rank [3].

3 Robustness of Polynomially Stable Closed-Loop System

The results in Section 2 can be applied to the study of robust output regulation of linear distributed parameter systems with infinite-dimensional exosystems. The internal model principle [5] states that any feedback controller achieving robust output tracking and disturbance rejection must contain a

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given number of copies of the dynamics of the exosystem. An unfortunate consequence of this is that — since the exosystem has an infinite number of eigenvalues on the imaginary axis — it is impossible to stabilize the resulting closed-loop system exponentially. It is possible, however, to construct an observer based error feedback controller that achieves polynomial closed-loop stability. For a closed-loop system with such a controller we can use Theorem 2.1 to study the preservation of the closed-loop stability under perturbations of the operators of the plant. The robustness of the controller implies that the output tracking and disturbance rejection are achieved for any perturbations for which the closed-loop system remains polynomially stable.

References

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