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CATEGORIES OF ALGEBRAIC LOGIC

$$((a \vee b) \wedge \neg a) \rightarrow b$$

a	b	$a \rightarrow b$
0	0	1
0	1	1
1	0	0
1	1	1

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Preface

This monograph, structured on 5 chapters, contains some notions of classical algebra, lattices theory, universal algebras, theory of categories which will be useful for the last part of this book and which will be devoted to the study of some algebraic categories that have their origin in mathematical logic. In writing this book I have mainly used papers [12]-[34] (revised and improved); the first germ of this book is my monograph [30]. Several sections on advanced topics have been added, and the References have been considerably expanded.

This book also contains some taken over results in a new context for some reference papers of mathematical literature (see References); when some results or proofs were taken *ad literam* from other papers I have mentioned.

The title of this monograph **Categories of algebraic logic** is justified because here we have included many algebras with origin in mathematical logic (it is the case of Boole algebras, Heyting algebras, residuated lattices, Hilbert algebras, Hertz algebras and Wajsberg algebras).

As the title indicates, the main emphasis is on algebraic methods.

Taking into consideration the algebraic character of this monograph, I have not insisted much on their origin, the reader could easily clarify the aspects by consulting the papers [2], [8], [35], [37], [49], [58], [73], [75], [76], [80] and [81].

Concerning the citations of some results in this book ,I have to say that if I have mentioned for example **Result x**.**y**.**z** it means that I refer to the result **z** contained in the paragraph **y** of the chapter **x.**

This book is self-containing, thus no previous knowledge in algebra or in logic is requested.The reader should, however, be familiar with standard mathematical reasoning and denotation.

Chapter 1 (**Preliminary notions**) is dedicated to some very often used notions in any mathematical branch. So, I have included notions about sets, binary relations, equivalence relations, functions and others.

In Chapter 2 (**Ordered sets**) we have presented basic notions on ordered sets (semilattices, lattices) and there are also presented Boole's elementary algebras notions.

Chapter 3 (**Topics on Universal algebra**) contains the basic notions of Universal Algebra, necessary for presenting some mathematical results in their own language. The presentation of the main results on the varieties of algebras will have an important role because they will permit, in the following chapters, the presentation of many results from the *equational categories* (often met in algebra).

Chapter 4 (**Topics on theory of categories**) contains basic results from Category's theory. I have included this chapter for presenting some results from previous chapters from the category's theory view point and because they will be needed to present in the same spirit some results from Chapter 5.

Chapter 5 (**Algebras of logic**), the main chapter of this monograph, contains algebraic notions relative to algebras with origin in mathematical logic; here I have included Heyting ,Hilbert, Hertz algebras, residuated lattices and Wajsberg algebras. This chapter contains classical results and my original results relative to these categories of algebras(more of these results have their origins in my Ph. D. Thesis: **Contributions to the study of Hilbert algebras).**

This book is didactic in its spirit, so it is mainly addressed to the students in the mathematical and computer science faculties (including post-graduate students, as well as the Ph.D. students in this field of mathematics); it could also be used by math teachers and also by everybody who works in algebraic logic.

Preliminary versions have been tested in several graduate courses in algebra which I teach to the students from the Faculty of Mathematical and Computer Science in Craiova.

Taking into consideration that order relation appears not only in algebra but also in other mathematical domains, we consider that this monograph is useful to a large category of mathematical users.

It is a pleasure for me to thank Professor **Constantin Năstăsescu,** Correspondent member of the Romanian Academy, and Professor **George Georgescu** from the Faculty of Mathematical and Computer Science, University of Bucharest, for the discussions which led to this book structure.

We also thank to Dr.Doc.**Nicolae Popescu**, Correspondent member of the Romanian Academy and the Official referee for this book on behalf of the Mathematical Section of the Romanian Academy, for his careful and competent reading and for sugessting several improvements.

This monograph (like other published books) was not possible without the effort of my colleague **Dana Piciu** (who was not only a precious collaborator, with whom I took several discussions concerning this book, but she also assured the typewriting and correction procedures); I use this moment to thank her for the collaboration in achieving this book and also in achieving, in the future, some other necessary algebraic books for mathematical study.

I would also like to thank my colleague **Mihai Coşoveanu** from the English Department of the University of Craiova for his precious help in supervising the English text and my son **Cătălin Buşneag** for his assistance in the manuscript preparation process.

Craiova, February 17, 2006 . Prof. univ. Dr. Dumitru Buşneag

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Chapter 3 : TOPICS IN UNIVERSAL ALGEBRA

Chapter 4 : TOPISC IN THEORY OF CATEGORIES

c. Residuated lattices

d. Wajsberg algebras

Index of Symbols

Iff	: abbreviation for *if and only if*
i.e.	: that is
$\Rightarrow$ ($\Leftrightarrow$)	: logical implication (equivalence)
$\forall$ ($\exists$)	: universal (existential) quantifier
$x\in A$	: the element x belongs to the set A
$A\subseteq B$	: the set A is included in B
$A\subsetneq B$	: the set A is strictly included in the set B
$A\cap B$	: intersection of the sets A and B
$A\cup B$	: union of the sets A and B
$A \setminus B$	: difference of the sets A and B
$A\Delta B$	: symmetrical difference of the sets A and B
P(M)	: the power set of M
$C_M A$	: the complementary subset of A relative to M
$A\times B$	: the cartesian product of the sets A and B
Δ_A	: the diagonal of cartesian product $A\times A$
∇_A	: the cartesian product $A\times A$
Rel(A)	: the set of all binary relations on A
Echiv(A)	: the set of all equivalence relations on A
A/ρ	: the quotient set of A by equivalence relation ρ
$\|M\|$ (or card(M))	: the cardinal of set M (if M is finite $\|M\|$ is the number of elements of M)
1_A	: identity function of the set A
$\mathbb{N}(\mathbb{N}^*)$	: the set of natural numbers (non nulles)
$\mathbb{Z}(\mathbb{Z}^*)$	: the set of integer numbers (non nulles)
$\mathbb{Q}(\mathbb{Q}^*)$	: the set of rational numbers (non nulles)
$\mathbb{Q}_+^*$	: the set of strictly positive rational numbers
$\mathbb{R}(\mathbb{R}^*)$	: the set of real numbers (non nulles)
$\mathbb{R}_+^*$	: the set of strictly positive real numbers
$\mathbb{C}(\mathbb{C}^*)$	: the set of complex numbers (non nulles)
$\approx$	: relation of isomorphism
$\leq$	: relation of order
$(A, \leq)$	: ordered set A
0	: the smallest (bottom) element in an ordered set
1	: the greatest (top) element ia an ordered set
inf (S)	: the infimum of the set S
sup(S)	: the supremum of the set S

$x \wedge y$	: inf {x,y}
$x \vee y$	: sup{x,y}
$(L, \wedge, \vee)$	: lattice L
(S]	: the ideal generated by S
[S)	: the filter generated by S
I(L)	: the set of all ideals of lattice L
F(L)	: the set of all filters of lattice L
Spec(L)	: the spectrum of lattice L (the set of all prime ideals of L)
a*	: the pseudocomplement of the element a
a′	: the complement of the element a
$a \to b$	: the pseudocomplement of a relative to b
L/I	: the quotient lattice of lattice L by ideal I
Con(A)	: the set of all congruences of A
[S]	: the subalgebra generated by S
$\ominus$ (Y)	: the congruence generated by Y
Hom(A, B)	: the set of all morphisms from A to B in a category
$A \approx B$	: the objects A and B are isomorphic
$A \not\approx B$	: the objects A and B are not isomorphic
$\mathbf{C}^{\mathbf{0}}$	: the dual of category **C**
Sets	: the category of sets
Pre	: the category of preordered sets
Ord	: the category of ordered sets
Ld(0,1)	: the category of bounded distributive lattices
B	: the category of Boole algebras
Top	: the category of topological spaces
Ker(f,g)	: the kernel of couple of morphisms (f,g)
Coker(f,g)	: the cokernel of couple of morphisms (f,g)
Ker(f)	: the kernel of the morphism f
Coker(f)	: the cokernel of the morphism f
$h_A(h^A)$	: the functor (cofunctor) associated with A
$\coprod_{i\in I} A_i$	: the coproducts of the family $(A_i)_{i\in I}$ of objects
$\prod_{i\in I} A_i$	: the products of the family $(A_i)_{i\in I}$ of objects
$\underset{\overrightarrow{i\in I}}{\lim} A_i$	: the inductive limit of the family $(A_i)_{i\in I}$ of objects
$\underset{\overleftarrow{i\in I}}{\lim} A_i$	: the colimit of the family $(A_i)_{i\in I}$ of objects
$M \coprod_P N$	: the fibred coproduct of M with N over P
$M \prod_P N$	: the fibred product of M with N over P
Ds(A)	: the set of all deductive systems of a Hilbert algebra A

Chapter 1

SETS AND FUNCTIONS

1.1. Sets. Operations on sets

In this book we will consider the sets in the way they were seen by GEORG CANTOR - the first mathematician who initiated their systematical study (known in mathematics as the *naive theory of sets*).

We ask the reader to consult books [59] and [79] to find more information about the paradoxes which imply this view point and the way they could be eliminated.

Definition 1.1.1. If A and B are two sets, we say that A is *included* in B (or A is a *subset* of B) if all elements of A are in B; in this case we write $A\subseteq B$; in the opposite case we write $A\nsubseteq B$.

So, we have: $A\subseteq B$ iff $x\in A \Rightarrow x\in B$

$A\nsubseteq B$ iff there is $x\in A$ such that $x\notin B$.

We say that the sets A and B are *equal* if for every x, $x\in A\Leftrightarrow x\in B$. So, $A=B \Leftrightarrow A\subseteq B$ and $B\subseteq A$.

We say that A is *strictly included* in B (we write $A\subset B$) if $A\subseteq B$ and $A\neq B$.

It is accepted the existence of a set which doesn't contain elements denoted by $\emptyset$ and it is called the *empty set.*It is immediate to deduce that for every set A, $\emptyset\subseteq A$ (because if by contrary we suppose $\emptyset\nsubseteq A$, then there is $x\in\emptyset$ such that $x\notin A$ – which is a contradiction!).

A different set from the empty set will be called *non-empty.*

For a set T, we denote by **P**(T) the set of all his subsets (clearly, $\emptyset$, $T\in$ **P**(T)); **P**(T) is called *power set* of T .

The following result is immediate:

If T is a set and A, B, C∈P (T), then

(i) $A\subseteq A$;

(ii) If $A\subseteq B$ and $B\subseteq A$, then $A=B$;

(iii) If $A\subseteq B$ and $B\subseteq C$, then $A\subseteq C$.

In this book we will use the notion of *family* of elements, indexed by a index set I .

So, by $(x_i)_{i\in I}$ we will denote a family of elements and by $(A_i)_{i\in I}$ a family of sets indexed by the *index set* I.

For a set T and A, B∈**P**(T) we define

$$A\cap B = \{x\in T : x\in A \text{ and } x\in B\},$$

$$A\cup B = \{x\in T : x\in A \text{ or } x\in B\},$$

$$A\setminus B = \{x\in T : x\in A \text{ and } x\notin B\},$$

$$A\triangle B = (A\setminus B) \cup (B\setminus A).$$

If $A\cap B=\emptyset$, the sets A and B will be called *disjoints*.

The operations ∩, ∪, \ and △ are called *intersection, union, difference* and *symmetrical difference*,respectively.

In particular, $T\setminus A$ will be denoted by $\complement_T(A)$ (or $\complement(A)$ if there is no danger of confusion) and will be called the *complementary* of A in T.

Clearly, for A, B∈**P**(T) we have

$A\setminus B = A\cap\complement_T(B)$,

$A\triangle B = (A\cup B) \setminus (A\cap B) = (A\cap\complement_T(B))\cup(\complement_T(A)\cap B)$,

$\complement_T(\emptyset) = T$, $\complement_T(T) = \emptyset$,

$A\cup\complement_T(A) = T$, $A\cap\complement_T(A) = \emptyset$ and $\complement_T(\complement_T(A)) = A$.

Also, for x∈T we have

$x\notin A\cap B \Leftrightarrow x\notin A$ or $x\notin B$,

$x\notin A\cup B \Leftrightarrow x\notin A$ and $x\notin B$,

$x\notin A\setminus B \Leftrightarrow x\notin A$ or $x\in B$,

$x\notin A\triangle B \Leftrightarrow (x\notin A$ and $x\notin B)$ or $(x\in A$ and $x\in B)$,

$x\notin\complement_T(A)\Leftrightarrow x\in A$.

From the above, it is immediate that if A, B∈**P**(T), then $\complement_T(A\cap B)=\complement_T(A)\cup\complement_T(B)$ and $\complement_T(A\cup B)=\complement_T(A)\cap\complement_T(B)$.

These two last equalities are known as *De Morgan's relations.*

For a non-empty family $(A_i)_{i\in I}$ of subsets of T we define

$$\bigcap_{i\in I} A_i = \{x\in T : x\in A_i \text{ for every } i\in I\} \text{ and}$$

$$\bigcup_{i\in I} A_i = \{x\in T : \text{there exists } i\in I \text{ such that } x\in A_i\}.$$

So, in a general context the De Morgan's relations are true.

If $(A_i)_{i\in I}$ is a family of subsets of T, then

$$C_T\left(\bigcap_{i\in I} A_i\right)=\bigcup_{i\in I} C_T(A_i) \text{ and } C_T\left(\bigcup_{i\in I} A_i\right)=\bigcap_{i\in I} C_T(A_i).$$

The following result is immediate :

Proposition 1.1.2. If T is a set and A, B, C∈P(T), then
(i) A∩(B∩C)=(A∩B)∩C and A∪(B∪C)=(A∪B)∪C ;
(ii) A∩B=B∩A and A∪B=B∪A ;
(iii) A∩T=A and A∪∅=A;
(iv) A∩A=A and A∪A=A.

Remark 1.1.3. 1. From (i) we deduce that the operations ∪ and ∩ are *associative*, from (ii) we deduce that both are *commutative*, from (iii) we deduce that T and ∅ are neutral elements for ∩ and respectively ∪, and by (iv) we deduce that ∩ and ∪ are *idempotent* operations on **P**(T).

2. By double inclusion we can prove that if A, B, C ∈ **P**(T) then

A∩(B∪C)=(A∩B)∪(A∩C) and A∪(B∩C)=(A∪B)∩(A∪C) ,

that is, the operations of intersection and union are *distributive* one relative to another.

Proposition 1.1.4. If A, B, C∈P(T), then
(i) A△(B△C)=(A△B)△C;
(ii) A△B=B△A;
(iii) A△∅=A and A△A=∅;
(iv) A∩(B△C)=(A∩B)△(A∩C).

Proof. (i). By double inclusion we can immediately prove that

$A\triangle(B\triangle C)=(A\triangle B)\triangle C=[A\cap \complement_T(B)\cap \complement_T(C)]\cup[\complement_T(A)\cap B\cap \complement_T(C)]\cup$
$\cup[\complement_T(A)\cap \complement_T(B)\cap C]\cup(A\cap B\cap C)$.

Another proof is in [32] by using the characteristic function (see also Proposition 1.3.12).

(ii), (iii). Clearly.

(iv). By double inclusion or using the distributivity of the intersection relative to union. ■

Definition 1.1.5. For two objects x and y, by *ordered pair* of these objects we mean the denoted set by (x, y) and defined by (x, y)={{x}, {x, y}}.

It is immediate that if x and y are two objects such that $x \neq y$, then $(x, y) \neq (y, x)$ and if (x, y) and (u, v) are two ordered pairs, then $(x, y) = (u, v) \Leftrightarrow x = u$ and $y = v$; in particular we have $(x, y) = (y, x) \Leftrightarrow x = y$.

Definition 1.1.6. If A and B are two sets, the set denoted by $A \times B = \{(a, b) : a \in A$ and $b \in B\}$ is called the *cartesian product* of the sets A and B.

Clearly :

$$A \times B \neq \emptyset \Leftrightarrow A \neq \emptyset \text{ and } B \neq \emptyset,$$
$$A \times B = \emptyset \Leftrightarrow A = \emptyset \text{ or } B = \emptyset,$$
$$A \times B = B \times A \Leftrightarrow A = B,$$
$$A' \subseteq A \text{ and } B' \subseteq B \Rightarrow A' \times B' \subseteq A \times B.$$

If A, B, C are three sets, we will define $A \times B \times C = (A \times B) \times C$.

The element ((a, b), c) from $A \times B \times C$ will be denoted by (a, b, c).

More generally, if $A_1, A_2, ..., A_n$ $(n \geq 3)$ are sets we will write

$$A_1 \times A_2 \times ... \times A_n = ((...((A_1 \times A_2) \times A_3) \times ...) \times A_n).$$

If A is a finite set, we denote by |A| the numbers of elements of A. Clearly, if A and B are finite subsets of a set M, then $A \cup B$ is a finite subset of M and $|A \cup B| = |A| + |B| - |A \cap B|$.

Now, we will present a general result known as *principle of inclusion and exclusion*:

Proposition 1.1.7. Let M be a finite set and $M_1, M_2, ..., M_n$ subsets of M. Then

$$\left|\bigcup_{i=1}^{n} M_i\right| = \sum_{1 \leq i \leq n} |M_i| - \sum_{1 \leq i < j \leq n} |M_i \cap M_j| + \sum_{1 \leq i < j < k \leq n} |M_i \cap M_j \cap M_k| -$$
$$- .... + (-1)^{n-1} |M_1 \cap ... \cap M_n|.$$

Proof. By mathematical induction relative to n. For $n = 1$ the equality from enounce is equivalent with $|M_1| = |M_1|$, which is true. For $n = 2$ we must show that

$$(1) \quad |M_1 \cup M_2| = |M_1| + |M_2| - |M_1 \cap M_2|$$

which is also true, because the elements from $M_1 \cap M_2$ are commune in M_1 and M_2.

Suppose that the equality from the enounce is true for every m subsets of M with $m < n$ and we will prove it for n subsets $M_1, M_2, ..., M_n$.

If we denote $N = \bigcup_{i=1}^{n-1} M_i$, then from (1) we have

$$(2) \quad \left|\bigcup_{i=1}^{n} M_i\right| = |N \cup M_n| = |N| + |M_n| - |N \cap M_n|.$$

But $N \cap M_n = \left(\bigcup_{i=1}^{n-1} M_i\right) \cap M_n = \bigcup_{i=1}^{n-1} (M_i \cap M_n)$, so we apply mathematical induction for $\bigcup_{i=1}^{n-1} (M_i \cap M_n)$. Since

$$(M_i \cap M_n) \cap (M_j \cap M_n) = (M_i \cap M_j) \cap M_n ,$$

$(M_i \cap M_n) \cap (M_j \cap M_n) \cap (M_k \cap M_n) = (M_i \cap M_j \cap M_k) \cap M_n$, etc, we obtain

(3)

$$|N \cap M_n| = \left|\bigcup_{i=1}^{n-1} (M_i \cap M_n)\right| = \sum_{i=1}^{n-1} |M_i \cap M_n| - \sum_{1 \le i < j \le n-1} |M_i \cap M_j \cap M_n| +$$

$$+ \sum_{1 \le i < j < k \le n-1} |M_i \cap M_j \cap M_k \cap M_n| - .... + (-1)^{n-2} \left|\bigcap_{i=1}^{n} M_i\right|.$$

If we apply mathematical induction for $|N|$ we obtain

$$(4) \quad \begin{aligned} |N| &= \left|\bigcup_{i=1}^{n-1} M_i\right| = \sum_{i=1}^{n-1} |M_i| - \sum_{1 \le i < j \le n-1} |M_i \cap M_j| + \\ &+ \sum_{1 \le i < j < k \le n-1} |M_i \cap M_j \cap M_k| - .... + (-1)^{n-2} \left|\bigcap_{i=1}^{n-1} M_i\right| \end{aligned}$$

so, by (3) and (4) the relation (2) will become

$$\left|\bigcup_{i=1}^{n} M_i\right| = |N| + |M_n| - |N \cap M_n| =$$

$$= \left(\sum_{i=1}^{n-1}|M_i| + |M_n|\right) - \left(\sum_{1\le i<j\le n-1}|M_i \cap M_j| + \sum_{i=1}^{n-1}|M_i \cap M_n|\right) +$$

$$+ \left(\sum_{1\le i<j<k\le n-1}|M_i \cap M_j \cap M_k| + \sum_{1\le i<j\le n-1}|M_i \cap M_j \cap M_n|\right) - \ldots +$$

$$+ \left[(-1)^{n-2}\left|\bigcap_{i=1}^{n-1} M_i\right|\right] -$$

$$- (-1)^{n-3} \sum_{1\le i_1<i_2<\ldots<i_{n-2}\le n-1}\left|M_{i_1} \cap M_{i_2} \cap \ldots \cap M_{i_{n-2}} \cap M_n\right| -$$

$$- (-1)^{n-2}\left|\bigcap_{i=1}^{n} M_i\right| = \sum_{i=1}^{n}|M_i| - \sum_{1\le i<j\le n}|M_i \cap M_j| +$$

$$+ \sum_{1\le i<j<k\le n}|M_i \cap M_j \cap M_k| - \ldots + (-1)^{n-1}\left|\bigcap_{i=1}^{n} M_i\right|.$$

By the principle of mathematical induction, the equality from enounce is true for every natural number n . ■

1.2. Binary relation on a set. Equivalence relations

Definition 1.2.1. If A is a set, by *binary relation* on A we mean every subset ρ of the cartesian product A×A. If a, b∈A and (a, b)∈ρ we say that the element a *is in relation* ρ with b.

We will also write aρb to denote that (a, b)∈ρ.

For a set A we denote by **Rel**(A) the set of all binary relations on A (that is, **Rel**(A)=**P**(A×A)).

The relation $\triangle_A$={(a, a) : a∈A} will be called the *diagonal* of a cartesian product A×A; we also denote ∇_A = A×A .

For ρ∈**Rel**(A) we define ρ^{-1}={(a, b)∈A×A : (b, a)∈ρ}. Clearly, $(\rho^{-1})^{-1}$=ρ, so, if we have θ∈**Rel** (A) such that $\rho\subseteq\theta \Rightarrow \rho^{-1}\subseteq\theta^{-1}$.

Definition 1.2.2. For ρ, ρ′∈Rel(A) we define his *composition* ρ∘ρ′ by ρ∘ρ′ = {(a, b)∈A×A : there is c∈A such that (a, c)∈ρ′ and (c, b)∈ρ}.

It is immediate the following result :

Proposition 1.2.3. Let ρ, ρ′, ρ′′∈Rel(A). Then

(i) $\rho\circ\Delta_A=\Delta_A\circ\rho=\rho$;

(ii) $(\rho\circ\rho')\circ\rho''=\rho\circ(\rho'\circ\rho'')$;

(iii) $\rho\subseteq\rho'\Rightarrow\rho\circ\rho''\subseteq\rho'\circ\rho''$ **and** $\rho''\circ\rho\subseteq\rho''\circ\rho'$;

(iv) $(\rho\circ\rho')^{-1}=\rho'^{-1}\circ\rho^{-1}$;

(v) $(\rho\cup\rho')^{-1}=\rho^{-1}\cup\rho'^{-1}$ **; more general, if** $(\rho_i)_{i\in I}$ **is a family of binary relations on A, then**

$$\left(\bigcup_{i\in I}\rho_i\right)^{-1}=\bigcup_{i\in I}\rho_i^{-1}.$$

For n∈ℕ and ρ∈**Rel** (A) we define

$$\rho^n=\begin{cases}\Delta_A & for\ n=0\\ \underbrace{\rho\circ\rho\circ\ldots\circ\rho}_{n\ times} & for\ n\geq 1\end{cases}.$$

It is immediate that for every m, n ∈ℕ, then $\rho^m\circ\rho^n=\rho^{m+n}$.

Definition 1.2.4. A relation ρ∈Rel (A) will be called

(i) ***reflexive*, if** $\Delta_A\subseteq\rho$;

(ii) ***symmetric*, if** $\rho\subseteq\rho^{-1}$;

(iii) ***anti-symmetric*, if** $\rho\cap\rho^{-1}\subseteq\Delta_A$;

(iv) ***transitive*, if** $\rho^2\subseteq\rho$.

It is immediate the following result

Proposition 1.2.5. A relation ρ∈Rel(A) is reflexive (symmetric, anti-symmetric, transitive) iff ρ^{-1} **is reflexive (symmetric, anti-symmetric, transitive).**

Definition 1.2.6. A relation ρ∈Rel(A) will be called an *equivalence* on A if it is reflexive, symmetric and transitive.

By **Echiv**(A) we denote the set of all equivalence relations on A; clearly, $\triangle_A$, $\nabla_A = A\times A \in$ **Echiv**(A).

Proposition 1.2.7. If $\rho\in$Echiv(A), then $\rho^{-1}=\rho$ and $\rho^2=\rho$.

Proof. Since ρ is symmetric, then $\rho\subseteq\rho^{-1}$. If $(a, b)\in\rho^{-1}$, then $(b, a)\in\rho\subseteq\rho^{-1} \Rightarrow (b, a)\in\rho^{-1} \Rightarrow (a, b)\in\rho$, hence $\rho^{-1}\subseteq\rho$, that is, $\rho^{-1}=\rho$. Since ρ is transitive we have $\rho^2\subseteq\rho$. Let $(x,y)\in\rho$. From $(x,x)\in\rho$ and $(x,y)\in\rho\Rightarrow(x,y)\in\rho\circ\rho=\rho^2$, hence $\rho\subseteq\rho^2$, that is, $\rho^2=\rho$. ■

Proposition 1.2.8. Let $\rho_1, \rho_2 \in$ Echiv (A).
Then $\rho_1\circ\rho_2\in$Echiv (A) iff $\rho_1 \circ \rho_2=\rho_2 \circ \rho_1$.
In this case $\rho_1 \circ\rho_2= \bigcap\limits_{\substack{\rho'\in Echiv(A)\\ \rho_1,\rho_2\subseteq\rho'}} \rho'$.

Proof. If ρ_1, ρ_2, $\rho_1\circ\rho_2\in$**Echiv**(A), then $(\rho_1\circ\rho_2)^{-1}=\rho_1\circ\rho_2$ (by Proposition 1.2.7.). By Proposition 1.2.3 we have $(\rho_1\circ\rho_2)^{-1}= \rho_2^{-1}\circ\rho_1^{-1}=\rho_2\circ\rho_1$, so $\rho_1\circ\rho_2=\rho_2\circ\rho_1$.

Conversely, suppose that $\rho_1\circ\rho_2=\rho_2\circ\rho_1$.

Since $\Delta_A\subseteq\rho_1, \rho_2\Rightarrow\Delta_A = \Delta_A\circ\Delta_A \subseteq \rho_1\circ\rho_2$, that is, $\rho_1\circ\rho_2$ is reflexive. Since $(\rho_1\circ\rho_2)^{-1}=\rho_2^{-1}\circ\rho_1^{-1}=\rho_2\circ\rho_1=\rho_1\circ\rho_2$, we deduce that $\rho_1\circ\rho_2$ is symmetric. From $(\rho_1\circ\rho_2)^2= (\rho_1\circ\rho_2)\circ(\rho_1\circ\rho_2) = \rho_1\circ(\rho_2\circ\rho_1)\circ\rho_2 = \rho_1\circ(\rho_1\circ\rho_2)\circ\rho_2 = \rho_1^2\circ\rho_2^2 = \rho_1\circ\rho_2$ we deduce that $\rho_1\circ\rho_2$ is transitive, so there is an equivalence relation on A.

Suppose now that $\rho_1\circ\rho_2=\rho_2\circ\rho_1$ and let $\rho'\in$**Echiv** (A) such that $\rho_1, \rho_2\subseteq\rho'$.

Then $\rho_1\circ\rho_2 \subseteq\rho'\circ\rho'=\rho'$, hence $\rho_1 \circ \rho_2 \subseteq \bigcap\limits_{\substack{\rho'\in Echiv(A)\\ \rho_1,\rho_2\subseteq\rho'}} \rho' \stackrel{def}{=} \theta$.

Since ρ_1, $\rho_2\in$**Echiv** (A) and $\rho_1\circ\rho_2\in$**Echiv** (A) $\Rightarrow\rho_1, \rho_2\subseteq\rho_1\circ\rho_2 \Rightarrow\theta\subseteq\rho_1\circ\rho_2$, that is, $\theta = \rho_1\circ\rho_2$.■

For $\rho\in$**Rel** (A), we define the *equivalence relation on A generated by ρ* by

$$\langle\rho\rangle = \bigcap\limits_{\substack{\rho'\in Echiv(A)\\ \rho\subseteq\rho'}} \rho'.$$

Clearly, the relation $<\rho>$ is characterized by the conditions: $\rho\subseteq<\rho>$ and if $\rho'\in$**Echiv**(A) such that $\rho\subseteq\rho' \Rightarrow <\rho>\subseteq\rho'$ (that is, $<\rho>$ is the lowest equivalence relation, relative to inclusion, which contains ρ).

Lemma 1.2.9. Let $\rho\in$Rel(A) and $\overline{r}=\Delta_A\cup\rho\cup\rho^{-1}$. Then the relation $\overline{r}$ has the following properties:

(i) $\rho\subseteq\overline{r}$;

(ii) $\overline{r}$ is reflexive and symmetric;

(iii) If ρ' is another reflexive and symmetric relation on A such that $\rho\subseteq\rho'$, then $\overline{r}\subseteq\rho'$.

Proof. (i). Clearly .

(ii). From $\Delta_A\subseteq\overline{r}$ we deduce that $\overline{r}$ is reflexive; since $\overline{r}^{-1}=(\Delta_A\cup\rho\cup\rho^{-1})^{-1}=\Delta_A^{-1}\cup\rho^{-1}\cup(\rho^{-1})^{-1}=\Delta_A\cup\rho\cup\rho^{-1}=\overline{r}$ we deduce that $\overline{r}$ is symmetric.

(iii). If ρ' is reflexive and symmetric such that $\rho\subseteq\rho'$, then $\rho^{-1}\subseteq\rho'^{-1}=\rho'$. Since $\Delta_A\subseteq\rho'$ we deduce that $\overline{r}=\Delta_A\cup\rho\cup\rho^{-1}\subseteq\rho'$.■

Lemma 1.2.10. Let $\rho\in$Rel(A) reflexive and symmetric and $\overline{r}=\bigcup_{n\geq 1} r^n$. Then $\overline{r}$ has the following properties:

(i) $\rho\subseteq\overline{r}$;

(ii) $\overline{r}$ is an equivalence relation on A;

(iii) If $\rho'\in$Echiv(A) such that $\rho\subseteq\rho'$, then $\overline{r}\subseteq\rho'$.

Proof. (i). Clearly .

(ii). Since $\Delta_A\subseteq\rho\subseteq\overline{r}$ we deduce that $\Delta_A\subseteq\overline{r}$, hence $\overline{r}$ is reflexive. Since ρ is symmetric and for every $n\in\mathbb{N}^*$, $(\rho^n)^{-1}=(\rho^{-1})^n=\rho^n$, we deduce that

$$\overline{r}^{-1}=\left(\bigcup_{n\geq 1} r^n\right)^{-1}=\bigcup_{n\geq 1}\left(r^n\right)^{-1}=\bigcup_{n\geq 1} r^n=\overline{r}\ ,$$

hence $\overline{r}$ is symmetric. Let now $(x, y)\in\overline{r}\circ\overline{r}$; then there is $z\in A$ such that $(x, z), (z, y)\in\overline{r}$, hence there exist $m, n\in\mathbb{N}^*$ such that $(x, z)\in\rho^m$ and $(z, y)\in\rho^n$. It is immediate that $(x, y)\in\rho^n\circ\rho^m=\rho^{n+m}\subseteq\overline{r}$, so $\overline{r}^2\subseteq\overline{r}$, hence $\overline{r}$ is transitive, that is, $\overline{r}\in$**Echiv** (A).

(iii). Let now $\rho'\in$**Echiv**(A) such that $\rho\subseteq\rho'$. Since $\rho^n\subseteq(\rho')^n=\rho'$ for every $n\in\mathbb{N}^*$ we deduce that $\overline{r}=\bigcup_{n\geq 1} r^n\subseteq\rho'$. ■

From Lemmas 1.2.9 and 1.2.10 we deduce :

Theorem 1.2.11. If $\rho\in Rel(A)$, then

$$\langle r \rangle = \bigcup_{n\geq 1}\left(\Delta_A \cup r \cup r^{-1}\right)^n .$$

Proposition 1.2.12. Let $\rho, \rho'\in Rel(A)$. Then

(i) $(\rho\cup\rho')^2=\rho^2\cup\rho'^2\cup(\rho\circ\rho')\cup(\rho'\circ\rho)$;

(ii) If $\rho, \rho'\in Echiv(A)$, then $\rho\cup\rho'\in Echiv(A)$ iff $\rho\circ\rho', \rho'\circ\rho \subseteq \rho\cup\rho'$.

Proof. (i). We have $(x, y)\in(\rho\cup\rho')^2 = (\rho\cup\rho')\circ(\rho\cup\rho') \Leftrightarrow$ there is $z\in A$ such that $(x, z)\in\rho\cup \rho'$ and $(z, y)\in\rho\cup\rho' \Leftrightarrow$ [$(x, z)\in\rho$ and $(z, y)\in\rho$] or [$(x, z)\in\rho'$ and $(z, y)\in\rho'$] or [$(x, z)\in\rho'$ and $(z, y)\in\rho$] or $(x, z)\in\rho$ and $(z, y)\in\rho'$] $\Leftrightarrow$ $(x, y)\in\rho^2$ or $(x, y)\in\rho'^2$ or $(x, y)\in\rho\circ\rho'$ or $(x, y)\in\rho'\circ\rho \Leftrightarrow (x, y)\in\rho^2\cup\rho'^2\cup(\rho\circ\rho')\cup(\rho'\circ\rho)$, hence $(\rho\cup\rho')^2 = \rho^2\cup\rho'^2\cup(\rho\circ\rho')\cup(\rho'\circ\rho)$.

(ii).,,$\Rightarrow$''. We have $\rho^2 = \rho$, $\rho'^2 = \rho'$ and $(\rho\cup\rho')^2 = \rho\cup\rho'$. So, the relation from (i) is equivalent with $\rho\cup\rho'=\rho\cup\rho'\cup(\rho\circ\rho')\cup(\rho'\circ\rho)$, hence $\rho\circ\rho'\subseteq\rho\cup\rho'$ and $\rho'\circ\rho\subseteq\rho\cup\rho'$.

,,$\Leftarrow$''. By hypothesis and relation (i) we deduce $(\rho\cup\rho')^2=\rho^2\cup\rho'^2\cup(\rho\circ\rho')\cup(\rho'\circ\rho)=\rho\cup\rho'\cup(\rho\circ\rho')\cup(\rho'\circ\rho)\subseteq\rho\cup\rho'$, hence $\rho\cup\rho'$ is transitive. Since $\Delta_A\subseteq\rho$ and $\Delta_A\subseteq\rho'\Rightarrow\Delta_A\subseteq\rho\cup\rho'$, hence $\rho\cup\rho'$ is reflexive. If $(x, y)\in\rho\cup\rho'\Rightarrow (x, y)\in\rho$ or $(x, y)\in\rho' \Rightarrow (y, x)\in\rho$ or $(y, x)\in\rho'\Rightarrow (y, x)\in\rho\cup\rho'$, hence $\rho\cup\rho'$ is symmetric, that is, an equivalence on A. ■

Proposition 1.2.13. Let A be a set and $\rho\in Rel(A)$ with the following properties:

(i) For every $x\in A$, there is $y\in A$ such that (y,x) ((x, y))$\in\rho$;

(ii) $\rho\circ\rho^{-1}\circ\rho =\rho$.

Then $\rho\circ\rho^{-1}$ ($\rho^{-1}\circ\rho$) $\in Echiv(A)$.

Proof.

We have $\rho\circ\rho^{-1}=\{(x, y) :$ there is $z\in A$ such that $(x, z)\in\rho^{-1}$ and $(z, y)\in\rho\}$.

So, to prove $\Delta_A\subseteq\rho\circ\rho^{-1}$ we must show that for every $x\in A$, $(x, x)\in\rho\circ\rho^{-1}$ $\Leftrightarrow$there is $z\in A$ such that $(z, x)\in\rho$ (which is assured by(i)). We deduce that $\rho\circ\rho^{-1}$ is reflexive (analogous for $\rho^{-1}\circ\rho$).

If $(x, y)\in \rho\circ\rho^{-1} \Rightarrow$ there is $z\in A$ such that $(x, z)\in\rho^{-1}$ and $(z, y)\in\rho \Leftrightarrow$ there is $z\in A$ such that $(y, z)\in\rho^{-1}$ and $(z, x)\in\rho \Leftrightarrow (y, x)\in\rho\circ\rho^{-1}$, hence $\rho\circ\rho^{-1}$ is symmetric (analogous for $\rho^{-1}\circ\rho$). Since $(\rho\circ\rho^{-1})\circ(\rho\circ\rho^{-1}) = (\rho\circ\rho^{-1}\circ\rho)\circ\rho^{-1} = \rho\circ\rho^{-1}$ we deduce that $\rho\circ\rho^{-1}$ is also transitive, so is an equivalence. Analogous for $\rho^{-1}\circ\rho$. ■

Definition 1.2.14. If $\rho\in$Echiv (A) and $a\in A$, by the *equivalence class of a relative to* ρ we understand the set $[a]_\rho = \{x\in A : (x, a)\in\rho\}$ (since ρ is in particular reflexive, we deduce that $a\in[a]_\rho$, so $[a]_\rho\neq\varnothing$ for every $a\in A$).

The set $A/\rho =\{[a]_\rho : a\in A\}$ is called *the quotient set* of A by relation ρ.

Proposition 1.2.15. If $\rho\in$Echiv (A), then

(i) $\bigcup_{a\in A}[a]_\rho=A$;

(ii) If $a, b\in A$ then $[a]_\rho=[b]_\rho \Leftrightarrow (a, b)\in\rho$;

(iii) If $a, b\in A$, then $[a]_\rho=[b]_\rho$ or $[a]_\rho\cap[b]_\rho=\varnothing$.

Proof. (i). Because for every $a\in A$, $a\in[a]_\rho$ we deduce the inclusion from right to left; because the other inclusion is clear, we deduce the requested equality.

(ii). If $[a]_\rho = [b]_\rho$, since $a\in[a]_\rho$ we deduce that $a\in[b]_\rho$ hence $(a, b)\in\rho$.

Let now $(a,b)\in\rho$ and $x\in[a]_\rho$; then $(x,a)\in\rho$. By the transitivity of ρ we deduce that $(x, b)\in\rho$, hence $x\in[b]_\rho$, so we obtain the inclusion $[a]_\rho \subseteq[b]_\rho$. Analogous we deduce that $[b]_\rho\subseteq[a]_\rho$, that is, $[a]_\rho=[b]_\rho$.

(iii). Suppose that $[a]_\rho\cap[b]_\rho\neq\varnothing$. Then there is $x\in A$ such that (x, a), $(x, b)\in\rho$, hence $(a, b)\in\rho$, that is, $[a]_\rho = [b]_\rho$ (by (ii)). ■

Definition 1.2.16. By *partition* of a set M we understand a family $(M_i)_{i\in I}$ of subsets of M which satisfies the following conditions :

(i) For every $i, j\in I$, $i\neq j \Rightarrow M_i \cap M_j=\varnothing$;

(ii) $\bigcup_{i\in I} M_i = M$.

Remark 1.2.17. From Proposition 1.2.15 we deduce that if ρ is an equivalence relation on the set A, then the set of equivalence classes relative to ρ determine a partition of A.

1.3. Functional relations. Notion of function. Classes of functions

Definition 1.3.1. Let A, B be two sets. A subset $R\subseteq A\times B$ will be called *functional relation* if

(i) for every $a\in A$ there is $b\in B$ such that $(a, b)\in R$;

(ii) $(a, b), (a, b')\in R \Rightarrow b=b'$.

We call *function* (or *mapping*) a triple $f=(A, B, R)$ where A and B are two non-empty sets and $R \subseteq A\times B$ is a functional relation .

In this case, for every $a\in A$ there is an unique element $b\in B$ such that $(a, b)\in R$; we denote $b=f(a)$ and the element b will be called the *image of a by f*. The set A will be called the *domain* (or *definition domain of* f) and B will be called the *codomain of* f ; we usually say that f is a function defined on A with values in B, writing by $f : A \to B$ or $A \xrightarrow{f} B$.

The functional relation R will be also called the *graphic of f* (we denote R by G_f, so $G_f =\{(a, f(a)) : a\in A\}$).

If $f : A\to B$ and $f ' : A'\to B'$ are two functions, we say that they are *equal* (and we write $f=f '$) if $A=A'$, $B=B'$ and $f(a)=f '(a)$ for every $a\in A$. For a set A, the function $1_A:A \to A$, $1_A(a)=a$ for every $a\in A$ is called *identity function on A* (in particular it is possible to talk about *identity function* on the empty set $1_\emptyset$).

If $A=\emptyset$ then there is an unique function $f : \emptyset\to B$ (which is the inclusion of $\emptyset$ in B). If $A\neq\emptyset$ and $B=\emptyset$, then it is clear that there doesn't exist a function from A to B.

If $f : A \to B$ is a function, $A'\subseteq A$ and $B'\subseteq B$ then we denote: $f(A')=\{f (a) : a\in A'\}$ and $f^{-1}(B')=\{a\in A:f (a)\in B'\}$, ($f(A')$ will be called the *image* of A' by f and $f^{-1}(B')$ *contraimage* of B' by f).

In particular, we denote $\mathbf{Im}(f)=f (A)$. Clearly, $f(\emptyset)=\emptyset$ and $f^{-1}(\emptyset)=\emptyset$.

Definition 1.3.2. For two functions $f:A \to B$ and $g:B \to C$ we call their *composition* the function denoted by $g\circ f :A \to C$ and defined by $(g\circ f)(a)=g(f(a))$ for every $a\in A$.

Proposition 1.3.3. If we have three functions $A\xrightarrow{f} B\xrightarrow{g} C\xrightarrow{h} D$ then

(i) $h\circ(g\circ f)=(h\circ g)\circ f$;

(ii) $f\circ 1_A=1_B\circ f=f$.

Proof.(i). Indeed, $h\circ(g\circ f)$ and $(h\circ g)\circ f$ have A as domain of definition, D as codomain and for every $a\in A$, $(h\circ(g\circ f))(a)=((h\circ g)\circ f)(a)=h(g(f(a)))$.

(ii).Clearly. ■

Proposition 1.3.4. Let f:A →B, A′, A″ ⊆A, B′, B″⊆B and $(A_i)_{i\in I}$, $(B_j)_{j\in J}$ two families of subsets of A and respective B. Then

(i) **A′ ⊆ A″ ⇒ f(A′) ⊆ f(A″);**

(ii) **B′⊆B″⇒f^{-1}(B′)⊆f^{-1}(B″);**

(iii) $f\left(\bigcap_{i\in I} A_i\right)\subseteq \bigcap_{i\in I} f(A_i)$**;**

(iv) $f\left(\bigcup_{i\in I} A_i\right)= \bigcup_{i\in I} f(A_i)$**;**

(v) $f^{-1}\left(\bigcap_{j\in J} B_j\right)= \bigcap_{j\in J} f^{-1}(B_j)$**;**

(vi) $f^{-1}\left(\bigcup_{j\in J} B_j\right)= \bigcup_{j\in J} f^{-1}(B_j)$.

Proof. (i). If b∈f(A′), then b=f(a) with a∈A′; since A′ ⊆ A″ we deduce that b∈f(A″), that is, f(A′)⊆f(A″).

(ii). Analogous as in the case of (i).

(iii). Because for every k∈I, $\bigcap_{i\in I} A_i \subseteq A_k$, by (i) we deduce that $f\left(\bigcap_{i\in I} A_i\right)\subseteq f(A_k)$, hence $f\left(\bigcap_{i\in I} A_i\right)\subseteq \bigcap_{i\in I} f(A_i)$.

(iv). The equality follows immediately from the following equivalences : b∈ $f\left(\bigcup_{i\in I} A_i\right)$⇔ there is a∈ $\bigcup_{i\in I} A_i$ such that b=f(a) ⇔ there is i_0∈I such that a∈ A_{i_0} and b=f(a)⇔ there is i_0∈I such that b∈f(A_{i_0})⇔ b∈ $\bigcup_{i\in I} f(A_i)$.

(v). Follows immediately from the equivalences: a∈ $f^{-1}\left(\bigcap_{j\in J} B_j\right)$⇔ f(a)∈ $\bigcap_{j\in J} B_J$ ⇔for every j∈J, f(a)∈B_j ⇔ for every j∈J, a∈$f^{-1}(B_j)$ ⇔a∈ $\bigcap_{j\in J} f^{-1}(B_j)$.

(vi). Analogous as for (iv). ■

Definition 1.3.5. A function $f : A \to B$ will be called

(i) *injective* or *one-to-one*, if for every $a, a' \in A$, $a \neq a' \Rightarrow f(a) \neq f(a')$ (equivalent with $f(a) = f(a') \Rightarrow a = a'$);

(ii) *surjective* or *onto*, if for every $b \in B$, there is $a \in A$ such that $b = f(a)$;

(iii) *bijective*, if it is simultaneously injective and surjective.

If $f : A \to B$ is bijective, the function $f^{-1} : B \to A$ defined by $f^{-1}(b) = a \Leftrightarrow b = f(a)$ ($b \in B$ and $a \in A$) will be called the *inverse* of f.

It is immediate to see that $f^{-1} \circ f = 1_A$ and $f \circ f^{-1} = 1_B$.

Proposition 1.3.6. Let $f : A \to B$ and $g : B \to C$ two functions.

(i) If f and g are injective (surjective; bijective) then $g \circ f$ is injective (surjective, bijective; in this last case, $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$);

(ii) If $g \circ f$ is injective (surjective, bijective) then f is injective, (g is surjective; f is injective and g is surjective).

Proof. (i). Let $a, a' \in A$ such that $(g \circ f)(a) = (g \circ f)(a')$. Then $g(f(a)) = g(f(a'))$. Since g is injective we deduce that $f(a) = f(a')$; since f is injective we deduce that $a = a'$, that is, $g \circ f$ is injective.

We suppose f and g are surjective and let $c \in C$; since g is surjective, $c = g(b)$ with $b \in B$. By the surjectivity of f we deduce that $b = f(a)$ with $a \in A$, so $c = g(b) = g(f(a)) = (g \circ f)(a)$, that is, $g \circ f$ is surjective.

If f and g are bijective, then the bijectivity of $g \circ f$ is immediate. To prove the equality $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$, let $c \in C$. We have $c = g(b)$ with $b \in B$ and $b = f(a)$ with $a \in A$. Since $(g \circ f)(a) = g(f(a)) = g(b) = c$, we deduce that $(g \circ f)^{-1}(c) = a = f^{-1}(b) = f^{-1}(g^{-1}(c)) = (f^{-1} \circ g^{-1})(c)$, that is, $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

(ii). We suppose that $g \circ f$ is injective and let $a, a' \in A$ such that $f(a) = f(a')$. Then $g(f(a)) = g(f(a')) \Leftrightarrow (g \circ f)(a) = (g \circ f)(a') \Rightarrow a = a'$, that is, f is injective.

If $g \circ f$ is surjective, for $c \in C$, there is $a \in A$ such that $(g \circ f)(a) = c \Leftrightarrow g(f(a)) = c$, that is, g is surjective.

If $g \circ f$ is bijective, then in particular $g \circ f$ is injective and surjective, hence f is injective and g surjective. ■

Proposition 1.3.7. **Let M and N two sets and $f:M\to N$ a function. Between the sets P(M) and P(N) we define the functions $f_* : P(M)\to P(N)$ and $f^* : P(N)\to P(M)$ by $f_*(A)=f(A)$, for every $A \in P(M)$ and $f^*(B) = f^{-1}(B)$, for every $B\in P(N)$.**

The following are equivalent :

(i) f is injective;

(ii) f_* is injective;

(iii) $f^*\circ f_*=1_{P(M)}$;

(iv) f^* is surjective;

(v) $f(A\cap B) = f(A)\cap f(B)$, for every $A, B\in P(M)$;

(vi) $f(\complement_M A)\subseteq\complement_N f(A)$, for every $A\in P(M)$;

(vii) If $g, h:L \to M$ are two functions such that $f\circ g = f\circ h$, then $g = h$;

(viii) There is a function $g :N \to M$ such that $g\circ f = 1_M$.

Proof. We will prove the implications using the following schema: (i)⇒(ii)⇒(iii)⇒(iv)⇒(v)⇒(vi)⇒(vii)⇒(i) and the equivalence (i)⇔(viii) .

(i)⇒(ii). Let $A, A'\in \mathbf{P}(M)$ such that $f_*(A)=f_*(A')\Leftrightarrow f(A)=f(A')$.

If $x\in A$, then $f(x)\in f(A)\Rightarrow f(x)\in f(A')\Rightarrow$ there is $x'\in A'$ such that $f(x)=f(x')$. Since f is injective, then $x=x'\in A'$, that is, $A\subseteq A'$; analogous $A'\subseteq A$, hence $A=A'$, that is , f_* is injective.

(ii)⇒(iii). For $A\in\mathbf{P}(M)$ we must show that $(f^*\circ f_*)(A) = A\Leftrightarrow f^{-1}(f(A))=A$. The inclusion $A\subseteq f^{-1}(f(A))$ is true for every function f. For the converse inclusion, if $x\in f^{-1}(f(A))\Rightarrow f(x)\in f(A)\Rightarrow$ there is $x'\in A$ such that $f(x)=f(x')\Rightarrow f_*(\{x\})=f_*(\{x'\})$ $\Rightarrow \{x\}=\{x'\}\Rightarrow x = x'\in A$, that is, $f^{-1}(f(A))\subseteq A$, hence $f^{-1}(f(A)) = A$.

(iii)⇒(iv). Since $f^*\circ f_*=1_{P(M)}$, for every $A\in\mathbf{P}(M)$, $f^*(f_*(A))=A$, so, if we denote by $B=f_*(A)\in\mathbf{P}(N)$, then $f^*(B)=A$, which means f^* is surjective.

(iv)⇒(v). Let $A, B\in\mathbf{P}(M)$ and $A', B'\in\mathbf{P}(N)$ such that $A=f^{-1}(A')$ and $B=f^{-1}(B')$. Then $f(A\cap B)=f(f^{-1}(A)\cap f^{-1}(B'))=f(f^{-1}(A'\cap B'))$.

We want to show that $f(f^{-1}(A'))\cap f(f^{-1}(B'))\subseteq f(f^{-1}(A'\cap B'))$.

If $y\in f(f^{-1}(A'))\cap f(f^{-1}(B'))\Rightarrow y\in f(f^{-1}(A'))$ and $y\in f(f^{-1}(B'))\Rightarrow$ there exist $x'\in f^{-1}(A')$ and $x''\in f^{-1}(B')$ such that $y=f(x')=f(x'')$. Since $x'\in f^{-1}(A')$ and $x''\in f^{-1}(B')\Rightarrow f(x')\in A'$ and $f(x'')\in B'$, hence $y\in A'\cap B'$. Since $y = f(x') \Rightarrow x'\in f^{-1}(A'\cap B')$, that is, $y\in f(f^{-1}(A'\cap B'))$.

So, $f(A\cap B)\supseteq f(A)\cap f(B)$; since the inclusion $f(A\cap B)\subseteq f(A)\cap f(B)$ is clearly true for every function f, we deduce that $f(A\cap B)=f(A)\cap f(B)$.

(v)⇒(vi). For $A\in\mathbf{P}(M)$ we have $f(A)\cap f(\complement_M A)=f(A\cap\complement_M A)=f(\emptyset)=\emptyset$, hence $f(\complement_M A)\subseteq\complement_N f(A)$.

(vi)⇒(vii). Let g, h : L→M two functions such that $f\circ g=f\circ h$ and suppose by contrary that there is $x\in L$ such that $g(x)\neq h(x)$, which is, $g(x)\in\complement_M\{h(x)\}$; then $f(g(x))\in f(\complement_M\{h(x)\})\subseteq\complement_N f(h(\{x\}))=\complement_N\{f(h(x))\}$ hence $f(g(x))\neq f(h(x)) \Leftrightarrow (f\circ g)(x)\neq (f\circ h)(x) \Leftrightarrow f\circ g\neq f\circ h$, a contradiction!.

(vii)⇒(i). Let $x, x'\in M$ such that $f(x)=f(x')$ and suppose by contrary that $x\neq x'$. We denote $L = \{x, x'\}$ and define $g, h : L\to M$, $g(x)=x$, $g(x')=x'$, $h(x)=x'$, $h(x')=x$, then $g\neq h$ and $f\circ g=f\circ h$, a contradiction!.

(i)⇒(viii). If we define $g:N\to M$, $g(y)=x$ if $y=f(x)$ with $x\in M$ and y_0 if $y\notin f(M)$, then by the injectivity of f, we deduce that g is correctly defined and clearly $g\circ f=1_M$.

(viii)⇒(i). If $x, x'\in M$ and $f(x)=f(x')$, then $g(f(x))=g(f(x'))\Rightarrow x=x'$, which means f is injective. ■

Proposition 1.3.8. With the notations from the above proposition, the following assertions are equivalent :

(i) f is surjective ;

(ii) f_* is surjective ;

(iii) $f_*\circ f^*=1_{P(N)}$;

(iv) f^* is injective ;

(v) $f(\complement_M A)\supseteq\complement_N f(A)$, for every $A\in\mathbf{P}(M)$;

(vi) If g, h:N→P are two functions such that $g\circ f = h\circ f$, then g = h ;

(vii) There is a function g:N→M such that $f\circ g=1_N$.

Proof. I will prove the implications (i)⇒(ii)⇒(iii)⇒(iv)⇒(v)⇒(vi)⇒(i) and the equivalence (i)⇔(vii).

(i)⇒(ii). Let $B\in\mathbf{P}(N)$ and $y\in B$; then there is $x_y\in M$ such that $f(x_y) = y$.

If we denote $A=\{x_y : y\in B\}\subseteq M$, then $f(A)=B\Leftrightarrow f_*(A)=B$.

(ii)⇒(iii). We need to prove that for every $B\in\mathbf{P}(N)$, $f(f^{-1}(B))=B$. The inclusion $f(f^{-1}(B))\subseteq B$ is true for every function f. Let now $y\in B$; since f_* is surjective, there is $A\subseteq M$ such that $f_*(A)=\{y\}\Leftrightarrow f(A)=\{y\}$, hence there is $x\in A$ such that $y=f(x)$; since $y\in B\Rightarrow x\in f^{-1}(B)\Rightarrow y=f(x)\in f(f^{-1}(B))$, so we also have the contrary inclusion $B\subseteq f(f^{-1}(B))$, hence the equality $B = f(f^{-1}(B))$.

(iii)$\Rightarrow$(iv). If B_1, $B_2 \in P(N)$ and $f^*(B_1)=f^*(B_2)$, then $f_*(f^*(B_1))=f_*(f^*(B_2)) \Leftrightarrow 1_{P(N)}(B_1)=1_{P(N)}(B_2) \Leftrightarrow B_1=B_2$, that means f^* is injective.

(iv)$\Rightarrow$(v). Let $A\subseteq M$; to prove that $f(\complement_M A)\supseteq \complement_N f(A)$, we must show that $f(\complement_M A)\cup f(A)=N \Leftrightarrow f(\complement_M A\cup A)=N \Leftrightarrow f(M)=N$. Suppose by contrary that there is $y_0\in N$ such that for every $x\in M$, $f(x)\neq y_0$, that means, $f^{-1}(\{y_0\})=\emptyset \Leftrightarrow f^*(\{y_0\})=\emptyset$. Since $f^*(\emptyset)=\emptyset \Rightarrow f^*(\{y_0\})=f^*(\emptyset)$; but f^* is supposed injective, hence $\{y_0\}=\emptyset$, a contradiction!.

(v)$\Rightarrow$(vi). In particular for A=M we have

$f(\complement_M M)\supseteq \complement_N f(M) \Leftrightarrow f(\emptyset)\supseteq \complement_N f(M) \Leftrightarrow \emptyset \supseteq \complement_N f(M) \Leftrightarrow f(M)=N$.

If g, $h:N\rightarrow P$ are two functions such that $g\circ f=h\circ f$, then for every $y\in N$, there is $x\in M$ such that $f(x)=y$ (because $f(M)=N$), and so $g(y)=g(f(x))=(g\circ f)(x)=(h\circ f)(x)=h(f(x)) = h(y)$, which means $g = h$.

(vi)$\Rightarrow$(i). Suppose by contrary that there is $y_0\in N$ such that $f(x)\neq y_0$, for every $x\in M$. We define g, h : $N\rightarrow\{0, 1\}$ by: $g(y)=0$, for every $y\in N$ and
$$h(y)=\begin{cases}0, & \text{for } y\in N-\{y_0\}\\ 1, & \text{for } y=y_0\end{cases}.$$

Clearly $g\neq h$ and $g\circ f=h\circ f$, a contradiction, hence f is surjective.

(i)$\Rightarrow$(vii). If for every $y\in N$ we consider a unique $x_y\in f^{-1}(\{y\})$, we obtain a function $g : N\rightarrow M$, $g(y) = x_y$, which clearly verifies the equality $f\circ g =1_N$.

(vii)$\Rightarrow$(i). For $y\in N$, if we write that $f(g(y)) = y$, then $y = f(x)$, with $x = g(y)\in M$, which means that f is surjective.■

From the above propositions we deduce

Corollary 1.3.9. With the notations from Proposition 1.3.7, the following assertions are equivalent :

(i) f is bijective;

(ii) $f(\complement_M A)=\complement_N f(A)$, for every $A\in P(M)$;

(iii) f_* and f^* are bijective;

(iv) There is a function $g:N\rightarrow M$ such that $f\circ g =1_N$ and $g\circ f =1_M$.

Proposition 3.10. Let M be a finite set and $f:M\rightarrow M$ a function. The following assertions are equivalent :

(i) f is injective;

(ii) f is surjective;

(iii) f is bijective.

Proof. We prove the implications: (i)⇒(ii)⇒(iii)⇒(i).

(i)⇒(ii). If f is injective, then f(M) and M have the same number of elements; since f (M)⊆M we deduce that f (M) = M, which means f is surjective.

(ii)⇒(iii). If f is surjective, then for every y∈M there is a unique x_y∈M such that $f(x_y) = y$, that means f is injective.

(iii)⇒(i). Clearly. ■

Proposition 1.3.11. Let M and N be two finite sets with m, respective n elements. Then

(i) The number of functions from M to N is equal with n^m;

(ii) If m = n, the number of bijective functions from M to N is equal with m!;

(iii) If m ≤ n, the number of injective functions from M to N is equal with A_n^m;

(iv) If m ≥ n, the number of surjective functions from M to N is equal with $n^m - C_n^1(n-1)^m + C_n^2(n-2)^m - ... + (-1)^{n-1}C_n^{n-1}$.

Proof. (i). By mathematical induction relative to m; if m=1, then the set M contain only one element, so we have $n = n^1$ functions from M to N. Supposing the enounce true for M sets with maximum m-1 elements.

If M is a set with m elements, it is possible to write $M = M'\cup\{x_0\}$, with $x_0\in M$ and M′ a subset of M with m-1 elements such that $x_0\notin M'$.

For every y∈N and g : M′→N a function, we consider $f_{g,\,y}$: M→N, $f_{g,\,y}(x)=g(x)$ if x∈M′ and y if $x=x_0$, we deduce that to every function g: M′→N we could assign n distinct functions from M to N whose restrictions to M′ are equal with g. By applying hypothesis of induction for the functions from M′ to N, we deduce that from M to N we could define $n\cdot n^{m-1} = n^m$ functions.

(ii). Mathematical induction relative to m; if m=1, the sets M and N have only one element, so there is only a bijective function from M to N.

Suppose the enounce true for all sets M′ and N′ both having almost m-1 elements and let M and N sets both having m elements. If we write $M=M'\cup\{x_0\}$, with $x_0\in M$ and M′ subset of M with m-1 elements $x_0\notin M'$, then every bijective function f: M→N is determined by $f(x_0)\in N$ and a bijective function g:M′→N′, where $N'=N \setminus \{f(x_0)\}$. Because we can choose $f(x_0)$ in m kinds and g in (m-1)! kinds (by induction hypothesis) we deduce that from M to N we can define (m-1)!· m =m! bijective functions.

(iii). If $f:M\to N$ is injective, taking $f(M)\subseteq N$ as codomain for f, we deduce that f determines a bijective function $\overline{f}:M\to f(M)$, $\overline{f}(x)=f(x)$, for every $x\in M$, and f(M) has m elements. Conversely, if we choose in N a part N′ of its with m elements, then we can establish m! bijective functions from M to N′ (by (ii)). Because the numbers of subsets N′ of N with m elements are equal with C_n^m, we deduce that we can construct m! $C_n^m = A_n^m$ injective functions from M to N.

(iv). Let's consider $M=\{x_1, x_2, ..., x_m\}$, $N=\{y_1, y_2, ..., y_n\}$ and M_i the set of all functions from M to N such that y_i is not an image of any elements of M, i =1,2,...,n.

So, if we denote by F_m^n the set of functions from M to N, the set of surjective functions S_m^n from M to N will be the complementary of $M_1\cup M_2\cup ... \cup M_n$ in F_m^n, so by Proposition 1.1.7 we have:

$$(1)\quad \left|S_m^n\right| = \left|F_m^n\right| - \left|\bigcup_{i=1}^{n} M_i\right| = n^m - \left|\bigcup_{i=1}^{n} M_i\right| = n^m - \sum_{i=1}^{n}\left|M_i\right| + \sum_{1\le i<j\le n}\left|M_i\cap M_j\right| - \\ - \sum_{1\le i<j<k\le n}\left|M_i\cap M_j\cap M_k\right| + ... + (-1)^n\left|M_1\cap M_2\cap ....\cap M_n\right|.$$

Because M_i is in fact the set of all functions defined on M with values in $N\setminus\{y_i\}$, $M_i\cap M_j$ the set of all functions defined on M with values in $N\setminus\{y_i, y_j\}$ ..., by (i) we obtain

(2) $|M_i| = (n-1)^m$, $|M_i\cap M_j| = (n-2)^m$, ..., etc,

($|M_1\cap M_2 \cap ...\cap M_n| = 0$, because $M_1\cap M_2 \cap ...\cap M_n = \emptyset$).

Since the sums which appear in (1) have, respective, C_n^1, C_n^2, ..., C_n^n equal terms, from (2), we obtain for relation (1)

$$|S_m^n| = n^m - C_n^1(n-1)^m + C_n^2(n-2)^m - ... + (-1)^{n-1}C_n^{n-1}. \quad \blacksquare$$

For a non-empty set M and $A\in \mathbf{P}(M)$ we define $\varphi_A : M\to\{0,1\}$,

$$\varphi_A(x) = \begin{cases} 0, & for\ x\notin A \\ 1, & for\ x\in A \end{cases}$$

for every $x\in M$; the function φ_A will be called the *characteristic function* of A .

Proposition 1.3.12. If $A, B\in P(M)$, then

(i) $A=B \Leftrightarrow \varphi_A=\varphi_B$;

(ii) $\varphi_\emptyset=0$, $\varphi_M=1$;

(iii) $\varphi_{A\cap B}=\varphi_A\varphi_B$, $\varphi_A^2=\varphi_A$;

(iv) $\varphi_{A\cup B}=\varphi_A+\varphi_B-\varphi_A\varphi_B$;

(v) $\varphi_{A\setminus B}=\varphi_A-\varphi_A\varphi_B$, $j_{C_M A}=1-\varphi_A$;

(vi) $\varphi_{A\Delta B}=\varphi_A+\varphi_B-2\varphi_A\varphi_B$.

Proof.

(i).”⇒”. Clearly.

“⇐”. Suppose that $\varphi_A=\varphi_B$ and let $x\in A$; then $\varphi_A(x)=\varphi_B(x)=1$, hence $x\in B$, that is, $A\subseteq B$. Analogous $B\subseteq A$, hence $A=B$.

(ii). Clearly.

(iii). For $x\in M$ we have the cases: ($x\notin A$, $x\notin B$) or ($x\in A$, $x\notin B$) or ($x\notin A$, $x\in B$) or ($x\in A$, $x\in B$). In every above situations we have $\varphi_{A\cap B}(x)=\varphi_A(x)\varphi_B(x)$.

Since $A\cap A=A \Rightarrow \varphi_A=\varphi_A\varphi_A=\varphi_A^{\,2}$.

(iv), (v). Analogous with (iii).

(vi). We have $\varphi_{A\Delta B}=\varphi_{(A\setminus B)\cup(B\setminus A)}=\varphi_{A\setminus B}+\varphi_{B\setminus A}-\varphi_{A\setminus B}\varphi_{B\setminus A}=$

$$=\varphi_A-\varphi_A\varphi_B+\varphi_B-\varphi_B\varphi_A-\varphi_{(A\setminus B)\cap(B\setminus A)}=\varphi_A+\varphi_B-2\varphi_A\varphi_B$$

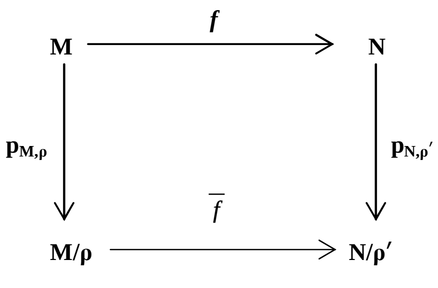
(since $(A\setminus B)\cap(B\setminus A)=\emptyset$). ■

Let M be a set and $\rho\in$**Echiv** (M). The function $p_{\rho,M}: M\to M/\rho$ defined by $p_{\rho,M}(x)=[x]_\rho$ for every $x\in M$ is surjective; $p_{\rho,M}$ will be called *canonical surjective function*.

Proposition 1.3.13. Let M and N two sets, $\rho\in$Echiv (M), $\rho'\in$Echiv (N) and $f: M\to N$ a function with the following property :

$$(x, y)\in\rho \Rightarrow (f(x), f(y))\in\rho',\ \text{for every } x, y\in M.$$

Then, there is a unique function $\overline{f}: M/\rho\to N/\rho'$ such that the diagram



is commutative (i.e, $p_{N,\rho'}\circ f=\overline{f}\circ p_{M,\rho}$, where $p_{M,\rho}$, $p_{N,\rho'}$, are canonical surjective functions).

Proof. For $x\in M$, we denote by $[x]_\rho$ the equivalence class of x modulo the relation ρ. For $x\in M$, we define $\overline{f}([x]_\rho) = [f(x)]_{\rho'}$. If $x, y\in M$ such that $[x]_\rho=[y]_\rho$ $\Leftrightarrow$ $(x, y)\in\rho \Rightarrow (f(x), f(y))\in\rho'$ (from enounce) $\Rightarrow [f(x)]_{\rho'}=[f(y)]_{\rho'}$, that means, $\overline{f}$ is correctly defined.

If $x\in M$, then $(\overline{f}\circ p_{M,\rho})(x)=\overline{f}(p_{M,\rho}(x)) = \overline{f}([x]_\rho)=[f(x)]_{\rho'}=p_{N,\rho'}(f(x))=(p_{N,\rho'}\circ f)(x)$, that is, $p_{N,\rho'}\circ f=\overline{f}\circ p_{M,\rho}$.

To prove the uniqueness of $\overline{f}$, suppose that we have another function $\overline{f}': M/\rho\to N/\rho'$ such that $p_{N,\rho'}\circ f=\overline{f}'\circ p_{M,\rho}$, and let $x\in M$.

Thus $\overline{f}'([x]_\rho)=\overline{f}'(p_{M,\rho}(x))=(\overline{f}'\circ p_{M,\rho})(x)=(p_{N,\rho'}\circ f)(x) = p_{N,\rho'}(f(x)) = [f(x)]_{\rho'} = \overline{f}([x]_\rho)$, that is , $\overline{f}=\overline{f}'$. ■

Proposition 1.3.14. Let M and N two sets and f :M→N a function; we denote by ρ_f the relation of M defined by

$$(x, y)\in\rho_f \Leftrightarrow f(x)=f(y) \quad (x, y\in M).$$

Then

(i) ρ_f is an equivalence relation on M;

(ii) There is a unique bijective function $\overline{f}: M/\rho_f\to Im(f)$ such that $i\circ\overline{f}\circ p_{M,r_f} = f$, where $i:Im(f)\to N$ is the inclusion.

Proof. (i). Clearly.

(ii). With the notations from Proposition 1.3.13, for $x\in M$ we define $\bar{f}([x]_{r_f}) = f(x)$. The function $\overline{f}$ is correcttly defined because if $x, y\in M$ and $[x]_{r_f}=[y]_{r_f} \Leftrightarrow (x, y)\in\rho_f \Leftrightarrow f(x)=f(y)$ (we will deduce immediately the injectivity of $\overline{f}$). Since $\overline{f}$ is clearly surjective, we deduce that $\overline{f}$ is bijective. To prove the uniqueness of $\overline{f}$, let $f_1 : M/\rho_f\to Im(f)$ another bijective function such that $i\circ f_1\circ p_{M,r_f}=f$ and $x\in M$. Then, $(i\circ f_1\circ p_{M,r_f})(x)=f(x) \Leftrightarrow f_1([x]_{r_f})=f(x)\Leftrightarrow f_1([x]_{r_f})=f(x)=\bar{f}([x]_{r_f})$, that is, $f_1=\overline{f}$. ■

Proposition 1.3.15. Let M be a finite set with m elements. Then the number $N_{m,k}$ of all equivalence relations defined on M such that the factor set has k elements ($k\le m$) is equal with

$$N_{m,k} = (1/k!)\cdot[k^m - C_k^1(k-1)^m + C_k^2(k-2)^m - ... + (-1)^{k-1}C_k^{k-1}].$$

So, the number of equivalence relations defined on M is equal with $N = N_{m,1} + N_{m,2} + ... + N_{m,m}$.

Proof. If ρ∈Echiv (M), we have the canonical surjective function $p_{M,\rho} : M \to M / \rho$.

If $f : M \to N$ is a surjective function, then following Proposition 1.3.14, we obtain an equivalence relation on M : $(x, y) \in \rho_f \Leftrightarrow f(x) = f(y)$. More, if $g : N \to N'$ is a bijective function , then the relations ρ_f and $\rho_{g \circ f}$ coincide because $(x,y) \in \rho_{g\circ f} \Leftrightarrow (g\circ f)(x) = (g\circ f)(y) \Leftrightarrow g(f(x)) = g(f(y)) \Leftrightarrow f(x) = f(y) \Leftrightarrow (x, y) \in \rho_f$.

So, if N has k elements, then k! surjective functions from M to N will determine the same equivalence relation on M. In particular for $N = M/\rho$, by Proposition 1.3.11 we deduce that

$$N_{m,k} = (1/k!) \cdot [k^m - C_k^1 (k-1)^m + C_k^2 (k-2)^m - ... + (-1)^{k-1} C_k^{k-1}].\ ■$$

Proposition 1.3.16. Let M be a non-empty set. Then the function which assign to an equivalence relation ρ on M the partition $\{[x]_\rho : x \in M\}$ of M generated by ρ is bijective.

Proof. We denote by **Part** (M) the set of all partitions of M and consider f : **Echiv**(M)→**Part**(M) the function which assign to every congruence relation ρ of M, the partition of M relative to ρ: $f(\rho) = \{[x]_\rho : x \in M\}$.

Also, we define g : **Part**(M)→**Echiv**(M) by : if $P = (M_i)_{i \in I}$ is a partition of M, we define the relation g(P) on M by : (x, y)∈ g(P)⇔ there is i∈I such that x, $y \in M_i$.

The reflexivity and symmetry of g(P) is immediate. Let (x, y),(y, z)∈g(P). So, there exist $i_1, i_2 \in I$ such that x, $y \in M_{i_1}$ and y, $z \in M_{i_2}$; if $i_1 \neq i_2$ then $M_{i_1} \cap M_{i_2} = \emptyset$, a contradiction (because y is a commune element), hence $i_1 = i_2$, so, x, $z \in M_i$, hence (x, z)∈ g(P). So, g(P) is transitive, hence g(P)∈ **Echiv**(M), that means g is correctly defined.

For every $x \in M_i$, the equivalence class $\bar{x}$ of x modulo g(P) is equal with M_i. Indeed, $y \in M_i \Leftrightarrow (x, y) \in g(P) \Leftrightarrow y \in \bar{x} \Leftrightarrow M_i = \bar{x}$.

So, we obtain that g is the inverse function of f, hence f is bijective. ■

Now we can mark some considerations relative to the *set of natural numbers* .

Definition 1.3.17. A *Peano triple* is a triple (N, 0, s), where N is a non-empty set, $0\in N$ and $s:N \to N$ is a function such that :

P_1 : $0\notin s(N)$;
P_2 : s is an injective function ;
P_3 : If $P\subseteq N$ is such that $(n\in P\Rightarrow s(n)\in P)$, then P=N.

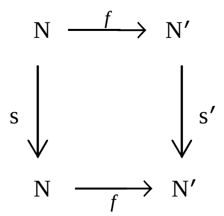
Next, we accept as axiom the existence of a Peano triple (see [59] for more information relative to this aspect).

Lemma 1.3.18. If (N, 0, s) is a Peano triple, then $N=\{0\}\cup s(N)$.

Proof. If we denote $P=\{0\}\cup s(N)$, then $P\subseteq N$ and since P verifies P_3, we deduce that P=N .■

Theorem 1.3.19. Let (N, 0, s) be a Peano triple and (N′, 0′, s′) another triple with N′ non-empty set, $0'\in N'$ and $s':N' \to N'$ a function. Then

(i) There is a unique function $f:N\to N'$ such that f(0) = 0′, and the diagram



is commutative (i.e. $f \circ s = s'\circ f$);

(ii) If (N′, 0′, s′) is another Peano triple, then f is bijective.

Proof. (i). To prove the existence of f, we will consider all relations $R\subseteq N\times N'$ such that:

r_1: $(0, 0') \in R$;

r_2: If $(n, n')\in R$, then $(s(n), s'(n'))\in R$ and by R_0 we will denote the intersection of all these relations.

We shall prove that R_0 is a functional relation, so f will be the function with the graphic R_0 (then, from $(0, 0')\in R_0$ we deduce that f (0)=0′ and if $n\in N$ and $f(n)=n'\in N'$, $(n , n')\in R_0$, hence $(s(n), s'(n'))\in R_0$, that is, $f(s(n))=s'(n')= s'(f(n))$).

To prove that R_0 is a functional relation, we will prove that for every $n\in N$, there is $n'\in N'$ such that $(n, n')\in R_0$ and if we have $n\in N$, $n', n''\in N'$ such that $(n, n')\in R_0$ and $(n, n'')\in R_0$, then $n' = n''$.

For the first part, let $P=\{n\in N$: there is $n'\in N'$ such that $(n, n')\in R_0 \}\subseteq N$.

Since $(0, 0')\in R_0$ we deduce that $0\in P$. Let now $n\in P$ and $n'\in N'$ such that $(n, n')\in R_0$. From the definition of R_0 we deduce that $(s(n), s'(n'))\in R_0$; so, we obtain that $s(n)\in P$, and because $(N, 0, s)$ is a Peano triple, we deduce that $P=N$.

For the second part, let

$Q = \{n\in N$: if $n', n''\in N'$ and $(n, n'), (n, n'')\in R_0 \Rightarrow n' = n''\}\subseteq N$; we will prove that $0\in Q$.

For this, we prove that if $(0, n')\in R_0$ then $n'=0'$. If by contrary, $n' \neq 0'$, then we consider the relation $R_1=R_0 \setminus\{(0, n')\}\subseteq N\times N'$. From $n' \neq 0'$ we deduce that $(0, 0')\in R_1$ and if for $m\in N'$ we have $(n, m)\in R_1$, then $(n, m)\in R_0$ and $(n , m) \neq (0, n')$. So , $(s(n), s'(m))\in R_0$ and since $(s(n), s'(m)) \neq (0, n')$ (by P_1), we deduce that $(s(n), s'(m))\in R_1$. Since R_1 verifies r_1 and r_2, then we deduce that $R_0\subseteq R_1$ – a contradiction (since the inclusion of R_1 in R_0 is strict).

To prove that $0\in Q$, let $n', n''\in N'$ such that $(0, n'), (0, n'')\in R_0$. Then, by the above, we deduce that $n'=n''=0'$, hence $0\in Q$.

Let now $n\in Q$ and $n'\in N'$ such that $(n, n')\in R_0$; we shall prove that if $(s(n),n'')\in R_0$, then $n''=s'(n')$. Suppose by contrary that $n''\neq s'(n')$ we consider the relation $R_2 = R_0 \setminus\{(s (n), n'')\}$. We will prove that R_2 verifies r_1 and r_2.

Indeed, $(0, 0')\in R_2$ (because $0 \neq s(n)$) and if $(p, p')\in R_2$, then $(p, p') \in R_0$ and $(p, p') \neq (s(n), n'')$.

We deduce that $(s(p), s'(p'))\in R_0$ and if suppose $(s(p), s'(p')) = (s(n), n'')$, then $s(p) =s(n)$, hence $p=n$. Also, $s'(p')=n''$. Then $(n,n')\in R_0$ and $(n, p')\in R_0$; because $n\in Q\Rightarrow n'=p'$, hence $n''=s'(p')=s'(n')$, in contradiction with $n'' \neq s(n')$. So, $(s(p), s'(p')) \neq (s(n), n'')$, hence $(s(p), s'(p'))\in R_2$, that means, R_2 satisfies r_1 and r_2. Again we deduce that $R_0\subset R_2$ – which is a contradiction!

Hence $(s(n), n'')\in R_0 \Rightarrow n''=s'(n')$, so, if $r, s\in N'$ and $(s(n), r), (s(n), s)\in R_0$, then $r = s = s'(n')$, hence $s(n)\in Q$, that is, $Q=N$.

For the uniqueness of f, suppose that there is $f':N\rightarrow N'$ such that $f'(0)=0'$ and $s'(f'(n)) = f'(s(n))$ for every $n\in N$.

If we consider $P=\{n\in N : f(n)=f'(n)\}\subseteq N$, then $0\in P$ and if $n\in P$ (hence $f(n) = f'(n)$), then $s'(f(n)) = s'(f'(n)) \Rightarrow f(s(n)) = f'(s(n))\Rightarrow s(n)\in P$, so $P=N$, that is , $f = f'$.

(ii).To prove the injectivity of f, we consider $P=\{n\in N :$ if $m\in N$ and $f(m)=f(n)\Rightarrow m=n\}\subseteq N$ and we shall firstly prove that $0\in P$. Let us consider $m\in N$ such that $f(0)=f(m)$ and we shall prove that $m=0$. If by contrary $m \neq 0$, then $m=s(n)$ with $n\in N$ and by equality $f(m)=f(0)$ we deduce $f(s(n))=f(0)=0'$, hence $s'(f(n))=0'$, which is a contradiction because by hypothesis $(N', 0', s')$ is a Peano triple.

Let now $n\in P$; to prove $s(n)\in P$, let $m\in N$ such that $f(m)=f(s(n))$.

Then $m \neq 0$ (by contrary we obtain that $0'=f(0)=f(s(n))=s'(f(n))$, which is a contradiction), so, by Lemma 1.3.18, $m=s(p)$ with $p\in N$ and the equality $f(m)=f(s(n))$ implies $f(s(p))=f(s(n))\Leftrightarrow s'(f(p))=s'(f(n))$, hence $f(p)=f(n)$; because $n\in P$, then $n=p$ hence $m=s(p)=s(n)$.

To prove the surjectivity of f, we consider

$P'=\{n'\in N' :$ there is $n\in N$ such that $n'=f(n)\}\subseteq N'$.

Since $f(0)=0'$ we deduce that $0'\in P'$. Let now $n'\in P'$; then there is $n\in N$ such that $n'=f(n)$. Since $s'(n')=s'(f(n))=f(s(n))$, we deduce that $s'(n')\in P'$ and because $(N', 0', s')$ is a Peano triple, we deduce that $P'=N'$, hence f is surjective, hence bijective. ■

Remark 1.3.20. Following Theorem 1.3.19 (called the *theorem of recurrence*) a Peano triple is unique up to a bijection.

In what follows by $(\mathbb{N}, 0, s)$ we will denote a Peano triple; the elements of $\mathbb{N}$ will be called *natural numbers*.

The element 0 will be called *zero*.

We denote by $1=s(0)$, $2=s(1)$, $3=s(2)$, hence $\mathbb{N}=\{0, 1, 2, \dots\}$. The function s will be called *successor function*. The axioms P_1 – P_3 are known as *Peano axiom's* (the axiom P_3 will be called the *mathematical induction axiom*).

1.4. The kernel (equalizer) and cokernel (coequalizer) of a couple of functions

Definition 1.4.1. Let f, g : A→B a couple of functions. A pair (K, i) with K a set and i: K→A a function will be called the *kernel (equalizer)* of the couple (f, g) if the following conditions are verified:

(i) f∘i = g∘i;

(ii) For every pair (K′, i′) with K′ set and i′ : K′→A such that f∘i′= g∘i′, there is a unique function u : K′→K such that i∘u = i′.

Theorem 1.4.2. For every couple of functions f, g : A→B there is the kernel of the couple (f, g) unique up to a bijection (in the sense that if (K, i) and (K′, i′) are two kernels for the couple (f, g), then there is a bijective function u : K→K′ such that i′∘u = i).

Proof. To prove the existence of kernel, we consider K={x∈A: f(x)=g(x)} and i : K→A the inclusion function (K will be possible to be the empty set ∅).

Clearly f∘i=g∘i. Let now (K′, i′) with i′ : K′→A such that f∘i′=g∘i′. For a∈K′, since f (i′(a))=g (i′(a)) we deduce that i′(a)∈K. If we define u:K′→K by u(a) = i′(a), for every a∈K′, then i∘u=i′.

If u′:K′→K is another function such that i∘u′=i′, then for every a∈K′ we have i(u′(a))=u(a), hence u′(a)=i′(a)=u(a), that is , u=u′.

To prove the uniqueness of kernel, let (K, i) and (K′, i′) be two kernels for couple (f, g).

Since (K′, i′) is a kernel for couple (f, g) we deduce the existence of a function u:K→K′ such that i′∘u=i. Analogous, we deduce the existence of another function u′:K′→K such that i∘u′=i′.

We deduce that i′∘(u∘u′)=i′ and i∘(u′∘u)=i. Since i′∘$1_{K'}$=i′ and i∘1_K=i, by the uniqueness from Definition 1.4.1, we deduce that u∘u′=$1_{K'}$ and u′∘u=1_K, that is, u is bijective and i′∘u=i. ■

Remark 1.4.3. We will denote (K, i) = Ker (f, g) (or only K=Ker(f, g) if there is no danger of confusion).

Definition 1.4.4. Let f, g :A→B a couple of functions. A pair (P, p) with P a set and p : B→P a function will be called the *cokernel (coequalizer)* of the couple (f,g) if the following conditions are verified :

(i) p∘f=p∘g ;

(ii) For every pair (P ′, p′) with P′ set and p′ : B→P′ such that p′∘f = p′∘g, there is a unique function v :P→P′ such that v∘p=p′.

Theorem 1.4.5. For every pair of functions f, g : A→B, there is the cokernel of the pair (f, g) unique up to a bijection (in the sense that if (P, p) and (P′, p′) are two cokernels for the couple (f, g), then there is a bijection u : P→P′ such that p′∘u=p).

Proof. We prove only the existence of cokernel of pair (f, g) because the uniqueness will be proved in the same way as in the case of kernel.

We consider the binary relation on B : ρ = {(f(x), g(x)) : x∈A} and let <ρ> the equivalence relation of B generated by ρ (see Theorem 1.2.11).

We will prove that the pair (B / <ρ>, $p_{<\rho>, B}$) is the cokernel of the couple (f,g). Since for every x∈A we have (f(x), g(x))∈ρ⊆<ρ>, we deduce that (f(x), g(x))∈<ρ>, hence $p_{<\rho>, B}(f(x))=p_{<\rho>, B}(g(x))$, that is, $p_{<\rho>, B}\circ f=p_{<\rho>, B}\circ g$.

Let now a pair (P′, p′) with P′ a set and p′:B→P′ such that p′∘f=p′∘g. Then for every x∈A, p′(f(x))=p′(g(x)), hence (f(x), g(x))∈$\rho_{p'}$ (see Proposition 1.3.14), so $\rho\subseteq\rho_{p'}$. Since $\rho_{p'}$ is an equivalence relation on B, by the definition of <ρ> we deduce that <ρ>⊆$\rho_{p'}$.

By Proposition 1.3.13 there is a function α : B/<ρ>→B/$\rho_{p'}$ such that $\alpha\circ p_{<\rho>, B}= p_{r_{p'},B}$. Let β:B/$\rho_{p'}$→Im(p′) the bijection given by Proposition 1.3.14. We have $p'=i'\circ\beta\circ p_{r_{p'},B}$, where i′: Im (p′)→P′ is the inclusion mapping.

If we denote v=i′∘β∘α, then $v\circ p_{\langle r\rangle,B}=(i'\circ\beta\circ\alpha)\circ p_{\langle r\rangle,B}=$ $=(i'\circ\beta)\circ(\alpha\circ p_{\langle r\rangle,B})=(i'\circ\beta)\circ p_{r_{p'},B}= i'\circ(\beta\circ p_{r_{p'},B})=p'$.

If we also have v′:B/<ρ>→P′ such that $v'\circ p_{\langle r\rangle,B}=p'$, then $v'\circ p_{\langle r\rangle,B}= v\circ p_{\langle r\rangle,B}$; since $p_{\langle r\rangle,B}$ is surjective, we deduce that v′=v (by Proposition 1.3.8). ■

Remark 4.6. We denote (B/<ρ>, $p_{\langle r\rangle,B}$)=Coker (f, g) (or B/<ρ>= **Coker**(f,g) if there is no danger of confusion).

1. 5. Direct product (coproduct) of a family of sets

Definition 1.5.1. Let $(M_i)_{i\in I}$ be a non-empty family of sets. We call the *direct product* of this family a pair $(P, (p_i)_{i\in I})$, where P is a non-empty set and $(p_i)_{i\in I}$ is a family of functions $p_i : P \to M_i$ for every $i \in I$ such that :

For every other pair $(P', (p_i')_{i\in I})$ composed by the set P' and a family of functions $p_i' : P' \to M_i$ $(i \in I)$, there is a unique function $u:P'\to P$ such that $p_i \circ u = p_i'$, for ever $i \in I$.

Theorem 1.5.2. For every non-empty family of sets $(M_i)_{i\in I}$ there is his direct product which is unique up to a bijection.

Proof. The uniqueness of direct product. If $(P, (p_i)_{i\in I})$ and $(P', (p_i')_{i\in I})$ are two direct products of the family $(M_i)_{i\in I}$, then by the universality property of direct product there exist $u:P'\to P$ and $v:P\to P'$ such that $p_i\circ u = p_i'$ and $p_i'\circ v = p_i$ for every $i\in I$.

We deduce that $p_i\circ (u\circ v) = p_i$ and $p'_i \circ (v\circ u) = p_i'$ for every $i\in I$. Since $p_i\circ 1_P = p_i$, $p_i' \circ 1_{P'} = p_i'$ for every $i\in I$, by the uniqueness of direct product we deduce that $u\circ v = 1_P$ and $v\circ u=1_{P'}$, hence u is a bijection.

The existence of direct product. Let $P=\{f : I\to \bigcup_{i\in I} M_i : f(i) \in M_i$ for every $i\in I \}$ and $p_i : P \to M_i$ $p_i (f) = f (i)$ for $i\in I$ and $f \in P$.

It is immediate that the pair $(P, (p_i)_{i\in I})$ is the direct product of the family $(M_i)_{i\in I}$. ∎

Remark 1.5.3. The pair $(P, (p_i)_{i\in I})$ which is the direct product of the family of sets $(M_i)_{i\in I}$ will be denoted by $\prod_{i\in I} M_i$.

For every $j\in I$, $p_j: \prod_{i\in I} M_i \to M_j$ is called j-th *projection*. Usually, by direct product we understand only the set P (omitting the explicit mention of projections).

Since every function $f:I\to \bigcup_{i\in I} M_i$ is determined by f(i) for every $i\in I$, if we denote $f (i) = x_i \in M_i$, then

$$\prod_{i\in I} M_i =\{(x_i)_{i\in I} : x_i\in M_i \text{ for every } i\in I\}.$$

If $I=\{1, 2, ..., n\}$, then $\prod_{i\in I} M_i$ coincides with $M_1 \times .... \times M_n$ defined in §1.1.

Thus, $p_j:\prod_{i\in I} M_i \to M_j$ is defined by $p_j((x_i)_{i\in I}) = x_j$, $j\in I$.

Let now $(M_i)_{i\in I}$ and $(M_i')_{i\in I}$ two non-empty families of non-empty sets and $(f_i)_{i\in I}$ a family of functions $f_i : M_i\to M_i'$, $(i\in I)$.

The function $f:\prod_{i\in I} M_i \to \prod_{i\in I} M_i'$, $f((x_i)_{i\in I})=(f_i(x_i))_{i\in I}$ for every $(x_i)_{i\in I}\in \prod_{i\in I} M_i$ is called the *direct product* of the family $(f_i)_{i\in I}$ of functions; we denote $f=\prod_{i\in I} f_i$. The function is unique with the property that $p_i' \circ f = f_i\circ p_i$ for every $i\in I$.

It is immediate that $\prod_{i\in I} 1_{M_i} = 1_{\prod_{i\in I} M_i}$ and so, if we have another family of sets $(M_i'')_{i\in I}$ and a family of functions $(f_i')_{i\in I}$ with $f_i' : M_i'\to M_i''$, $(i\in I)$, then

$$\prod_{i\in I}\left(f_i' \circ f_i\right)=\left(\prod_{i\in I} f_i'\right)\circ\left(\prod_{i\in I} f_i\right).$$

Proposition 1.5.4. If for every $i\in I$, f_i is an injective (surjective, bijective) function, then $f=\prod_{i\in I} f_i$ is injective (surjective, bijective).

Proof. Indeed, suppose that for every $i\in I$, f_i is injective and let $\alpha, \beta\in \prod_{i\in I} M_i$ such that $f(\alpha)=f(\beta)$.

Then for every $j\in I$, $f(\alpha)(j)=f(\beta)(j) \Leftrightarrow f_j(\alpha(j))=f_j(\beta(j))$. Since f_j is injective, we deduce that $\alpha(j)=\beta(j)$, hence $\alpha=\beta$, that means f is injective.

Suppose now that for every $i\in I$, f_i is surjective and let $\varphi\in \prod_{i\in I} M_i'$, that is, $\varphi:I\to \bigcup_{i\in I} M_i'$ and $\varphi(j)\in M_j'$ for every $j\in J$. Since f_i is surjective, there is $x_j\in M_j$ such that $f_j(x_j)=\varphi(j)$. If we consider $\psi:I\to \bigcup_{i\in I} M_i$ defined by $\psi(j)=x_j$ for every $j\in I$, then $f(\psi)=\varphi$, that is, f is surjective. ■

In the theory of sets, the dual notion of direct product is the notion of *coproduct* of a family of sets (later we will talk about the notion of *dualization* - see Definition 4.1.4).

Definition 1.5.5. We call *coproduct* of a non-empty family of sets $(M_i)_{i\in I}$, a pair $(S, (\alpha_i)_{i\in I})$ with S a non-empty set and $\alpha_i:M_i\to S$ $(i\in I)$ a family of functions such that :

For every set S' and a family $(\alpha'_i)_{i\in I}$ of functions with $\alpha'_i:M_i\to S'$ ($i\in I$), there is a unique function $u:S\to S'$ such that $u\circ\alpha_i=\alpha'_i$ for ever $i\in I$.

Theorem 1.5.6. For every non-empty family $(M_i)_{i\in I}$ of functions there is its coproduct which is unique up to a bijection.

Proof. The proof of the uniqueness is analogous as in the case of direct product .

To prove the existence, for every $i\in I$ we consider $\overline{M_i}=M_i\times\{i\}$ and $S=\bigcup_{i\in I}\overline{M_i}$ (we observe that for $i\neq j$, $\overline{M_i}\cap\overline{M_j}=\emptyset$). We define for every $i\in I$, $\alpha_i : M_i \to S$ by $\alpha_i(x) = (x, i)$ ($x\in M_i$) and it is immediate that the pair $(S, (\alpha_i)_{i\in I})$ is the coproduct of the family $(M_i)_{i\in I}$. ∎

Remark 1.5.7. The coproduct of the family $(M_i)_{i\in I}$ will be denoted by $\coprod_{i\in I} M_i$ and will be called *disjunctive union* of the family $(M_i)_{i\in I}$.

The functions $(\alpha_i)_{i\in I}$, which are injective, will be called *canonical injections* (as in the case of direct product, many times when we speak about the direct sum we will mention only the subjacent set, the canonical injections are implied).

As in the case of direct product, if we have a family of functions $(f_i)_{i\in I}$ with $f_i : M_i \to M'_i$, ($i\in I$), then the function $f:\coprod_{i\in I} M_i\to\coprod_{i\in I} M'_i$ defined by $f((x, i))=(f_i(x), i)$ for every $i\in I$ and $x\in M_i$ is the unique function with the property that $\alpha'_i\circ f_i=f\circ\alpha_i$ for every $i\in I$; we denote $f=\coprod_{i\in I} f_i$ which will be called the coproduct of $(f_i)_{i\in I}$.

It is immediate that $\coprod_{i\in I} 1_{M_i}=1_{\coprod_{i\in I} M_i}$ and if we have another family of functions $(f'_i)_{i\in I}$ with $f'_i: M'_i\to M''_i$ ($i\in I$), then $\coprod_{i\in I}\left(f'_i\circ f_i\right)=\left(\coprod_{i\in I} f'_i\right)\circ\left(\coprod_{i\in I} f_i\right)$.

As in the case of direct product of a family of functions $(f_i)_{i\in I}$ we have an analogous result and for $f=\coprod_{i\in I} f_i$:

Proposition 1.5.8. If for every $i\in I$, f_i is an injective (surjective, bijective) function, then $f=\coprod_{i\in I} f_i$ is injective (surjective, bijective) function.

Proposition 1.5.9. Let $(A_i)_{i\in I}$, $(B_i)_{i\in I}$ be two families of functions such that for every i, j∈I, i ≠ j, $A_i\cap A_j$ = $B_i\cap B_j$= Ø. If for every i∈I there is a bijection $f_i : A_i\to B_i$, then there is a bijection $f:\bigcup_{i\in I} A_i \to \bigcup_{i\in I} B_i$.

Proof. For every $x\in \bigcup_{i\in I} A_i$ there is a unique i∈I such that $x\in A_i$. If we define $f(x)=f_i(x)$, then it is immediate that f is a bijection. ■

Chapter 2

ORDERED SETS

2.1. Ordered sets. Semilattices. Lattices

Definition 2.1.1. An *ordered set* is a pair (A, £) where A is a non-empty set and £ is a binary relation on A which is reflexive, anti-symmetric and transitive. The relation £ will be called an *order* on A. For x, y Î A we write x < y if x £ y and x ¹ y. If the relation £ is only reflexive and transitive, the pair (A, £) will be called a *partially ordered* set (or a *poset*).

If for x, y∈A we define x ≥ y iff y ≤ x, we obtain a new relation of order on A. The pair (A, ≥) will be denoted by A° and will be called the *dual* of (A, ≤). As a consequence of this result we can assert that to every statement that concerns an order on a set A there is a dual statement that concerns the corresponding dual order on A; this remark is the basic for the next very utile principle:

Principle of duality **:*To every theorem that concerns an ordered set A there is a corresponding theorem that concerns the dual set A°; this is obtained by replacing each statement that involves £, explicitly or implicitly, by its dual .***

Let (A, ≤) be a poset and *r* an equivalence relation on A. We say that *r* is *compatible with the order* ≤ of A (or that *r* is a *congruence on* (A, ≤)) if for every x, y, z, t ∈ A such that $(x,y)\in r,(z,t)\in r$ and $x\le z \Rightarrow y\le t$.

If *r* is a relation of equivalence on A compatible with the preorder ≤, then on the factor set A/*r* there will be possible to define a partial order by $[x]_\rho \le [y]_\rho \Leftrightarrow x \le y$.

Indeed, if we have x´, y´∈A such that $[x']_\rho = [x]_\rho$ and $[y']_\rho = [y]_\rho$ then (x, x´)∈ρ, (y, y´)∈ρ; since ρ is a congruence on (A, ≤) and x ≤ y we deduce that x´ ≤ y´, that is, the order on A/ρ is correctly defined.

The order defined on A/ρ will be called the *preorder quotient*.

In what follows by (A, $\leq$) we shall denote an ordered set.

If there is no danger of confusion, in the case of an ordered set (A, $\leq$) we mention only the subjacent set A (without mentioning the relation $\leq$, because it is implied).

Definition 2.1.2. Let m, M Î A and S Í A, S ¹ Æ.

(i) m is said to be the *lower bound* of S if for every s Î S, m £ s; by inf (S) we will denote the top element (when such exists) of the lower bounds of S. The lower bound for A will be called the *bottom element* or the *minimum element* of A (usually denoted by 0);

(ii) M is said to be the *upper bound* of S if M is the lower bound for S in A°, that means, for every s Î S, s £ M; by sup (S) we will denote the bottom element (when such element exists) of the upper bounds of S; the upper bound for A will be called the *top element* or the *maximum element* of A (usually denoted by 1).

A poset A with 0 and 1 will be called *bounded.*

If $S=\{s_1, s_2, ..., s_n\}\subseteq A$ then we denote inf (S) = $s_1\wedge s_2\wedge ...\wedge s_n$ and sup (S) = $s_1\vee s_2 \vee ...\vee s_n$ (of course, if these exist!).

We say that two elements a, b of A are *comparable* if either $a \leq b$ or $b \leq a$; if all pairs of elements of A are comparable then we say that A forms a *chain*, or that $\leq$ is a *total order* on A. In contrast, we say that a,b $\in$A are *incomparable* when $a \nleq b$ and $b \nleq a$.

For a, b $\in$ A we denote

$(a, b)=\{x\in A: a<x<b\}$

$[a, b]=\{x\in A: a\leq x\leq b\}$

$(a, b]=\{x\in A: a<x\leq b\}$

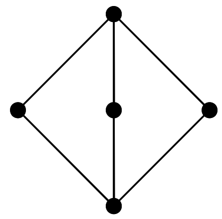
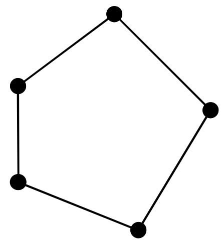
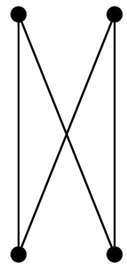
$[a, b)=\{x\in A: a\leq x<b\}$; these subsets of A will be called *intervals* in A.

For a, b$\in$A we say that a is *covered* by b (or that b *covers* a) if $a < b$ and if we have $a \leq c \leq b$, then a=c or c=b; we denote this by using the notation $a \prec b$.

Many ordered sets A can be represented by their means of a *Hasse diagram*; in such a diagram we represent the elements of A by small circles "∘" in such way that if $a \prec b$ then the circle representing ***a*** is lower in the diagram than that representing *b*; now connect these two circles with straight lines (we remark that the intersections of two straight lines can't be an element of the set A).

This procedure can always be carried out when the set A is finite, and even in the infinite case the structure of A can sometimes be indicated.

Below are some examples of Hasse diagrams:

$\mathbf{M_5}$ $\mathbf{N_5}$

Definition 2.1.3. We say that an ordered set A is

(i) ***meet–semilattice,*** **if for every two elements a, b Î A there is a∧b=inf{a, b};**

(ii) ***join–semilattice,*** **if for every two elements a, b Î A there is a∨b=sup{a, b};**

(iii) ***lattice,*** **if it is both meet and join-semilattice (that is, for every two elements a, b Î A there exist a ∧ b and a Ú b in A);**

(iv) ***inf–complete,*** **if for every subset S Í A there is inf (S);**

(v) ***sup–complete,*** **if for every subset S Í A there is sup(S);**

(vi) ***complete*** **if it is both inf and sup-complete (in this case A will be called** ***complete lattice*****);**

The weaker notion of *conditional completeness* refers to a poset in which sup(S) exists if S is non-empty and S has an upper bound, and dually.

Remark 2.1.4.

(i) If A is a complete lattice, then inf (∅) = **1** and sup (∅) = **0**.

(ii) Every ordered set A which is inf-complete or sup-complete is a complete lattice.

Suppose that A is inf-complete. If M ⊆ A, then sup(M) = inf (M′), where M′ is the set of all upper bounds of M (M′ is non-empty since **1**= inf (∅)∈M′). Indeed, for every x∈M and y ∈ M′ we have x ≤ y, hence x ≤ m = inf(M′), hence m∈ M′, that means, m = sup (M). Analogous if we suppose L is sup-complete.

Theorem 2.1.5. Let L be a set endowed with two binary operations ∧,∨ : L ´ L ® L associative, commutative, idempotent and with the absortion property (which is, for every x, y Î L we have x ∧ (x ∨ y) = x and x ∨ (x ∧ y) = x).

Then

(i) For every x, y Î L, x ∧ y = x Û x ∨ y = y;

(ii) If we define for x, y Î L

x £ y Û x ∧ y = x Û x ∨ y = y,

then (L, £) is a lattice where ∧ and ∨ plays the role of infimum and respective supremum.

Proof. (i). If x ∧ y = x, since y ∨ (x ∧ y) = y ⇒ y ∨ x = y ⇒ x ∨ y = y. Dually, if x ∨ y = y ⇒ x ∧ y = x.

(ii). Since x ∧ x = x ⇒ x ≤ x. If x ≤ y and y ≤ x ⇒ x ∧ y = x and y ∧ x = y ⇒ x = y. If x ≤ y and y ≤ z ⇒ x ∧ y = x and y ∧ z = y. Then x ∧ z = (x ∧ y) ∧ z = x ∧ (y ∧ z) = x ∧ y = x, hence x ≤ z. So, (L, ≤) is an ordered set.

We have to prove that for every x, y∈L, inf{x,y} = x ∧ y and sup{x,y}= x ∨ y.

Since x ∨ (x ∧ y) = x ⇒ x ∧ y ≤ x. Analogous x ∧ y ≤ y. If we have t∈L such that t ≤ x and t ≤ y ⇒ t ∧ x = t, t ∧ y = t and t ∧ (x ∧ y) = (t ∧ x) ∧ y = t ∧ y = t ⇒ t ≤ x ∧ y.

Analogous we will prove that sup{x,y} = x ∨ y. ■

Definition 2.1.6. An element m Î A will be called :

(i) *minimal*, if we have a Î A such that a £ m, then m = a;

(ii) *maximal*, if we have a Î A such that m £ a we deduce that m = a.

Definition 2.1.7. If A is a meet-semilattice (respective, join-semilattice) we say that A′⊆A is a *meet-sub-semilattice* (respective, *join -sub-semilattice*), if for every a, b∈A′ we have a∧b∈A′ (respective, a∨b∈A′).

If A is a lattice, A′⊆A will be called *sublattice*, if for every a, b∈A′ we have a∧b, a∨b∈A′.

Examples

1. Let ℕ be the set of natural numbers and "|" the relation of divisibility on ℕ. Then "|" is an order relation on ℕ; with respect to this order ℕ is a lattice, where for m, n∈ℕ, m ∧ n = (m, n) (the greatest common divisor of m and n) and m ∨ n = [m, n] (the least common multiple of m and n).

Clearly, for the relation of divisibility the number 1∈ℕ is the initial element and the number 0∈ℕ is the final element. This order is not a total one, since if we have two natural numbers m, n such that (m,n) = 1 (as the examples 2 and 3) dones not have m | n or n | m.

2. If **K** is one of the sets ℕ, ℤ, ℚ or ℝ, then **K** become lattice relative to the natural ordering and the natural ordering is total .

3. Let M be a set and **P**(M) the set of all subsets of M; then (**P**(M), ⊆) is a complete lattice (called the *lattice of power sets* of M ; clearly, in this lattice **0** = ∅ and **1** = M).

Let now A, A′ be two ordered sets (if there is no danger of confusion, we will denote by ≤ the same relations of order from A and A′) and f : A→A′ a function.

Definition 2.1.8. The function f is said to be a morphism *of ordered set* or *isotone* (*anti-isotone) function* if for every a, b∈A, a ≤ b implies f(a) ≤ f(b) (f(b) ≤ f(a)) (alternative f is said *monotone increasing (decreasing)*)).

If A, A′ are meet (join) – semilattices, f will be called *morphism of meet (join) semilattices* if for every a, b∈A, f (a ∧ b) = f (a) ∧ f (b) (respective f (a ∨ b) = f (a) ∨ f (b)).

If A, A′ are lattices, f will be called *morphism of lattices* if for every a, b ∈A we have f (a ∧ b) = f (a) ∧ f (b) and f (a ∨ b) = f (a) ∨ f (b).

Clearly, the morphisms of meet (join) – semilattices are isotone mappings and the composition of two morphism of the same type is also a morphism of the same type.

The morphism of ordered sets f:A→A′ will be called *isomorphism of ordered set* if there is g:A′→A a morphism of ordered sets such that $f\circ g = 1_{A'}$ and $g\circ f = 1_A$; in this case we write A≈A′.

Since the definition of isomorphism of ordered set implies that f is bijective, an isomorphism f of ordered set is a bijective function for which f and g are order preserving.

We note that simply choosing f to be an isotone bijection is not suffice to imply that f is an isomorphism of ordered sets (see [9], p.13)

Analogous we define the notions of isomorphism for meet (join) – semilattices and lattices.

Next we will establish the way how partially ordered sets determine ordered sets (see Definition 2.1.1); for this let (A,≤) be a poset.

It is immediate that the relation r defined on A by: $(x, y)\in r \Leftrightarrow x\le y$ and $y\le x$ is an equivalence on A.

If x, y, x′, y′∈A such that (x, x′)∈ρ, (y, y′)∈ρ and x ≤ y, then x ≤ x′, x′ ≤ x, y ≤ y′ and y′ ≤ y. From x ≤ y, y ≤ y′ ⇒ x ≤ y′ and from x′ ≤ x and x≤ y′ ⇒ x′ ≤ y′, that is, ρ is a congruence on (A,≤).

We consider $\overline{A} = A/r$ together with preorder quotient (defined at the beginning of the paragraph) we have to prove that this preorder is in fact an order on $\overline{A}$ (that means, r is anti-symmetric).

Indeed, let $[x]_r, [y]_r \in \overline{A}$ such that $[x]_r \le [y]_r$, $[y]_r \le [x]_r$ and we have to prove that $[x]_r = [y]_r$. We have x ≤ y and y ≤ x, hence $(x, y) \in r$, therefore $[x]_r = [y]_r$.

Therefore, the canonical surjection $p_A : A \to \overline{A}$ is an isotone function.

Following Proposition 1.3.13 it is immediate that the quotient set $(\overline{A}, \le)$ together with the canonical surjective function $p_A : A \to \overline{A}$ verify the following property of universality:

For every ordered set $(B, \le)$ and every isotone function $f : A \to B$ there is an unique isotone function $\bar{f} : \overline{A} \to B$ such that $\bar{f} \circ p_A = f$.

Let (I, ≤) be a chain and $(A_i, \le)_{i\in I}$ o family of ordered sets (mutually disjoint) . Then $A = \bigcup_{i\in I} A_i = \coprod_{i\in I} A_i$ (see Definition 1.5.5).

We define on A an order ≤ by : x ≤ y iff $x \in A_i$, $y \in A_j$ and i < j or $\{x, y\} \subseteq A_k$ and x ≤ y in A_k (i,j,k∈I).

Definition 2.1.9. The ordered set (A, ≤) defined above will be called the *ordinal sum* of the family of ordered sets $(A_i, \le)_{i\in I}$.

In some books, (A, ≤) will be denoted by $\bigoplus_{i\in I} A_i$.

If I = {1,2, …,n}, $\bigoplus_{i\in I} A_i$ is replaced by $A_1 \oplus \ldots \oplus A_n$.

Consider now a set I and $P = \prod_{i\in I} A_i$ (see Definition 1.5.1).

For two elements x, y∈P, $x = (x_i)_{i\in I}$, $y = (y_i)_{i\in I}$ we define: $x \le y \Leftrightarrow x_i \le y_i$ for every i∈I. It is immediate that (P, ≤) become an ordered set and canonical projections $(p_i)_{i\in I}$ (with $p_i : P \to A_i$ for every i∈I) are isotone functionss .This order on P will be called *direct product order* .

As in the case of the sum between (ordinal) the pair form from ordered set (P, ≤) and the family of projections $(p_i)_{i\in I}$ verifies the following property of universality:

Theorem 2.1.10. For every ordered set $(P', \leq)$ and every family of isotone functions $(p'_i)_{i\in I}$ with $p'_i : P'\to A_i$ $(i\in I)$ there is a unique isotone function $u:P'\to P$ such that $p_i \circ u = p'_i$, for every $i\in I$.

Proof. As in the case of direct product of sets (see Theorem 1.5.2) it is immediate that $u : P'\to P$, $u(x) =(p'_i(x))_{i\in I}$ for every $x\in P'$ verifies the conditions of the enounce . ■

Definition 2.1.11. The pair $(P, (p_i)_{i\in I})$ will be called the *direct product* of the family $(A_i, \leq)_{i\in I}$.

Suppose that $I = \{1, 2, \ldots, n\}$. On the direct product $P = P_1\times\ldots\times P_n$ we can define a new order on P: if $x = (x_i)_{1\leq i\leq n}$, $y = (y_i)_{1\leq i\leq n} \in P$: $x \leq y \Leftrightarrow$ there is $1\leq s\leq n$ such that $x_1 = y_1,\ldots, x_{s-1} = y_{s-1}$ and $x_s < y_s$.

This order will be called *lexicographical order* (clearly if $x, y \in P$ and $x \leq y$ in lexicographical order, then $x \leq y$ relative to product order).

Theorem 2.1.12. (Knaster [54]) Let L be a complete lattice and $f:L\to L$ an isotone function. Then there is $a\in L$ such that $f(a) = a$.

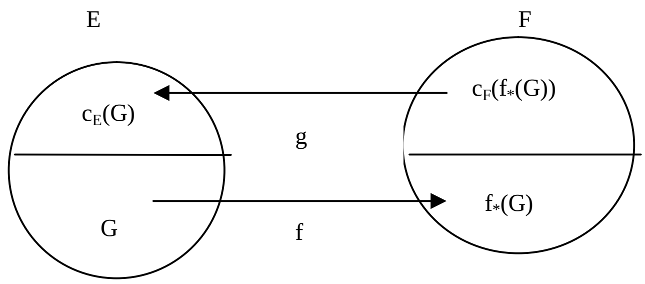
Proof. Let $A=\{x\in L: x \leq f(x)\}$. Since $0\in A$ we deduce that $A \neq \emptyset$; let $a = \sup(A)$. For every $x\in A$, $x \leq a$, hence $x \leq f(x) \leq f(a)$, so we deduce that $a \leq f(a)$. Then $f(a) \leq f(f(a))$, hence $f(a)\in A$ and so $f(a) \leq a$, which is $a = f(a)$. ■

An interesting application of Theorem 2.1.12 is the proof of the following important set-theoretic result:

Corollary 2.1.13. (Bernstein [4]) Let E and F two sets such that there are two injections $f:E\to F$ and $g:F\to E$. Then E and F are equipotent.

Proof. For a set M we consider $c_M : \mathbf{P}(M)\to\mathbf{P}(M)$, $c_M(N)=C_M(N)$ (complementary of N in M). We recall the functions defined in Proposition 1.3.7 : $f_* : \mathbf{P}(E)\to\mathbf{P}(F)$, $f_*(G)=f(G)$, for every $G\subseteq E$ and $g_* : \mathbf{P}(F)\to\mathbf{P}(E)$, $g_*(H)=g(H)$, for every $H\subseteq F$ and consider the function $h:\mathbf{P}(E)\to\mathbf{P}(E)$, $h=c_E\circ g_*\circ c_F\circ f_*$, which is isotone (because if $G,K\subseteq E$ and $G\subseteq K\Rightarrow f(G)\subseteq f(K) \Rightarrow c_F(f(K))\subseteq c_F(f(G))\Rightarrow g(c_F(f(K)))\subseteq g(c_F(f(G)))\Rightarrow c_E(g(c_F(f(G))))\subseteq c_E(g(c_F(f(K)))) \Rightarrow h(G)\subseteq h(K)$). Since $(\mathbf{P}(E), \subseteq)$ is a complete lattice, then by Theorem of Knaster (Theorem 2.1.12), there is $G\subseteq E$ such that $h(G)=G$, and therefore $c_E(G) =$

$(g_* \circ c_F \circ f_*)(G)$. We have that $E = G \cup c_E(G)$ and $F = f_*(G) \cup c_F(f_*(G))$, so $f: G \to f_*(G)$ and $g: c_F(f_*(G)) \to c_E(G)$ are bijections as in the next figure :



Then $t: E \to F$, $t(x) = \begin{cases} f(x), & if\ x \in G, \\ y, & if\ x \notin G\ and\ g(y) = x \end{cases}$ is a bijection, hence E and F are equipotent. ■

2.2. Ideals (filters) in a lattice

Definition 2.2.1. Let A be a meet - semilattice and F $\subseteq$A a non – empty subset. F will be called *filter* of A if F is a meet -sub-semilattice of A and for every a, b Î A, if a £ b and a Î F, then b Î F.

We denote by **F**(A) the set of filters of A.
The dual notion for filter is the notion of *ideal* for a join-semilattice:

Definition 2.2.2. Let A be a join - semilattice and I Í A a nonempty subset of A. I will be called an *ideal* of A if I is a join-sub-semilattice of A and for every a, b Î A with a £ b, if b Î I, then a Î I.

We denote by **I**(A) the set of ideals of A.

Remark 2.2.3. If A is a lattice, then the notions of filter and ideal have a precise definition in A (since A is simultaneous meet and join-semilattice), so A $\in$ **F**(A) $\cap$ **I**(A).

Since the intersection of a family of filters (ideals) is also a filter (ideal), we can define the notion of *filter (ideal) generated by a non-empty set* (which is, the intersections of all filters (ideals) of A which contains S).

If A is a meet (join) - semilattice, for $\emptyset \neq S \subseteq A$ we denote by [S) ((S]) the *filter (ideal) generated* by S.

Proposition 2.2.4. If A is a meet -semilattice and S Í A a non-empty subset of A, then

$$[S)=\{a\in A: \text{there exist } s_1, s_2, ..., s_n\in S \text{ such that } s_1\wedge s_2\wedge\ldots\wedge s_n\leq a\}.$$

Proof. Let $F_S=\{a\in A$: there exist $s_1, s_2, ..., s_n\in S$ such that $s_1\wedge s_2 \wedge\ldots\wedge s_n\leq a\}$. It is immediate that $F_S \in \mathbf{F}(A)$ and $S\subseteq F_S$, hence $[S)\subseteq F_S$. If $F'\in\mathbf{F}(A)$ such that $S\subseteq F'$ then $F_S\subseteq F'$, hence $F_S\subseteq\cap F'=[S)$,that is, $[S)=F_S$. ∎

By the Principle of duality we have:

Proposition 2.2.5. If A is a join-semilattice and S Í A is a non-empty subset of A, then

$$(S]=\{a\in A: \text{there exist } s_1, s_2, ..., s_n\in S \text{ such that } a\leq s_1\vee s_2\vee\ldots\vee s_n\}.$$

So, $(\mathbf{F}(A),\subseteq)$ and $(\mathbf{I}(A),\subseteq)$ are lattices, where for $F_1, F_2\in\mathbf{F}(A)$ (respective $I_1, I_2\in\mathbf{I}(A)$) we have $F_1\wedge F_2=F_1\cap F_2$ and $F_1\vee F_2=[F_1\cup F_2)$ (respective $I_1\wedge I_2=I_1\cap I_2$ and $I_1\vee I_2=(I_1\cup I_2]$).

In facts, these two lattices are complete.

If A is a meet (join)-semilattice and $a\in A$, we denote by [a) ((a]) the filter (ideal) generated by {a}.

It is immediate that: $[a)=\{x\in A:a\leq x\}$ and $(a]=\{x\in A : x\leq a\}$; [a), ((a]) is called the *principal filter* (*ideal*) generated by a.

Corollary 2.2.6. Let L be a lattice, $a\in L$, $I, I_1, I_2\in I(L)$ and $F, F_1, F_2\in F(L)$. Then

(i) $I(a)\overset{def}{=}(I\cup\{a\}]=I\vee(a]=\{x\in L: x\leq y\vee a \text{ with } y\in I\}$;

(ii) $F(a)\overset{def}{=}[F\cup\{a\})=F\vee[a)=\{x\in L:y\wedge a\leq x \text{ with } y\in F\}$;

(iii) $I_1\vee I_2 = \{x\in L:x\leq i_1\vee i_2 \text{ with } i_1\in I_1 \text{ and } i_2\in I_2\}$;

(iv) $F_1\vee F_2 = \{x\in L:f_1\wedge f_2\leq x \text{ with } f_1\in F_1 \text{ and } f_2\in F_2\}$.

Theorem 2.2.7. Let (A, £) be an ordered set. Then A is isomorphic with a set of subsets of some set (ordered by inclusion).

Proof. For every a∈A we consider M_a={x∈A|x≤a}⊆A. Since for every a, b∈A, a ≤ b we have $M_a \subseteq M_b$, we deduce that the isomorphism of ordered set a → M_a for a∈A yells to the result. **n**

Definition 2.2.8.

(i) An ordered set A with the property that every non-empty subset of A have an initial element is called *well ordered* (clearly, a well ordered set is inf-complete and total ordered);

(ii) An ordered set A with the property that every total ordered non-empty subset of A have an upper bound (lower bound) is called *inductive (co-inductive) ordered* set.

In [31] (§1 of Chapter 3, Theorem 1.21) it is proved that (ℕ, ≤) is an example of well ordered set.

Next, we accept that for every set M the *axiom of choice* is true:

There is a function s : P(M) ® M such that s(S) Î S for every non-empty subset S of M.

We recall a main result of Bourbaki and some important corollaries (for the proof of these corollaries see [70]).

Lemma 2.2.9. (*Bourbaki*). If (A, £) is a non-empty ordered set, inductive ordered and f : A ® A is a function such that f (a) £ a for every a Î A, then there exists u Î A such that f (u) =u.

Corollary 2.2.10. (*Hansdorf principle of maximality*). Every ordered set contain a maximal chain.

Corollary 2.2.11. (*Zorn's lemma*). Every non-empty set which is inductive (co inductive) ordered set has a maximal (minimal) element.

Corollary 2.2.12. (*Principle of maximal (minimal) element)*). Let (A, £) be an inductive (co inductive) ordered set and a Î A. Then there exists a maximal (minimal) element m_a Î A such that a £ m_a (m_a £ a).

Corollary 2.2.13. (*Kuratowski lemma)*. Every total ordered subset of an ordered set is contained in a maximal chain.

Corollary 2.2.14. (*Zermelo theorem*). On every non-empty set A one can introduce an order such that the set A become well ordered.

Corollary 2.2.15. (*Principle of transfinite induction*). Let (A, £) be an infinite well ordered set and P a given property. To prove that all elements of a have the property P, it is suffice to prove that:

(i) The initial element 0 of A has property P;

(ii) If for a Î A, all elements x Î A such that x < a has property P, then the element a has property P.

2.3. Modular lattices. Distributive lattices

Proposition 2.3.1. Let $(L,\wedge,\vee)$ be a lattice. The following identities in L are equivalent :

(i) $\mathbf{x\wedge(y\vee z)=(x\wedge y)\vee(x\wedge z)}$**;**

(ii) $\mathbf{x\vee(y\wedge z)=(x\vee y)\wedge(x\vee z)}$**.**

Proof. (i)⇒(ii). Suppose that (i) is true. Then

$x\vee(y\wedge z) = (x\vee(x\wedge z))\vee(y\wedge z) = x\vee[(x\wedge z)\vee(y\wedge z)] = x\vee[z\wedge(x\vee y)] =$

$= (x\wedge(x\vee y))\vee(z\wedge(x\vee y)) = (z\vee x)\wedge(x\vee y) = (x\vee y)\wedge(x\vee z).$

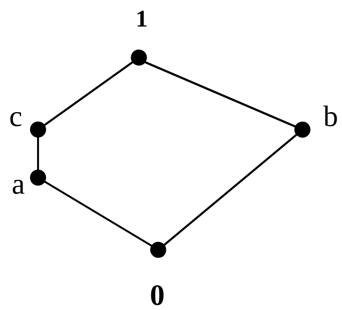
(ii)⇒(i). Analogous. ■

Definition 2.3.2. We say that a lattice $(L,\leq)$ is *distributive* if L verifies one of the equivalent conditions of Proposition 2.3.1.

Definition 2.3.3. We say that a lattice $(L,\leq)$ is *modular* if for every x, y, z Î L with $z \leq x$ we have x Ù (y Ú z) = (x Ù y) Ú z.

We note that we have lattices which are not modular.

Indeed, if we consider the lattice usually denoted by $\mathbf{N_5}$:



we remark that $a \leq c$, but $a \vee (b \wedge c) = a \vee \mathbf{0} = a$ and $(a \vee b) \wedge c = \mathbf{1} \wedge c = c \neq a$, hence $c \wedge (b \vee a) \neq (c \wedge b) \vee a$, that is, $\mathbf{N_5}$ is not a modular lattice.

A classical example of modular lattice is the lattice $\mathbf{L_0}$(G) of normal subgroups of a group G (which is a sublattice of the lattice **L**(G) of the subgroups of G - see [31]).

Theorem 2.3.4. (*Dedekind*). For every lattice L the following assertions are equivalent :

(i) L is modular;

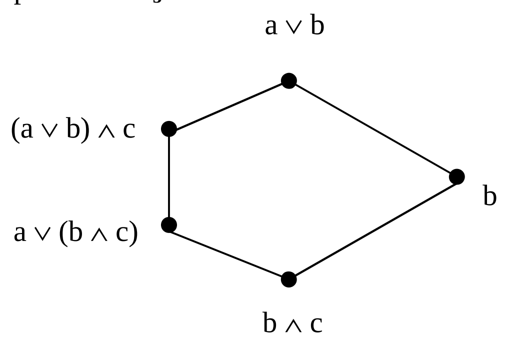
(ii) for every a, b, c Î L, if c £ a, then a Ù(b Ú c) £ (a Ù b)Ú c;

(iii) for every a, b, c Î L we have ((aÙc) Ú b) Ù c = (aÙc) Ú (bÙc);

(iv) for every a, b, c Î L, if a £ c, then from a Ùb =c Ùb and a Ú b = c Ú b we deduce that a = c;

(v) L doesn't contain sublattices isomorphic with $\mathbf{N_5}$.

Proof. Since in every lattice, if $c \leq a$, then $(a \wedge b) \vee c \leq a \wedge (b \vee c)$, the equivalence (i) $\Leftrightarrow$ (ii) it is immediate.
(i) $\Rightarrow$ (iii). Follows from $a \wedge c \leq c$.
(iii) $\Rightarrow$ (i). Let a, b, c $\in$ L such that $a \leq c$. Then $a = a \wedge c$, hence $(a \vee b) \wedge c = ((a \wedge c) \vee b) \wedge c = (a \wedge c) \vee (b \wedge c) = a \vee (b \wedge c)$.
(i) $\Rightarrow$ (iv). We have $a=a\vee(a\wedge b)=a\vee(c\wedge b)=a \vee (b \wedge c)=(a \vee b)\wedge c= (c \vee b) \wedge c = c$.
(iv) $\Rightarrow$ (v). Clearly (by the above remark).
(v)$\Rightarrow$ (i). Suppose by contrary that L is not modular. Then we have a, b, c in L such that $a \leq c$, and $a \vee (b \wedge c) \neq (a \vee b) \wedge c$. We remark that $b\wedge c < a \vee (b \wedge c) < (a \vee b) \wedge c < a\vee b$, $b \wedge c < b < a \vee b$, $a \vee (b \wedge c) \leq b$ and $b \leq (a \vee b) \wedge c$. In this way we obtain a Hasse diagram for a sublattice of L isomorphic with $\mathbf{N_5}$:



(we remark that $(a \vee (b\wedge c)) \vee b = a\vee ((b\wedge c) \vee b) = a\vee b$ and $((a\vee b) \wedge c) \wedge b = ((a \vee b) \wedge b) \wedge c = b \wedge c$), which is a contradiction!. **n**

Theorem 2.3.5. (*Scholander*). Let L be a set and Ù, Ú : L ´ L ® L two binary operations. The following assertions are equivalent:

(i) (L, Ù, Ú) is a distributive lattice;

(ii) In L we have the following identities true:

1) x Ù (x Ú y) = x;
2) x Ù (y Ú z) = (z Ù x) Ú (y Ù x).

Proof. (i) ⇒ (ii). Clearly .

(ii) ⇒ (i). From (1) and (2) we deduce that $x = x \wedge (x \vee x) = (x\wedge x)\vee(x \wedge x)$; $x \wedge x = (x \wedge x)\wedge((x \wedge x) \vee (x \wedge x)) = (x \wedge x) \wedge x$; $x \wedge x = x \wedge((x \wedge x)\vee (x \wedge x)) = ((x \wedge x) \wedge x)\vee((x \wedge x) \wedge x) = (x \wedge x) \vee (x \wedge x) = x$; $x \vee x = (x \wedge x) \vee (x \wedge x) = x$, so we deduce the idempotence of $\wedge$ and $\vee$.

For commutativity and dual absortion:

$x \wedge y = x \wedge (y \vee y) = (y \wedge x) \vee (y \wedge x) = y \wedge x$;

$(x \wedge y) \vee x = (y \wedge x) \vee (x \wedge x) = x \wedge (x \vee y) = x$;

$x \wedge (y \vee x) = (x \wedge x)\vee(y \wedge x) = x \vee (x \wedge y) = x \vee ((x \wedge y)\wedge ((x \wedge y)\vee x))= =(x \wedge x) \vee ((x \wedge y) \wedge x)=x \wedge((x \wedge y) \vee x)=x \wedge x= x$;

$x \vee y = (x \wedge(y \vee x))\vee(y\wedge(y \vee x)) = (y \vee x)\wedge(y \vee x) =y \vee x$.

Associativity:

$x \wedge ((x \vee y) \vee z) = (x \wedge (x \vee y))\vee (x \wedge z) = x \vee (x \wedge z) = x$;

$x \vee (y \vee z) = (x \wedge ((x \vee y) \vee z)) \vee (y \wedge ((y \vee x) \vee z)) \vee (z \wedge ((x \vee y) \vee z)) = = (x \wedge ((x \vee y) \vee z))\vee[((x \vee y) \vee z) \wedge (y \vee z)] = ((x \vee y) \vee z) \wedge (x \vee (y \vee z))$;

$(x \vee y)\vee z = z \vee (y\vee x) = ((z \vee y) \vee x) \wedge (z\vee (y\vee x)) =$
$= [(x \vee y) \vee z) \wedge (x \vee (y\vee z)] = x \vee (y \vee z)$.

So, by Theorem 2.1.5, (L, $\wedge$, $\vee$) is a lattice and from 2) we can deduce its distributivity. **n**

Theorem 2.3.6. (*Ferentinou-Nicolacopoulou*). Let L be a set, 0 Î L and Ù, Ú : L ´ L ® L two binary operations. The following assertions are equivalent :

(i) (L, Ù, Ú) is a distributive lattice with 0;

(ii) In L we have the following identities :

1) x Ù (x Ú y) = x;
2) x Ù (y Ú z) = (z Ù (x Ú 0)) Ú (y Ù (x Ú 0)).

Proof. (i) ⇒ (ii). Clearly.

(ii) ⇒ (i). We shall prove that $x \vee 0 = x$ and then we apply Theorem 2.3.5.

Indeed, $x \vee x = (x \wedge (x \vee 0))\vee(x \wedge (x \vee 0)) = x \wedge (x \vee x) = x$; $x \wedge x = x \wedge (x \vee x) = x$; $x \wedge y = x \wedge (y \vee y)=(y\wedge(x \vee 0))\vee(y \wedge (x \vee 0))=y\wedge(x \vee 0)$; $x \vee 0 = =(x \vee 0) \wedge (x \vee 0) = x \wedge (x \vee 0) = x$. **n**

Clearly, every distributive lattice is modular.

*In what follows by **Ld** we denote the class of distributive lattices and by **Ld (0, 1)** the class of all bounded distributive lattices.*

Examples

1. If L is a chain, then L∈**Ld** (**0**, **1**).

2. (ℕ, |), (**P** (M), ⊆) ∈ **Ld** (**0**, **1**).

Remark 2.3.7. Reasoning inductively relative to $n\in\mathbb{N}^*$, we deduce that if $S_1, S_2, ..., S_n$ are non-empty subsets of a distributive lattice L, then $\overset{n}{\underset{i=1}{\vee}}(\wedge S_i)=\wedge\left\{\overset{n}{\underset{i=1}{\vee}} f(i) \;\middle|\; f\in S_1\times\ldots\times S_n\right\}$.

Theorem 2.3.8. For a lattice L the following assertions are equivalent :

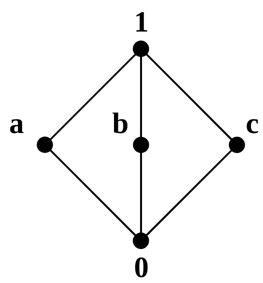
(i) L Î Ld;

(ii) a Ù (b Ú c) £ (a Ù b) Ú (a Ù c) for every a, b, c Î L;

(iii) (a Ù b) Ú (b Ù c) Ú (c Ù a) = (a Ú b) Ù (b Ú c) Ù (c Ú a) for every a, b, c Î L;

(iv) For every a, b, c Î L, if a Ù c = b Ù c and a Ú c = b Ú c, then a = b;

(v) L doesn't contain sublattices isomorphic with N_5 or M_5, where we recall that M_5 has the following Hasse diagram



Proof. (i) ⇔ (ii). Follows from the remark that for every elements a, b, c ∈ L, (a ∧ b) ∨ (a ∧ c) ≤ a ∧ (b ∨ c).

(i) ⇒ (iii). Suppose that L ∈ **Ld** and let a, b, c ∈ L. Then (a ∨ b) ∧ (b ∨ c) ∧ (c ∨ a) = (((a ∨ b) ∧ b) ∨ ((a ∨ b) ∧ c)) ∧ (c ∨ a) = (b ∨ ((a ∧ c) ∨ (b ∧ c))) ∧ (c ∨ a) = (b ∨ (a ∧ c)) ∧ (c ∨ a) = (b ∧(c ∨ a)) ∨ ((a ∧ c) ∧ (c ∨ a)) = ((b ∧ c) ∨ (b ∧ a)) ∨ (a ∧ c) = (a ∧ b) ∨ (b ∧ c) ∨ (c ∧ a).

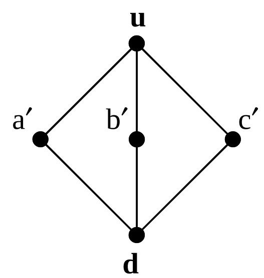
(iii) ⇒ (i). We deduce immediate that L is modular, because if a, b, c ∈ L and a ≤ c, then (a ∨ b) ∧ c = (a ∨ b) ∧ ((b ∨ c) ∧ c) = (a ∨ b) ∧ (b ∨ c) ∧ (c ∨ a) = (a ∧ b) ∨ (b ∧ c) ∨ (c ∧ a) = (a ∧ b) ∨ (b ∧ c) ∨ a = ((a ∧ b) ∨ a) ∨ (b ∧ c) = a ∨ (b ∧ c).

With this remark, the distributivity of L follows in the following way: a ∧ (b ∨ c) = (a ∧ (a ∨ b)) ∧ (b ∨ c) = ((a ∧ (c ∨ a)) ∧ (a ∨ b) ∧ (b ∨ c) = a ∧ (a ∨ b) ∧ (b ∨ c) ∧ (c ∨ a) = a ∧ ((a ∧ b) ∨ (b ∧ c) ∨ (c ∧ a)) = (a ∧ ((a ∧ b) ∨ (b ∧ c))) ∨ (c ∧ a) = (by modularity) = a ∧ (b ∧ c)) ∨ (a ∧ b) ∨ (c ∧ a) = (by modularity) = (a ∧ b) ∨ (a ∧ c).

(i) ⇒ (iv). If a ∧ c = b ∧ c and a ∨ c = b ∨ c, then a = a ∧ (a ∨ c) = a ∧ (b ∨ c) = (a ∧ b) ∨ (a ∧ c) = (a ∧ b) ∨ (b ∧ c) = b ∧ (a ∨ c) = b ∧ (b ∨ c) = b.

(iv) $\Rightarrow$ (v). Suppose by contrary that $\mathbf{N_5}$ or $\mathbf{M_5}$ are sublattices of L. In the case of $\mathbf{N_5}$ we observe that $b \wedge c = b \wedge a = \mathbf{0}$, $b \vee c = b \vee a = \mathbf{1}$ but $a \neq c$ and in the case of $\mathbf{M_5}$, $b \wedge a = b \wedge c = \mathbf{0}$, $b \vee a = b \vee c = \mathbf{1}$ but $a \neq c$ – which is a contradiction!

(v) $\Rightarrow$ (i). By Theorem 2.3.4, if L doesn't have isomorphic sublattices with $\mathbf{N_5}$ then L is modular. Since for every a, b, c $\in$ L we have $(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) \leq (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$, suppose by contrary that there are a, b, c $\in$ L such that $(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) < (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$. We denote $d = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a)$, $u = (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$, $a' = (d \vee a) \wedge u$, $b' = (d \vee b) \wedge u$ and $c' = (d \vee c) \wedge u$. The Hasse diagram of the set $\{d, a', b', c', u\}$ is :



Since $\{d, a', b', c', u\} \subseteq L$ is a sublattice, if we verify that the elements d, a′, b′, c′, u are distinct, then the sublattice $\{d, a', b', c', u\}$ will be isomorphic with $\mathbf{M_5}$ - a contradiction !.

Since $d < u$, we will verify the equalities $a' \vee b' = b' \vee c' = c' \vee a' = u$, $a' \wedge b' = b' \wedge c' = c' \wedge a' = d$ and then we will have that the 5 elements d, a′, b′, c′, u are distinct.

By the modularity of L we obtain $a' = d \vee (a \wedge u)$, $b' = d \vee (b \wedge u)$, $c' = d \vee (c \wedge u)$ and by symmetry it is suffice to prove only the equality $a' \wedge c' = d$.

Indeed, $a' \wedge c' = ((d \vee a) \wedge u) \wedge ((d \vee c) \wedge u) = (d \vee a) \wedge (d \vee c) \wedge u = ((a \wedge b) \vee (b \wedge c) \vee (c \wedge a) \vee a) \wedge (d \vee c) \wedge u = ((b \wedge c) \vee a) \wedge (d \vee c) \wedge u = ((b \wedge c) \vee a) \wedge ((a \wedge b) \vee c) \wedge (a \vee b) \wedge (b \vee c) \wedge (c \vee a) = ((b \wedge c) \vee a) \wedge ((a \wedge b) \vee c) = (b \wedge c) \vee (a \wedge ((a \wedge b) \vee c))$ = (by modularity) $= (b \wedge c) \vee (((a \wedge b) \vee c) \wedge a) = (b \wedge c) \vee ((a \wedge b) \vee (c \wedge a))$ = (by modularity) = d. ∎

Corollary 2.3.9. A lattice L is distributive iff for every two ideals I, J Î I (L), I Ú J = {i Ú j : i Î I and j Î J}.

Proof. Suppose that L is distributive. By Corollary 2.2.6, for t∈I ∨ J we have i ∈ I, j ∈ J such that t ≤ i ∨ j, so t = (t ∧ i) ∨ (t ∧ j) = i′ ∨ j′ with i′ = t ∧ i ∈ I and j′ = t ∧ j ∈ J.

To prove the converse assertion, suppose by contrary that L is not distributive and we have to prove that there are I, J∈**I** (L) which doesn't verify the hypothesis.

By Theorem 2.3.8, L contains a, b, c which together with 0 and **1** determine the lattices $\mathbf{N_5}$ or $\mathbf{M_5}$.

Let I = (b], J = (c]. Since a ≤ b ∨ c we deduce that a∈I ∨ J. If we have a = i ∨ j with i∈I and j∈J, then j ≤ c, hence j ≤ a ∧ c < b. We deduce that j∈I and a = i ∨ j ∈I – which is a contradiction! **n**

Corollary 2.3.10. Let L Î Ld and I, J Î I(L). If IÙJ and IÚJ are principal ideals, then I and J are principal ideals.

Proof. Let I ∧ J = (x] and I ∨ J = (y]. By Corollary 2.3.9, y = i ∨ j with i∈I and j∈J. If c = x ∨ i and b = x ∨ j, then c∈I and b∈J. We have to prove that I = (c] and J = (b].

If by contrary J ≠ (b], then we have a∈J, a > b and {x, a, b, c, y} is isomorphic with $\mathbf{N_5}$ – which is a contradiction!

Analogous, if I ≠ (c], we find a sublattice of L isomorphic with $\mathbf{M_5}$, which is a new contradiction! **n**

Corollary 2.3.11. Let L be a lattice and x, y∈L. Then (x] Ù (y] = (x Ù y] and (x Ú y] ⊆ (x] Ú (y]; if L∈Ld, then (x] Ú (y] = (x Ú y].

Proof. The equality (x] ∧ (y] = (x ∧ y] is immediate by double inclusion; the inclusion (x ∨ y]⊆(x] ∨ (y] follows from Corollary 2.2.6. If L∈**Ld**, then by Corollary 2.3.9, (x] ∨ (y] = {i ∨ j : i ∈ (x] and j ∈ (y]} = {i ∨ j : i ≤ x and j ≤ y}, hence (x] ∨ (y] ⊆ (x ∨ y], that is, (x ∨ y] = (x] ∨ (y]. **n**

Definition 2.3.12. Let L be a lattice. An element a∈L is called *join (meet)-irreducible* (respective *join(meet)-prim*) if a = x∨y (a = x∧y) with x,y∈L, then a = x or a = y (respective, a ≤ x∨y (x∧y ≤ a) then a ≤ x or a ≤ y

(x $\leq$ a or y $\leq$ a)). If L has 0, (1) an element a$\in$L is called *atom* (*co-atom*) if a $\neq$ 0 and x $\leq$ a, then x = 0 or x = a (a $\neq$ 1 and a $\leq$ x, then x = a or x = 1).

Theorem 2.3.13. Let L be a distributive lattice. Then

(i) a$\in$L is join (meet) – irreducible iff is, respective, join (meet)-prim ;
(ii)If L have 0, (1) then every atom (co-atom) is join(meet)- irreducible.

Proof.(i). ,"$\Rightarrow$". Let a$\in$L join–irreducible and a $\leq$ x$\vee$y. Then a = a$\wedge$(x$\vee$y) = (a$\wedge$x)$\vee$(a$\wedge$y), hence a = a$\wedge$x or a = a$\wedge$y, that is, a $\leq$ x or a $\leq$ y.

"$\Leftarrow$". Suppose a = x$\vee$y. Then x $\leq$ a and y $\leq$ a. Since a $\leq$ x$\vee$y, by hypothesis a $\leq$ x or a $\leq$ y, hence a = x or a = y, that is, a is join-irreducible. Analogous for the case meet-irreducible equivalent with meet-prim.

(ii). Suppose L has **0** and let a$\in$L an atom such that a $\leq$ x$\vee$y. Since a$\wedge$x $\leq$ a and a$\wedge$y $\leq$ a, then a$\wedge$x = a$\wedge$y = **0** or a$\wedge$x = a or a$\wedge$y= a. The first case is impossible because **0** $\neq$ a = (a$\wedge$x)$\vee$(a$\wedge$y), hence a $\leq$ x or a $\leq$ y. Analogous for the case of co-atoms. ■

Proposition 2.3.14. Let L be a distributive lattice x, y$\in$L, I$\in$I(L) and I(x) = (I$\cup${x}]. Then

(i) If x$\wedge$y$\in$ I, I(x)$\cap$I(y) = I;
(ii) The following assertions are equivalent :

1) I is an meet-irreducible element in the lattice (I(L),$\subseteq$);

2) If x, y$\in$L such that x$\wedge$y$\in$I, then x$\in$I or y$\in$I.

Proof. (i).Let x, y$\in$L such that x$\wedge$y$\in$I. If z$\in$I(x)$\cap$I(y), then by Corollary 2.3.9 there are t, r$\in$I such that z $\leq$ x$\vee$t and z $\leq$ y$\vee$r. We deduce that z $\leq$ (x$\vee$t)$\wedge$(y$\vee$r) = (x$\wedge$y)$\vee$(x$\wedge$r)$\vee$(t$\wedge$y)$\vee$(t$\wedge$r)$\in$I, hence z$\in$I, that is, I(x)$\cap$I(y)$\subseteq$I. Since the another inclusion is immediate, we deduce that I(x)$\cap$I(y)=I.

(ii). 1)$\Rightarrow$2). Since I is supposed to be an meet-irreducible element in **I**(L) and by (i), I(x)$\cap$I(y) = I, we deduce that I = I(x) or I = I(y), hence x$\in$I or y$\in$I.

2)$\Rightarrow$1). Let I_1, $I_2$$\in$**I**(L) such that I=$I_1 \cap I_2$.

Suppose by contrary that I $\neq$ I_1 and I $\neq$ I_2, hence there exist $x_1 \in I_1$ such that $x_1 \notin$I and $x_2 \in I_2$ such that $x_2 \notin$I. Then $x_1 \wedge x_2 \in I_1 \cap I_2$ =I; by hypothesis, $x_1 \in$I or $x_2 \in$I – which is a contradiction, hence I=I_1 or I=I_2. ■

Dually, we deduce

Proposition 2.3.15. Let L be a distributive lattice x, y∈L, F∈F(L) and F(x) = [F∪{x}). Then

(i) If x∨y∈F, F(x)∩F(y) = F;
(ii) The following assertions are equivalent :

1) F is meet-irreducible element in the lattice (F(L),⊆);

2) If x, y∈L such that x∨y∈F, then x∈F or y∈F.

Definition 2.3.16. Let L be a distributive lattice. A proper ideal (filter) of L will be called *prime* if it verifies one of the equivalent conditions from Proposition 2. 3.14 (Proposition 2. 3.15).

Definition 2.3.17. Let L be a lattice (distributive). A proper ideal (filter) of L will be called *maximal* if it is maximal element in the lattice I(L) (F(L)). Maximal filters are also called *ultrafilters*.

Corollary 2.3.18. (M.H. Stone). In a distributive lattice, every maximal ideal (filter) is prime.

Proof. It is an immediate consequence of Propositions 2.3.14 and 2.3.15 because one maximal ideal (filter) is an inf-irreducible element in the lattice (**I**(L), ⊆) (**F**(L), ⊆). ■

The following result is immediate :

Proposition 2.3.19. If L is a distributive lattice, then I∈I(L) is a prime ideal iff L \ I is a prime filter.

2.4. The prime ideal (filter) theorem in a distributive lattice

Theorem 2.4.1. Let L be a distributive lattice, I∈I(L) and F∈F(L) such that I∩F=Ø. Then there is a prime ideal (filter) P such that I⊆P (F⊆P) and P∩F = Ø (P∩I = Ø).

Proof. By duality principle, it is suffice to prove the existence of the prime ideal P such that I⊆P and P∩F=Ø .

Let $\mathcal{F}_I$={I′∈I(L): I⊆I′ and I′∩F=Ø}. Since I∈$\mathcal{F}_I$ we deduce that $\mathcal{F}_I$≠Ø. It is immediate that ($\mathcal{F}_I$,⊆) is an inductive set, so by Zorn's Lemma (see Corollary 2.2.11) in $\mathcal{F}_I$ we have a maximal element P with properties I⊆P and P∩F=Ø. Since

$F \neq \emptyset$ we deduce that $P \neq L$. We shall prove that P is a prime ideal, hence let $x,y \in L$ such that $x \wedge y \in P$. Suppose by contrary that $x \notin P$ and $y \notin P$. Then $I \subseteq P \subseteq P \vee (x] = P(x)$ and by the maximality of P we deduce that $P(x) \cap F \neq \emptyset$.

By Corollary 2.2.6 we have $z \in F$ such that $z \leq t \vee x$ with $t \in P$. Analogous we deduce that there is $z' \in F$ such that $z' \leq t' \vee y$ with $t' \in P$. Then $z \wedge z' \leq (t \vee x) \wedge (t' \vee y) = (t \wedge t') \vee (t \wedge y) \vee (x \wedge t') \vee (x \wedge y) \in P$, hence $z \wedge z' \in P$. Since $z \wedge z' \in F$ we deduce that $P \cap F \neq \emptyset$, - which is a contradiction!. Hence $x \in P$ or $y \in P$, that is, P is a prime ideal. ■

Corollary 2.4.2. Let L be a distributive lattice, $I \in \mathbf{I}(L)$ and $a \in L$ such that $a \notin I$. Then there is a prime ideal P such that $I \subseteq P$ and $a \notin P$.

Proof. It is an immediate consequence of Theorem 2.4.1 for F=[a), because if $a \notin I$, then $I \cap F = \emptyset$. ■

Analogous we deduce

Corollary 2.4.3. Let L be a distributive lattice, $F \in \mathbf{F}(L)$ and $a \in L$ such that $a \notin F$. Then there is a prime filter P such that $F \subseteq P$ and $a \notin P$.

Corollary 2.4.4. In a distributive lattice L, every ideal (filter) is the intersection of all prime ideals (filters) containing it.

Proof. It will suffice to prove for ideals.For $I \in \mathbf{I}(L)$ we consider $I_1 = \cap\{P: I \subseteq P$ and P is prime ideal in L$\}$. If $I \neq I_1$, then there is $a \in I_1 \setminus I$ and by Corollary 2.4.2 there is a prime ideal P in L such that $I \subseteq P$ and $a \notin P$. Since $I_1 \subseteq P$ and $a \in I_1$ we deduce that $a \in P$, a contradiction !. ■

Corollary 2.4.5. Let L be a distributive lattice and $x,y \in L$ such that $x \not\leq y$. Then there is an prime ideal (filter) P such that $x \in P$ and $y \notin P$.

Proof. We apply Theorem 2.4.1 for I = (y], F = [x). ■

Definition 2.4.6. A family $\mathcal{R}$ of subsets of a set X will be called *ring of sets* if for every A, B $\in \mathcal{R}$ then $A \cap B \in \mathcal{R}$ and $A \cup B \in \mathcal{R}$.

For a distributive lattice L we denote by **Spec**(L) the set of all prime ideals of L; **Spec**(L) will be called the *spectrum of L*.

We define φ_L:L→**P**(**Spec**(L)) by $\varphi_L(x)$ = {P∈**Spec**(L): x∉P}.

Proposition 2.4.7. Let L be a distributive lattice. Then
(i) $\varphi_L(0) = \emptyset$ **and** $\varphi_L(1)$ **= Spec(L);**
(ii) $\varphi_L(x\vee y) = \varphi_L(x) \cup \varphi_L(y)$**, for any x,y∈L;**
(iii) $\varphi_L(x\wedge y) = \varphi_L(x) \cap \varphi_L(y)$**, for any x,y∈L;**
(iv) φ_L **is an injective function.**

Proof.(i) Straigtforward.

(ii). For P∈**Spec**(L), by Definition 2.3.16 we have $P\in\varphi_L(x)\cup\varphi_L(y) \Leftrightarrow P\in\varphi_L(x)$ or $P\in\varphi_L(y) \Leftrightarrow x\notin P$ or $y\notin P \Leftrightarrow x\vee y\notin P \Leftrightarrow P\in\varphi_L(x\vee y)$, hence $\varphi_L(x\vee y) = \varphi_L(x) \cup \varphi_L(y)$.

(iii). Analogous.

(iv). It follows from Corollary 2.4.5. ■

Corollary 2.4.8. (Birkhoff, Stone) A lattice L is distributive iff it is isomorphic with a ring of sets.

Proof. ”⇒”.For X = **Spec**(L), by Proposition 2.4.7 we deduce that L is isomorphic (as a lattice) with $\varphi_L(L)$ and $\varphi_L(L)$ is a ring of subsets of X.

“⇐”.Clearly. ■

Theorem 2.4.9. Let L be a distributive lattice with 1 and I∈I(L), I≠L. Then I is contained in a maximal ideal of L.

Proof. It is immediate that if we denote $\mathcal{F}_I$ = {J∈I(L), J ≠ L: I⊆J} then $\mathcal{F}_I \neq \emptyset$ (because $I\in\mathcal{F}_I$) and ($\mathcal{F}_I$,⊆) is an inductive set, so we apply Zorn's Lemma. ■

Analogous we deduce

Theorem 2.4.10. Let L be a distributive lattice with 0 and F∈F(L), F≠L. Then F is contained in an ultrafilter.

Theorem 2.4.11. Let L be a distributive lattice with 0. Every element x ≠ 0 of L is contained in an ultrafilter.

2.5. The factorization of a bounded distributive lattice by an ideal (filter)

Let L be a bounded distributive lattice, $I\in\mathbf{I}(L)$ and $F\in\mathbf{F}(L)$.

Lemma 2.5.1. The following assertions are equivalent:
(i) For every x, $y\in L$ there is $i\in I$ such that $x\vee i = y\vee i$;
(ii) For every x, $y\in L$ there is i, $j\in I$ such that $x\vee i = y\vee j$.

Proof.(i)⇒(ii). Clearly.

(ii)⇒(i). Let $x,y\in L$; by hypothesis there are i, $j\in I$ such that $x\vee i = y\vee j$. If we consider $k = i\vee j\in I$, then $(x\vee i)\vee k = (y\vee j)\vee k \Leftrightarrow x\vee k = y\vee k$. ■

Analogous we deduce

Lemma 2.5.2. The following assertions are equivalent:
(i) For every x, $y\in L$ we have $i\in F$ such that $x\wedge i = y\wedge i$;
(ii) For every x, $y\in L$ we have i, $j\in F$ such that $x\wedge i = y\wedge j$.

We consider on L the binary relations

θ_I: $(x,y)\in \theta_I \Leftrightarrow$ there is $i\in I$ such that $x\vee i = y\vee i \Leftrightarrow$ there are i, $j\in I$ such that $x\vee i = y\vee j$;

θ_F: $(x,y)\in \theta_F \Leftrightarrow$ there is $i\in F$ such that $x\wedge i = y\wedge i \Leftrightarrow$ there are i, $j\in F$ such that $x\wedge i = y\wedge j$.

Proposition 2.5.3. θ_I and θ_F are congruences on L.

Proof. It will suffice to prove only for θ_I. Since for every $x\in L$, $x\vee 0=x\vee 0$ and $0 \in I$ we deduce that θ_I is reflexive. The symmetry of θ_I is clear. To prove the transitivity of θ_I, let x, y, $z\in L$ such that (x,y), $(y,z)\in\theta_I$. Thus there are i, $j\in I$ such that $x\vee i=y\vee i$ and $y\vee j=z\vee j$. If we denote $k=i\vee j\in I$, we have $x\vee k=x\vee(i\vee j)=(x\vee i)\vee j=(y\vee i)\vee j=(y\vee j)\vee i=(z\vee j)\vee i=z\vee k$, hence $(x,z)\in \theta_I$.

To prove the compatibility of θ_I with $\wedge$ and $\vee$, let x, y, z, $t\in L$ such that (x,y), $(z,t) \in \theta_I$. Then there are i, $j\in I$ such that $x\vee i=y\vee i$ and $z\vee j=t\vee j$. If we denote $k=i\vee j\in I$, then $(x\vee z)\vee k=(y\vee t)\vee k$, hence $(x\vee z, y\vee t)\in \theta_I$.

Also, we obtain: $(x\vee i)\wedge(z\vee j)=(y\vee i)\wedge(t\vee j)$

$$\Leftrightarrow(x\wedge z)\vee(x\wedge j)\vee(z\wedge i)\vee(i\wedge j)=(y\wedge t)\vee(y\wedge j)\vee(t\wedge i)\vee(i\wedge j).$$

If we denote $k = (x\wedge j)\vee(z\wedge i)\vee(i\wedge j)\in I$, $r = (y\wedge j)\vee(t\wedge i)\vee(i\wedge j)\in I$, then $(x\wedge z)\vee k=(y\wedge t)\vee r$, hence $(x\wedge z, y\wedge t)\in \theta_I$. ■

For $x\in L$ we denote by x/I (x/F) the equivalence class of x modulo θ_I (θ_F) and by L/I (L/F) the factor set L/θ_I (L/θ_F) which in a natural way becomes a bounded distributive lattice (because θ_I and θ_F show congruence on L).

We also denote by $p_I:L\to L/I$ ($p_F:L\to L/F$) the canonical surjective morphism defined by $p_I(x)=x/I$ ($p_F(x)=x/F$), for every $x\in L$. The lattice L/I (L/F) will be called *quotient lattice* (we say that we have *factorized* L by ideal I (filter F)).

Theorem 2.5.4. Let L be a distributive lattice with 0, $I\in \mathbf{I}(L)$ and $x, y\in L$. Then

(i) $x/I \le y/I \Leftrightarrow x \le y\vee i$ for some $i\in I$;

(ii) $x/I = \mathbf{0} = 0/I \Leftrightarrow x\in I$;

(iii) If $F\in \mathbf{F}(L)$ and $I\cap F=\emptyset$, then $p_I(F)$ is a proper filter of L/I.

Proof.(i). We have $x/I \le y/I \Leftrightarrow x/I\wedge y/I = x/I \Leftrightarrow (x\wedge y)/I = x/I \Leftrightarrow (x\wedge y, x)\in \theta_I \Leftrightarrow (x\wedge y)\vee i=x\vee I$ for some $i\in I \Leftrightarrow (x\vee i)\wedge(y\vee i) = x\vee i \Leftrightarrow x\vee i\le y\vee i \Leftrightarrow x \le y\vee i$ for some $i\in I$.

(ii). If $x/I=0/I$ then there is $i\in I$ such that $x\vee i = 0\vee i = i\in I$. Since $x \le x\vee i$ we deduce that $x\in I$. Conversely, if $x\in I$, since $x\vee x = x = x\vee 0$ we deduce that $(x, 0)\in \theta_I$, hence $x/I = \mathbf{0} = 0/I$.

(iii). Firstly we have to prove that $p_I(F)\in \mathbf{F}(L/I)$.

Clearly, if $\alpha,\beta\in p_I(F)$, $\alpha = x/I$, $\beta = y/I$ with $x,y\in F$ then $\alpha\wedge\beta = (x\wedge y)/I\in p_I(F)$ (because $x\wedge y\in F$). Now let $\alpha, \beta\in L/I$, $\alpha \le \beta$ and suppose that $\alpha = x/I$ with $x\in F$; let $\beta = y/I$ with $y\in L$. From $\alpha \le \beta$ we deduce that $x/I \le y/I$ and by (i) we obtain that $x \le y\vee i$ for some $i\in I$. Then $y\vee i\in F$; since $(y\vee i)/I = y/I\vee i/I = y/I\vee \mathbf{0} = y/I$ we deduce that $y/I\in p_I(F)$, hence $p_I(F)$ is a filter in L/I.

We shall prove that $p_I(F)\ne L/I$; if by contrary $p_I(F) = L/I$, then $\mathbf{0}\in p_I(F)$, hence $\mathbf{0} = x/I\in p_I(F)$ with $x\in I$.

We deduce that $x/I = y/I$ with $y\in F$, hence there is $i\in I$ such that $x\vee i = y\vee i$. Since $y \le y\vee i = x\vee i$ and $x,i\in I$ we deduce that $y\in I$, hence $I\cap F\ne \emptyset$, which is a contradiction!. ■

Analogous we deduce

Theorem 2.5.5. Let L be a distributive lattice with 1, $F\in\mathbf{F}(L)$ and $x, y\in L$. Then

(i) $x/F \le y/F \Leftrightarrow x \le y\wedge i$ for some $i\in F$;

(ii) $x/F = \mathbf{1} = 1/F \Leftrightarrow x\in F$;

(iii) If $I\in I(L)$ and $I\cap F=\emptyset$, then $p_F(I)$ is a proper ideal of L/F.

Remark 2.5.6. Although L doesn't have **0**, if $I\in I(L)$ then L/I has **0**. Indeed if we take $x_0\in I$ then, since for every $x\in L$, $x_0 \le x\vee x_0$ we deduce that $x_0/I \le x/I$, hence $x_0/I = \mathbf{0}$ in L/I.

Analogous, if $F\in\mathbf{F}(L)$ and $y_0\in F$, then $y_0/F = \mathbf{1}$ (in L/F).

2. 6. Complement and pseudocomplement in a lattice. Boolean lattices. Boolean algebras.

Definition 2.6.1. Let L be a bounded lattice. We say that an element $a\in L$ is *complemented* if there is $a'\in L$ such that $a \wedge a' = \mathbf{0}$ and $a \vee a' = \mathbf{1}$ (a' will be called the *complement* of a).

If every element of L is complemented, we say that L is *complemented*.

If L is a lattice and $a, b \in L$, $a \le b$, the *relative complement* for an element $x\in[a, b]$ is the element $x'\in[a, b]$ (if it exists!) such that $x \wedge x' = a$ and $x \vee x' = b$.

We say that a lattice L is *relatively complemented* if every element x of L is complemented in every interval [a,b] which contain x.

A *relatively complemented distributive lattice* is a distributive lattice such that every element is relatively complemented; such a lattice with 0 is called *a generalized Boolean algebra.*

Lemma 2.6.2. If $L\in Ld(0, 1)$, then the complement of an element $a\in L$ (if it exists) is unique.

Proof. Let $a\in L$ and a', a'' two complements of a. Then $a \wedge a' = a \wedge a'' = \mathbf{0}$ and $a \vee a' = a \vee a'' = \mathbf{1}$, hence $a' = a''$ (by Theorem 2.3.8 (iv)). ■

Lemma 2.6.3. Every modular, bounded and complemented lattice L is relatively complemented.

Proof. Let $b, c \in L$, $b \leq c$, $a \in [b, c]$ and $a' \in L$ the complement of a in L. If we consider $a'' = (a' \vee b) \wedge c \in [b, c]$, then by $a \geq b$ and by the modularity of L we obtain $a \wedge a'' = a \wedge [(a' \vee b) \wedge c] = [(a \wedge a') \vee (a \wedge b)] \wedge c = (a \wedge b) \wedge c = b \wedge c = b$ and $a \vee a'' = a \vee [(a' \vee b) \wedge c] = (a \vee a' \vee b) \wedge (a \vee c) = \mathbf{1} \wedge c = c$, hence a'' is the relative complement of a in [b, c]. **n**

Theorem 2.6.4. Let L be a relatively complemented distributive lattice. Then in L an ideal I is prime iff I is maximal.

Proof. If I is maximal, then by Corollary 2.3.18, I is prime. Suppose I is a prime ideal. If we consider $J \in \mathbf{I}(L)$ such that $I \subset J$ if we prove that $J = L$, then I will be maximal. We choose $x \in J \setminus I$, $y \in I$ and $z \in L$. By hypothesis x have a complement x' in $[x \wedge y, x \vee z]$. Then $x \wedge x' = x \wedge y \in I$; since I is prime we deduce that $x' \in I$ (because $x \notin I$). Since $I \subset J$ we deduce that $x' \in J$. Since $x \vee x' = x \vee z$ and $x \vee x' \in J$ we deduce that $x \vee z \in J$, therefore $z \in J$, hence $L = J$. ■

Theorem 2.6.5. ***(Nachbin).*****Let L be a distributive lattice such that every prime ideal of L is maximal. Then L is relatively complemented.**

Proof. Suppose by contrary that there are $a_0, a_1, a_2 \in L$, $a_0 \leq a_1 \leq a_2$ and a_1 doesn't have a complement in $[a_0, a_2]$. Then $a_0 < a_1 < a_2$. Let $I_0 = \{x \in L : a_1 \wedge x \leq a_0\}$ and $I_1 = \{x \in L : a_2 \wedge x \leq a_1 \vee y$, for some $y \in I_0\}$. It is immediate that $I_0, I_1 \in I(L)$, $a_0 \in I_0$, $a_1 \notin I_0$, $a_1 \in I_1$ and $I_0 \subseteq I_1$. We remark that $a_2 \notin I_1$, since by contrary, then $a_2 \leq a_1 \vee y$ for some $y \in I_0$; thus if we denote $y' = (y \wedge a_2) \vee a_0$ we have $a_1 \vee y' = a_2$ and $a_1 \wedge y' = a_0$, hence a_1 has a complement in $[a_0, a_2]$ – which is a contradiction!.

By the Theorem of prime ideal (Theorem 2.4.1), there is a prime ideal J_0 such that $a_2 \notin J_0$ and $I_1 \subseteq J_0$. If we denote $F = [(L \setminus J_0) \cup \{a_1\})$ we shall prove that $F \cap I_0 = \emptyset$. Indeed, if $F \cap I_0 \neq \emptyset$ then there is $x \in F \cap I_0$ such that $y \wedge a_1 \leq x$ for some $y \notin J_0$.

But $a_1 \wedge x \leq a_0$, hence $a_1 \wedge y \leq a_1 \wedge x \leq a_0$. Then $y \in I_0$, hence $y \in J_0$, which is a contradiction!. By applying again the Theorem of prime ideal, there is a prime ideal J_1 such that $I_0 \subseteq J_1$ and $F \cap J_1 = \emptyset$. Thus $J_1 \subseteq J_0$, hence J_1 is not maximal, which is a contradiction! ■

Lemma 2.6.6. (*De Morgan*). Let L Î Ld(0, 1), a, b Î L having the complements a¢, b¢ Î L. Then a Ù b and a Ú b have also complements in L and (a Ù b)¢ = a¢ Ú b¢, (a Ú b)¢ = a¢ Ù b¢.

Proof. By Lemma 2.6.2 and duality principle, it is suffice to prove that $(a \wedge b) \wedge (a' \vee b') = \mathbf{0}$ and $(a \wedge b) \vee (a' \vee b') = \mathbf{1}$.

Indeed, $(a \wedge b) \wedge (a' \vee b') = (a \wedge b \wedge a') \vee (a \wedge b \wedge b') = \mathbf{0} \vee \mathbf{0} = \mathbf{0}$ and $(a \wedge b) \vee (a' \vee b') = (a \vee a' \vee b') \wedge (b \vee a' \vee b') = \mathbf{1} \wedge \mathbf{1} = \mathbf{1}$. ■

Theorem 2.6.7. Let L be a bounded distributive lattice, $(a_i)_{i \in I} \subseteq L$ and $c \in L$ a complemented element.

(i) If $\bigvee_{i\in I} a_i$ exists in L, then $c \wedge (\bigvee_{i\in I} a_i) = \bigvee_{i\in I} (c \wedge a_i)$;

(ii) If $\bigwedge_{i\in I} a_i$ exists in L, then $c \vee (\bigwedge_{i\in I} a_i) = \bigwedge_{i\in I} (c \vee a_i)$.

Proof .(i). Suppose that $a = \bigvee_{i\in I} a_i$ in L. Then $a \geq a_i$, hence $c \wedge a \geq c \wedge a_i$, for every $i \in I$. Let $b \geq c \wedge a_i$, for every $i \in I$; then $c' \vee b \geq c' \vee (c \wedge a_i) = (c' \vee c) \wedge (c' \vee a_i) = 1 \wedge (c' \vee a_i) = c' \vee a_i \geq a_i$, for every $i \in I$, hence $c' \vee b \geq a$. Then $c \wedge (c' \vee b) \geq c \wedge a \Rightarrow (c \wedge c') \vee (c \wedge b) \geq c \wedge a \Rightarrow 0 \vee (c \wedge b) \geq c \wedge a \Rightarrow c \wedge b \geq c \wedge a \Rightarrow b \geq c \wedge a$, hence $c \wedge a = \bigvee_{i\in I} (c \wedge a_i)$.

(ii). By (i) using the principle of duality. ■

Remark 2.6.8. If $L \in \mathbf{Ld}(\mathbf{0}, \mathbf{1})$ and $a \in L$ have a complement $a' \in L$, then a' is the greatest element of L such that $a \wedge a' = \mathbf{0}$ (that is, $a' = \sup(\{x \in L : a \wedge x = \mathbf{0}\})$.

Following this remark we have

Definition 2.6.9. Let L be join-semilattice with 0 and $a \in L$. An element $a^* \in L$ will be called the *pseudocomplement* of a, if $a^* = \sup(\{x \in L : a \wedge x = \mathbf{0}\})$.

L will be called *pseudocomplemented* if every element of L has a pseudocomplement.

A lattice L with 0 is called *pseudocomplemented,* if the join-semilattice $(L, \wedge, 0)$ is pseudocomplemented.

Lemma 2.6.10. Let L be a bounded modular lattice, $a \in L$ and a' a complement of a. Then a' is a pseudocomplement for a.

Proof. Indeed, if $b \in L$ such that $a' \leq b$ and $b \wedge a = \mathbf{0}$, then $b = b \wedge \mathbf{1} = b \wedge (a' \vee a) = a' \vee (b \wedge a) = a' \vee \mathbf{0} = a'$. ■

Theorem 2.6.11. Let L be a pseudocomplemented distributive lattice with 0, R (L) = {**a* : a** $\in$ L} **and D**(L) = {**a** $\in$ L **: a* = 0**}.

Then, for every a, b $\in$ **L we have:**

(i) $a \wedge a^* = 0$ and $a \wedge b = 0 \Leftrightarrow a \leq b^*$;

(ii) $a \leq b \Rightarrow a^* \geq b^*$;

(iii) $a \leq a^{**}$;

(iv) $a^* = a^{***}$;

(v) $(a \vee b)^* = a^* \wedge b^*$;

(vi) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$;

(vii) $a \wedge b = 0 \Leftrightarrow a^{**} \wedge b^{**} = 0$;

(viii) $a \wedge (a \wedge b)^* = a \wedge b^*$;

(ix) $0^* = 1$, $1^* = 0$;

(x) $a \in R(L) \Leftrightarrow a = a^{**}$;

(xi) $a, b \in R(L) \Rightarrow a \wedge b \in R(L)$

(xii) $\sup_{R(L)}\{a, b\} = (a \vee b)^{**} = (a^* \wedge b^*)^*$;

(xiii) $0, 1 \in R(L)$, $1 \in D(L)$ and $R(L) \cap D(L) = \{1\}$;

(xiv) $a, b \in D(L) \Rightarrow a \wedge b \in D(L)$;

(xv) $a \in D(L)$ and $a \leq b \Rightarrow b \in D(L)$;

(xvi) $a \vee a^* \in D(L)$.

Proof. **(i).** It follows from the definition of a^*; the equivalence follows from the definition of b^*.

(ii). Since $b \wedge b^* = \mathbf{0}$, then for $a \leq b$, we deduce that $a \wedge b^* = \mathbf{0}$, hence $b^* \leq a^*$.

(iii). From $a \wedge a^* = \mathbf{0}$ we deduce that $a^* \wedge a = \mathbf{0}$, hence $a \leq (a^*)^* = a^{**}$.

(iv). From $a \leq a^{**}$ and ii) we deduce that $a^{***} \leq a^*$, hence using (iii) we deduce that $a^* \leq (a^*)^{**} = a^{***}$, so $a^* = a^{***}$.

(v). We have $(a \vee b) \wedge (a^* \wedge b^*) = (a \wedge a^* \wedge b^*) \vee (b \wedge a^* \wedge b^*) = \mathbf{0} \vee \mathbf{0} = \mathbf{0}$. Let now $x \in L$ such that $(a \vee b) \wedge x = \mathbf{0}$. We deduce that $(a \wedge x) \vee (b \wedge x) = \mathbf{0}$, hence $a \wedge x = b \wedge x = \mathbf{0}$. So, $x \leq a^*$, $x \leq b^*$, hence $x \leq a^* \wedge b^*$. Analogous for the rest of assertions. ∎

Remark 2.6.12.

(i) The elements of **R** (L) are called *regular* and the elements of **D**(L) are called *dense*.

(ii) From iv) and x) we deduce that $\mathbf{R}(L) = \{a \in L : a^{**} = a\}$.

(iii) From xiv) and xv) we deduce that $\mathbf{D}(L) \in \mathbf{F}(L)$.

Theorem 2.6.13. Let L Î Ld and a Î L.

Then f_a : L ® (a] ´ [a), f_a (x) = (x Ù a, x Ú a) for x Î L is injective morphism in Ld. If L Î Ld(0, 1), then f_a is an isomorphism in Ld (0, 1) iff a is complemented.

Proof. It is immediate that f_a is a morphism in **Ld**. Let now x, y ∈ L such that f_a (x) = f_a (y); then x ∧ a = y ∧ a and x ∨ a = y ∨ a. Since L∈**Ld**, then x = y (by Theorem 2.3.8), hence f_a is injective.

Suppose that L∈**Ld**(**0**, **1**). If f_a is an isomorphism in **Ld**(**0**, **1**), then for (**0**, **1**) ∈ (a] × [a) there is x∈L such that f (x) = (**0**, **1**), hence a ∧ x = **0** and a ∨ x = **1**, therefore x = a′.

Conversely , if a′ ∈ L is the complement of a, for (u, v) ∈ (a] × [a) if consider x = (u∨a′) ∧v , then f_a (x) = (u, v), hence f_a is surjective , that is, an isomorphism in **Ld**(**0**,**1**). ∎

Definition 2.6.14. A *Boolean lattice* is a complemented bounded distributive lattice.

Examples

1. The trivial chain **1** = {∅} and the chain **2** = {0, 1} (where 0′ = 1 and 1′ = 0); in fact, **1** and **2** are the only chains which are Boolean lattices.

2. For every set M, (**P**(M), ⊆) is a Boolean lattice, where for every X ⊆ M, X′ = M \ X = C_M (X).

3. Let n∈ℕ, n ≥ 2 and D_n the set of all natural divisors of n.

The ordered set (D_n, |) is a Boolean lattice iff n is square free (thus for p, q ∈ D_n, p ∧ q = (p, q), p ∨ q = [p, q], **0** = 1, **1** = n and p′ = n / p).

4. For a set M, let $\mathbf{2^M}$ = {f : M → **2**}. We define on $\mathbf{2^M}$ an order by f ≤ g ⇔ f (x) ≤ g (x) for every x∈M. Thus ($\mathbf{2^M}$, ≤) become a Boolean lattice (where for f ∈ $\mathbf{2^M}$, f ′ = **1** - f).

Definition 2.6.15. From Universal Algebra's point of view (see Chapter 3), a *Boolean algebra* is an algebra (B, Ù, Ú, ¢, 0, 1) of type (2, 2, 1, 0,0) such that

B_1: (B, Ù, Ú) Î Ld ;

B_2: In B the following identities are true

x Ù 0 = 0, x Ú 1 = 1, x Ù x¢ = 0, x Ú x¢ = 1.

We denote by **B** the class of Boolean algebras.

If $B_1, B_2 \in$ **B**, $f : B_1 \to B_2$ is called *morphism of Boolean algebras* if f is a morphism in **Ld** (**0**, **1**) and $f(x') = (f(x))'$ for every $x \in B_1$.

The bijective morphisms from **B** will be called *isomorphism.*

Definition 2.6.16. By an ideal (filter) in a Boolean lattice B we understand the corresponding notions from the lattice (B, Ù, Ú). By I(B) (F(B)) we denote the set of ideals (filters) of B.

Proposition 2.6.17. (*Glivenko*). Let (L, Ù, *, 0) be an pseudocomplemented join-semilattice and R(L) = {a* : a Î L}. Then, relative to the induced order from L, R(L) is becomes a Boolean algebra.

Proof. By Theorem 2.6.11 we deduce that L is bounded ($1 = 0^*$) and for $a, b \in$ **R**(L), $a \wedge b \in$ **R**(L) and $\sup_{R(L)} \{a, b\} = (a^* \wedge b^*)^*$, hence **R**(L) is a bounded lattice and sub-meet-semilattice of L.

Since for every $a \in$ **R** (L), $a \vee a^* = (a^* \wedge a^{**})^* = \mathbf{0}^* = \mathbf{1}$ and $a \wedge a^* = \mathbf{0}$ we deduce that a^* is the complement of a in **R**(L). To prove the distributivity of **R**(L), let x, y, z $\in$ **R**(L). Then $x \wedge z \le x \vee (y \wedge z)$ and $y \wedge z \le x \vee (y \wedge z)$, hence $x \wedge z \wedge [x \vee (y \wedge z)]^* = \mathbf{0}$ and $(y \wedge z) \wedge [x \vee (y \wedge z)]^* = \mathbf{0}$, hence $z \wedge [x \vee (y \wedge z)]^* \le x^*, y^*$, therefore $z \wedge [x \vee (y \wedge z)]^* \le x^* \wedge y^*$ and $z \wedge [x \vee (y \wedge z)]^* \wedge (x^* \wedge y^*)^* = \mathbf{0}$ which implies that $z \wedge (x^* \wedge y^*)^* \le [x \vee (y \wedge z)]^{**}$.

Since the left side of the above inequality is $z \wedge (x \vee y)$ and the right side is $x \vee (y \wedge z)$ (in **R**(L)), we deduce that $z \wedge (x \vee y) \le x \vee (y \wedge z)$, that is, **R**(L) is distributive. ∎

Lemma 2.6.18. Let B be a Boolean algebra and a, bÎB such that a Ù b = 0 and aÚ b = 1. Then b = a¢.

Proof. It is immediate from Lemma 2.6.2. ■

Lemma 2.6.19. If B is a Boolean algebra and a, b Î B, then (a¢)¢ = a, (a Ù b)¢=a¢ Ú b¢ and (a Ú b)¢ = a¢ Ù b¢.

Proof. It is immediate from Lemma 2.6.6. ■

Proposition 2.6.20. For every set M, the Boolean algebras 2^M and P(M) are isomorphic.

Proof. Let $X \in P(M)$ and $a_X : M \to \mathbf{2}$,

$$a_X(x) = \begin{cases} 0, & for\ x \notin X \\ 1, & for\ x \in X \end{cases}.$$

Then the assignment $X \to \alpha_X$ defines an isomorhism of Boolean algebras $\alpha : \mathbf{P}(M) \to \mathbf{2}^{\mathbf{M}}$. ∎

For a Boolean algebra B and $a \in B$, we denote $I[a] = [\mathbf{0}, a]$.

Proposition 2.6.21. For every $a \in B$:

(i) $(I[a], \wedge, \vee, {}^*, 0, a)$ is a Boolean algebra, where for $x \in I[a]$, $x^* = x' \wedge a$;

(ii) $\alpha_a : B \to I[a]$, $\alpha_a(x) = a \wedge x$, for $x \in B$, is a surjective morphism of Boolean algebras;

(iii) $B \approx I[a] \times I[a']$.

Proof. (i). $I[a] \in \mathbf{Ld}(\mathbf{0}, \mathbf{1})$ (as sublattice of B). For $x \in I[a]$, $x \wedge x^* = x \wedge (x' \wedge a) = (x \wedge x') \wedge a = \mathbf{0} \wedge a = \mathbf{0}$ and $x \vee x^* = x \vee (x' \wedge a) = (x \vee x') \wedge (x \vee a) = \mathbf{1} \wedge (x \vee a) = x \vee a = a$.

(ii). If $x, y \in B$, then $\alpha_a(x \vee y) = a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y) = \alpha_a(x) \vee \alpha_a(y)$, $\alpha_a(x \wedge y) = a \wedge (x \wedge y) = (a \wedge x) \wedge (a \wedge y) = \alpha_a(x) \wedge \alpha_a(y)$, $\alpha_a(x') = a \wedge x' = (a \wedge a') \vee (a \wedge x') = a \wedge (a' \vee x') = a \wedge (a \wedge x)' = (\alpha_a(x))^*$, $\alpha_a(\mathbf{0}) = \mathbf{0}$ and $\alpha_a(\mathbf{1}) = a$, hence α_a is an surjective morphism in **B**.

(iii). It is immediate that $\alpha : B \to I[a] \times I[a']$, $\alpha(x) = (a \wedge x, a' \wedge x)$ for $x \in B$ is a morphism in **B**.

For $(y, z) \in I[a] \times I[a']$, since $\alpha(y \vee z) = (a \wedge (y \vee z), a' \wedge (y \vee z)) = ((a \wedge y) \vee (a \wedge z), (a' \wedge y) \vee (a' \wedge z)) = (y \vee \mathbf{0}, \mathbf{0} \vee z) = (y, z)$ we deduce that α is surjective. Let now $x_1, x_2 \in B$ such that $\alpha(x_1) = \alpha(x_2)$. Then $a \wedge x_1 = a \wedge x_2$ and $a' \wedge x_1 = a' \wedge x_2$, hence $(a \wedge x_1) \vee (a' \wedge x_1) = (a \wedge x_2) \vee (a' \wedge x_2) \Leftrightarrow (a \vee a') \wedge x_1 = (a \vee a') \wedge x_2 \Leftrightarrow \mathbf{1} \wedge x_1 = \mathbf{1} \wedge x_2 \Leftrightarrow x_1 = x_2$, hence α is an isomorphism in **B**. ∎

2.7. The connections between Boolean rings and Boolean algebras

Definition 2.7.1. A ring $(A, +, \cdot, -, 0, 1)$ is called *Boolean* if $a^2 = a$ for every $a \in A$.

Exemples

1. **2** is a Boolean ring (where 1 + 1 = 0).

2. (**P**(X), Δ, ∩, ′ , ∅, X) with X a non-empty set and Δ the symmetrical difference of sets.

Lemma 2.7.2. If A is a Boolean ring, then for every a Î A, a + a = 0 and A is commutative.

Proof. From $a + a = (a + a)^2$ we deduce that a + a = a + a + a + a, hence a + a = **0**, that is, -a = a.

For a, b ∈ A, from $a + b = (a + b)^2$ we deduce that $a + b = a^2 + ab + ba + +b^2$, hence ab + ba = **0**, so ab = **-** (ba) = ba. ∎

The connections between Boolean algebras and Boolean rings are given by:

Proposition 2.7.3. (i) If (A, +, x , -, 0, 1) is a Boolean ring, then the relation relative to the order £ on A defined by a £ b Û ab = a, A become a Boolean lattice, where for a, b Î A, a Ù b = ab, a Ú b = a + b + ab and a¢ = 1 + a.

(ii) Conversely, if (A, Ù, Ú, ¢, 0, 1) is a Boolean algebra, then A become a Boolean ring relative to operations +, x defined for a, b Î A by a + b = (a Ù b¢) Ú (a¢ Ù b), axb = a Ù b and -a = a.

Proof. (i) The fact that (A, ≤) is a poset is routine. Let now a, b ∈ A. Since $a (ab) = a^2 b = ab$ and $b (ab) = ab^2 = ab$ we deduce that ab ≤ a and ab ≤ b. Let c ∈ A such that c ≤ a and c ≤ b, hence ca = c and cb = c. Then $c^2 ab = c^2$ ⇔ cab = c ⇔ c ≤ ab, hence the conclusion that ab = a ∧ b.

Analogous we prove that a ∨ b = a + b + ab.

Since a ∧ (b ∨ c) = a (b + c + bc) = ab + ac + abc and (a ∧ b)∨ (a ∧ c) = =(ab) ∨ (ac) = $ab + ac + a^2 bc$ = ab + ac + abc we deduce that a∧ (b ∨ c) = (a ∧ b) ∨ (a ∧ c), hence A ∈ **Ld**. Since for a ∈ A, a ∧ a′ = a ∧ (**1** + a) = a (**1** +a) =$a + a^2$ = a + a = **0** and a ∨ a′ = a ∨ (**1** + a) = a + **1** + a + a (**1** + a) = a + **1** + a + a + $+a^2$ = **1** + a+ a + a + a = **1**, a ∧ **0** = a ·**0** = **0** and a ∨ **1** = a + **1** + a ·**1** = a + a + **1** = **1**, we deduce that (A, ∧, ∨, ′, **0**, **1**) is a Boolean lattice.

(ii) For a, b, c ∈ A we have

1. a + (b + c) = [a ∧ (b + c)′] ∨ [a′ ∧ (b + c)] =
= {a ∧ [(b ∧ c′) ∨ (b′ ∧ c)]′} ∨ {a′ ∧ [(b ∧ c′) ∨ (b′ ∧ c)]} =
= {a ∧ [(b′ ∨ c) ∧ (b ∨ c′)]} ∨ {(a′ ∧ b ∧ c′) ∨ (a′ ∧ b′ ∧ c)} =
= {a ∧ [(b′ ∧ c′) ∨ (c ∧ b)]} ∨ {(a′ ∧ b ∧ c′) ∨ (a′ ∧ b′ ∧ c)} =
= (a ∧ b′ ∧ c′) ∨ (a ∧ b ∧ c) ∨ (a′ ∧ b ∧ c′) ∨ (a′ ∧ b′ ∧ c) =
= (a ∧ b ∧ c) ∨ (a ∧ b′ ∧ c′) ∨ (b ∧ c′ ∧ a′) ∨ (c ∧ a′ ∧ b′)

Since the final form is symmetric in a, b and c we deduce that $a + (b + c) = (a + b) + c$.

2. $a + b = (a \wedge b') \vee (a' \wedge b) = (b \wedge a') \vee (a \wedge b') = b + a$.

3. $a + \mathbf{0} = (a \wedge \mathbf{0}') \vee (a' \wedge \mathbf{0}) = (a \wedge \mathbf{1}) \vee \mathbf{0} = a$.

4. $a + a = (a' \wedge a) \vee (a \wedge a') = \mathbf{0} \vee \mathbf{0} = \mathbf{0}$, deci $-a = a$.

5. $a (bc) = a \wedge (b \wedge c) = (a \wedge b) \wedge c = (ab) c$

6. $a \cdot \mathbf{1} = a \wedge \mathbf{1} = a$.

7. $a (b + c) = a \wedge [(b \wedge c') \vee (b' \wedge c)] = (a \wedge b \wedge c') \vee (a \wedge b' \wedge c)$ iar $(ab) + (ac) = (a \wedge b) + (a \wedge c) = [(a \wedge b) \wedge (a \wedge c)'] \vee [(a \wedge b)' \wedge (a \wedge c)] = [a \wedge b \wedge (a' \vee c')] \vee [(a' \vee b') \wedge (a \wedge c)] = [(a \wedge b \wedge a') \vee (a \wedge b \wedge c')] \vee [(a \wedge c \wedge a') \vee (a \wedge c \wedge b')] =$
$= (a \wedge b \wedge c') \vee (a \wedge c \wedge b')$, so $a (b + c) = ab + ac$.

From 1-7 we deduce that $(A, +, \cdot, -, \mathbf{0}, \mathbf{1})$ is a Boolean ring (clearly, $a^2 = a \wedge a = a$ for every $a \in A$). ■

Theorem 2.7.4. Let $(B_1, +, \cdot)$, $(B_2, +, \cdot)$ two Boolean rings and $(B_1, \wedge, \vee, ', 0, 1)$, $(B_2, \wedge, \vee, ', 0, 1)$ the Boolean algebras induced (by Proposition 2.7.3).

Then $f : B_1 \to B_2$ is a morphism of rings iff f is a morphism of Boolean algebras.

Proof. Routine by using Proposition 2.7.3. ■

Theorem 2.7.5. Let B_1, B_2 Boolean algebras and $f : B_1 \to B_2$ a mapping. The following are equivalent :

(i) f is a morphism of Boolean algebras;

(ii) f is a morphism of bounded lattices;

(iii) f is a morphism of meet-semilattices and $f(x') = (f(x))'$ for every $x \in B_1$;

(iv) f is a morphism of join-semilattices and $f(x') = (f(x))'$ for every $x \in B_1$.

Proof. (i) ⇒ (ii). Clearly.

(ii)⇒(i). $f(x) \wedge f(x') = f(x \wedge x') = f(\mathbf{0}) = \mathbf{0}$ and analogous $f(x) \vee f(x') = f(x \vee x') = f(\mathbf{1}) = \mathbf{1}$, hence $f(x') = (f(x))'$.

(iii) ⇒ (i). f is a morphism of join – semilattices since $f(x \vee y) = f(x'' \vee y'') = f((x' \wedge y')') = (f(x' \wedge y'))' = (f(x') \wedge f(y'))' = ((f(x))' \wedge (f(y))')' = f(x)'' \vee f(y)'' = f(x) \vee f(y)$.

Thus $f(\mathbf{0}) = f(x \wedge x') = f(x) \wedge (f(x))' = \mathbf{0}$ and analogous $f(\mathbf{1}) = \mathbf{1}$, hence f is a morphism of Boolean algebras.

(i) ⇒ (iii). Clearly.

(iv). It is the duale of (iii), so the equivalence (iv) ⇔ (i) will be proved analogously as the equivalence (i) ⇔ (iii). ■

Theorem 2.7.6. Let $f : B_1 \to B_2$ a morphism of Boolean algebras and $Ker(f) = f^{-1}\{0\} = \{x \in B_1 : f(x) = 0\}$. Then $Ker(f) \in I(B_1)$ and f is injective iff $Ker(f) = \{0\}$.

Proof. Let $x \in$ **Ker**(f) and $y \in B_1$ such that $y \le x$. Then, since f is isotone $\Rightarrow$ $f(y) \le f(x) = \mathbf{0} \Rightarrow f(y) = \mathbf{0} \Rightarrow y \in$ **Ker**(f). If x, $y \in$ **Ker**(f), then clearly $x \vee y \in$ **Ker** (f), hence **Ker**(f) $\in$ **I**(B_1).

Suppose that **Ker**(f) = {**0**} and let x, y $\in$ **Ker**(f) such that f(x) = f(y). Then $f(x \wedge y') = f(x) \wedge f(y') = f(x) \wedge f(y)' = f(x) \wedge f(x)' = \mathbf{0}$, hence $x \wedge y' \in$ Ker (f), which is, $x \wedge y' = \mathbf{0}$, hence $x \le y$. Analogous we deduce $y \le x$, hence x = y.

The converse implication is clear since f(**0**) = **0**. ■

Theorem 2.7.7. Let $f : B_1 \to B_2$ be a morphism of Boolean algebras. The following are equivalent:

(i) f is a isomorphism of Boolean algebras;

(ii) f is surjective and for every x, $y \in B_1$ we have $x \le y \Leftrightarrow f(x) \le f(y)$;

(iii) f is invertible and f^{-1} is a morphism of Boolean algebras.

Proof. (i) $\Rightarrow$ (ii). If f is a isomorphism, then in particular f is onto.

Since every morphism of Boolean algebras is an isotone function, if $x \le y \Rightarrow f(x) \le f(y)$.

Suppose $f(x) \le f(y)$. Then $f(x) = f(x) \wedge f(y) = f(x \wedge y)$; since f is injective then $x = x \wedge y$, hence $x \le y$.

(ii) $\Rightarrow$ (iii). We shall prove that f is injective. If $f(x) = f(y) \Rightarrow f(x) \le f(y)$ and $f(y) \le f(x) \Rightarrow x \le y$ and $y \le x \Rightarrow x = y$. Since f is surjective, there result that f is bijective, hence invertible. We shall prove for example that $f^{-1}(x \wedge y) = f^{-1}(x) \wedge f^{-1}(y)$ for every x, $y \in B_2$. From x, $y \in B_2$ and f onto we deduce that there are x_1, $y_1 \in B_1$ such that $f(x_1) = x$ and $f(y_1) = y$, hence $f^{-1}(x \wedge y) = f^{-1}(f(x_1) \wedge f(y_1)) = f^{-1}(f(x_1 \wedge y_1)) = x_1 \wedge y_1 = f^{-1}(f(x_1)) \wedge f^{-1}(f(y_1)) = f^{-1}(x) \wedge f^{-1}(y)$.

Analogous $f^{-1}(x \vee y) = f^{-1}(x) \vee f^{-1}(y)$ and $f^{-1}(x') = (f^{-1}(x))'$.

(iii) $\Rightarrow$ (i). Clearly. ■

In a Boolean algebra $(B, \wedge, \vee, ', 0, 1)$, for $x, y \in B$ we define
$x \to y = x' \vee y$ and $x \leftrightarrow y = (x \to y) \wedge (y \to x) = (x' \vee y) \wedge (y' \vee x)$.

Theorem 2.7.8. Let B be a Boolean algebra.
Then for every x, y, $z \in B$ we have:

(i) $x \le y \Leftrightarrow x \to y = 1$;

(ii) $\mathbf{x \rightarrow 0 = x', 0 \rightarrow x = 1, x \rightarrow 1 = 1, 1 \rightarrow x = x, x \rightarrow x = 1, x' \rightarrow x = x, x \rightarrow x' = x'}$;

(iii) $\mathbf{x \rightarrow (y \rightarrow x) = 1}$;

(iv) $\mathbf{x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z)}$;

(v) $\mathbf{x \rightarrow (y \rightarrow z) = (x \wedge y) \rightarrow z}$;

(vi) **If** $\mathbf{x \leq y}$**, then** $\mathbf{z \rightarrow x \leq z \rightarrow y}$ **şi** $\mathbf{y \rightarrow z \leq x \rightarrow z}$;

(vii) $\mathbf{(x \rightarrow y) \wedge y = y, x \wedge (x \rightarrow y) = x \wedge y}$;

(viii) $\mathbf{(x \rightarrow y) \wedge (y \rightarrow z) \leq x \rightarrow z}$;

(ix) $\mathbf{((x \rightarrow y) \rightarrow y) \rightarrow y = x \rightarrow y}$;

(x) $\mathbf{(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x = x \vee y}$;

(xi) $\mathbf{x \rightarrow y = \sup \{ z \in B : x \wedge z \leq y\}}$;

(xii) $\mathbf{x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)}$;

(xiii) $\mathbf{(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)}$;

(xiv) $\mathbf{x \wedge (y \rightarrow z) = x \wedge [(x \wedge y) \rightarrow (x \wedge z)]}$;

(xv) $\mathbf{x \leftrightarrow y = 1 \Leftrightarrow x = y}$.

Proof. **(i).** If $x \rightarrow y = \mathbf{1}$, then $x' \vee y = \mathbf{1} \Leftrightarrow x \leq y$.

(iii). $x \rightarrow (y \rightarrow x) = x' \vee (y' \vee x) = \mathbf{1} \vee y' = \mathbf{1}$

Analogous for the other relations. ∎

2.8. Filters in a Boolean algebra

We recall that by a filter in a Boolean algebra $(B, \wedge, \vee, ', 0, 1)$ we understand a filter in the lattice $(B, \wedge, \vee, 0, 1)$. As in the case of lattices, by **F(**B**)** we denote the set of filters of a Boolean algebra B.

A maximal (hence proper) filter of B will be called *ultrafilter*.

As in the case of lattices (see Theorems 2.4.9 and 2.4.10) we deduce

Theorem 2.8.1. (i) In every Boolean algebra B there exist ultrafilters;

(ii) Every element $\mathbf{x \neq 0}$ of B is contained in an ultrafilter.

Corollary 2.8.2. Let B be a Boolean algebra and $\mathbf{x, y \in B}$, $\mathbf{x \neq y}$. Then we have an ultrafilter U of B such that $\mathbf{x \in U}$ and $\mathbf{y \notin U}$.

Proof. If $x \neq y$, then $x \nleq y$ and $y \nleq x$.

If $x \nleq y$, then $x \wedge y' \neq \mathbf{0}$ (because if $x \wedge y' = \mathbf{0}$, then $x \leq y$). By Theorem 2.8.1, (ii), since $x \wedge y' \neq 0$ there is an ultrafilter U of B such that $x \wedge y' \in U$. Since x ∧ y′ ≤ x, y′ and U is in particular a filter, we deduce that x, y′∈ U. Clearly y∉ U (because U ≠ B). **n**

Theorem 2.8.3. Let (B, Ù, Ú, ¢, 0, 1) be a Boolean algebra and F Î F(B). On B we define the following binary relations:

x ~$_F$ y Û there is f Î F such that x Ù f = y Ù f,

x »$_F$ y Û x « y Î F.

Then

(i) ~$_F$ = »$_F$ $\overset{not}{=}$ r$_F$;

(ii) r$_F$ **is a congruence on B;**

(iii) If for every x Î B we denote by x/F the equivalence class of x relative to r$_F$**, B/F = {x/F : x Î B}, and we define for x,y Î B, x/F Ù y/F = (xÙy)/F, x/F Ú y/F = (xÚy)/F and (x/F)¢ = x¢/F, (B/F, Ù, Ú, ¢, 0, 1) becomes a Boolean algebra (where 0 = {0}/F = { x Î B : x¢ Î F} and 1= {1}/F = F).**

Proof.(i). Let x ~$_F$ y; then there is f∈ F such that x ∧ f = y ∧ f. .

Then x′ ∨ (x ∧ f) = x′ ∨ (y ∧ f) ⇒ (x′ ∨ x) ∧ (x′∨ f) = (x′∨ y) ∧ (x′ ∨ f) ⇒ x′ ∨ f = (x′ ∨ y) ∧ (x′ ∨ f). Since f∈ F and f ≤ x′ ∨ f, then x′ ∨ f ∈ F ⇒ x′∨y∈ F. Analogous x ∨ y′∈ F, hence x ↔ y ∈ F, that is, x ≈$_F$ y.

Conversely, if x ≈$_F$ y ⇒ x ↔ y∈ F ⇒(x′ ∨ y) ∧ (x ∨ y′)∈ F ⇒ x′ ∨ y, x ∨ y′ ∈ F.If we denote x′ ∨ y = f_1 and x ∨ y′= f_2, then f_1, f_2∈ F and x ∧ f_1 = x ∧ (x′ ∨ y) = (x ∧ x′) ∨ (x ∧ y) = x ∧ y , y ∧ f_2 = x ∧ y, so, if f = f_1 ∧ f_2∈ F, then x ∧ f = y ∧ f.

(ii). We shall prove that ρ$_F$ is a congruence on B (see Lemma 2.5.2).

Since x′ ∨ x = **1**∈ F, then x ρ$_F$ x, hence ρ$_F$ is reflexive. As the symmetry is immediate, to prove the transitivity of ρ$_F$, let x,y,z ∈ F such that x ρ$_F$ y and y ρ$_F$ z hence x′ ∨ y , x ∨ y′, y′ ∨ z , y ∨ z′ ∈ F. Then x′ ∨ z = x′ ∨ z ∨ (y ∧ y′)= (x′ ∨ z ∨ y)∧ (x′ ∨ z ∨ y′) ≥ (x′ ∨ y) ∧ (y′ ∨ z). Since x′ ∨ y, y′ ∨ z ∈ F, then x′ ∨ z ∈ F. Analogous x ∨ z′ ∈ F, hence x ρ$_F$ z, that is ρ$_F$ is an equivalence on B.

To probe the compatibility of ρ$_F$ with the operations ∨ ,∧,′, let x,y,z,t ∈ B such that x ρ$_F$ y and z ρ$_F$ t. Then x′ ∨ y, z′ ∨ t∈ F ⇒ (x′ ∨ y) ∧ (z′ ∨ t) ∈ F. We have (x′ ∨ y)∧(z′ ∨ t) ≤ (x′∨ y ∨ t)∧(z′ ∨ t ∨ y) = (x′ ∧ z′) ∨ (y ∨ t) = (x ∨ z)′ ∨ (y ∨ t), hence (x ∨ z)′ ∨ (y ∨ t) ∈ F.

Analogous (y ∨ t)′ ∨ (x ∨ z), hence (x ∨ z) ρ$_F$ (y ∨ t).

Suppose that x ρ$_F$ y. Then x ↔ y∈ F and x′↔y′ = (x′′∨ y′)∧(y′′∨ x′) = (x ∨ y′) ∧ (x′ ∨ y) = x ↔ y, hence x′ ρ$_F$ y′.

To prove the compatibility of ρ$_F$ with ∧, suppose x ρ$_F$ y and z ρ$_F$ t. Then x′ ρ$_F$ y′, z′ ρ$_F$ t′ and (x′ ∨ z′)ρ$_F$ (y′ ∨ t′) ⇔ (x ∧ z)′ ρ$_F$ (y ∧ t)′ ⇔ (x ∧ z) ρ$_F$ (y ∧ t).

(iii). Clearly, since ρ$_F$ is a congruence on B. **n**

Theorem 2.8.4. Let B_1, B_2 two Boolean algebras and $f : B_1 \to B_2$ a morphism of Boolean algebras. We denote $F_f = f^{-1}(\{1\}) = \{x \in B_1 : f(x) = 1\}$. Then

(i) $F_f \in F(B_1)$;
(ii) f is an injective function iff $F_f = \{1\}$;
(iii) $B_1/F_f \approx Im(f)$ (where $Im(f) = f(B_1)$).

Proof. (i). It follows from Theorem 2.7.6 and from Principle of duality.

(ii). If f is injective and we have $x \in F_f$, then from $f(x) = \mathbf{1} = f(\mathbf{1}) \Rightarrow x = \mathbf{1}$. If $F_f = \{\mathbf{1}\}$ and $f(x) = f(y)$, then $f(x' \vee y) = f(x \vee y') = \mathbf{1}$, hence $x' \vee y = x \vee y' = \mathbf{1}$, therefore $x \le y$ and $y \le x$, hence $x = y$.

(iii). We consider the function $\alpha : B_1/F_f \to f(B_1)$ defined by $\alpha(x/F_f) = f(x)$, for every $x/F_f \in B_1/F_f$.

Since for $x, y \in B_1$: $x/F_f = y/F_f \Leftrightarrow x \sim_{F_f} y \Leftrightarrow (x' \vee y) \wedge (x \vee y') \in F_f$ (by Theorem 2.8.3) $\Leftrightarrow f((x' \vee y) \wedge (x \vee y')) = \mathbf{1} \Leftrightarrow f(x' \vee y) = f(x \vee y') = \mathbf{1} \Leftrightarrow f(x) = f(y) \Leftrightarrow \alpha(x/F_f) = \alpha(y/F_f)$, we deduce that α is correctly defined and injective.

We have: $\alpha(x/F_f \vee y/F_f) = \alpha((x \vee y) / F_f) = f(x \vee y) = f(x) \vee f(y) = \alpha(x/F_f) \vee \alpha(x/F_f)$; analogous we have $\alpha(x/F_f \wedge y/F_f) = \alpha(x/F_f) \wedge (y/F_f)$ and $\alpha(x'/F_f) = (\alpha(x/F_f))'$, hence α is a morphism of Boolean algebras.

Let $y = f(x) \in f(B_1)$, $x \in B_1$; then $x/F_f \in B_1/F_f$ and $\alpha(x/F_f) = f(x) = y$, hence α is surjective, that is, an isomorphism of Boolean algebras. ∎

Theorem 2.8.5. For a proper filter F of a Boolean algebra B the following assertions are equivalent:

(i) F is an ultrafilter;
(ii) For every $x \in B \setminus F$, then $x' \in F$.

Proof. We remark that it is not possible to have $x, x' \in F$, because then $x \wedge x' = \mathbf{0} \in F$, hence $F = B$, which is a contradiction!.

(i) $\Rightarrow$ (ii). Suppose F is an ultrafilter and let $x \notin F$. Then $[F \cup \{x\}) = B$. Since $\mathbf{0} \in B$, there are $x_1, \ldots, x_n \in F$ such that $x_1 \wedge \ldots \wedge x_n \wedge x = \mathbf{0}$, hence $x_1 \wedge \ldots \wedge x_n \le x'$, therefore $x' \in F$.

(ii) $\Rightarrow$ (i). Suppose by contrary that there is a filter F_1 in B such that $F \subsetneq F_1$; hence we have $x \in F_1 \setminus F$. Then $x' \in F$, hence $x' \in F_1$; since $x \in F_1$ we deduce that $0 \in F_1$, hence $F_1 = B$, that is, F is an ultrafilter. ∎

Theorem 2.8.6. For a proper filter F of a Boolean algebra B, the following assertions are equivalent:

(i) F is an ultrafilter;
(ii) $0 \notin F$ and for every $x, y \in B$, if $x \vee y \in F$ then $x \in F$ or $y \in F$ (that is, F is *prime* filter).

Proof. (i) $\Rightarrow$ (ii). Suppose $x \vee y \in F$ and $x \notin F$.

Then $[F \cup \{x\}) = B$; since $\mathbf{0} \in B$ there is $z \in F$ such that $z \wedge x = \mathbf{0}$. Since $z, x \vee y \in F$ there results that $z \wedge (x \vee y) = (z \wedge x) \vee (z \wedge y) = \mathbf{0} \vee (z \wedge y) = z \wedge y \in F$. Since $z \wedge y \leq y$ we deduce that $y \in F$.

(ii) $\Rightarrow$ (i). Since for every $x \in B$, $x \vee x' = 1$, we deduce that $x \in F$ or $x' \in F$ hence by Theorem 2.8.5, F is an ultrafilter. ■

Theorem 2.8.7. For a proper filter F of a Boolean algebra B, the following assertions are equivalent:

(i) F is an ultrafilter;

(ii) $B/F \approx \mathbf{2}$.

Proof. (i) $\Rightarrow$ (ii). We recall that $B/F = \{x/F : x \in B\}$ (see Theorem 2.8.3). Let $x \in B$ such that $x/F \neq \mathbf{1}$. Then $x \notin F$ and by Theorem 2.8.5, $x' \in F$, hence $x'/F = \mathbf{1}$. But $(x/F)' = x'/F$, hence $x/F = (x/F)'' = \mathbf{1}' = \mathbf{0}$, so $B/F = \{\mathbf{0},\mathbf{1}\} \approx \mathbf{2}$.

(ii) $\Rightarrow$ (i). If $x \notin F$ then $x/F \neq \mathbf{1}$, hence $x/F = \mathbf{0}$ and $x'/F = (x/F)' = \mathbf{0}' = \mathbf{1}$, therefore $x' \in F$, so, by Theorem 2.8.5 we deduce that F is an ultrafilter. ■

Theorem 2.8.8. (*Stone*). For every Boolean algebra B there is a set M such that B is isomorphic with a Boolean subalgebra of the Boolean algebra (P(M), $\subseteq$).

Proof. We consider $M = \mathbf{FM}(B)$ the set of all ultrafilters of B and $u_B : B \to \mathbf{P}(\mathbf{FM}(B))$, $u_B(x) = \{F \in \mathbf{FM}(B) : x \in F\}$. We shall prove that u_B is an injective morphism of Boolean algebras; then B will be isomorphic with $u_B(B)$.

If $x,y \in B$ and $x \neq y$ then by Corollary 2.8.2 we have $F \in \mathbf{FM}(B)$ such that $x \in F$ and $y \notin F$, hence $F \in u_B(x)$ and $F \notin u_B(y)$, therefore $u_B(x) \neq u_B(y)$.

Clearly, $u(0) = \emptyset$ and $u(1) = \mathbf{FM}(B)$.

Let now $x,y \in B$ and $F \in \mathbf{FM}(B)$. We have: $F \in u_B(x \wedge y) \Leftrightarrow x \wedge y \in F \Leftrightarrow x \in F$ and $y \in F$, hence $u_B(x \wedge y) = u_B(x) \wedge u_B(y)$.

By Theorem 2.8.6 we deduce that $u_B(x \vee y) = u_B(x) \vee u_B(y)$, and by Theorem 2.8.5 we deduce that $u_B(x') = (u_B(x))'$, that is, u_B is a morphism of Boolean algebras. ■

Definition 2.8.9. By *field of sets* on a set X we understand a ring of sets $\mathcal{F}$ of X such that for every $A \in \mathcal{F}$, then $X \setminus A \in \mathcal{F}$.

Clearly, a field of sets of a set X is a Boolean subalgebra of the Boolean algebra $(\mathbf{P}(X), \cap, \cup, ', \emptyset, X)$.

So, Theorem 2.8.8 of Stone has the following form for Boolean algebras :

Every Boolean algebra is isomorphic with a field of sets.

Remark. For the proof of Theorem 2.8.8 we can use the proof of Proposition 2.4.7 and Corollary 2.4.8 (by working with ideals). This explains why the forms of φ_L (from Proposition 2.4.8) and u_B (from Theorem 2.8.8) are different.

From Corollary 2.4.7 and Theorem 2.8.8 we deduce :

Corollary 2.8.10. Every bounded distributive lattice can be embedded by an one-to-one morphism of bounded lattices in a Boolean algebra.

Theorem 2.8.11. (*Glivenko*). Let (L, Ù, *, 0) a pseudocomplemented meet–semilatice. Then, relative to induced order from L, R(L) becomes a Boolean algebra and L / D(L) » R(L) (as Boolean algebras).

Proof. By Theorem 2.6.11, **R**(L) = { a∈L : a = a**} and it is a bounded sublattice of L. If a∈**R**(L), then a = a** and a*∈**R**(L). Since a ∧ a* = **0** ∈**R**(L) and a ∨ a* (in **R**(L)) = (a* ∧ a**)* = **0*** = **1**, we deduce that a* is (in **R**(L)) the complement of a. **n**

Theorem 2.8.12. (*Nachbin*). A bounded distributive lattice L is a Boolean lattice iff every prime filter of L is maximal.

Proof. ([45]).“⇒”.It follows from Theorem 2.8.6.

“⇐”. Suppose that L contains an uncomplemented element a. Take the filters F_0 = {x∈L : a ∨ x = **1**} and F_1 = { x∈L : a ∧ y ≤ x for some y∈F_0}. Since a is uncomplemented, then 0∉F_1. By Theorem 2.4.1 there is a prime filter P_1 such that $F_1 \subseteq P_1$. Let I = ((L \ P_1)∪{a}]. We remark that L \ P_1 ⊂ I, since a∈I and a∈$F_1 \subseteq P_1$ implies a∉L \ P_1.

We have to prove that $F_0 \cap I = \emptyset$. If by contrary there is x∈$F_0 \cap I$, then x∈F_0 and because L \ P_1 is an ideal, then x ≤ a ∨ y for some y∈L \ P_1. Then **1** = a ∨ x ≤ a ∨ y hence y∈$F_0 \subseteq F_1 \subseteq P_1$ – which is a contradiction!. Thus, $F_0 \cap I = \emptyset$ and by Theorem 2.4.1 there is a prime filter P such that $F_0 \subseteq P$ and P∩I=∅. Then P ⊆ L \ I⊂ P_1, therefore P is not maximal. **n**

Theorem 2.8.13. (*Nachbin*). A bounded lattice L is a Boolean lattice iff (Spec(L), Í) is *unordered* (that is, for every P,Q Î Spec(L), P ¹ Q, P ⊄ Q and Q ⊄ P).

Proof. “⇐”. Suppose that L is a Boolean lattice and that there exist P, Q∈Spec(L) such that Q ⊂ P, hence there is a∈P such that a∉Q. Then a′∉P hence a′∉Q. So, we obtain that a, a′∉Q and a ∧ a′ = **0** ∈Q – which is a contradiction (because Q is prime ideal).

"⇒". Suppose that (**Spec**(L), ⊆) is unordered and that there is an element $a \in L$ which has no complement in L (clearly $a \neq \mathbf{0}, \mathbf{1}$).

Set $F_a = \{x \in L: a \vee x = \mathbf{1}\} \in \mathbf{F}(L)$. Clearly, $a \notin F_a$ and take $D_a = [F_a \cup \{a\}) = \{x \in L: x \geq d \wedge a$ for some $d \in F_a\}$ (see Corollary 2.2.6).

D_a does not contain **0**, since if by contrary $\mathbf{0} \in D_a$ then we have $d \in F_a$ (hence $d \vee a = \mathbf{1}$) such $d \wedge a = \mathbf{0}$, would mean that d is a complement of a – which is a contradiction!.

By Theorem 2.4.1 we have $P \in$ **Spec**(L) such that $P \cap D_a = \emptyset$. Then $\mathbf{1} \notin (a] \vee P$, otherwise we have $\mathbf{1} = a \vee p$ for some $p \in P$, hence $p \in D_a$, contradicting $P \cap D_a = \emptyset$.

By Theorem 2.4.1 there is an ideal $Q \in$ **Spec**(L) such that $(a] \vee P \subseteq Q$; then $P \subset Q$ which is impossible since (**Spec**(L), ⊆) is supposed unordered. ■

2.9. Algebraic lattices

Definition 2.9.1. Let L be a complete lattice and $a \in L$. The element a is called *compact* if we have $X \subseteq L$ such that $a \leq \sup(X)$, then there is a finite subset $X_1 \subseteq X$ such that $a \leq \sup(X_1)$.

The complete lattice L is called *algebraic* (or *compact generated*) if every element of H is the supremum of some compact elements.

Theorem 2.9.2. Let (L, ∨, 0) be a join-semilattice. Then (I(L), ⊆) is an algebraic lattice.

Proof. The lattice (**I**(L), ⊆) is complete since for every $(I_k)_{k \in K}$, $\bigwedge_{k \in K} I_k = \bigcap_{k \in K} I_k$ and $\bigvee_{k \in K} I_k = (\bigcup_{k \in K} I_k] = \{x \in L: x \leq x_1 \vee \ldots \vee x_n$ with $x_i \in I_{k_i}$, $1 \leq i \leq n$ and $\{k_1, \ldots, k_n\} \subseteq K\}$ (see Proposition 2.2.5).

We have to prove that for every $a \in L$, (a] is a compact element in the lattice (**I**(L), ⊆); so we suppose that we have $X \subseteq \mathbf{I}(L)$ such that $(a] \subseteq \vee\{I \in \mathbf{I}(L): I \in X\}$. By Proposition 2.2.5, $a \leq x_1 \vee \ldots \vee x_n$ with $x_k \in I_k$, $I_k \in X$, $1 \leq k \leq n$. If we consider $X_1 = \{I_1, \ldots, I_n\}$, we deduce that $(a] \subseteq \vee\{I \in \mathbf{I}(L) : I \in X_1\}$, that is, (a] is compact. Since for every $I \in \mathbf{I}(L)$ we have $I = \vee\{(a]: a \in I\}$, we deduce that (**I**(L), ⊆) is algebraic. ■

Theorem 2.9.3. Let (L, ∧, ∨, 0, 1) be an algebraic lattice and C_L the set of compact elements of L. Then C_L is a sub-join-semilattice of L and L is isomorphic (latticeal) with $I(C_L)$.

Proof. Clearly, $0 \in C_L$. Let now a, $b \in C_L$ and suppose that $a \vee b \leq \sup(X)$ with $X \subseteq L$. Then $a \leq a \vee b \leq \sup(X)$, hence there is a finite subset $X_a \subseteq X$ such that $a \leq \sup(X_a)$. Analogous we deduce the existence of a finite subset $X_b \subseteq X$ such that $b \leq \sup(X_b)$. Since $X_a \cup X_b \subseteq X$ is finite and $a \vee b \leq \sup(X_a \cup X_b)$, we deduce that $a \vee b \in C_L$.

We consider $\varphi: L \to \mathbf{I}(C_L)$ defined for $a \in L$ by $\varphi(a) = \{x \in C_L : x \leq a\} = (a]$ (in C_L) and we have to prove that φ is a latticeal isomorphism.

From the definition of algebraic lattice, we deduce that $a = \sup \varphi(a)$, hence φ is injective. To prove the surjectivity of φ, let $I \in \mathbf{I}(C_L)$ such that $a = \sup(I)$. Then $I \subseteq \varphi(a)$ and let $x \in \varphi(a)$. We have that $x \leq \sup I$, and by the compacity of x, $x \leq \sup I_1$ with $I_1 \subseteq I$ finite. We deduce that $x \in I$, hence $\varphi(a) = I$. By Corollary 2.3.11 we deduce that φ is a morphism of lattices, so φ is an isomorphism of lattices. ■

Corollary 2.9.4. A lattice L is algebraic iff it is isomorphic with the lattice of ideals of a join-semilattice with 0.

Corollary 2.9.5. If L is a lattice, then (I(L),⊆) and (F(L),⊆) are algebraic lattices.

Definition 2.9.6. A complete lattice L will be called *Brouwerian* if $a \wedge (\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (a \wedge b_i)$ **, for every $a \in L$ and every family $(b_i)_{i \in I}$ of elements of L.**

Theorem 2.9.7. For every distributive lattice L, the lattices (I(L), ⊆) and (F(L), ⊆) are Brouwerian algebraic lattices.

Proof. By the Principle of duality it will suffice to prove only for ($\mathbf{I}$(L), ⊆); so, let I, $(I_k)_{k \in K}$ ideals of L.

The inclusion $I \wedge (\bigvee_{k \in K} I_k) \supseteq \bigvee_{k \in K} (I \wedge I_k)$ is clear.

Let now $x \in I \wedge (\bigvee_{k \in K} I_k) = I \cap (\bigcup_{k \in K} I_k]$. Then $x \in I$ and we have a finite subset $K' \subseteq K$ such that $x_k \in I_k$ ($k \in K'$) and $x \leq \bigvee_{k \in K'} x_k$. Then $x = x \wedge (\bigvee_{k \in K'} x_k) = \bigvee_{k \in K'} (x \wedge x_k)$; since $x \wedge x_k \in I \cap I_k$ for every $k \in K'$ we deduce that $x \in \bigvee_{k \in K} (I \wedge I_k)$, therefore $I \wedge (\bigvee_{k \in K} I_k) = \bigvee_{k \in K} (I \wedge I_k)$. ■

Let L be a distributive lattice with 0 and 1; for I, J∈**I**(L) we define I → J = {x∈L: [x) ∩I ⊆ J}.

Lemma 2.9.8. I → J = {x∈L: x∧i∈J, for every i∈I}.

Proof. If x∈I → J and i∈I, since x∧i ∈ [x)∩I ⊆ J, we deduce that x∧i∈J, so we have an inclusion.

Suppose now that x∈L and x∧i∈J for every i∈I. If y∈[x)∩I, then y ≤ x and y∈I. We deduce that y=y∧x∈J, hence [x)∩I⊆J, therefore x∈ I → J, which is, the other inclusion, hence we have the equality from the enounce. ■

2.10. Closure operators

Definition 2.10.1. For a set A, a function C : P(A) → P(A) is called *closure operator* on A if for every X, Y ⊆ A we have

Oc_1 : X ⊆ C (X) ;
Oc_2 : C^2 (X) = C (X);
Oc_3 : X ⊆ Y implies C (X) ⊆ C (Y).

A subset X of A is called *closed subset* if **C**(X) = X; we denote by $L_{\mathbf{C}}$ the set of all closed subsets of A.

Theorem 2.10.2. If C is a closure operator on a set A, then ($L_{\mathbf{C}}$, ⊆) is a complete lattice.

Proof. It is immediate that if $(A_i)_{i\in I}$ is a family of closed subsets of A, then inf $((A_i)_{i\in I}) = C(\bigcap_{i\in I} A_i)$ and sup $((A_i)_{i\in I}) = \mathbf{C}(\bigcup_{i\in I} A_i)$. ■

Theorem 2.10.3. Every complete lattice L is isomorphic to the lattice of closed subsets of some set A with a closure operator C.

Proof. For X ⊆ L if we define **C** : **P** (L) → **P** (L), **C** (X) = {a ∈ L : a ≤ sup (X)}, then **C** is a closure operator on L and the assignment a → {b ∈ L : b ≤ a} = C({a}), for a ∈ L gives the desired isomorphism between the lattices L and $L_{\mathbf{C}}$. ■

Definition 2.10.4. A closure operator C on the set A is said to be *algebraic closure operator* if for every X ⊆ A we have

Oc_4: C(X) = ∪ {C(Y) : Y ⊆ X and Y is finite}.

Theorem 2.10.5. If C is an algebric closure operator on A, then the lattice L_C is an algebraic lattice (see Definition 2.9.14).

The compact elements of L_C are precisely the closed sets C (X) with X ⊆ A a finite subset of A.

Proof. ([11]). If we prove that **C** (X) is compact, with X ⊆ A a finite subset then, by Oc_4 and Theorem 2.10.2 we deduce that L_C is algebraic.

Let X = {a_1, a_2, ... , a_n} ⊆ A such **C**(X) ⊆ $\bigvee_{i\in I}$ C(A_i) = **C**($\bigcup_{i\in I} A_i$).

For each a_j ∈ X, by Oc_4, we have a finite $X_j \subseteq \bigcup_{i\in I} A_i$ such that a_j ∈ **C** (X_j).

Since there are finitely many A_j, say $A_{j_1},...,A_{j_{n_j}}$ such that $X_j \subseteq A_{j_1} \cup ... \cup A_{j_{n_j}}$, then a_j ∈ **C** ($A_{j_1} \cup ... \cup A_{j_{n_j}}$).

Since $X \subseteq \bigcup_{1\le j\le k} C(A_{j_1} \cup ... \cup A_{j_{n_j}})$, then $X \subseteq C(\bigcup_{\substack{1\le j\le k\\ 1\le i\le n_j}} A_{j_i})$, hence

C$(X) \subseteq C(\bigcup_{\substack{1\le j\le k\\ 1\le i\le n_j}} A_{j_i}) = \bigvee_{\substack{1\le j\le k\\ 1\le i\le n_j}} C(A_{j_i})$, which means **C**(X) is compact.

Suppose now that C(Y) is not equal to C(X) for any finite subset X of A.

Since **C**(Y) = ∪ {**C**(X) : X ⊆ Y and X finite} we deduce that **C**(Y) can not be contained in any finite union of C(X), **C**(Y) is not compact. ■

Definition 2.10.6. If C is a closure operator on A and Y ⊆ A a closed subset of A, Y = C(X), then we say that X is a *generating set* for Y. If X is finite we say that Y is *finitely generated*.

Corollary 2.10.7. If C is a closure operator on A, then the finitely generated subsets of A are precisely the compacts elements of L_C.

Theorem 2.10.8. Every algebraic lattice L is isomorphic to the lattice of closed subsets of some set A relative to an algebraic closure operator C on A.

Proof. Let A ⊆ L the subset of compacts elements of A. For X ⊆ A we define **C**(X) = {a ∈ A : a ≤ sup (X)}. It is immediate that **C** is an algebraic closure operator on A and the assignment a → {b ∈ A : b ≤ a}, a ∈A gives the desired isomorphism as L is compactly generated. ■

Chapter 3

TOPICS IN UNIVERSAL ALGEBRA

In this chapter we will present the fundamental concepts and results of Universal Algebra (some of them more or less studied, depending on the way they will be necessary for the following chapters).

The introduction of this chapter was necessary because the semilattices, lattices, Boolean algebras, as other algebraic structures will be considered in most part of this book as algebras.

3.1. Algebras and morphisms

For a non-empty set A and a natural number n we define $A^0 = \{\emptyset\}$ and for n > 0, $A^n = \underbrace{A \times ... \times A}_{n\ times}$.

Definition 3.1.1. By *n – ary algebraic operation* on set A we understand a function $f : A^n \rightarrow A$ (n will be called the *arity* or *rank* of f).

An operation $f : A^0 = \{\emptyset\} \rightarrow A$ will be called *nullary* operation (or *constant*), $f : A \rightarrow A$ will be called *unary*, $f : A^2 \rightarrow A$ will be called binary, etc.

By similarity type (or *type*) we understand an m - tuple $\tau = (n_1,n_2,...,n_m)$ of natural numbers; m will be called the *order* of τ (in symbols we write $m = o(\tau)$).

By an *algebra of type* $\tau = (n_1, n_2, ... , n_{o(\tau)})$ we understand a pair $A = (A, F)$ where A is a non-empty set (called the universe or *underlying set of* algebra *A*) and *F* is an $o(\tau)$ – tuple $(f_1, f_2, ... , f_{o(\tau)})$ of algebraic operations on A such that for every $1 \leq i \leq o(\tau)$, f_i is n_i –ary algebraic operation on A.

Remark 3.1.2. (i). Usually we use for all algebras of type τ the same notation f_i for n_i – ary operation, $1 \leq i \leq o(\tau)$.

(ii). In what follows, if there is no danger of confusion, by *algebra* we understand only its universe (without mentioning every time the algebraic operations) and when in general we speak relative to an algebra A *algebra of type* τ we understand an algebra of type $(n_1, n_2, ... , n_{o(\tau)})$.

(iii). Giving a nullary operation on A is equivalent with putting in evidence an element of A.

Definition 3.1.3. An algebra A = (A, *F*) is called *unary* if all of its operations are unary and *mono-unary* if it has just one unary operation.

A is called *grupoid* if it has just one binary operation, *finite* if A is a finite set and *trivial* if A has just one element.

Examples

1. *Groups*. A *group* is an algebra $(G, \cdot, ^{-1}, \mathbf{1})$ of type (2, 1, 0), such that the following identities are true:

G_1: $x \cdot (y \cdot z) = (x \cdot y) \cdot z$;
G_2: $x \cdot \mathbf{1} = \mathbf{1} \cdot x = x$;
G_3: $x \cdot x^{-1} = x^{-1} \cdot x = \mathbf{1}$.

A group is called *commutative* (or *abelian*) if the following identity is true:

G_4: $x \cdot y = y \cdot x$.

2. *Semigroups* and *monoids*. By a *semigroup* we understand an algebra (G,.) in which G_1 is true; a *monoid* is an algebra $(M, \cdot, \mathbf{1})$ of type (2, 0), satisfying **G_1** and **G_2**.

3. *Rings.* A *ring* is an algebra $(A, +, \cdot, -, \mathbf{0})$ of type (2,2,1,0) satisfying the following condition:

R_1: $(A, +, -, \mathbf{0})$ is an abelian group;
R_2: $(A, \cdot)$ is a semigroup;
R_3: the next identities are true:
$x \cdot (y + z) = x \cdot y + x \cdot z$
$(x + y) \cdot z = x \cdot z + y \cdot z$.

By a *ring with identity* we understand an algebra $(A, +, \cdot, -, \mathbf{0}, \mathbf{1})$ of type (2, 2, 1, 0, 0) such that $(A, +, \cdot, -, \mathbf{0})$ is a ring, $\mathbf{1} \in A$ is a nullary operation such that **G_2** is true.

4. *Semilattices.* From Universal algebra view point, a *semilattice* (see Chapter 2) is a semigroup $(S,\wedge)$ which satisfies **G_4** and the loin of *idempotence*

S_1: $x \wedge x = x$.

5. *Lattices.* From Universal algebra view point, a *lattice* (see Chapter 2) is an algebra $(L, \wedge, \vee)$ of type (2, 2) such that the following identities are verified:

L_1: $x \vee y = y \vee x$, $x \wedge y = y \wedge x$
L_2: $(x \vee y) \vee z = x \vee (y \vee z)$, $(x \wedge y) \wedge z = x \wedge (y \wedge z)$
L_3: $x \vee x = x$, $x \wedge x = x$
L_4: $x \vee (x \wedge y) = x$, $x \wedge (x \vee y) = x$.

A *bounded lattice* is an algebra $(L, \wedge, \vee, \mathbf{0}, \mathbf{1})$ of type (2, 2, 0, 0) such that $(L, \wedge, \vee)$ is a lattice, $\mathbf{0}, \mathbf{1} \in L$ are nullary operations such that the following identities are verified:

$$x \wedge \mathbf{0} = \mathbf{0},\ x \vee \mathbf{1} = \mathbf{1}.$$

In the following chapters we will consider other examples of algebras.

Definition 3.1.4. Let A and B two algebras of the same type τ. A function f : A → B is called a *morphism of algebras of type τ* (or simple *morphism*) if for every 1 ≤ i ≤ o(τ) and $a_1, a_2, ..., a_{n_i} \in A^{n_i}$ **we have: f(f$_i$(**$a_1, a_2, ..., a_{n_i}$**)) = f$_i$(** $f(a_1), f(a_2), ..., f(a_{n_i})$ **).**

Remark 3.1.5. In what follows, for abbreviating the writing, when we say that "f : A → B is a morphism" we understand that A and B are the universe of two algebras of same type τ and f is a morphism of algebras of type τ.

Examples

1. If (G, · , $^{-1}$, **1**) and (G′, · , $^{-1}$, **1**) are two groups, a *morphism of groups* from G to G′ is a function f : G → G′ such that for every x, y ∈ G, f (x · y) = = f (x) · f (y), f (x^{-1}) = $(f(x))^{-1}$ and f(**1**) = **1** (it is immediate to see that f is a morphism of groups iff f(x · y) = f(x) · f(y) for every x, y ∈ G).

2. If (S, ∧) and (S′, ∧) are two semilattices, then a morphism of semilattices from S to S′ is a function f : S → S′ such that for every x, y ∈ S, f (x ∧ y) = f (x) ∧ f (y) (see Chapter 2).

3. If (L, ∧, ∨) and (L′, ∧, ∨) are two lattices, f : S → S′ is a *morphism of lattices* if f (x ∧ y) = f (x) ∧ f (y) and f(x ∨ y) = f(x) ∨ f(y), for every x, y ∈ L.

In the case of bounded lattices, by a *morphism of bounded lattices* we understand a morphism of lattices f such that f (**0**) = **0** and f (**1**) = **1**.

Remark 3.1.6. The composition of two morphisms of the same type is also by the same type.

The morphisms i : A → B which are injective functions will be called *embeddings*. The morphisms f : A → B with the property that there is a morphism g : B → A such that g ∘ f = 1_A and f ∘ g = 1_B will be called *isomorphisms*; in this case we say that A and B are *isomorphic*, written A ≈ B (see Chapter 4, §2).

It is immediate that if the morphism f : A → B is a bijective mapping, then f^{-1} : B → A is also a morphism, hence isomorphisms are just bijective morphisms.

An isomorphism f : A → A will be called *automorphism* of A.

Remark 3.1.7. For two algebras A and B of the same type, we denote by **Hom**(A, B) the set of all morphisms from A to B.

Definition 3.1.8. Let A be an algebra of type τ and B ⊆ A a non-empty subset. We say that B is *subalgebra* of A if for every 1 ≤ i ≤ o(τ) and $b_1, b_2, ..., b_{n_i} \in B^{n_i}$ **, then f_i (**$b_1, b_2, ..., b_{n_i}$**) ∈ B.**

Clearly, the subalgebras of A (together with the restrictions of the operations from A) are algebras of the same type τ. If B ⊆ A is a subalgebra of A (and if there is no danger of confusion) we simply write B ≤ A.

If A and B are two algebras of same type and f : A → B is morphism, then f(A) is a subalgebra of B; if B ⊆ A, then the inclusion mapping $1_{B, A}$: B → A is a morphism iff B is a subalgebra of A.

Definition 3.1.9. Let A be an algebra and S ⊆ A a subset. If there is the smallest subalgebra of A which contains S, then it is called the *subalgebra of A generated by S* and it will be denoted by [S] (the elements of S will be called *generators* of A). An algebra A is said to be *finitely generated* if there is a finite subset S of A such that [S] = A.

Remark 3.1.10. Since the intersection of a set of subalgebras of A is again a subalgebra of A (except when the intersection is empty!), [S] exists whenever S is non-empty. If S = ∅, then [S] exists if the intersection of all of the subalgebras of A is non-empty.

Lemma 3.1.11. If A and B are two algebras of same type, S ⊆ A is a subset for which there is [S], and f, g : [S] → B are two morphisms such that $f_{|S} = g_{|S}$, then f = g.

Proof. Indeed, let K = {x ∈ [S] : f(x) = g(x)}. Then K is a subalgebra of [S] since for every 1 ≤ i ≤ o(τ) and $(x_1, ..., x_{n_i}) \in K^{n_i}$ then $f(f_i(x_1, ..., x_{n_i})) = f_i(f(x_1), ..., f(x_{n_i})) = f_i(g(x_1), ..., g(x_{n_i})) = g(f_i(x_1, ..., x_{n_i}))$,

that is, $f_i(x_1, ..., x_{n_i})$ ∈ K.

But S ⊆ K ⊆ [S] and [S] contains no proper subalgebras that contains S, hence K = [S]. ■

Let A be an algebra and we consider the operator **Sg** : **P**(A) → **P**(A), **Sg**(X) = [X], for every X ⊆ A.

Theorem 3.1.12. For every algebra A, the operator Sg defined before is an algebraic closure operator on A.

Proof.([11]). It is immediate that **Sg** is a closure operator on A whose closed sets are precisely the subalgebras of A.

For any X ⊆ A we define E (X) = X ∪ {f (a_1, ... , a_n) : f is n – ary operation on A and a_1, ... , a_n ∈ X} and recursive E^n (X) for n ∈ ℕ by $E^0(X) = X$ and $E^{n+1}(X) = E(E^n(X))$.

Since X ⊆ E(X) ⊆ E^2 (X) ⊆ ... we deduce that **Sg**(X) = X ∪ E(X) ∪ E^2 (X) ∪ ..., so, if a ∈ **Sg** (X), then a ∈ $E^n(X)$ for some n ∈ ℕ, hence for some finite subset Y ⊆ X, a ∈ $E^n(Y)$. Then a ∈ **Sg** (Y), hence **Sg** is an algebraic closure operator on A. ■

Corollary 3.1.13. For every algebra A, then L_{Sg} (the lattice of subalgebras of A) is an algebraic lattice; if there is no danger of confusion we denote this lattice by P [A] to be distinguished by the power set P(A) of A.

Theorem 3.1.14. (*Birkhoff - Frink*) If L is an algebraic lattice, then there is an algebra A such that L is isomorphic with P[A].

Proof. ([11]). By Theorem 2.10.8, there is a set A and an algebraic operator of closure **C** on A such that L ≈ $L_{\mathbf{C}}$.

For every finite subset B ⊆ A and b ∈ **C**(B) we define an n-ary operation $f_{B,\,b}$(n = |B|) on A, by:

$$f_{B,b}(a_1,...,a_n)=\begin{cases} b, & if\ B=\{a_1,...,a_n\}, \\ a_1 & otherwise \end{cases}.$$

We also denote by A the resulting algebra.

Then $f_{B,\,b}(a_1, ... , a_n)$ ∈ **C**({a_1, ... , a_n}), hence for X ⊆ A, **Sg**(X) ⊆ **C**(X). Also, **C**(X) = ∪ {**C**(B) : B ⊆ X and B is finite} and for B finite, **C**(B) = { $f_{B,\,b}(a_1, ... , a_n)$: B = {a_1, ... , a_n}, b ∈ **C**(B)} ⊆ **Sg**(B) ⊆ **Sg**(X), which imply **C** (X) ⊆ **Sg** (X), hence **C** (X) = **Sg** (X).

Thus $L_{\mathbf{C}}$ = P [A], hence P[A] ≈ L. ■

3.2. Congruence relations. Isomorphism theorems

Let A be an algebra of type $\tau = (n_1, n_2, \ldots, n_{o(\tau)})$.

Definition 3.2.1. We call the *congruence relation* on A any equivalence relation $\theta \in$ Echiv(A) which verifies the *substitution property*:

For every $i \in \{1, 2, \ldots, o(\tau)\}$, if $(a_j, a'_j) \in \theta$ for $j = 1, 2, \ldots, n_i$, then $(f_i(a_1, a_2, \ldots, a_{n_i}), f_i(a'_1, a'_2, \ldots, a'_{n_i})) \in \theta$.

We denote by **Con**(A) the set of all congruence relations on A (clearly $\Delta_A, \nabla_A \in$ **Con**(A), where we recall that $\Delta_A = \{(x, x) : x \in A\}$ and $\nabla_A = A \times A$).

Let $\theta \in$ **Con** (A); for any $a \in A$ we denote by a / θ the equivalence class of a modulo θ and by A/θ the quotient set of all equivalence classes.

Then A / θ becomes in a natural way an algebra of type τ if we define the n_i – ary operation A / θ by:

$f_i^q : (A/q)^{n_i} \to A/q$, $f_i^q(a_1/q, \ldots, a_n/q) = (f_i(a_1, \ldots, a_n))/q$, (where f_i is n_i – ary algebraic operation on A, $1 \le i \le o(\tau)$).

Since $\theta \in$ **Con**(A), then f_i^q is correctly defined; the canonical mapping $\pi_\theta : A \to A / \theta$, $\pi_\theta(a) = a / \theta$ $(a \in A)$ is clearly a surjective morphism.

Examples

1. Let $(G, \cdot)$ be a group, **L**(G) the lattice of subgroups of G and $\mathbf{L_0}$(G) the modular sublattice of **L**(G) of normal subgroups of G. For $H \in \mathbf{L_0}(G)$, then the binary relation θ_H on G defined by $(a,b) \in \theta_H \Leftrightarrow a \cdot b^{-1} \in H$ is a congruence on G and the assignment $H \to \theta_H$ is a bijective and isotone function between the lattices $\mathbf{L_0}$(G) and **Con**(G) (see [31]).

2. Let R be a commutative ring and **Id**(R) the lattice of ideals of R. For $I \in$ **Id**(R), the binary relation θ_I on R defined by $(a,b) \in \theta_I \Leftrightarrow a - b \in I$ is a congruence relation on R and the assignment $I \to \theta_I$ is a bijective and isotone function between the lattices **Id**(R) and **Con**(R) (see [31]).

Definition 3.2.2. Let A, B be algebras of type $\tau = (n_1, n_2, \ldots, n_{o(\tau)})$ and f ∈ Hom (A, B). Then the *kernel* of f, written ker(f) is defined as a binary relation on A by: (a, b) ∈ Ker(f) ⇔ f(a) = f(b).

Proposition 3.2.3. For f ∈ Hom(A, B), Ker(f) ∈ Con(A).

Proof. Let $1 \le i \le o(\tau)$ and $(a_j, a'_j) \in$ **Ker**(f) for $1 \le j \le n_i$; then $f(a_j) = f(a'_j)$. we deduce that $f(f_i(a_1, a_2, \ldots, a_{n_i})) = f_i(f(a_1), f(a_2), \ldots, f(a_{n_i})) = f_i(f(a'_1), f(a'_2), \ldots, f(a'_{n_i}))$ ⇔ $f(f_i(a_1, a_2, \ldots, a_{n_i})) = f(f_i(a'_1, a'_2, \ldots, a'_{n_i}))$ ⇔ $(f_i(a_1, a_2, \ldots, a_{n_i}), f_i(a'_1, a'_2, \ldots, a'_{n_i})) \in$ **Ker**(f), hence **Ker**(f) ∈ **Con**(A). ∎

Theorem 3.2.4. For every algebra A, (Echiv(A), ⊆) is a complete lattice and Con(A) is a complete sublattice of Echiv(A).

Proof. Clearly (**Echiv**(A), ⊆) is a lattice since for every ρ, ρ′ ∈ **Echiv**(A), $\rho \wedge \rho' = \rho \cap \rho' \in$ **Echiv**(A) and $\rho \vee \rho'$ = the equivalence relation of A generated by $\rho \cup \rho'$ (see Proposition 1.2.8).

We have the following description for $\rho \vee \rho'$: $(a, b) \in \rho \vee \rho'$ iff there is a sequence of elements $a_1, a_2, \ldots, a_n \in A$ such that $a = a_1$, $b = a_n$ and for every $1 \le i \le n - 1$, $(a_i, a_{i+1}) \in \rho$ or $(a_i, a_{i+1}) \in \rho'$.

More, **Echiv**(A) is a complete lattice since for a family $(\theta_i)_{i\in I}$ of elements of **Echiv**(A), $\bigwedge_{i\in I} \theta_i = \bigcap_{i\in I} \theta_i$ and $\bigvee_{i\in I} \theta_i = \cup\{\theta_{i_0} \circ \theta_{i_1} \circ \ldots \circ \theta_{i_k} : i_0, i_1, \ldots, i_k \in I\}$.

Since the intersection of any family of congruence relations on A is also an equivalence relation on A we deduce that **Con**(A) is a complete meet-semilattice and meet-sublattice of **Echiv**(A).

Let now $(\theta_i)_{i\in I}$ be a family of congruence relations on A and f an n-ary algebraic operation on A. If $(a_1, b_1), \ldots, (a_n, b_n) \in \bigvee_{i\in I} \theta_i$, then there are $i_0, i_1, \ldots, i_k \in I$ such that $(a_i, b_i) \in \theta_{i_0} \circ \theta_{i_1} \circ \ldots \circ \theta_{i_k}$, $1 \le i \le n$, so $(f(a_1,\ldots,a_n), f(b_1,\ldots,b_n)) \in \theta_{i_0} \circ \theta_{i_1} \circ \ldots \circ \theta_{i_k}$, hence $\bigvee_{i\in I} \theta_i \in$ **Con**(A), that is, **Con**(A) is a joint-complete sublattice (hence complete) of **Echiv**(A). ∎

Remark 3.2.5. By Remark 2.1.5 it will suffice to prove that **Echiv**(A) (as for **Con**(A)) is a meet–complete to obtain the conclusion that these lattices are complete; we have proved and the join-completitude to give the characterization for $\vee_{i\in I} q_i$ with $\theta_i \in$ **Con**(A): (x, y) $\in \vee_{i\in I} q_i$ iff there is a sequence of elements of A, x = a_1, ..., a_n = y such that for every $1 \leq j \leq n-1$, $(a_j, a_{j+1}) \in q_{i_j}$ with $i_j \in I$.

Theorem 3.2.6. For every algebra A there is an algebraic closure operator on A × A such that the closed subsets of A × A are precisely the congruence on A.

Proof. ([11]). Let us start by organize A × A as an algebra.

Firstly, for every n–ary operation f on A we consider the n–ary operation g on A × A defined by g ((a_1, b_1), ... , (a_n, b_n)) = (f (a_1, ... , a_n), f (b_1, ... , b_n)).

Then we add to these operations the nullary operations (a,a) for each a∈A, an unary operations *s* defined by s((a,b)) = (b,a) and a binary operation *t* defined by

$$t((a,b),(c,d)) = \begin{cases} (a,d) & if\ b = c, \\ (a,b) & otherwise \end{cases}$$

then it is immediate that θ is a subalgebra of A × A iff θ ∈ **Con**(A), so, if we denote by **C** the operator **Sg** (above defined), then **Con**(A) = $(A \times A)_{\mathbf{C}}$. ■

Corollary 3.2.7. For every algebra A, Con(A) is an algebraic lattice.

Proof. Follows from Theorems 2.2.6 and 2.10.8. ■

Definition 3.2.8. For an algebra A and a_1, ... , a_n ∈ A we denote by ⊖(a_1, ... , a_n) the congruence relation on A generated by {(a_i, a_j) : 1 ≤ i, j ≤ n} (i.e, the smallest congruence on A such that a_1, a_2, ... , a_n are in the same equivalence class).

The congruence ⊖(a_1, a_2) is called the *principal congruence*.

For a subset Y ⊆ A, by ⊖(Y) we denote the congruence generated by Y × Y.

Examples

1. If G is a group and a, b, c, d ∈G, then (a, b) ∈ ⊖(c, d) iff ab^{-1} is a finite product of conjugates of cd^{-1} and conjugates of dc^{-1}.

2. If R is a ring with unity and a, b, c, d $\in$ R, then (a, b) $\in \ominus$(c, d) iff a-b = $\sum_{i=1}^{n} r_i(c-d)s_i$, with r_i, $s_i \in$ R, $1 \le i \le n$.

3. If L is a distributive lattice and a,b,c,d $\in$ L, then (a, b) $\in \ominus$ (c, d) iff $c \wedge d \wedge a = c \wedge d \wedge b$ and $c \vee d \vee a = c \vee d \vee b$.

Theorem 3.2.9. Let A be an algebra, $a_1, b_1, \ldots, a_n, b_n \in$ A and $\theta \in$ Con(A). Then

(i) $\ominus(a_1, b_1) = \ominus(b_1, a_1)$;

(ii) $\ominus((a_1, b_1), \ldots, (a_n, b_n)) = \ominus(a_1, b_1) \vee \ldots \vee \ominus(a_n, b_n)$;

(iii) $\ominus(a_1, \ldots, a_n) = \ominus(a_1, a_2) \vee \ominus(a_2, a_3) \vee \ldots \vee \ominus(a_{n-1}, a_n)$;

(iv) $\theta = \cup \{\ominus(a, b) : (a, b) \in \theta\} = \sup \{\ominus(a, b) : (a, b) \in \theta\}$;

(v) $\theta = \cup\{\ominus((a_1, b_1), \ldots, (a_n, b_n)) : (a_i, b_i) \in \theta, n \ge 1\}$.

Proof. ([11]). (i). Since $(b_1, a_1) \in \ominus(a_1, b_1)$ we deduce that $\ominus(b_1, a_1) \subseteq \ominus(a_1, b_1)$ and analogous $\ominus(a_1, b_1) \subseteq \ominus(b_1, a_1)$, hence $\ominus(a_1, b_1) = \ominus(b_1, a_1)$.

(ii). For $1 \le i \le n$, $(a_i, b_i) \in \ominus((a_1, b_1), \ldots, (a_n, b_n))$ (since $\ominus((a_1, b_1), \ldots, (a_n, b_n))$ is a congruence relation on A generated by the set $\{(a_1, b_1), \ldots, (a_n, b_n)\}$), hence $\ominus(a_i, b_i) \subseteq \ominus((a_1, b_1), \ldots, (a_n, b_n))$, so we obtain the inclusion $\ominus((a_1, b_1), \ldots, (a_n, b_n)) \supseteq \ominus(a_1, b_1) \vee \ldots \vee \ominus(a_n, b_n)$.

On the other hand, for $1 \le i \le n$, $(a_i, b_i) \in \ominus(a_i, b_i) \subseteq \ominus(a_1, b_1) \vee \ldots \vee \ominus(a_n, b_n)$; so $\{(a_1, b_1), \ldots, (a_n, b_n)\} \subseteq \ominus(a_1, b_1) \vee \ldots \vee \ominus(a_n, b_n)$, hence $\ominus((a_1, b_1), \ldots, (a_n, b_n)) \subseteq \ominus(a_1, b_1) \vee \ldots \vee \ominus(a_n, b_n)$, so we have the desired equality.

(iii). For $1 \le i \le n-1$, $(a_i, a_{i+1}) \in \ominus(a_1, \ldots, a_n)$, hence $\ominus(a_i, a_{i+1}) \subseteq \ominus(a_1, \ldots, a_n)$, so $\ominus(a_1, a_2) \vee \ominus(a_2, a_3) \ldots \vee \ominus(a_{n-1}, a_n) \subseteq \ominus(a_1, \ldots, a_n)$.

Conversely, for $1 \le i, j \le n$, $(a_i, a_j) \in \ominus(a_i, a_{i+1}) \circ \ldots \circ \ominus(a_{j-1}, a_j)$; so $(a_i, a_j) \in \ominus(a_i, a_{i+1}) \vee \ldots \vee \ominus(a_{j-1}, a_j)$, hence $(a_i, a_j) \in \ominus(a_1, a_2) \vee \ominus(a_2, a_3) \vee \ldots \vee \ominus(a_{n-1}, a_n)$.

In viewing (i), $\ominus(a_1, \ldots, a_n) \subseteq \ominus(a_1, a_2) \vee \ominus(a_2, a_3) \vee \ldots \vee \ominus(a_{n-1}, a_n)$, so $\ominus(a_1, \ldots, a_n) = \ominus(a_1, a_2) \vee \ominus(a_2, a_3) \vee \ldots \vee \ominus(a_{n-1}, a_n)$.

(iv). For $(a, b) \in \theta$, clearly $(a, b) \in \ominus(a, b) \subseteq \theta$, so $\theta \subseteq \cup \{\ominus(a, b) : (a, b) \in \theta\} \subseteq \vee \{\ominus(a, b) : (a, b) \in \theta\} \subseteq \theta$, hence $\theta = \cup \{\ominus(a, b) : (a, b) \in \theta\} = \vee \{\ominus(a, b) : (a, b) \in \theta\}$.

(v). Similarly as in the case of (iv). ■

Let **A** be an algebra of type τ, univers A and $n \in \mathbf{N}^*$.

Since many classes of algebras are defined by "*identities*" we will make this concept in a precise way .

Definition 3.2.10. The *n-ary polynomials* of type τ are functions from A^n to A, defined recursively in the following way:

(i) The projections $p_{i,n} : A^n \to A$, $p_{i,n}(a_1, \ldots, a_n) = a_i$ $(1 \le i \le n)$ are n–ary polynomials on A;

(ii) If $p_1, \ldots, p_{n_i}$ are n-ary polynomials and f_i is n_i – ary algebraic operation, then the function $f_i(p_1, \ldots, p_{n_i}) : A^n \to A$, defined by

$$f_i(p_1,\ldots,p_{n_i})(a_1,\ldots,a_n) = f_i(p_1(a_1,\ldots,a_n),\ldots,p_{n_i}(a_1,\ldots,a_n))$$

is an n–ary polynomial on A;

(iii) The n-ary polynomials on A are exactly those functions which can be obtained by a finite numbers of applications of (i) and (ii).

If $p : A^n \to A$, $(1 \le k \le n)$ is an n–ary polynomial and k variables of p where replaced with some constants from A, we obtain a function from A^{n-k} to A, called *algebraic function*.

Examples

1. If (L, ∨, ∧) is a lattice, then the only unary polynomial on L is 1_L.

Let now have an example of binary polynomials: $p : A^2 \to A$, $p(x, y) = x$, $q : A^2 \to A$, $q(x, y) = x \wedge y$.

2. If (R, +, · , 0, 1) is a ring with identity, then every unary polynomial on R has the form $p(x) = n_0 + n_1x + \ldots + n_mx^m$ where $m \in \mathbf{N}$ and n_i is zero or 1 + ...+ 1 for a finite number of time.

3. If (G, ·) is a semigroup, then every unary polynomial of G has the form $p(x) = x^n$ (with $n \in \mathbf{N}$).

We will present now a characterization for the congruence of the form $\theta(H)$ with $H \subseteq A$.

Theorem 3.2.11. Let *A* be an algebra of univers A **and H** ⊆ A **a non-empty subset.**

Then (c, d) ∈ ⊖(H) iff there is n ∈ N, a sequence of elements $c = z_0, z_1, \ldots, z_n = d$, $(a_i, b_i) \in H \times H$ and algebraic unary functions p_i such that $\{p_i(a_i), p_i(b_i)\} = \{z_{i-1}, z_i\}$, for $1 \le i \le n$.

Proof. ([11]). Following Theorem 3.2.9 it will suffice to prove this theorem only in the case H = {a, b}, and for this we denote by ρ the binary relation on A defined by the right conditions of the equivalence from the enounce.

Since the polynomials have the substitution property, if ρ ∈ **Con**(A) and (a, b) ∈ ρ, then for the sequence $(z_i)_{0\le i \le n}$ of elements in A chosen as in the enounce of the theorem we have $\{z_{i-1}, z_i\} \in \rho$, hence (c, d) ∈ ρ.

So, to prove the equality ⊖(a, b) = ρ (using the fact that ⊖(a, b) is the congruence generated by (a, b)) it is suffice to prove that ρ ∈ **Con**(A) and (a, b) ∈ ρ (then ⊖(a, b) ⊆ ρ and since ρ ⊆ ⊖(a, b) we obtain the desired equality).

Clearly (a, b) ∈ ρ (we can choose the sequence a, b and unary function $p_{1,1}(x) = x$, x ∈ A) and ρ ∈ **Echiv**(A).

We have to prove that ρ has the substitution property.

Let now f_i be the n_i–ary operation and $(a_0,b_0),\ldots,(a_{n_i-1}, b_{n_i-1}) \in \rho (1\le i \le o(\tau))$.

By the definition of ρ we have the sequences

$$a_0 = z_0^0, \ldots, z_{n_0}^0 = b_0$$

- - - - - - - - - - - - - - - - -

$a_{n_i-1} = z_0^{n_i-1}, \ldots, z_{n_{(n_i-1)}}^{n_i-1} = b_{n_i-1}$ of elements in A and

$$p_0^0, \ldots, p_{n_0-1}^0$$

- - - - - - - - - - - - -

$p_0^{n_i-1}, \ldots, p_{n_{(n_i-1)}}^{n_i-1}$ unary algebraic functions which verify the conditions from the enounce in definition of ρ.

We will use mathematical induction relative to i for proving that $(f_i(a_0,\ldots,a_{n_i-1}), f_i(b_0,\ldots,b_{n_i-1})) \in \rho$.

This is clear for i = 0; suppose it is true for $i < n_i$.

Since $(a_i, b_i) \in \rho$, there is the sequence $a_1 = z_0, \ldots, z_m = b_1$ of elements in A and unary polynomials $p_0, \ldots, p_{m-1}$ on A such that $(z_j, z_{j+1}) = \{p_j(a), p_j(b)\}$, for $0 \le j \le m - 1$.

We consider now the sequence

$$t_0 = f_i(b_0,\ldots,b_{i-1}, z_0, a_{i+1},\ldots,a_{n_i-1})$$
$$t_1 = f_i(b_0,\ldots,b_{i-1}, z_1, a_{i+1},\ldots,a_{n_i-1})$$

\- -

$$t_m = f_i(b_0,...,b_{i-1}, z_m, a_{i+1},..., a_{n_i-1})$$

of elements in A and

$$q_0 = f_i(b_0,...,b_{i-1}, p_0, a_{i+1},..., a_{n_i-1})$$

$$q_1 = f_i(b_0,...,b_{i-1}, p_1, a_{i+1},..., a_{n_i-1})$$

\- -

$$q_{m-1} = f_i(b_0,...,b_{i-1}, p_{m-1}, a_{i+1},..., a_{n_i-1})$$

unary algebraic functions on A; by induction hypothesis we deduce immediate that $(f_i(b_0,...,b_{i-1}, a_i, a_{i+1},..., a_{n_i-1}), f_i(b_0,...,b_{i-1}, b_i, a_{i+1},..., a_{n_i-1})) \in \rho$ and by the transitivity of ρ that $(f_i(a_0,..., a_{n_i-1}), f_i(b_0,..., b_{n_i-1})) \in \rho$, so ρ ∈ **Con**(A) and the proof is complete. ■

Corollary 3.2.12. (c, d) ∈ ⊜(a, b) iff there is n ∈ N*, a sequence of elements in A, c = z_0, ... ,z_n = d and a sequence of unary algebraic functions p_0, p_1, ... , p_{n-1} such that {p_i(a), p_i(b)} = {z_i, z_{i+1}}, for 0 ≤ i ≤ n - 1.

Examples

1. If (G, ·) is a group and a, b, c, d ∈ G, then (c, d) ∈ ⊜(a, b) iff there is an unary algebraic function p on G such that p(a) = c and p(b) = d.

2. If R is a ring, since a congruence on A is also a congruence on the group (R , +), we deduce for ⊜(a, b) on the ring R the same characterization as in the case of groups.

Definition 3.2.13. An algebra A is called *congruence-modular (distributive)* if (Con(A), ⊆) is a modular (distributive) lattice.

A is *congruence–permutable* if every pair of congruence on A permutes.

Lemma 3.2.14. For an algebra A and θ, θ′∈Con(A), the following are equivalent:

(i) θ ∘ θ′ = θ′ ∘ θ;

(ii) θ ∨ θ′ = θ ∘ θ′;

(iii) θ ∘ θ′ ⊆ θ′ ∘ θ.

Proof. It follows from Proposition 1.2.3. and Theorem 3.2.4. ■

Theorem 3.2.15. (*Birkoff*). If A is a congruence–permutable, then A is congruence-modular.

Proof. Let $\theta_1, \theta_2, \theta_3 \in$ **Con**(A) with $\theta_1 \subseteq \theta_2$.

To prove the modularity loin, it is suffice to prove the inclusion $\theta_2 \cap (\theta_1 \vee \theta_3) \subseteq \theta_1 \vee (\theta_2 \cap \theta_3)$. If $(a, b) \in \theta_2 \cap (\theta_1 \vee \theta_3)$, by Lemma 2.14, $\theta_1 \vee \theta_3 = \theta_1 \circ \theta_3$, hence there is $c \in A$ such that $(a, c) \in \theta_1$ and $(c,b) \in \theta_3$. Then $(c, a) \in \theta_1 \subseteq \theta_2$, hence $(c, a) \in \theta_2$ and since $(a, b) \in \theta_2$ we deduce that $(c, b) \in \theta_2$, hence $(c, b) \in \theta_2 \cap \theta_3$.

From $(a, c) \in \theta_1$ and $(c, b) \in \theta_2 \circ \theta_3$, we deduce that $(a, b) \in (\theta_2 \cap \theta_3) \circ \theta_1$, hence $(a, b) \in \theta_1 \vee (\theta_2 \cap \theta_3)$, so we obtain the modularity equality. ■

In what follows we will present some known theorems in *Universal Algebra* with the name of de *theorems of isomorphism* (next we will still use the convention that when we say that a mapping f : A → B is a morphism of algebras we will understand that A and B are algebras of type τ and f is a morphism of algebras of type τ).

For $f \in$ **Hom** (A, B) we denote by **Im**(f) the image of A by f, that is, $Im(f) = \{f(a) : a \in A\} \leq B$.

Theorem 3.2.16. *(The first theorem of isomorphism)*. Let A, B be two algebras and f∈ Hom(A, B). Then A / Ker(f) ≈ Im(f).

Proof. Let $\theta =$ **Ker**(f) $\in$ **Con**(A) and φ : A / **Ker**(f) → **Im**(f), $\varphi(a / \theta) = f(a)$. We have to prove that φ is an isomorphism.

Indeed, for $a, b \in A$ from the equivalences: $a / \theta = b / \theta \Leftrightarrow (a, b) \in \theta \Leftrightarrow f(a) = f(b)$ we deduce that φ is correctly defined and is an injective function.

Since φ is clearly surjective, to prove that φ is an isomorphism we have only to prove that φ is a morphism.

If f_i is n_i–ary operation on A ($1 \leq i \leq o(\tau)$, where τ is the type of A and B) and $a_1,\ldots, a_{n_i} \in A$, then

$$f_i^q (j(a_1),..., j(a_{n_i})) = f_i^q (a_1 / q,..., a_{n_i} / q) = (f_i(a_1,..., a_{n_i})) / q = j(f_i^q (a_1,..., a_{n_i}))$$

hence φ is a morphism. ■

Corollary 3.2.17. If the morphism f : A → B is surjective, then A / Ker(f) ≈ B.

Let A be an algebra and $\rho, \theta \in$ **Con**(A) with $\theta \subseteq \rho$.

If we denote ρ / θ = {(a / θ, b / θ)} ∈ (A / θ)^2 : (a, b) ∈ ρ}, it is immediate to see that ρ / θ ∈ **Con**(A / θ).

Theorem 3.2.18. (***The second theorem of isomorphism***)
If θ, ρ ∈ Con(A) and θ ⊆ ρ, then (A / θ) / (ρ / θ) ≈ A / ρ.

Proof. Define φ : A / θ → A / ρ by φ(a / θ) = a / ρ, a ∈ A. If a, b ∈ A and a / θ = b / θ, then (a, b)∈ θ ⊆ ρ, hence a / ρ = b / ρ, that is, φ is correctly defined.

If f_i is the n_i–ary operation on A and a_1, …, a_{n_i} ∈A (1 ≤ i ≤ o(τ)), then

$$j\,(f_i^q(a_1/q,...,a_{n_i}/q)) = j\,((f_i(a_1,...,a_{n_i}))/q) =$$

$$= (f_i(a_1,...,a_{n_i}))/r = f_i^q(a_1/r,...,a_{n_i}/r) = f_i^q(j\,(a_1/r),...,j\,(a_{n_i}/r)),$$

hence φ is a morphism (clearly, surjective).

Since for a, b ∈ A we have (a / θ, b / θ) ∈ **Ker**(φ) ⇔ φ(a / θ) = φ (b / θ) ⇔ a / ρ = b / ρ ⇔ (a, b) ∈ ρ ⇔ (a / θ, b / θ) ∈ ρ / θ, we deduce that **Ker**(φ) = ρ / θ and all that results from Corollary 3.2.17. ■

Let now A an algebra, B ⊆ A and θ ∈ **Con**(A).

We denote by B^θ the subalgebra of A generated by {a ∈ A : B ∩ (a / θ) ≠ ∅} and by $θ_{|B}$ = θ ∩ B^2 (if B ≤ A, then $θ_{|B}$ ∈ **Con**(B)).

Theorem 3.2.19. (***The third theorem of isomorphism***)
If B ≤ A and θ ∈ Con(A), then B / $θ_{|B}$ ≈ B^θ / $q_{|B^q}$.

Proof. It is immediate that the desired isomorphism is the mapping φ : B / $θ_{|B}$ → B^θ / $q_{|B^q}$, φ (b / $θ_{|B}$) = b/ $q_{|B^q}$, for every b ∈ B. ■

Theorem 3.2.20. (***Theorem of correspondence***)
Let A be an algebra and θ ∈ Con(A).
Then Con(A / θ) ≈ [θ, $∇_A$] (as lattices).

Proof. We will prove that α : [θ, $∇_A$] → **Con**(A / θ), α (ρ) = ρ / θ(θ ⊆ ρ), is the latticeal isomorphism desired. If ρ, ρ′∈ [θ, $∇_A$], ρ ≠ ρ′, then we can suppose that there are a, b ∈A such that (a, b) ∈ ρ \ ρ′ (difference of sets!).

Then $(a / \theta, b / \theta) \in (\rho/\theta) \setminus (\rho'/\theta)$, so, $\alpha(\rho) \neq \alpha(\rho')$, hence α is injective.

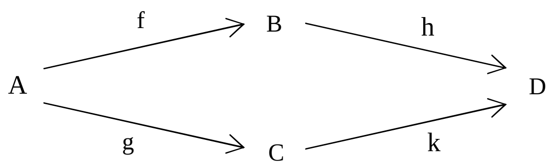
For $\rho' \in \mathbf{Con}(A / \theta)$ if we consider $\rho = \mathrm{Ker}\,(\pi_{\rho'} \circ \pi_\theta) = \{(a, b) \in A^2 : (a / \theta, b / \theta) \in \rho'\} \in \mathbf{Con}\,(A / \theta)$, then $(a / \theta, b / \theta) \in \rho / \theta \Leftrightarrow (a, b) \in \rho \Leftrightarrow (a / \theta, b / \theta) \in \rho' \Leftrightarrow \rho / \theta = \rho' \Leftrightarrow \alpha(\rho) = \rho'$, that is, α is surjective.

Since the fact that α is latticeal morphism is immediate, we deduce that α is a latticeal isomorphism. ■

Remark 3.2.21. It is easy to translate the above theorems of isomorphism and the correspondence theorem into the usual theorems used for example in groups and rings theory (see [31]).

Definition 3.2.22. Let K be a class of algebras of the same type. We say that K has the *congruence extension property* if for every $A \in K$, $B \leq A$ and $\theta \in$ Con (B), there is $\rho \in$ Con(A) such that $\rho \cap B^2 = \theta$.

Remark 3.2.23.([2]). An equation class **K** (see the final of this chapter) has the congruence extension property iff for every injective morphism $f : A \to B$ and surjective morphism $g : A \to C$ there is a surjective morphism $h : B \to D$ and injective morphism $k : C \to D$ such that $h \circ f = k \circ g$, that is, the diagram



is commutative.

3.3. Direct product of algebras. Indecomposable algebras

Let $(A_j)_{j\in I}$ a non-empty indexed family of algebras of the same type τ.

For every $1 \leq i \leq o(\tau)$, on the set $\underset{j\in I}{\times} A_j$ we define the n_i–ary algebraic operation $\bar{f}_i$ by : $(\bar{f}_i(a_1,...,a_{n_i}))(j) = (f_i(a_1(j),...,a_{n_i}(j))$, $j \in I$, and $(a_1,...,a_{n_i}) \in (\underset{j\in I}{\times} A_j)^{n_i}$.

Definition 3.3.1. An algebra of type τ and universe $\underset{j\in I}{\times} A_j$ above described is denoted by $\prod_{j\in I} A_j$ and is called the *direct product* of the family $(A_j)_{j\in I}$ of algebras.

The functions $p_k : \prod_{j\in I} A_j \to A_k$ (k ∈ I) defined by $p_k((a_i)_{i\in I}) = a_k$, are called *projections* (clearly, these are surjective morphisms).

Theorem 3.3.2. The pair ($\prod_{j\in I} A_j$, $(p_j)_{j\in I}$) verifies the following property of universality:

For any algebra A of type τ and every family $(p'_j)_{j\in I}$ of morphisms with $p'_j \in$ Hom (A, A_j)($j \in I$), there is a unique $u\in$ Hom (A, $\prod_{j\in I} A_j$) such that $p_j \circ u = p'_j$, for every $j \in I$.

Proof. It is easy to see that the desired morphism is u : A → $\prod_{j\in I} A_j$ defined for a ∈ A by $u(a) = (p'_j(a))_{j\in I}$.

For the rest of details see the case of direct product of sets (§5 from Chapter 1).■

Proposition 3.3.3. If A_1, A_2, A_3 are algebras of the same type, then
(i) $A_1 \prod A_2 \approx A_2 \prod A_1$;
(ii) $A_1 \prod (A_2 \prod A_3) \approx A_1 \prod A_2 \prod A_3$.

Proof. It is immediate that the desired isomorphisms are $\alpha((a_1, a_2)) = (a_2, a_1)$ (for (i)), respective $\alpha((a_1, (a_2, a_3))) = (a_1, a_2, a_3)$ (for (ii)). ■

Lemma 3.3.4. If A_1, A_2 are two algebras of the same type, $A = A_1 \prod A_2$, then in Con(A) : $Ker(p_1) \cap Ker(p_2) = \Delta_A$, $Ker(p_1)$ and $Ker(p_2)$ permute and $Ker(p_1) \vee Ker(p_2) = \nabla_A$ (where p_1 , p_2 are the projections of A on A_1, respective A_2).

Proof. We have $((a_1, a_2), (b_1, b_2)) \in$ **Ker**$(p_1) \cap$ **Ker**$(p_2) \Leftrightarrow$ $p_1((a_1, a_2)) = p_1((b_1, b_2))$ and $p_2((a_1, a_2)) = p_2((b_1, b_2)) \Leftrightarrow a_1 = b_1$ şi $a_2 = b_2 \Leftrightarrow$ **Ker**$(p_1) \cap$ **Ker**$(p_2) = \Delta_A$.

Since for $(a_1, a_2), (b_1, b_2) \in A_1 \Pi A_2$, $((a_1, a_2), (b_1, b_2)) \in$ **Ker**(p_1) and $((a_1, a_2), (b_1, b_2)) \in$ **Ker**(p_2) we deduce that $((a_1, a_2), (b_1, b_2)) \in$ **Ker**$(p_2)\circ$**Ker**(p_1), hence **Ker**$(p_2) \circ$ **Ker**$(p_1) = \nabla_A$, so we obtain the conclusion that **Ker**(p_1) and **Ker**(p_2) permute and **Ker**$(p_1) \vee$ **Ker**$(p_2) = \nabla_A$ (see Lemma 3.2.14). ■

Definition 3.3.5. $\theta \in$ Con(A) is called a *factor congruence* if there is $\theta^* \in$ Con(A) such that $\theta \cap \theta^* = \Delta_A$, $\theta \vee \theta^* = \nabla_A$ and θ permute with θ^*. In this case the pair (θ, θ^*) is called a *pair of factor congruence* on A.

Corollary 3.3.6. If A_1, A_2 are two algebras of the same type, then $(Ker(p_1), Ker(p_2))$ is a pair of factor congruence on $A_1 \Pi A_2$.

Proof. See Lemma 3.3.4. ■

Theorem 3.3.7. If (θ, θ^*) is a pair of factor congruence on an algebra A, then $A \approx (A / \theta) \Pi (A / \theta^*)$.

Proof. We have to prove that $f : A \to (A / \theta) \Pi (A / \theta^*)$, $f(a) = (a / \theta, a / \theta^*)$ $(a \in A)$ is the desired isomorphism.

If $a, b \in A$ and $f(a) = f(b)$, then $a / \theta = b / \theta$ and $a / \theta^* = b / \theta^*$, hence $(a, b) \in \theta \cap \theta^* = \Delta_A$, so $a = b$, that is, f is injective.

Also, for $a, b \in A$, since $\theta \vee \theta^* = \nabla_A$, then $\theta \circ \theta^* = \theta^* \circ \theta = \nabla_A$, hence there is $c \in A$ such that $(a, c) \in \theta$ and $(c, b) \in \theta^*$. Then $f(c) = (c / \theta, c / \theta^*) = (a / \theta, b / \theta^*)$, hence f is surjective, that is, f is bijective. Since it is immediate that f is morphism, we deduce that f is isomorphism. ■

Definition 3.3.8. An algebra A is (*directly*) *indecomposable* if A is not isomorphic to a direct product of two nontrivial algebras.

For example, any finite algebra with a prime number of elements must be directly indecomposable.

By Theorem 3.3.7 we deduce

Corollary 3.3.9. An algebra A is (directly) indecomposable iff the only factor congruence on A is (Δ_A, ∇_A).

Theorem 3.3.10. Every finite algebra A is isomorphic to a direct product of indecomposable algebras.

Proof. We proceed by mathematical induction on the cardinality |A| of A. If A is trivial (that is, |A| = 1), then clearly A is indecomposable. Suppose A is a nontrivial finite algebra. If A is not indecomposable, then $A = A_1 \prod A_2$ with $|A_1|, |A_2| > 1$. Since $|A_1|, |A_2| < |A|$, then by the induction hypothesis, $A_1 \approx B_1 \prod\ldots\prod B_m$, $A_2 \approx C_1 \prod\ldots\prod C_n$, with B_i, C_j indecomposable (i = 1, n, j = 1, m), so $A \approx B_1 \prod\ldots\prod B_m \prod C_1 \prod\ldots\prod C_n$. ■

Remark 3.3.11. Following the universality property of direct product of algebras (see Theorem 3.3.2) we obtain that for any two families $(A_i)_{i\in I}$, $(B_i)_{i\in I}$ of algebras of the same type and any family $(f_i)_{i\in I}$ of morphisms with $f_i \in$ **Hom**(A_i, B_i) ($i \in I$), there is a unique morphism $u : \prod_{i\in I} A_i \to \prod_{i\in I} B_i$ such that for every $i \in I$, $f_i \circ p_i = q_i \circ u$, where $(p_i)_{i\in I}$ şi $(q_i)_{i\in I}$ are canonical projections.

We denote $u = \prod_{i\in I} f_i$ and will be called the *direct product* of the family $(f_i)_{i\in I}$ of morphisms.

Clearly u is defined by $u((a_i)_{i\in I}) = (f_i(a_i))_{i\in I}$ for every $(a_i)_{i\in I} \in \prod_{i\in I} A_i$.

Also, if A is another algebra by the same type with the algebras $(A_i)_{i\in I}$ and $f_i \in$ **Hom**(A, A_i) for every $i \in I$, then there is $v \in$ **Hom**$(A, \prod_{i\in I} A_i)$ such that $p_i \circ v = f_i$, for every $i \in I$.

The morphism v is defined by $v(a) = (f_i(a))_{i\in I}$ ($a \in A$).

Definition 3.3.12. Let A, B and $(A_i)_{i\in I}$ sets and $f : A \to B$, $f_i : A \to A_i$ ($i \in I$) be functions.

We say that

(i) f *separates* the elements $a_1, a_2 \in A$ if $f(a_1) \neq f(a_2)$;

(ii) $(f_i)_{i\in I}$ *separates* the elements of A if for every $a_1, a_2 \in A$ there is $i\in I$ such that f_i separate a_1 and a_2.

Theorem 3.3.13. Let A, $(A_i)_{i\in I}$ algebras of the same type and $(f_i)_{i\in I}$ a family of morphisms with $f_i \in Hom(A, A_i)$, $i \in I$. If we consider the morphism $v \in Hom(A, \prod_{i\in I} A_i)$ above defined, then the following assertions are equivalent:

(i) v is injective morphism;

(ii) $\bigcap_{i\in I} Ker(f_i) = \Delta_A$;

(iii) The maps $(f_i)_{i\in I}$ separate the elements of A.

Proof. We recall that for $a \in A$, $v(a) = (f_i(a))_{i\in I}$, hence for $a, b \in A$, $v(a) = v(b) \Leftrightarrow f_i(a) = f_i(b)$ for every $i \in I \Leftrightarrow (a, b) \in \bigcap_{i\in I} Ker(f_i)$, so we obtain the equivalence (i) $\Leftrightarrow$ (ii).

The equivalence (i) $\Leftrightarrow$ (iii) is immediate. ■

3.4. Subdirect products. Subdirectly irreducible algebras. Simple algebras

Definition 3.4.1. Let $(A_i)_{i\in I}$ be an indexed non-empty family of algebras of type τ. We say that an algebra A of type τ is a *subdirect product* of the family $(A_i)_{i\in I}$ if

(i) $A \le \prod_{i\in I} A_i$;

(ii) $p_i(A) = A_i$, for each $i \in I$ (where $(p_i)_{i\in I}$ are the canonical projections of $\prod_{i\in I} A_i$).

An embedding $u: A \to \prod_{i\in I} A_i$ is called *subdirect* if u(A) is a subdirect product of the family $(A_i)_{i\in I}$.

Lemma 3.4.2. Let $(\theta_i)_{i\in I}$ be a family of elements of Con(A) such that $\bigcap_{i\in I} \theta_i = \Delta_A$. Then the natural morphism $u : A \to \prod_{i\in I} A_i/\theta_i$ defined by $u(a)(i) = a/\theta_i$ is a subdirect embedding.

Proof. From Theorem 3.3.13 we deduce that u is injective since if consider $p_{q_i} : A \to A/q_i$ the surjective canonical morphism, then $Ker(p_{q_i}) = q_i$, for every i ∈ I. Since every p_{q_i} is surjective, we deduce that u is a subdirect embedding. ■

Definition 3.4.3. We say that an algebra A of type τ is *subdirectly irreducible* if for every family $(A_i)_{i\in I}$ of algebras of type τ and every subdirect embedding u : A $\to \prod_{i\in I} A_i$ there is i ∈ I such that $p_i \circ$ u : A → A_i is an isomorphism.

Theorem 3.4.4. An algebra A is subdirectly irreducible iff A is trivial or there is a minimal congruence in Con(A) \ {Δ_A} (in the latter case the minimal element is the principal congruence ∩ (Con(A) \ {Δ_A})).

Proof. ([11]). "⇒". Suppose by contrary that A is non trivial and **Con**(A) \ {Δ_A} has no minimal element. Then ∩ (**Con**(A) \ {Δ_A} = Δ_A and if we consider I = **Con**(A) \ {Δ_A}, by Lemma 3.4.2, the natural morphism $u : A \to \prod_{q\in I}(A/q)$ is a subdirect embedding; since the natural map π_θ : A → A / θ is not injective for any θ ∈ I, then A is not subdirectly irreducible in contradiction with the hypothesis!

"⇐". If A is trivial and $u : A \to \prod_{i\in I} A_i$ is a subdirect embedding then every A_i is trivial, hence every $p_i \circ$ u is isomorphism.

Suppose A is non trivial, and let θ = ∩ (**Con**(A) \ {Δ_A} $\neq \Delta_A$. Let (a, b) ∈ θ with a ≠ b. If $u : A \to \prod_{i\in I} A_i$ is a subdirect embedding, then for some i ∈ I (u(a))(i) ≠ (u(b))(i), hence (p_i ∘ u)(a) ≠ (p_i ∘ u)(b). We deduce that (a, b) ∉ **Ker**(p_i ∘ u), hence θ ⊈ **Ker**(p_i ∘ u) which imply **Ker**(p_i ∘ u) = Δ_A, so p_i ∘ u : A → A_i is an isomorphism, that is, A is subdirectly irreducible.

If **Con**(A) \ {Δ_A} has a minimal element θ, then for a, b ∈ A, a ≠ b and (a, b) ∈ θ, we have ⊖(a, b) ⊆ θ, hence ⊖(a, b) = θ. ■

Remark 3.4.5. Using this last result we can put in evidence some classes of subdirectly irreducible algebras (see and [30]):

(i) A finite abelian group G is subdirectly irreducible iff it is cyclic and $|G| = p^n$ for some prime number p (that is, G it is a cyclic p-group);

(ii) The group C_{p^∞} is subdirectly irreducible;

(iii) Every simple group is subdirectly irreducible;

(iv) A vector space over a field **K** is subdirectly irreducible iff it is trivial or one-dimensional;

(v) An algebra with 2 elements is subdirectly irreducible.

A directly indecomposable algebra dones not need to be subdirectly irreducible (consider, for example, a three-element chain as a lattice).

The converse does indeed hold; since every congruence factor on a subdirectly irreducible algebra is the pair (Δ, ∇) by Theorem 3.4.4 we deduce that every subdirectly irreducible algebra is indecomposable.

Theorem 3.4.6. (*Birkhoff*). Every algebra A is isomorphic to a subdirect product of subdirectly irreducible algebras.

Proof .([11]). It will suffice to consider only the case of non trivial algebra A. For a, b ∈ A, with a ≠ b, using Zorn's lemma we can find a congruence $\theta_{a, b}$ of A which is maximal with respect to the property (a, b) ∉ $\theta_{a, b}$. Then $\ominus$(a, b) ∨ $\theta_{a, b}$ is the smallest congruence in $[\theta_{a, b}, \nabla_A] \setminus \{ \theta_{a, b} \}$, so by Theorems 3.2.20 and 3.4.4, A / $\theta_{a, b}$ is subdirectly irreducible.

As $\cap \{\theta_{a, b} : a \neq b\} = \{\Delta_A\}$, we can apply Lemma 3.4.2 to obtain that algebra A is subdirectly embeddable in $\prod_{a \neq b} (A/q_{a,b})$ (clearly A / $\theta_{a, b}$ with a ≠ b is subdirectly irreducible). ■

Corollary 3.4.7. Every finite algebra is isomorphic to a subdirect product of a finite number of subdirectly irreducible finite algebras.

Definition 3.4.8. An algebra A is called *simple* if Con(A) = {Δ_A, ∇_A}. A congruence θ ∈ Con(A) is *maximal* on A if the interval [θ, ∇_A] of Con(A) has exactly two elements.

Theorem 3.4.9. If θ ∈ Con(A), then A / θ is simple iff θ is a maximal congruence on A or θ = Δ_A.

Proof. Since by Theorem 3.2.20, **Con**$(A / \theta) \approx [\theta, \nabla_A]$, the theorem is an immediate consequence of Definition 3.4.8. ■

3.5. Class operators. Varieties

In this paragraph by *operator* we understand a mapping defined on a class of algebras (of same type) with values in another class of algebras (of same type).

By **K** we denote a class of algebras of the same type.

In what follows we introduce the operators **I**, **H**, **S**, **P**, $\mathbf{P_s}$ by:

Definition 3.5.1.

(i) A ∈ I(K) iff A is isomorphic to some algebra of K;

(ii) A ∈ S(K) iff A is isomorphic to a subalgebra of some algebra of K;

(iii) A ∈ H(K) iff A is homomorphic image of some algebra of K;

(iv) A ∈ P(K) iff A is isomorphic to a direct product of a non-empty family of algebras in K;

(v) A ∈ $\mathbf{P_s}$(K) iff A can be subdirectly embedded into a product of a non-empty family of algebras in K.

If $\mathbf{O_1}$, $\mathbf{O_2}$ are two operators, by $\mathbf{O_1O_2}$ we denote the composition of $\mathbf{O_1}$ and $\mathbf{O_2}$ (which is also an operator).

We write $\mathbf{O_1} \le \mathbf{O_2}$ iff $\mathbf{O_1(K)} \subseteq \mathbf{O_2(K)}$ for every class **K** of algebras.

An operator **O** is *idempotent* if $\mathbf{O}^2 = \mathbf{O}$.

A class **K** of algebras is *closed* under an operator **O** if $\mathbf{O(K)} \subseteq \mathbf{K}$.

If we denote by **O** one of the operators **I**, **S**, **H**, **P**, $\mathbf{P_s}$ above defined, we deduce that the restriction of **O** to some class of algebras (of same type) verifies the conditions: $\mathbf{K} \subseteq \mathbf{O(K)}$, $\mathbf{K_1} \subseteq \mathbf{K_2} \Rightarrow \mathbf{O(K_1)} \subseteq \mathbf{O(K_2)}$ and $\mathbf{O(O(K))} = \mathbf{K}$ for every classes of algebras of same type **K**, $\mathbf{K_1}$, $\mathbf{K_2}$, so we can consider **O** as a closure operator defined on the class of all algebras of some type (see §1).

Also, if A ∈ **K** we observe that every algebra isomorphic with A is also in **K**. Symbolically we write **O** = **IO**; we also have **OI** = **O**.

Lemma 3.5.2. The operators HS, SP, HP and HP$_s$ are closure operators on every class of algebras of same type.

Also the following inequalities hold: SH ≤ HS, PS ≤ SP, PH ≤ HP, $\mathbf{P_sH} \le \mathbf{HP_s}$, $\mathbf{P_sP} = \mathbf{P_s} = \mathbf{PP_s}$ and $\mathbf{P_sS} = \mathbf{SP} = \mathbf{SP_s}$.

Proof. ([58]). It is easy to see that the composition of two operators verifies the conditions **K** ⊆ **O**(**K**) and $\mathbf{K_1} \subseteq \mathbf{K_2} \Rightarrow \mathbf{O}(\mathbf{K_1}) \subseteq \mathbf{O}(\mathbf{K_2})$, that is, we obtain a new operator with the same properties. We also obtain that the composition of operators is associative and preserves the order ⊆.

So, the operators **HS**, **SP**, **HP** and $\mathbf{HP_s}$ verify the axioms for closure operators.

For the condition of idempotence we can use other relations (for example if we accept that **SH** ≤ **HS**, then $(\mathbf{HS})^2$ = (**HS**)(**HS**) = **H**(**SH**)**H** ≤ **H**(**HS**)**S** = **HHSS** = **HS** and on the other hand **HS** = (**HI**)(**IS**) ≤ (**HS**)(**HS**) = $(\mathbf{HS})^2$, so it is suffice to prove inequalities of the form **SH** ≤ **HS** (the others are analogous).

We have to prove for example that **PH** ≤ **HP**.

For this, let **K** be a class of algebras of the same type and A ∈ **K**. Then $\prod_{i\in I} A_i \overset{f}{\approx}$ A, with A_i ∈ **H**(**K**) for every i ∈ I. By the choice axiom we can find B_i ∈ **K** and onto morphisms $f_i : B_i \to A_i$ for any i ∈ I.

Then we have the onto morphism g: $\prod_{i\in I} B_i \to \prod_{i\in I} A_i$ defined by $g((b_i)_{i\in I}) = (f_i(b_i))_{i\in I}$. Since $f \circ g$: $\prod_{i\in I} B_i \to$ A is onto morphism, we deduce that A ∈ **HP**(**K**). ■

Definition 3.5.3. A non-empty class K of algebras of the same type is called a *variety* if it is closed under the operators H, S and P (that is, H(K) ⊆ K, S(K) ⊆ K and P(K) ⊆ K).

If **K** is a class of algebras of the same type, by **V**(K) we denote the smallest variety containing **K**; we say that **V**(K) is the *variety generated by* **K** (if **K** contains only an algebra A or a finite numbers A_1, ..., A_n of algebras we write **V**(A) or **V**(A_1, ..., A_n) for **V**(**K**)).

So, we obtain a new operator **V**.

Theorem 3.5.4. (*Tarski*). V = HSP.

Proof. By Lemma 3.5.2 we deduce that **HHSP** = **SHSP** = **PHSP** = **HSP**, hence **HSP**(**K**) is a variety which contains **K** for every **K**. On the other hand if $\overline{V}$ is a variety which contains **K**, then **HSP**(**K**) ⊆ **HSP**($\overline{V}$) = $\overline{V}$, hence **HSP**(**K**) is the smallest variety which contains **K**, that is, **HSP** = **V**. ■

More algebras which will be studied in this book form varieties. Others don't form varieties (as an example we have the algebraic lattices which are not closely related to **H** or **S**).

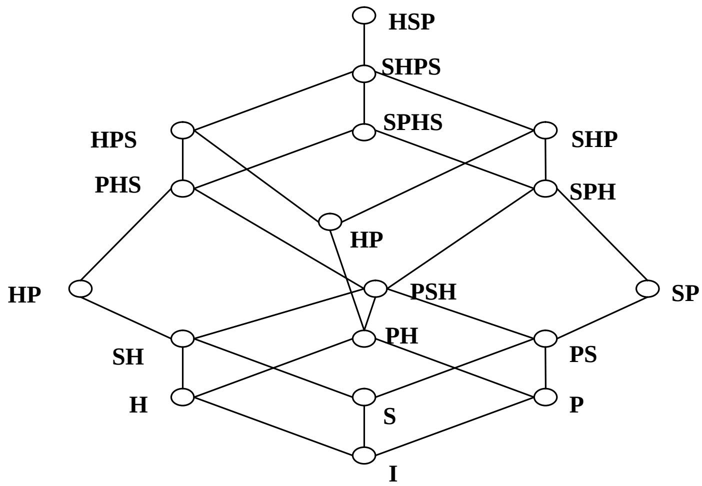
The following result will be very useful in the study of varieties, and it is easy to prove it.

Proposition 3.5.5. Let K be a class of algebras of some type and A an algebra of the same type. Then

(i) A ∈ SP(K) ⇔ there is a family of congruence $(\theta_i)_{i\in I}$ on A such that $\bigcap_{i\in I}\theta_i = \Delta_A$ and A / θ_i ∈ S(K) for every i ∈ I;

(ii) A ∈ HSP(K) ⇔ there is an algebra, the congruence $(\theta_i)_{i\in I}$ and θ on B such that B / θ ≈ A, θ ≥ $\bigcap_{i\in I}\theta_i$ and B/θ_i∈S(K) for every i∈I.

Remark 5.6. From the above, we deduce that the operators **I**, **H**, **S** and **P** generate an ordered monoid whose structure was determined in 1972 by D. Pigozzi [*On some operations on classes of algebras*, Algebra Universalis 2, 1972, 346-353] and have the following Hasse diagram



3. 6. Free algebras

Let **K** be a class of algebras of the same type τ.

Definition 3.6.1. An algebra A ∈ K is said to be *free* over K if there is a set X ⊆ A such that:

(i) [X] = A;

(ii) If B ∈ K and f : X → B is a function, then there is a morphism f′ : A → B such that f is the restriction of f′ to X (that is, $f'_{|X}$ = f).

In this case the set X is said to *freely generate* A and it is called a *free generating set.*

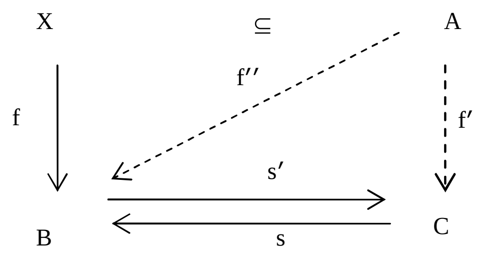
Note that by Lemma 3.1.11, f′ from the above definition is uniquely determined.

Lemma 3.6.2. If A is free over K, then A is free and over HSP(K).

Proof. It will suffice to prove that if A is free over **K**, then A is free over **H**(**K**), **S**(**K**) and **P**(**K**). We shall prove for example for **H**(**K**) (for the other it is similar).

Let now B ∈ **H**(**K**) and f : X → B be a function.

Since B ∈ **H**(**K**) there is C ∈ **K** and a surjective morphism s : C → B (so, we have s′ : B → C such that s ∘ s′ = 1_B).



Since A is free over **K**, [X] = A and there is a morphism f′ : A → C such that $f'_{|X}$ = s′ ∘ f. If we denote f′′ = s ∘ f′, then $f''_{|X}$ = f (since for every x ∈ X, f′′(x) = s(f′(x)) = s(s′(f(x))) = (s ∘ s′)(f(x)) = f(x)). ■

Lemma 3.6.3. If A_i is free in K over X_i (i = 1, 2) and $|X_1| = |X_2|$ then $A_1 \approx A_2$.

Proof . Let f : $X_1 \to X_2$ be a bijection. There are the morphisms f′ : $A_1 \to A_2$ and f′′ : $A_2 \to A_1$ such that $f'_{|X_1} = f$ and $f''_{|X_2} = f^{-1}$.

We deduce that f′′ ∘ f′ extends $f^{-1} \circ f = 1_{X_1}$; since 1_{A_1} also extend 1_{X_1} we deduce that $f'' \circ f' = 1_{A_1}$.

Analogous $f' \circ f'' = 1_{A_2}$, hence $A_1 \approx A_2$. ■

Following Lemma 3.6.3, an algebra A which is free over K is determined up to an isomorphism by the cardinality of any free generating set.

Definition 3.6.4. For every cardinal α, we pick any of the isomorphic copies of a free algebra over K with α free generators and call it the *free K-algebra on α free generators* and denote it by $F_K(\alpha)$ or if the free generating set X is specified, by $F_K(X)$ (with $|X| = \alpha$).

In [2, p.19] it is proved the following very important result:

Theorem 3.6.5. If K is a non-trivial variety, then $F_K(\alpha)$ exists for each cardinal $\alpha > 0$.

More algebras presented in this book are defined by the so called *identities* or *equations*; is the case of semilattices, lattices, Boolean algebras and in Chapter 5 we will present Heyting,Hilbert, Hertz, residuated lattices and Wajsberg algebras (for suplimentary information relative to the notions of *identity* or *equation* we recommend ,to the reader, the books [2], [11] and [58]).

In [2], [11] and [58] it is proved the followings results:

Proposition 3.6.6. If all algebras from a similar class K of algebras satisfies an identity, then every algebras from the variety generated by K satisfies that identity.

Corollary 3.6.7. If all subdirectly irreducible algebras from a variety K satisfy a identity, then every algebra from K satisfies that identity.

Theorem 3.6.8. (*Birkhoff*). A class K of similar algebras is a variety iff there is a set Ω of identities such that K is exactly the class of algebras that satisfies all the identities in Ω.

Corollary 3.6.9. Let K be a class of similar algebras and let Ω be a set of identities which are satisfied by every member of K. Then an algebra A is a member of the variety generated by K iff A satisfies every identity in Ω.

Remark 3.6.10. In some books of Universal Algebra (following Theorem 3.6.8) varieties are also called *equational classes*.

CHAPTER 4: TOPICS ON THE THEORY OF CATEGORIES

The notion of category and functor was introduced in an explicit way by S. Eilenberg and S. Mac Lane in 1945 (starting from the study of some constructions of objects in mathematics and for giving a precise sense for the notion of *duality*).

Till now, the general methods of the theory of categories are found in almost all branches of mathematics, so we can really say that the modern mathematics is in fact the study of some particular categories and functors.

4.1. The notion of a category. Examples. Subcategory. Dual category. Duality principle. Product of categories

Definition 4.1.1. We say that we have a *category* C if we have a class Ob(C), whose elements are called *objects* in C and for each ordered pair (M, N) of objects from C is given a set C(M, N), empty possible (called the set of *morphisms* of M to N), such that:

(i) For every ordered triple (M, N, P) of objects from C is given a function C(M, N) ´ C(N, P) ® C(M, P), (f, g) ® g∘f called the *composition* of morphisms;

(ii) The composition of morphisms is associative (i.e., for each M, N, P, Q objects from C and f Î C(M, N), g Î C(N, P), h Î C(P, Q), then h ∘ (g ∘ f) = (h ∘ g) ∘ f);

(iii) For every object M from C, there is an element 1_M Î C(M, M) (called the *identity morphism* or *identity* of M) such that for every objects N, P from C and f Î C(M, N), g Î C(P, M) we have f ∘ 1_M = f and 1_M ∘ g = g;

(iv) If the ordered pairs (M, N) and (M′, N′) of objects are distinct, then C(M, N) Ç C(M′, N′) = Ø.

Remark 4.1.2. (i). We will frequently write M ∈ **C** instead of M ∈ Ob(**C**)**;** if f ∈ **C**(M, N), we will frequently use the notation f : M → N or $M \xrightarrow{f} N$.

In this case, M is called the *domain* of f and N the *codomain* of f.

A category **C** is called *small* if Ob(**C**) is a set (for complete information about the notions of *set* and *class* we recommend to reader the book [79]).

(ii). For M ∈ **C**, $1_M : M \to M$ is unique in condition of (iii). Indeed, if $1'_M : M \to M$ is another identity morphism of M, then we have $1_M \circ 1'_M = 1'_M$ and $1_M \circ 1'_M = 1_M$, hence $1'_M = 1_M$.

Examples

1. The category **Set** (of *sets*). The objects of **Set** are the class of all sets.

For M, N ∈ **Set**, **Set**(M, N) = {f : M → N} and the composition of morphisms in **Set** is the usual compositions of functions.

For X ∈ **Set**, the function $1_X : X \to X$, $1_X(x) = x$ for every x ∈ X plays the role of identity morphism of X.

2. The category **Pre** (of *preordered sets*). The objects of **Pre** are the preordered sets. For (A, ≤), (A′, ≤′) ∈ **Pre**, **Pre**((A, ≤), (A′, ≤ ′)) = { f : A → A′ **:** x ≤ y ⇒ f (x) ≤′ f (y)} and the composition of morphisms in **Pre** (also called *isotone* maps) is the usual compositions of function (see Chapter 2).

For (X, ≤) ∈ **Pre**, the function $1_X : X \to X$, $1_X(x) = x$ for every x ∈ X plays the role of identity morphism of X.

3. The category **Gr** (of *groups*). The objects of **Gr** are the groups and for H, K ∈ **Gr**, **Gr**(H, K) = {f : H → K : f is a morphism of groups}, and the composition of morphisms in **Gr** is the usual composition of functions. For G ∈ **Gr** the function $1_G : G \to G$, $1_G(x) = x$ for every x ∈ G plays the role of identity morphism of G (see [31]).

4. The category **Rg** (of *unitary rings*). The objects of **Rg** are the rings with identity, for B ∈ **Rg**, **Rg**(A, B) = {f : A → B : f is morphism of unitary rings}, the composition of morphisms in **Rg** is the usual composition of functions and for a unitary ring A the function $1_A : A \to A$, $1_A(x) = x$ for every x ∈ A plays the role of identity of A (see [31]).

5. The category **Top** (of *topological spaces*). The objects of **Top** are the topological spaces, the morphisms are the continuous functions and the composition of morphisms in **Top** is the usual composition of functions.

For (X, τ) ∈ **Top**, the map $1_X : X \to X$, $1_X(x) = x$ for every x ∈ X plays the role of identity morphism of X.

6. The category $\mathbf{Mod_s}$(A) (of *left–modules over the unitary ring* A).

The objects of $\mathbf{Mod_s}$(A) are the left A–modules over a unitary ring A, the morphisms are the A-linear maps and the composition of morphisms in $\mathbf{Mod_s}$(A) is the usual composition of functions.

For M ∈ $\mathbf{Mod_s}$(A), the function $1_M : M \to M$, $1_M(x) = x$ for every x ∈ M plays the role of identity of M (see [31]).

Similarly we define the category $\mathbf{Mod_d}$(A) of *right modules over the unitary ring A.*

7. Let A be a unitary ring. We define a new category **A** by: Ob(**A**) = {A} and **A**(A, A) = A. The composition of morphisms in **A** is the multiplication on A and the identity of the ring A plays the role of identity of A.

8. Let $\mathbf{C}_\tau$ be a class (*equational*) of algebras of type τ. The category whose objects are the algebras from $\mathbf{C}_\tau$ and for A, B ∈ $\mathbf{C}_\tau$, $\mathbf{C}_\tau$(A, B) is the set of all morphisms of algebras of type τ from A to B, is called the category (*equational*) of algebras of type τ (see Chapter 3).

Definition 4.1.3. Let C be a category. A *subcategory of* C is a new category C′ which satisfies the following conditions:

(i) Ob(C′) ⊆ Ob(C);

(ii) If M, N ∈ C′, then C′(M, N) ⊆ C(M, N);

(iii) The composition of morphisms in C′ is the restriction of the composition of morphisms in C;

(iv) If M ∈C′, then 1_M (in C′) coincides with 1_M (in C).

A subcategory C′ of C with the property that for every M, N∈C′, C′(M, N) = C(M, N) is called a *full* subcategory.

Examples

1. If we denote by **Ab** the category whose objects are the abelian groups, then **Ab** is in canonical way a full subcategory of **Gr**.

2. If we denote by **Ord** the category whose objects are the ordered sets, then **Ord** is in canonical way a full subcategory of **Pre**.

3. Let **L** be the category of lattices (whose objects are all lattices and for two lattices L, L′, **L**(L, L′) = {f : L → L′ : f is a morphism of lattices} - see Chapter 2). Then in canonical way **L** becomes a subcategory of **Ord**.

If we denote by **L**(**0**, **1**) the category of bounded lattices (see Chapter 2) and for L, L′∈ **L**(**0**, **1**), **L**(**0**, **1**) (L, L′) = {f ∈ **L**(L, L′) : f(0) = 0 and f(1) = 1}, then **L**(**0**, **1**) become a subcategory of **L**.

4. If we denote by **Ld**(**0**, **1**) the category of bounded distributive lattices (whose objects are the bounded distributive lattices and the morphisms are defined as in the case of **L**(**0**, **1**)), then **Ld**(**0**, **1**) becomes a full subcategory of **L**(**0**, **1**) (see Chapter 2, §3).

5. If we denote by **Fd** the category of fields, then **Fd** becomes in a canonical way a subcategory of **Rg**.

Definition 4.1.4. **Let C be a category. We define a new category C^0 (called the *dual category* of C) in the following way: Ob(C^0) = Ob(C) and for M, N Î C^0, C^0(M, N) = C(N, M). The composition of morphisms is defined as follows: if $M \xrightarrow{f} N \xrightarrow{g} P$ are morphisms in C^0, then g * f = f∘g (we denoted by "*" the loin of composition in C^0). Clearly $(C^0)^0$ = C.**

Assigning each category **C** **with** its dual category $\mathbf{C}^0$ enables us to dualize each notion or statement concerning a category **C** into a corresponding notion or statement concerning the dual category $\mathbf{C}^0$. Thus we get the following *duality principle*:

Let P be a notion or statement about categories; then there is a dual notion or statement P^0 (called the *dual* of P) about categories.

In general, the characterization of the dual for a category proves to be a very complicated thing.

Let $(\mathbf{C}_i)_{i\in I}$ be a family of indexed categories (I ≠ ∅).

We define a new category **C** in the following way:

An object of **C** is a family $(M_i)_{i\in I}$ of objects, indexed by I, where $M_i \in \mathbf{C}_i$, for every i∈I. If M = $(M_i)_{i\in I}$, N = $(N_i)_{i\in I}$ are two objects in **C**, then we define **C**(M, N) = $\prod_{i\in I} C_i(M_i, N_i)$.

If we have P = $(P_i)_{i\in I}$ ∈ **C** and f = $(f_i)_{i\in I}$ ∈ **C**(M, N), g = $(g_i)_{i\in I}$ ∈ **C**(N, P), then we define the composition g∘f = $(g_i \circ f_i)_{i\in I}$.

Definition 4.1.5. The category C defined above is called the *direct product* of the family of categories $(C_i)_{i\in I}$; we write C = $\prod_{i\in I} C_i$.

If I = {1, 2, …, n} we write $C = C_1 \times \ldots \times C_n$**.**

4.2. Special morphisms and objects in a category. The kernel (equalizer) and cokernel (coequalizer) for a couple of morphisms

Definition 4.2.1. Let C be a category and u : M → N a morphism in C. The morphism u is called *monomorphism (epimorphism)* in C, if for every P ∈ C and f, g ∈ C(P, M) (respective f, g ∈ C(N, P)), from u ∘ f = u ∘ g (respective f ∘ u = g ∘ u) implies f = g.

We say that u is *bimorphism* if it is both monomorphism and epimorphism.

Remark 4.2.2. From Definition 4.2.1 we deduce that the morphism u is epimorphism in C iff u is a monomorphism in C^0**.**

Definition 4.2.3. We say that a morphism u : M → N from category C is an *isomorphism* if there is v : N → M a morphism such that v∘ u = 1_M **and u ∘ v =** 1_N**; in this case we say that the objects M and N are *isomorphic* (we write M ≈ N).**

Remark 4.2.4.

(i). If v, v′:N→M verify both conditions of Definition 4.2.3, then v = v′.

Indeed, we have the equalities (v∘u)∘v′ = 1_M ∘v′ = v′ and (v∘u)∘v′ = v∘(u∘v′) = v∘1_N = v, hence v = v′.

If such v exists, we say that v is the *inverse* of u and we write v = u^{-1}.

(ii). If **C′** is a subcategory of **C** and u is a monomorphism (epimorphism) in **C′**, it doesn't follow that u is a monomorphism (epimorphism) in **C**.

Indeed, let f : X → Y a morphism in **C** which is not a monomorphism or epimorphism in **C**, and **C′** the subcategory of **C** whose objects are X and Y and whose morphisms are 1_X, 1_Y and u. Clearly, u is a bimorphism in **C′**, but is not a bimorphism in **C**.

(iii). It is immediate that every isomorphism is bimorphism, but the converse is not true.

An example is offered by category **Top.** Indeed, let X be a set which contains at least two elements and $1_X : X \to X$ the identity function of X in **Set**. If we consider the codomain of 1_X equipped with the *rough topology* (Ø and X are all clopen's) and its domain with the *discrete topology* (for which all subsets of X are open sets), then 1_X becomes a bimorphism in **Top** which is not isomorphism. Indeed, if by contrary 1_X is an isomorphism, then $(1_X)^{-1} = 1_X$ which is not a continuous map from X (equipped with the rough topology) to X (equipped with the discrete topology).

In fact, the isomorphisms in **Top** are just the *homeomorphisms* of topological spaces.

Definition 4.2.5. A category C with the property that every bimorphism is isomorphism is called *balanced* (or *perfect*).

Following the above we deduce that the category **Top** is not balanced.

Definition 4.2.6. Let u : M ® N a morphism in a category C. A *section* (or *right inverse*) for u is a morphism v : N ® M such that $u \circ v = 1_N$. A *retraction* (or *left inverse)* of u is a morphism w : N ® M such that $w \circ u = 1_M$.

Proposition 4.2.7. Let $M \xrightarrow{f} N \xrightarrow{g} P$ **be two morphisms in the category C. Then:**

(i) If f has a section (retraction), then f is epimorphism (monomorphism);

(ii) If f and g are monomorphisms (epimorphisms), then $g \circ f$ is monomorphism (epimorphism);

(iii) If $g \circ f$ is a monomorphism (epimorphism), then f (respective g) is a monomorphism (respective epimorphism);

(iv) If f and g are isomorphisms, then $g \circ f$ is also an isomorphism and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$;

(v) If f and g have sections (retractions), then $g \circ f$ have section (retraction);

(vi) If $g \circ f$ has a section (retraction), then g has a section (f has a retraction);

(vii) A monomorphism (epimorphism) is a isomorphism iff it has a section (retraction);

(viii) If $g \circ f$ is an isomorphism, then g has a section and f has a retraction;

(ix) A bimorphism which has a section (retraction) is a isomorphism.

Proof. (i). We suppose that f has a section; then there is $h : N \to M$ such that $f \circ h = 1_N$.

Let now $r, s : N \to P$ such that $r \circ f = s \circ f$; we deduce that $(r \circ f) \circ h = (s \circ f) \circ h \Leftrightarrow r \circ (f \circ h) = s \circ (f \circ h) \Rightarrow r \circ 1_N = s \circ 1_N \Rightarrow r = s$, hence f is epimorphism. Analogous we prove that if f has a retraction, then f is a monomorphism.

(ii). Suppose that f and g are monomorphisms and let $r, s : Q \to M$ such that $(g \circ f) \circ r = (g \circ f) \circ s$. Then $g \circ (f \circ r) = g \circ (f \circ s)$; since g is a monomorphism $\Rightarrow$ $f \circ r = f \circ s \Rightarrow r = s$ (since f is a monomorphism). So, we deduce that $g \circ f$ is a monomorphism. Analogous we prove that if f and g are epimorphisms, then $g \circ f$ is an epimorphism.

(iii) - (ix). Analogous. ■

Applications

1. In the category **Set** the monomorphisms (epimorphisms, isomorphisms) are exactly the injective (surjective, bijective) functions – see Propositions 1.3.7, 1.3.8 and Corollary 1.3.9.

2. In the category **Gr** of groups, also, the monomorphisms (epimorphisms, isomorphisms) are exactly the injective (surjective, bijective) morphisms of groups (see [31]). So, **Gr** is a balanced category.

Let now a proof of **Eilenberg** for the characterization of the epimorphisms in **Gr.**

Clearly , every surjective morphism of groups is an epimorphism in **Gr**.

Conversely, suppose that G , G′ are groups, $f:G \to G'$ is a morphism of groups with the property that for every group G″ and every morphism of groups $\alpha, \beta: G' \to G''$, if $\alpha \circ f = \beta \circ f$, then $\alpha = \beta$ (that is , f is an epimorphism in **Gr**) and let's show that f is a surjective function.Let $H = f(G) \leq G'$ and suppose by contrary that $H \neq G'$. If $[G':H] = 2$, then $H \trianglelefteq G'$, and if we consider

G″=G′/H, α = p_H : G′→G″ the surjective canonical morphism and β : G′→G″ the nullary morphism, then α∘f=β∘f but α ≠ β - a contradiction !.

Suppose that [G′:H]>2 and let T= $(G'/H)_d$ the right classes set of G′ relative to H and G″ = $\sum(G')$ - the permutations group of G′.

We will also construct in this case two morphisms of groups α, β: G′→G″ , such that α ≠ β but α∘f=β∘f, in contradiction with f is an epimorphism.

Let α : G′→G″= $\sum(G')$ the Cayley morphism, (that is ,α(x)=θ_x, with θ_x : G′→G′, θ_x (y)=xy, for every x, y∈G′).

For the construction of β, let π : G′→T the canonical surjection, (that is , π(x)=Hx $\overset{def}{=}$ $\hat{x}$, for every x∈G′) and s : T→G′ a section of π (then π∘s=1_T , hence $s(\hat{x})\in \hat{x}$, for every $\hat{x}$ ∈T).

Since |T|=|G′:H|≥3, there exists a permutation σ : T→T such that $s\ (\hat{e}) = \hat{e}$ and σ≠1_T. If x∈G, since $s(\hat{x})\in \hat{x} \Rightarrow xs(\hat{x})^{-1}$ ∈H.

We define τ:G′→H by τ(x)= $xs(\hat{x})^{-1}$,for every x∈G′.Then λ:G′→G′, λ(x)=τ(x)·s $(s\ (\hat{x})) = xs(\hat{x})^{-1}s(s\ (\hat{x}))$ for every x∈G′ is a permutation of G′ (hence λ∈G″).

Indeed , if x, y∈G′ and λ(x)=λ(y), then

(1) $xs(\hat{x})^{-1}s(s\ (\hat{x})) = ys(\hat{y})^{-1}s(s\ (\hat{y}))$.

Since $xs(\hat{x})^{-1}$, $ys(\hat{y})^{-1}$ ∈H⇒ $s(s\ (\hat{x})) = s(s\ (\hat{y}))$ ⇒(π∘s)(σ($\hat{x}$))= (π∘s)(σ($\hat{y}$))⇒ σ($\hat{x}$)=σ($\hat{y}$)⇒ $\hat{x} = \hat{y}$ and by (1) we deduce that x=y.

Let now y∈G′; there exists $\hat{z}$ ∈T such that $\hat{y}$ =σ($\hat{z}$). Since $s(\hat{y})\in \hat{y}$ =Hy , then there exists h∈H such that $s(\hat{y}) = hy$ .If denote $x_1 = s(\hat{z})$ and x=$h^{-1}x_1$, then (since $\hat{x} = \hat{x}_1$ because xx_1^{-1} =h∈H) we have λx= $xs(\hat{x})^{-1}s(s\ (\hat{x})) = h^{-1}x_1s(\hat{x}_1)^{-1}s(s\ (\hat{x}_1)) = h^{-1}x_1s(\hat{x}_1)^{-1}s(s\ (\hat{z})) =$ $= h^{-1}x_1s(\hat{x}_1)^{-1}s(\hat{y}) = h^{-1}x_1s(\hat{x}_1)^{-1}hy$ .Since $x_1 = s(\hat{z})\in \hat{z},\ \ \hat{x}_1 = \hat{z}$ and $s(\hat{x}_1) = s(\hat{z}) = x_1$, then λ(x)= $h^{-1}x_1x_1^{-1}hy = y$, hence λ is surjective, that is , λ∈G″.

We define β : G′→G″ = $\sum(G')$ by β(x)=λ^{-1}∘α(x)∘λ for every x∈G′. Obviously β is a morphism of groups.We have α ≠ β because if α=β,then α(x)∘λ=λ∘α(x),for every x∈G′⇔ (α(x)∘λ)(y)=(λ∘α(x))(y) for every y∈G′⇔

xλ(y)=λ(xy) for every y∈G′ ⇔ $xys(\hat{y})^{-1}s(s(\hat{y}))=xys\left(\widehat{xy}\right)^{-1}s\left(s\left(\widehat{xy}\right)\right)$, for every y∈G′ ⇔ $s(\hat{y})^{-1}s(s(\hat{y}))=s\left(\widehat{xy}\right)^{-1}s\left(s\left(\widehat{xy}\right)\right)$ for every y∈G′.

For x=y^{-1} we obtain that $s(\hat{y})^{-1}s(s(\hat{y}))=s(\hat{e})^{-1}s(s(\hat{e}))=s(\hat{e})^{-1}s(\hat{e})=e$, hence $s(\hat{y})=s(s(\hat{y}))$. Since s is injective we deduce that $\hat{y}=s(\hat{y})$, that is, σ = 1_T - a contradiction !. Hence α ≠ β.

Let's show that α∘f=β∘f , a contradiction , hence we will deduce that f is surjective.

Indeed, α∘f=β∘f⇔(α∘f)x=(β∘f)x, for every x∈G ⇔α(f(x))=β(f(x)), for every x∈G ⇔ $q_{f(x)}=l^{-1}\circ a(f(x))\circ l$, for every x∈G⇔ $l\circ q_{f(x)}=q_{f(x)}\circ l$ for every x∈G⇔

⇔ $(l\circ q_{f(x)})(y)=(q_{f(x)}\circ l)(y)$, for any x ∈ G şi y ∈ G′ ⇔

⇔ λ (f (x) y) = f(x) λ(y), for every x ∈ G şi y ∈ G′ ⇔

⇔ $t\left(\widehat{f(x)y}\right)s(\sigma(\widehat{f(x)y}))=f(x)ys(\hat{y})^{-1}s(s(\hat{y}))$, for any x∈G and y∈G′ ⇔

$f(x)ys(\widehat{f(x)y})^{-1}s(\sigma(\widehat{f(x)y}))=f(x)ys(\hat{y})^{-1}s(s(\hat{y}))$, for any x∈G and y∈G′ ⇔

$s(\widehat{f(x)y})^{-1}s(\sigma(\widehat{f(x)y}))=s(\hat{y})^{-1}s(s(\hat{y}))$, for every x∈G and y∈G′ which is cleary because for x∈G, f(x)∈f(G)=H, so $\widehat{f(x)y}=\hat{y}$, for every y∈G′.

Remark 4.2.8. There are categories where not all the monomorphisms (epimorphisms) are injective (surjective) functions.

Indeed, let **Div** be the subcategory of **Ab** of all *divisible abelian groups* (we recall that an aditive group G is called *divisible,* if for any y ∈ G and any natural number n, there is x ∈ G such that y = nx).

We now consider the abelian divisible groups (**Q**,+), (**Q**/**Z**,+) and p:**Q**→**Q**/**Z** the onto canonical morphism of groups.

We have to prove that p is a monomorphism in the category **Div** (but clearly p is not an injective function).

Indeed, we consider in **Div** the diagram $G \underset{v}{\overset{u}{\rightrightarrows}} Q \xrightarrow{p} Q/Z$ such that

$u \neq v$ and we have to prove that $p \circ u \neq p \circ v$.

So, there is $a \in G$ such that $u(a)-v(a) = r/s \in$ $\mathbf{Q}^*$, with $s \neq \pm 1$ (we can suppose $s \neq \pm 1$, since if by contrary $s = \pm 1$, then if we consider $s' \neq \pm 1$, there is $a' \in G$ such that $s'a'=a$ and thus $u(a')-v(a') = r/s'$). If $b \in G$ such that $rb = a$, then $r(u(b)-v(b)) = u(a) - v(a) = r/s$, hence $p \circ u \neq p \circ v$, that is, p is a monomorphism in **Div**.

As a corollary we obtain that **Div** is not a balanced category.

We now consider the category **Rg** of unitary rings the inclusion morphism $i : \mathbf{Z} \to \mathbf{Q}$. We will prove that i is an epimorphism in **Rg** (but clearly it is not a surjective function).

Indeed, considering in **Rg** the diagram $Z \xrightarrow{i} Q \underset{v}{\overset{u}{\rightrightarrows}} A$ such that

$u \circ i = v \circ i \Leftrightarrow u_{|Z} = v_{|Z}$ we will prove that $u = v$.

If $x = m/n \in \mathbf{Q}$, then $u(x) = u(m/n) = mu(1/n) = m[u(n)]^{-1}$ (since u is a morphism of unitary rings), so $v(x) = v(m/n) = m[v(n)]^{-1}$; since $u(n) = v(n)$ we deduce that $u(x) = v(x)$, that is, $u = v$.

In [2, p.31], it is proved the following result:

Proposition 4.2.9. Let A be an equational category. Then in A the monomorphisms are just the injective morphisms.

Definition 4.2.10. Let C be a category. An object I(F) from C is called *initial* (*final*), if for every object X ∈ C, C(I, X) (C(X, F)) has only one element denoted by $a_X(w_X)$**.**

An object O from C which is simultaneously initial and final is called *nullary* object. By *subobject* of an object A Î C we understand a pair (B, u) with B Î C and u Î C(B, A) a monomorphism.

Two subobjects (B, u), (B′, u′) of an object A are called *isomorphic* if there is an isomorphism f∈C(B, B′) such that u′∘f = u.

Remark 4.2.11. (i). In general, in an algebraic category **C** the notions of subobject and subalgebra are different (it is possible as A ∈ **C**, B ≤ A and B ∉ **C**).

In the case of equational categories the two notions are identical.

(ii). I is the initial object (F is the final object) in category **C** iff I^0 (F^0) is the final (initial) object in $\mathbf{C^0}$.

(iii).If we have an initial (final, nullary) object in the category **C**, this is unique up to an isomorphism.

Indeed, if I, I′ are two initial objects in the category **C**, then there is a unique morphism u : I → I′ and a unique morphism v : I′ → I. Thus, u∘v = $1_{I'}$ and v∘u = 1_I, hence I≈I′. Analogous for final and nullary objects.

(iv). If I is an initial object in the category **C**, then every morphism u : X → I from **C** has a section (hence is an epimorphism) and if F is a final object, then every morphism v : F → X from **C** has a retraction (hence a monomorphism).

(v). If in a category **C** we have a nullary object **O**, then for every pair (X, Y) of objects of **C, C**(X, Y) ≠ ∅ (since **C**(X, Y) contains at least the composition of the morphisms $X \xrightarrow{w_X} O \xrightarrow{a_Y} Y$ denoted by $\mathbf{O_{YX}}$ and called the *nullary* morphism from X to Y). Clearly, for every u : X′ → X and v : Y → Y′, $\mathbf{O_{YX}}\circ u = \mathbf{O_{YX'}}$ and $v\circ \mathbf{O_{YX}} = \mathbf{O_{Y'X}}$.

Examples

1. In the category **Set**, the empty set ∅ is the only initial object and every set which contain only one element is a final object (clearly, these are isomorphic). We deduce that in **Set** we don't have nullary objects.

2. In the category **Fd** of fields we don't have initial or final objects.

Definition 4.2.12. A family $(G_i)_{i\in I}$ of objects in a category C is called family of *generators (cogenerators)* of C, if for every X, Y ∈ C and u, v ∈ C(X, Y), with u $\neq$ v, there is $f \in \bigcup_{i\in I} C(G_i, X)$ ($f \in \bigcup_{i\in I} C(Y, G_i)$) such that u∘f $\neq$ v∘f (f∘u $\neq$ f∘v).

If the family of generators (cogenerators) contains only an element G, then G is called *generator (cogenerator)* of **C.**

Clearly, the notions of generator and cogenerator are dual.

Examples

1. In the category **Set** every set which contains at least two elements is a cogenerator.

2. In the category **Top**, every discret, non-empty topological space, is a cogenerator for **Top** and every topological space containing at least two elements with trivial topology is a cogenerator for **Top**.

Let **C** be a category and f, g : X → Y a pair of morphisms in **C**.

Definition 4.2.13. The *kernel* or *equalizer* of a couple of morphisms (f, g), is a pair (K, i), with K Î C and i Î C(K, X) such that :

(i) f∘i = g∘i;

(ii) If (K′, i′) is another pair which verifies (i), then there is a unique morphism u : K′ ® K such that i∘u = i′.

Remark 4.2.14. If the kernel of a couple of morphisms exists, then it is unique up to an isomorphism.

Indeed, let (K′, i′) another kernel for the couple (f, g). Then there are athe morphisms u : K′ → K and u′ : K → K′ such that i′∘u′ = i. We deduce that i∘u∘u′ = i and i′∘u′∘u = i′; by the unicity from the definition of kernel we deduce that u∘u′ = 1_K and u′∘u = $1_{K'}$, that is, K≈K′.

In the case of existence, we denote the kernel of the couple of morphisms (f, g) by **Ker**(f, g).

The dual notion for kernel is the notion of *cokernel* for a couple of morphisms.

In fact, we have:

Definition 4.2.15. The *cokernel* or *coequalizer* of a couple (f, g) of morphisms is a pair (p, L) with L Î C and p Î C(Y, L) such that:

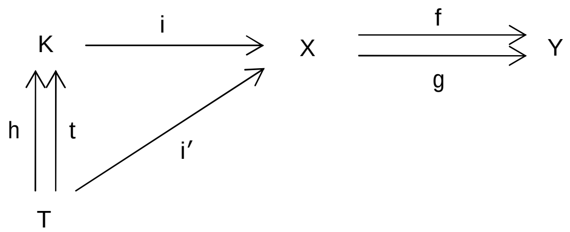
(i) p∘f = p∘g;

(ii) If (p′, L′) is another pair which verifies (i), then there is a unique morphism u : L ® L′ such that u∘p = p′.

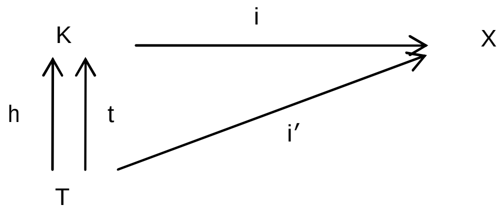
As in the case of kernel, the cokernel of a couple of morphisms (f,g) (which will be denoted by **Coker**(f, g)), if there exists, then it is unique up to an isomorphism.

Remark 4.2.16. If **Ker**(f, g) = (K, i), then i is a monomorphism in **C.**

Indeed, let T ∈ **C** and h, t : T → K morphisms such that $i \circ h = i \circ t = i'$.



Then $f \circ i' = g \circ i'$ and since t is closing the following diagram



we deduce from Definition 4.2.13 that h = t, hence i is a monomorphism in **C**.

Dually it is proved that if **Coker**(f, g) = (p, L), then p is an epimorphism in **C**.

Definition 4.2.17. We say that a category C is a category *with kernels (cokernels)* if every couple of morphisms in C has a kernel (cokernel).

Examples

1. The category **Set** is a category with kernels and cokernels (see §4 from Chapter 1).

2. The category **Top** is a category with kernels and cokernels.

Indeed, let f, g : (X, τ) → (Y, σ) a couple of morphisms in **Top** and (K, i) its kernel in the category **Set** for the couple f, g : X → Y.

If K is equipped with the topology $\bar{t}$ induced by the topology τ of X, then i: (K, $\bar{t}$) → (X, τ) is a continuous function and ((K, $\bar{t}$), i) = Ker(f, g) in **Top**.

If (p, L) is a cokernel in **Set** of couple (f, g) and if on L = Y / $\bar{R}$ (f, g) (see Remark 1.4.6 from, where $\bar{R}$ (f, g) is denoted by <ρ>) we consider the quotient topology $\bar{s}$, then p : (Y, s) → (L, $\bar{s}$) is continuous function and (p, (L, $\bar{s}$)) = **Coker**(f, g) in **Top**.

3. If we denote by **Set*** the subcategory of **Set** formed by non-empty sets and f, g : X → Y are morphisms in **Set*** such that {x ∈ X : f(x) = g(x)} = ∅, we deduce that in **Set*** doesn't exists **Ker**(f, g).

4. Let f, g : G → G′ be a couple of morphisms of groups, (K, i) = **Ker**(f, g) in **Set**, H the normal subgroup of G′ generated by the elements of the form $f(x)(g(x))^{-1}$, with x ∈ G (see [31]) and p : Y → G′/H is the canonical surjective morphism of groups. Then:

(i) K ≤ G, and (K, i) = **Ker**(f, g) in **Gr;**

(ii) (p, G′/H) = **Coker**(f, g) in **Gr**.

Conclusion: **The category Gr is a category with kernels and cokernels.**

Since **Gr** is a category with nullary object, if f : G → G′ is a morphism in **Gr**, then Ker(f) = **Ker**(f, $\mathbf{O}_{G'G}$) = {x ∈ G : f(x) = 0} (0 is the neutral element of G′!).

5. Let f, g :G → G′ a couple of morphisms in **Ab** and h : G → G′, $h(x) = f(x)\, g(x)^{-1}$, for every x ∈ G (clearly, h is a morphism in **Ab**).

Then:

(i) If K = **Ker**(h) and i : K → G is the inclusion morphism, then (K, i) = Ker(f, g) in **Ab**;

(ii) If H = **Im**(h) and p : G′ → G′/H is the surjective canonical morphism, then (p, G′ / H) = **Coker**(f, g) in **Ab**.

Conclusion: **The category Ab is a category with kernels and cokernels.**

6. Let f, g : A → A′ a couple of morphisms in the category $\mathbf{Rg_c}$ (of commutative unitary rings), (K, i) = **Ker**(f, g) in **Set** (clearly K is a subring of A and i is a morphism of unitary rings) and **a** the ideal of A′ generated by the elements of the form f(x) - g(x), with x ∈ A. If by p : A′ → A′ / **a** we denote the canonical surjective morphism, then:

(i) (K, i) = **Ker**(f, g) in $\mathbf{Rg_c}$;

(ii) (p, A′ / **a**) = **Coker**(f, g) in $\mathbf{Rg_c}$.

Conclusion: The category Rg_c is a category with kernels and cokernels.

The construction of cokernels in **Rg** is somewhat more complicated**;** in general, cokernels need not exist in the category **Fd** (see [72,p.51]).

7. Let f, g : (X, ≤) → (Y, ≤) be a couple of morphisms in **Pre** (respective **Ord**) and (K, i) = **Ker**(f, g) in **Set**.

If the set K will be equiped with the preorder (respective order) induced by the order of X, then there is a morphism in **Pre** and (K, i) = **Ker**(f, g) in **Pre** (respective **Ord**).

8. Let f, g : (X, ≤) → (Y, ≤) be a couple of morphisms in **Pre** (respective **Ord**) and (p, Z) = **Coker**(f, g) in **Set**.

Then

(i) If we consider on Z the preorder relation $\hat{y} \leq' \hat{y'}$ ⇔ there are $y_0,...,y_{n-1}$, y'_1 , …, y'_n in Y such that $\hat{y_0} = \hat{y}$, $\hat{y'_n} = \hat{y'}$, $\hat{y'_i} = \hat{y_i}$ for 1 ≤ i ≤ n-1 and $y_0 \leq y'_1$, $y_1 \leq y'_2$, …, $y_{n-1} \leq y'_n$, then p:(Y, ≤) → (Z, ≤′) is an isotone function and (p, Z) = **Coker**(f, g) in **Pre**.

(ii) If X, Y are ordered sets and $\overline{Z}$ is the ordered set associate to Z (that is, $\overline{Z} = Z/\sim$ where z∼z′ ⇔ z ≤ z′ and z′ ≤ z (see Chapter 2) and p_Z : Z → $\overline{Z}$ is the isotone canonical surjective function, then (p_Z , $\overline{Z}$) = **Coker**(f, g) in **Ord**.

Conclusion: **The categories Pre and Ord are categories with kernels and cokernels.**

Remark 4.2.18. If **C** has a nullary object O and f : X → Y is a morphism in **C**, we define the *kernel* of f (denoted by **Ker**(f)) as **Ker**(f, $\mathbf{O_{YX}}$) (of course, if it exists!), where, we recall that $\mathbf{O_{YX}}$: X → Y is the nullary morphism from X to Y.

Remark 4.2.19. More general, every equational categorie is a category with kernels and cokernels. The details are left for the reader (see the case of **Set,** Chapter 3 and [72]).

4.3. Functors. Examples. Remarkable functors. Functorial morphisms. Equivalent categories

Definition 4.3.1. If C and C¢ are two categories, we say that from C to C¢ is defined a *covariant (contravariant)* functor F (we write F : C ® C¢) if:

(i) For every object X∈C is defined a unique object F(X)∈C¢;

(ii) For every pair (X, Y) of objects in C and every f∈C(X, Y) is defined a unique F(f) ∈ C¢(F(X), F(Y)) (F(f) ∈ C¢(F(Y), F(X))) such that

a) $F(1_X) = 1_{F(X)}$ for every X ∈ C;

b) For every two morphisms f and g in C for which the composition g∘f is possible, then F(g) ∘ F(f) (F(f) ∘ F(g)) is defined and F(g∘f) = F(g) ∘ F(f) (F(g∘f) = F(f) ∘ F(g)).

Remark 4.3.2.

(i) If F : **C** ® **C¢** is a covariant (contravariant) functor, u is a morphism in **C** and s is a section (retract) of u in **C**, then F(s) is a section (retract) of F(u) in **C′**.

In particular, if u is an isomorphism in **C**, then F(u) is an isomorphism in **C¢** and $(F(u))^{-1} = F(u^{-1})$. So, F preserves the morphisms with section (retract) and isomorphisms. Also, F preserves identical morphisms and commutative diagrams.

(ii) To every contravariant functor F : $\mathbf{C}^{\mathbf{0}}$ ® **C¢** we can assign a covariant functor $\overline{F}$ **: C** ® **C¢**, where $\overline{F}$(X) = F(X), for every X ∈ $\mathbf{C}^{\mathbf{0}}$ and for every u^0 : X → Y in $\mathbf{C}^{\mathbf{0}}$ (that is, u : Y →X in **C**), $\overline{F}(u^0)$ = F(u) : F(X) → F(Y). Analogous to every contravariant functor F : **C** ® **C¢**^{0} we can assign a covariant functor $\overline{F}$ **:C** ® **C¢.**

Examples

1. For every category **C**, $1_{\mathbf{C}}$: **C** → **C**, defined by $1_{\mathbf{C}}(X) = X$, for every X ∈ **C** and $1_{\mathbf{C}}(u) = u$ for every morphism u in **C**, is a covariant functor (called the *identity* functor of **C**).

2. More general, if **C′** is a subcategory of **C**, then $1_{\mathbf{C}',\mathbf{C}}$: **C′** → **C** defined by: $1_{\mathbf{C}',\mathbf{C}}(X) = (X)$, for every X ∈ **C′** and $1_{\mathbf{C}',\mathbf{C}}(u) = u$ for every morphism u in **C′**, is a covariant functor (called *inclusion* functor).

3. If **C** is a category, then F : $\mathbf{C}^{\mathbf{0}} \times \mathbf{C}$ → **Set** defined by F(X, Y) = **C**(X, Y) and if (u, u′) : (X, Y) → (X′, Y′) is a morphism in $\mathbf{C}^{\mathbf{0}} \times \mathbf{C}$, then F(u, u′) : **C**(X, Y) → **C**(X′, Y′) is the function f → u′∘f∘u, is a covariant functor (denoted by **Hom**).

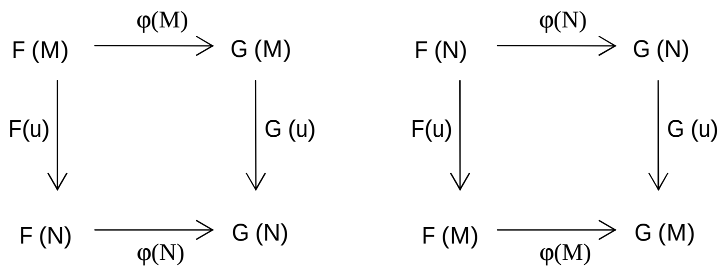
4. Let **C** be a category and A be a fixed object in **C**. We define the functor h^A : **C** → **Set** by: if M ∈ **C**, then $h^A(M)$ = **C**(A, M) and if u : M → N is a morphism in **C**, then $h^A(u) : h^A(M) \to h^A(N)$, $h^A(u)(f) = u \circ f$, for every f ∈ $h^A(M)$. The functor h^A is covariant.

Analogous we can define the contravariant functor h_A : **C** → **Set** by: $h_A(M)$ = C(M, A), for every M ∈ **C** and for u : M → N a morphism in **C**, $h_A(u) : h_A(N) \to h_A(M)$, $h_A(u)(f) = f \circ u$, for every f ∈ $h_A(N)$.

The functor h^A (h_A) is called the *functor(cofunctor) associated with* A.

Definition 4.3.3. If C, C′, C″ are three categories and F : C → C′, G : C′ → C″ are functors (covariants or contravariants), then we define GF : C → C″ by (GF)(M) = G(F(M)), for every M ∈ C and (GF)(u) = G(F(u)) for every morphism u in C. So, we obtain a new functor GF from C to C″ called the *composition* of G with F. Clearly, if F and G are covariants (contravariants), then GF is covariant, when if F is covariant and G is contravariant (or conversley), then GF is contravariant.

Definition 4.3.4. Let C, C′ be two categories and F, G : C → C′ be two covariants (contravariants) functors. We say that a *functorial morphism* φ is given from F to G (we write φ : F → G or $F \xrightarrow{\varphi} G$), if for every M ∈ C we have a morphism φ(M) : F(M) → G(M) such that for every morphism u : M → N in C, the diagrams



are commutative. We write j = (j(M))$_{M \in C}$ and we say that the functorial morphism j has the *components* j(M), M ∈ C.

If for every M ∈ C, j(M) is an isomorphism in C¢, we say that j is *functorial isomorphism* from F to G (in this case we say that F and G are *isomorphic* and we write F » G).

Remark 4.3.5. By 1_F : F → F we denote the functorial morphism of components $1_F(M) = 1_{F(M)}$: F(M) → F(M). Clearly, 1_F is a functorial isomorphism (called the *identical functorial morphism* of F).

In this book we will put in evidence other examples of functorial morphisms.

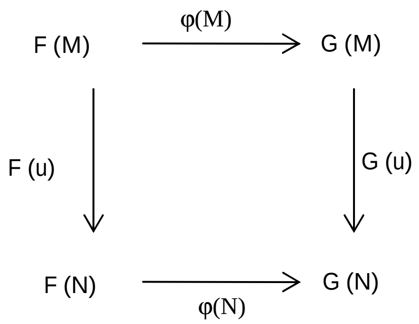
Definition 4.3.6. Let F, G, H three covariant functors from the category C to category C¢ and $F \xrightarrow{j} G \xrightarrow{y} H$ **two functorial morphisms. If for every M ∈ C, we define q(M) = y(M) ∘ j(M), we obtain in this way a functorial morphism q (denoted by y ∘ j) called the *composition* of functorial morphisms y and j.**

Analogous we can define the composition of two functorial morphisms if F, G and H are contravariants.

Proposition 4.3.7. Let F, G two covariant (contravariant) functors from the category C to the category C¢ and $F \xrightarrow{j} G$ **a functorial morphism. Then j is functorial isomorphism iff there is** $G \xrightarrow{y} F$ **a functorial morphism such that y ∘ j = 1_F and j ∘ y = 1_G (in this case we write y = j^{-1}).**

Proof. Suppose that φ is a functorial isomorphism. Then, if M ∈ **C**, φ(M) : F(M) → G(M) is an isomorphism in **C′**, hence we can consider the morphism $\psi(M) = (\varphi(M))^{-1}$:G(M) →F(M). The family $\{\psi(M)\}_{M \in \mathbf{C}}$ of morphisms determine a functorial morphism ψ : G → F.

Indeed, let u : M → N be a morphism in category **C**. We have the following commutative diagram:



hence $\varphi(N)\circ F(u) = G(u)\circ\varphi(M)$, so we obtain $F(u)\circ\varphi(M)^{-1} = \varphi(N)^{-1}\circ G(u)$ or $F(u)\circ\psi(M) = \psi(N)\circ G(u)$, which imply that ψ is a functorial morphism; clearly $\psi\circ\varphi = 1_F$ and $\varphi\circ\psi = 1_G$.

The converse assertion is clear. ∎

Definition 4.3.8. Let C, C' be two categories and $F : C \to C'$ be a covariant functor. We say that :

(i) F is *faithful (full)* if for every $X, Y \in C$, the function $F(X, Y) : C(X, Y) \to C'(X, Y)$ is injective (surjective);

(ii) F is *monofunctor* (or *embedding*) if for every $X, Y \in C$ such that $F(X) = F(Y)$, then $X = Y$;

(iii) F is *epifunctor* if for every $X' \in C'$ there is $X \in C$ such that $F(X) = X'$;

(iv) F is *bijective*, if it is simultaneously monofunctor and epifunctor;

(v) F is *representative* if for every $Y \in C'$ there is an object $X \in C$ such that $F(X) \approx Y$;

(vi) F is *conservative* if from F(f) is an isomorphism in C', then we deduce that f is an isomorphism in C;

(vii) F is an *equivalence of categories* if there is a covariant functor $G : C' \to C$ such that $GF \approx 1_C$ and $FG \approx 1_{C'}$; in this case we say that the categories C and C' are *equivalent* and that F and G is *quasi-inverse* one for another.

(viii) F is called an *isomorphism of categories* if F is an equivalence which produces a bijection between the objects of C and C' (i.e, F is bijective)

Remark 4.3.9.

(i). Let F : **C** ® **C¢ be** a covariant functor, X, Y ∈ **Ob(C)**, f ∈**C**(X, Y) and g ∈ **C**(Y, X). Then:

a) If F is faithful, F(g) is a section (retract) of F(f) iff g is a section (retract) of f;

b) If F is faithful and full, f has a section (retract) iff F(f) has a section (retract).

Indeed, if g is a section of f (that is, $f \circ g = 1_Y$), then $F(f) \circ F(g) = F(f \circ g) = F(1_Y) = 1_{F(Y)}$, hence F(g) is a section of F(f). Conversely, if F(g) is a section of F(f) (that is, $F(f) \circ F(g) = 1_{F(Y)}$), then $F(f \circ g) = F(1_Y)$, hence $f \circ g = 1_Y$ (since F is faithful). The rest is proved analogously.

(ii). From the above remark, we deduce that every faithful and full functor is conservative.

(iii). Every isomorphism of categories is an equivalence of categories, but conversely it is not true.

Theorem 4.3.10. Let C, C¢ be two categories and F : C ® C¢ be a covariant functor. The following assertions are equivalent:

(i) F is an equivalence of categories;

(ii) F is faithful, full and representative.

Proof. ([62]). (i) ⇒ (ii). We suppose that F is an equivalence of categories, hence there is a covariant functor G : **C′** → **C** such that $GF \overset{j}{\approx} 1_C$ and $FG \overset{y}{\approx} 1_{C'}$. Let now M, N ∈ **C**; we will prove that the function **C**(M, N) → **C′**(F(M), F(N)), f → F(f) is a bijection.

So, let f, f ′ ∈ **C**(M, N) such that F(f) = F(f ′).

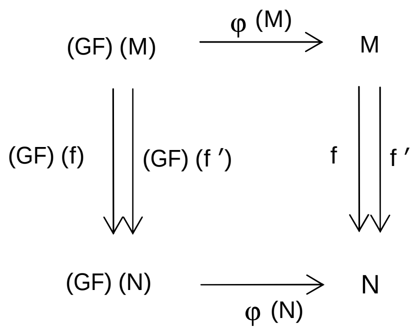
From the hypothesis we have two functorial isomorphisms φ : GF →$1_{\mathbf{C}}$ and ψ : FG →$1_{\mathbf{C¢}}$.

The pair of morphisms f, f′ induces a pair of morphisms

$$F(M) \underset{F(f')}{\overset{F(f)}{\rightrightarrows}} F(N) \text{ and } G(F(M)) \underset{G(F(f'))}{\overset{G(F(f))}{\rightrightarrows}} G(F(N)).$$

Since F(f) = F(f′), then G(F(f)) = G(F(f′)).

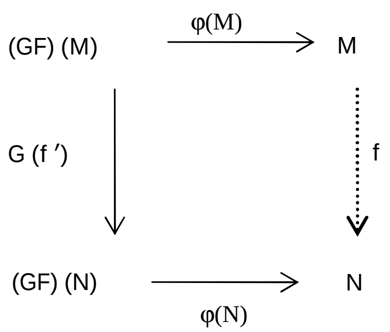
Consider the following commutative diagram:



From $f \circ \varphi(M) = \varphi(N) \circ (GF)(f)$ and $f' \circ \varphi(M) = \varphi(N) \circ (GF)(f')$ and from the fact that φ is a functorial isomorphism (hence all his components are isomorphisms), we deduce that $f = f'$, hence F is faithful.

To prove that F is full, let $f' \in \mathbf{C}'(F(M), F(N))$.

Then $G(f') : G(F(M)) \to G(F(N))$ and we consider the diagram:



We define $f \in \mathbf{C}(M, N)$ by $f = \varphi(N) \circ G(f') \circ \varphi(M)^{-1}$ (this is possible because φ(M) is an isomorphism).

We have to prove that $F(f) = f'$. From the equalities $f \circ \varphi(M) = \varphi(N) \circ (GF)(f)$ and $f \circ \varphi(M) = \varphi(N) \circ G(f')$, we deduce that $\varphi(N) \circ (GF)(f) = \varphi(N) \circ G(f') \Leftrightarrow G(F(f)) = G(f')$. Since G is an equivalence of categories we deduce (as before) that G is a faithful functor, that is, $F(f) = f'$.

So, we proved that F is faithful and full.

To prove this implication completely, let $X' \in \mathbf{C}'$ and denote $X = G(X')$. We have $F(X) = F(G(X')) = (FG)(X') \overset{\psi(X')}{\approx} X'$ (since $\psi(X')$ is an isomorphism).

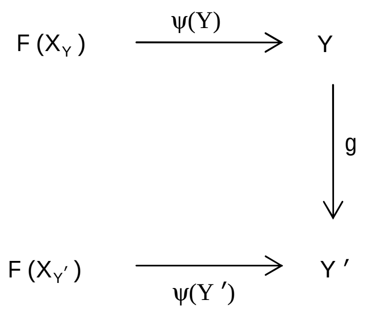
(ii) $\Rightarrow$ (i). Firstly, we have to prove that since F is faithful and full then from $F(X) \approx F(Y)$ we deduce that $X \approx Y$. Indeed, we have $\overline{f} : F(X) \to F(Y)$ and $\overline{g} : F(Y) \to F(X)$ such that $\overline{g} \circ \overline{f} = 1_{F(X)}$ and $\overline{f} \circ \overline{g} = 1_{F(Y)}$. Since the hypothesis F is full, there are $f : X \to Y$ and $g : Y \to X$ such that $F(f) = \overline{f}$ and $F(g) = \overline{g}$. From $\overline{g} \circ \overline{f} = 1_{F(X)}$ we deduce that $F(g) \circ F(f) = 1_{F(X)} \Rightarrow F(g \circ f) = F(1_X)$; since F is faithful, we deduce that $g \circ f = 1_X$. Analogous we deduce that $f \circ g = 1_Y$, hence $X \approx Y$.

Let's pass to the effective proof of implication (ii) $\Rightarrow$ (i).

Let $Y \in \mathbf{C}'$; by hypothesis there is $X_Y \in \mathbf{C}$ such that $Y \approx F(X_Y)$. Since the class of morphisms is a set using the axiom of choice we can select an isomorphism $\psi(Y) : F(X_Y) \to Y$.

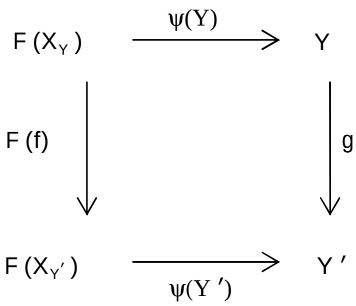
Analogously, if $Y' \in \mathbf{C}'$, then there is an isomorphism $\psi(Y') : F(X_{Y'}) \to Y'$.

Now let $g : Y \to Y'$ be a morphism in $\mathbf{C}'$ and we consider the diagram in $\mathbf{C}'$



We define $\overline{g} : F(X_Y) \to F(X_{Y'})$ by $\overline{g} = \psi(Y')^{-1} \circ g \circ \psi(Y)$. Since F is full (by the hypothesis), there is $f : X_Y \to X_{Y'}$, such that $F(f) = \overline{g}$.

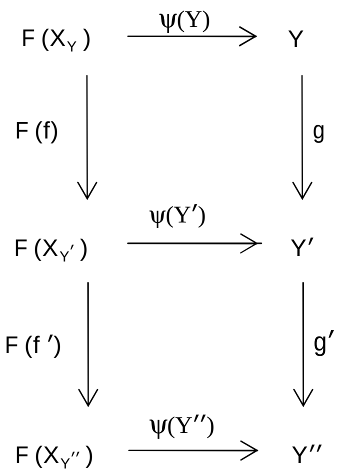
We have the following commutative diagram:



Define $G : \mathbf{C}' \to \mathbf{C}$ by $G(Y) = X_Y$ and $G(g) = f$ and we have to prove that G is a covariant functor, $FG \approx 1_{\mathbf{C}'}$ and $GF \approx 1_{\mathbf{C}}$.

If $g' : Y' \to Y''$ is another morphism in $\mathbf{C}'$, then as before, there is $f' : X_{Y'} \to X_{Y''}$ such that $G(g') = f'$.

From the diagram



we deduce that to $g' \circ g$ corresponds $F(f' \circ f)$, hence we deduce that $G(g' \circ g) = f' \circ f = G(g') \circ G(g)$ (since from $g \circ \psi(Y) = \psi(Y') \circ F(f)$ and $g' \circ \psi(Y') = \psi(Y'') \circ F(f')$ there results that $(g' \circ g) \circ \psi(Y) = \psi(Y'') \circ F(f' \circ f)$). Since $G(1_Y) = 1_{G(Y)}$, we deduce that G is a covariant functor.

So, $FG \approx 1_{\mathbf{C}'}$ (since $F(G(Y)) = F(X_Y)$ and $F(X_Y) \overset{y(Y)}{\approx} Y$). The fact that ψ is a functorial morphism results from the study of the above diagram.

From GF to $1_{\mathbf{C}}$ we construct φ in the following way: if $X \in \mathbf{C}$, then $F(X) \in \mathbf{C}'$ and by the hypothesis there is $X_F \in \mathbf{C}$ such that $F(X_F) \approx F(X)$.

According to a previous remark , $X_Y \overset{j(X)}{\approx} X$. It is easy to verify that φ is a functorial morphism and $GF \approx 1_{\mathbf{C}}$. So, the proof of theorem is complete. ■

Remark 4.3.11. In general a functor doesn't preserve a monomorphism or an epimorphism.

Indeed, let **C** be a category with at least two distinct objects X and Y and a morphism $u : X \to Y$ which is not monomorphism or epimorphism in **C**. We consider the subcategory **C′** of **C** which contains as objects only X and Y and as morphisms 1_X, 1_Y and u. We also consider $1_{\mathbf{C}',\mathbf{C}} : \mathbf{C}' \to \mathbf{C}$ the inclusion functor. Since u is bimorphism in **C′** and in **C**, $1_{\mathbf{C}',\mathbf{C}}(u) = u$ is not monomorphism or epimorphism we obtain the desired conclusion.

Definition 4.3.12. Let C, C′ be two categories and T : C → C′ be a contravariant functor. We say that T is a *duality of categories*, if there is a contravariant functor S : C′ → C such that TS ≈ $1_{\mathbf{C}'}$ and ST ≈ $1_{\mathbf{C}}$.

Remark 3.13. Following the above definition, to show that $\mathbf{C}^0 \approx \mathbf{C}'$ (in the sense of Definition 4.3.8, vii), the return to find two contravariant functors $T:\mathbf{C} \to \mathbf{C}'$ and $S : \mathbf{C}' \to \mathbf{C}$ such that $TS \approx 1_{\mathbf{C}'}$ and $ST \approx 1_{\mathbf{C}}$.

As an application, we will characterize the *dual categories* for **Set**, **Ld(0,1)** and **B** (of Boolean algebras).

4. 3.1. The dual category of Set

This subparagraph is drawn up after the paper [41].

Definition 4.3.14. A *normal lattice* is a bounded and join-complete lattice L which verifies the following axiom:

(N) For every x, y ∈ L, with x < y, there is an atom z ∈ L such that $x < x \vee z \le y$.

If L, L′ are two normal lattices, f : L → L′ is called a *morphism of normal lattices*, if f ∈ L(0,1)(L, L′) and f(sup A) = sup f(A), for every subset A of L.

We denote by **Lnr** the category of normal lattices.

Theorem 4.3.15. The dual category of Set is equivalent with Lnr (i.e. $\mathbf{Set}^{\mathbf{0}}$ ≈ Lnr).

Proof. To prove $\mathbf{Set}^{\mathbf{0}} \approx \mathbf{Lnr}$, it is necessary to construct two contravariant functors **P** : **Set** → **Lnr** and **a** : **Lnr** → **Set** (these notations are standard) such that $\mathbf{aP} \approx 1_{\mathbf{Set}}$ and $\mathbf{Pa} \approx 1_{\mathbf{Lnr}}$.

For every set X we consider **P**(X) the power set of X and for every function f : X → Y, the function f^* : **P**(Y) → **P**(X) (see Proposition 1.3.7). It is easy to prove that for X ∈ **Set**, **P**(X) ∈ **Lnr** and f^* : **P**(Y) → **P**(X) is a morphism of normal lattices, so we obtain by the assignments X → **P**(X) and f → f^* a contravariant functor **P** : **Set** → **Lnr**.

To define the contravariant functor **a**, let L∈**Lnr** and **a**(L) be the set of all atoms of L.

We have to prove that sup **a**(L) = **1**. If by contrary sup **a**(L) < **1**, then by axiom (N), there is x ∈ **a**(L) such that sup **a**(L) < x ∨ sup **a**(L) ≤ **1**, hence we deduce that x ∉ **a**(L) - which is a contradiction!

Let f : L → L′ be a morphism in **Lnr** and we can remark that for every y ∈ **a**(L′) there is a unique element x ∈ **a**(L) such that y ≤ f(x).

Indeed, for existence, suppose by contrary that there is y ∈ a(L′) such that for every x ∈ **a**(L), then f(x) < y. In these conditions, we deduce that y ≤ **1** = f(**1**) = f(sup **a**(L)) = sup f(**a**(L)); since y ∧ f(x) = **0** for every x ∈ **a**(L) ⇒ y = y ∧ sup f(**a**(L)) = y ∧ **1** = **0**, hence y = **0**, which is a contradiction.

Relative to uniqueness, suppose that for every y ∈ **a**(L′), there are x, x′ ∈ **a**(L), x ≠ x′ such that y ≤ f(x) and y ≤ f(x′). It is immediate that y ≤ f(x) ∧ f(x′) = f(x ∧ x′) = f(**0**) = **0**, so y = **0**, which is a contradiction!

Following the above, we can define **a**(f) : **a**(L′) → **a**(L) by **a**(f)(y) = x, where y ∈ **a**(L′) and x ∈ **a**(L) is the unique element with the property that y ≤ f(x).

To prove that **a** is a contravariant functor, we consider the morphisms of normal lattices $L \xrightarrow{f} L' \xrightarrow{g} L''$ and we will prove that **a**(g∘f) = **a**(f) ∘ **a**(g) (the equality **a**(1_L) = $1_{\mathbf{a}(L)}$) is clearly).

For this, let y ∈ **a**(L″) and **a**(g∘f)(y) = x, where x ∈ L and y ≤ (gf)(x) = g(f(x)).

We denote **a**(g)(y) = z (hence z ∈ L′ and y ≤ g(z)).

If **a**(f)(z) = x′ (with x′ ∈ L and z ≤ f(x′)), then **a**(f)(**a**(g)(y)) = **a**(f)(z) = x′ and since y ≤ g(z) ≤ g(f(x′)) = (g∘f)(x′) we deduce that x = x′, so **a**(g∘f)(y) = **a**(f)(**a**(g)(y)), hence **a**(g∘f) = **a**(f)∘**a**(g).

So, the assignments L → **a**(L) and f → **a**(f) define a contravariant functor **a** : **Lnr** → **Set**.

To prove that **Set**^{0} ≈ **Lnr** we have to prove the functorial isomorphisms **aP** ≈ $1_{\mathbf{Set}}$ and **Pa** ≈ $1_{\mathbf{Lnr}}$. The isomorphism **aP** ≈ $1_{\mathbf{Set}}$ is clear (since the atoms of **P**(X) coincides with the elements of X and if f : X → Y is a function, then **a**(f^*)(x) = f(x) for every x ∈ X, hence (**aP**)(f) = f).

To prove the isomorphism **Pa** ≈$1_{\mathbf{Lnr}}$, we consider the function α : L → **Pa**(L) and β : **Pa**(L) → L, with L ∈ **Lnr**, defined in the following way: for y ∈ L, α(y) = {x : x ∈ **a**(L), x ≤ y} and if A ∈ **Pa**(L), then β(A) = sup(A). It is easy to see that α and β are morphisms of a normal lattices.

We have to prove the equalities α∘β = $1_{\mathbf{Pa}(L)}$ and β∘α = 1_L.

For the first equality, let A = {x_i : x_i ∈ **a** (L), i ∈ I} and for an atom x ≤ β (A) = sup(A), x = **0** or x = x_{i_0} for an i_0 ∈ I.

If we denote $y_i = x \wedge x_i$, then y_i = **0** for every i ∈ I or i_0 ∈ I, doesn't exist such that x_{i_0} = x.

If y_i = **0** for every i ∈ I, we have **0** $= \sup_i\{y_i\} = \sup_i\{x \wedge x_i\} = x \wedge \sup_i\{x_i\} =$ x, which is not true. So, there is i ∈ I with x = x_i, hence x ∈ A and we obtain the equality (α∘β)(A) = A.

For y ∈ L, α(y) ≠ **0** and (β∘α)(y) = sup α(y) ≤ y. If we suppose that sup (**a**(L)) < y, then there is an atom x ∈ **a**(L) such that x ∉ α(y) and sup α(y) ∨ x ≤ y, hence x ≤ y – a contradiction, so (β∘α) (y) = y. Since (β∘α)(**0**) = **0**, we deduce that the second equality is true.

So, the proof is complete. ∎

4.3.2. The dual category of Ld(0, 1)

For $L \in$ **Ld(0,1)**, we will work in this subparagraph with the filters of L for which we have dual result for ideals contained in §4 from Chapter 2, and with the way we will use them without presenting for the dual proofs for every one.

So, for $L \in$ **Ld(0,1)** we denote by **FM**(L) the set of all maximal filters (ultrafilters) of L.

As in the case of ideals it is immediate to prove that if $L \in$ **Ld**(**0**, **1**) then $j_L : L \to P(\mathbf{FM}(L))$, $j_L(x) = \{F \in \mathbf{FM}(L) : x \in F\}$ for $x \in L$ is a monomorphism in **Ld**(**0**,**1**),that is,injective function and for every $x, y \in L$, $j_L(x \vee y) = j_L(x) \cup j_L(y)$, $j_L(x \wedge y) = j_L(x) \cap j_L(y)$, $j_L(\mathbf{0}) = \emptyset$ and $j_L(\mathbf{1}) =$ **FM**(L).

Definition 4.3.16. ([70, p.428]).**A T_0-quasicompact topological space is called *Stone space* if it verifies the following conditions:**

(s_1) The compact open sets form a basis of opens;

(s_2) The intersection of two open compacts is also an open compact;

(s_3) If D is a set of open compacts with the property of finite intersection and F is a closed set such that $F \cap C \neq \emptyset$ for every $C \in D$,then $F \cap \left(\bigcap_{C \in D} C \right) \neq \emptyset$.

For $L \in$ **Ld(0,1)** we consider **FM**(L) equiped with the topology τ_L generated by $\{j_L(x)\}_{x \in L}$ (called *Stone-Zariski topology*).

An element of τ_L will be an union of finite intersections of elements from the generating family $\{j_L(x)\}_{x \in L}$ and, since for $x_1, \dots, x_n \in L$, $j_L(x_1) \cap \dots \cap j_L(x_n) = j_L(x_1 \wedge \dots \wedge x_n)$ we deduce that an open set in the topological space (**FM**(L), τ_L) has the form $\bigcup_{x \in S} j_L(x)$ with $S \subseteq L$.

Theorem 4.3.17. **The topological space S_L = (FM(L), τ_L) is a Stone space.**

Proof. ([70]). The fact that $\mathbf{S_L}$ is T_0 follows from Corollary 2.4.5, by dualising the result to the case of filters.

To prove the compacity of **FM**(L) let $S \subseteq L$ such that **FM**(L) = $\bigcup_{x \in S} j_L(x)$. We will prove that (*) **FM**(L) = $\bigcup_{x \in (S]} j_L(x)$.

Since S ⊆ (S], the inclusion **FM**(L) ⊆ $\bigcup_{x\in(S]} j_L(x)$ is clear. Let now F∈ $\bigcup_{x\in(S]} j_L(x)$; then there is s_0 ∈ (S] such that F ∈ $j_L(s_0)$ ⇔ s_0 ∈ F. Since s_0 ∈ (S], there are s_1, ... , s_n ∈ S such that $s_0 \le s_1 \vee ... \vee s_n$. Since F is filter and s_0 ∈ F we deduce that $s_1 \vee ... \vee s_n$ ∈ F.

Since F is an ultrafilter, F is prime, so there is 1 ≤ i ≤ n such that s_i ∈ F ⇔ F ∈ $j_L(s_i)$ with s_i ∈ S, hence we obtain the equality (*).

We will prove that **1** ∈ (S].

If **1** ∉ (S], then there is a maximal ideal I such that (S] ⊂ I. The complementary set of I will be a prime filter with the property that for every s ∈ (S] we have s ∉ I which is in contradiction with relation (*). Hence **1**∈(S], so **1** = $s_1 \vee ... \vee s_n$ with s_1, ..., s_n ∈ S; thus **FM**(L) = j_L(**1**) = $j_L(s_1) \cup ... \cup j_L(s_n)$, that is, **FM**(L) is a compact set.

Analogously we prove that $j_L(x)$ is a compact set for each x ∈ L, so we have proved (s_1). The condition (s_2) follows immediately from the fact that for every x, y ∈ L, $j_L(x \wedge y) = j_L(x) \cap j_L(y)$.

Now we will prove the condition (s_3). By the above we can consider F = **FM**(L) \ $j_L(y)$ = ∁ $j_L(y)$ and **D** = $\{j_L(x)\}_{x\in S}$ with S ⊆ L.

The fact that ∁$j_L(y) \cap j_L(x) \neq \emptyset$ for every x ∈ S is equivalent with: for every x ∈ S and for every P ∈ ∁$j_L(y)$ we have P ∈ $j_L(x)$.

Supposing by contrary that ∁$j_L(y) \cap \left(\bigcap_{x\in S} j_L(x)\right) = \emptyset$ we deduce that for every P_0 ∈ ∁$j_L(y) \Rightarrow P_0 \notin \left(\bigcap_{x\in S} j_L(x)\right)$, hence there is P_0∈ ∁$j_L(y)$ and x_0 ∈ S such that P ∉ $j_L(x_0)$ which is contradictory, so (s_3) is true. ∎

Definition 4.3.18. For two Stone spaces X, Y, a function f : X → Y is called *strong continuous* if for every open compact D in Y ⇒ f^{-1}(D) is an open compact in X.

Next, by **St** we denote the category of Stone spaces (whose objects are Stone spaces and morphisms are the strong continuous functions).

Let now L, L′∈ **Ld**(**0**, **1**) and f ∈ **Ld**(**0**, **1**) (L, L′).

We consider the function **FM**(f): **FM**(L′) → **FM**(L) defined for F∈**FM**(L′) by **FM**(f)(F) = f^{-1}(F).

Proposition 4.3.19. The function **FM**(f) is strong continuous.

Proof. Clearly **FM**(f) is correctly defined (since F′ ∈ **FM**(L′) ⇒ f^{-1}(F′) ∈ **FM**(L)).

For x ∈ L, $(\mathbf{FM}(f))^{-1}(j_L(x))$ = {F′∈ **FM**(L′) : (**FM**)(f)(F′) ∈ $j_L(x)$} = {F′∈**FM**(L′) : f^{-1}(F′)∈ $j_L(x)$} = {F′∈ **FM**(L′) : x ∈ f^{-1}(F′)} = {F′ ∈ **FM**(L′) : f(x) ∈ F′} = $j_L(f(x))$ hence **FM**(f) is a strong continuous function (using the fact that $\{j_L(x)\}_{x\in L}$ are open compacts which form a basis for the Stone-Zariski topology). ∎

Now let X be a Stone space and T(X) the set of all open compacts of X. It is immediate that T(X) becomes relatively to union and intersection a lattice bounded (that is, T(X)∈**Ld(0,1)**).

If X, Y are two Stone spaces and f∈**St**(X, Y) then we denote by T(f) : T(Y) →T(X) the function defined by (T(f))(D) = f^{-1}(D), for every D∈T(Y). Clearly T(f) ∈ **Ld(0,1)**(T(Y), T(X)).

So, we obtained **FM** : **Ld(0,1)** → **St** and **T** : **St** → **Ld(0,1)** given by the assignments **L** → **FM(L)**, f → **FM**(f), respective X → **T**(X) and f → **T**(f).

It is immediate to prove that **FM** and **T** are contravariant functors.

Theorem 4.3.20. The dual category of Ld(0, 1) is equivalent with the category St of Stone spaces (i.e, $(\mathbf{Ld(0,1)})^0$ » St).

Proof. We will prove the existence of functorial isomorphisms

(1) **T** ∘ **FM** ≈ $1_{\mathbf{Ld(0,1)}}$;

(2) **FM** ∘ **T** ≈ $1_{\mathbf{St}}$.

Let X ∈ **St** and x ∈ X. The set {V ∈ T(X) : x ∈ V} is a prime filter of T(X). Conversely, we prove that every prime filter P = $(V_i)_{i\in I}$ of T(X) has the same form.

If F = $\bigcap_{i\in I} V_i$, then F ≠ ∅ and if we choose x ∈ F, then from the axiom (s_2) we deduce that P = {V ∈ T(X) : x∈ V}.

Now let L ∈ **Ld** (**0**, **1**); we shall prove that j_L(L) = T(**FM**(L)) which will imply the isomorphism (T∘**FM)**(L) ≈L.

For this it is suffice to prove that every open compact of **FM**(L) has the form $j_L(x)$ with x ∈ L.

If D ∈ T(**FM**(L)), then D = $\bigcup_{i\in I} j_L(x_i)$. Since D is compact, there are $x_1, \ldots, x_n$ such that D = $\bigcup_{i=1}^{n} j_L(x_i) = j_L\left(\bigvee_{i=1}^{n} x_i\right)$, hence D = $j_L(x)$ with $x = \bigvee_{i=1}^{n} x_i$.

The rest is small calculus (which mostly represents calculus techniques) so , we will left them for the reader. ■

4.3.3. The dual category of B (of Boolean algebras)

In the end of this paragraph let's characterize the dual category of **B** (the category of Boolean algebras).

For a Boolean algebra B by **FM**(B) we denote the set of all maximal filters (ultrafilters) of B (see §8 from Chapter 2) and by u_B : B → **P**(**FM**(B)), $u_B(a)$ = {F ∈ **FM**(B) : a ∈ F} for every a ∈ B (see Theorem 2.8.8).

Proposition 4.3.21. The function u_B is a monomorphism in B.

Proof. See the proof of Proposition 2.8.8 . ■

Proposition 4.3.22. ([70]) For every compact and Haussdorf topological space (X, σ) the following assertions are equivalent:

(i) For every x ∈ X, the intersection of all clopen sets which contain x is {x};

(ii) For every x, y ∈ X, x ≠ y, there is a clopen D such that x ∈ D and y ∉ D;

(iii) X is generated by its clopen sets;

(iv) The convex component of every element x is {x}.

Definition 4.3.23. We call *Boole space* every topological space (X, σ) which verifies one of the equivalent conditions of Proposition 4.3.22.

Now let B a Boolean algebra and σ_B the topology of **FM**(B) generated by $(u_B(a))_{a\in B}$.

Theorem 4.3.24. For every Boolean algebra B, (FM(B), σ_B) is a Boole space.

Proof. By Proposition 4.3.21 we deduce that an element of σ_B has the form $\bigcup_{x\in S} u_B(x)$ with S ⊆ B.

Firstly we will prove that **FM**(B) is separable.

Indeed, if $F_1, F_2 \in$ **FM**(B), and $F_1 \neq F_2$ then there is $x \in F_1$ such that $x \notin F_2$, hence $x' \in F_2$ (where x' is the complement of x in B).

Then $F_1 \in u_B(x)$, $F_2 \in u_B(x')$ and since $u_B(x) \cap u_B(x') = u_B(x \wedge x') = u_B(\mathbf{0}) = \emptyset$ we deduce that **FM**(B) is separable.

Since for every $x \in B$, $u_B(x)$ is a clopen set (because $\complement(u_B(x)) = u_B(x') \in \sigma_B$) we deduce that **FM**(B) is generated by the family of his clopen sets.

To prove that **FM**(B) is a compact set let's suppose that **FM**(B) = $\bigcup_{x \in S} u_B(x)$ with $S \subseteq B$.

We shall prove that $\mathbf{0} \in [\{x' : x \in S\})$. If we suppose the contrary, then $[\{x' : x \in S\})$ will be included into a maximal filter $U \in$ **FM**(B) (i.e, $[\{x' : x \in S\}) \subseteq U$). Since **FM**(B) = $\bigcup_{x \in S} u_B(x)$ there is $x_0 \in S$ such that $U \in u_B(x_0) \Leftrightarrow x_0 \in U$. But $x'_0 \in U$ – which is a contradiction!

Since $\mathbf{0} \in [\{x' : x \in S\})$, there are $x_1, ..., x_n \in S$ such that $\mathbf{0} = x'_1 \wedge ... \wedge x'_n \Leftrightarrow 1 = x_1 \vee ... \vee x_n$, hence **FM**(B) = $u_B(1) = u_B(x_1 \vee ... \vee x_n) = u_B(x_1) \vee ... \vee u_B(x_n)$, that is, **FM**(B) is a compact set. ∎

Now let B_1, B_2 be Boolean algebras, $f \in$ **B(**B_1, B_2) and **FM**(f) : **FM**(B_2) → **FM**(B_1), **FM**(f)(U) = $f^{-1}(U)$, for every $U \in$ **FM**(B_2).

Proposition 4.3.25. FM(f) is a continuous function.

Proof. For $x \in B_1$ we have $(\mathbf{FM}(f))^{-1}(u_{B_1}(x)) =$ $\{U \in \mathbf{FM}(B_2) : (\mathbf{FM}(f))(U) \in u_{B_1}(x)\} = \{U \in \mathbf{FM}(B_2) : f^{-1}(U) \in u_{B_1}(x)\} = \{U \in \mathbf{FM}(B_2) : x \in f^{-1}(U)\} = \{U \in \mathbf{FM}(B_2) : f(x) \in U\} = u_{B_2}(f(x))$, hence the function **FM**(f) is continuous. ∎

As in the cases of lattices, the assignments B → **FM**(B) and f → **FM**(f) define a contravariant functor **FM** : **B** → $\tilde{B}$ from the category **B** of a Boolean algebras to the category $\tilde{B}$ of Boole spaces (whose objects are Boole spaces and the morphisms are the continuous mappings).

This functor is called *Stone duality functor*.

Theorem 4.3.26. The dual category of category B of Boolean algebras is equivalent with the category $\tilde{B}$ of Boole spaces (i.e, $B^0 \approx \tilde{B}$).

Proof. ([70]). Firstly we will construct another contravariant functor T : $\tilde{B}$ → **B** which together with **FM** gives the desired equivalence.

For a Boole space X ∈ $\tilde{B}$ we denote by T(X) the Boolean algebra of clopen sets of X and for every morphism f : X → Y in $\tilde{B}$ we denote by T(f):T(Y) → T(X) the restriction of f^{-1} : P(Y) → P(X) to T(Y) (clearly this function is with values in T(X)). It is easy to prove that we have obtained a contravariant functor T : $\tilde{B}$ → **B**.

I want to prove that the pair (**FM**, T) of functors defines the equivalence of categories $\tilde{B}$ and **B** (so we obtain $B^0 \approx \tilde{B}$).

For this it is necessary to prove the existence of functorial isomorphisms T∘**FM** ≈ $\mathbf{1_B}$ and **FM**∘T ≈ $1_{\tilde{B}}$.

Firstly I remark that every ultrafilter of T(X) (with X Boole space) has the form {W ∈ T(X) : x ∈ W} with x ∈ X; see [70, p. 423]).

Now let B be a Boolean algebra. Since u_B is a monomorphism, B will be isomorphic with u_B(B).

So, to prove that B is isomorphic with T(**FM**(B)) it will suffice to prove that u_B(B) is equal with T(**FM**(B)). Since u_B(x)∈T(**FM**(B)) for every x ∈ B, we will prove that every clopen set in **FM**(B) has the form u_B(x) with x ∈ B.

If D ∈ T(**FM**(B)), then D= $\bigcup_{\substack{x\in S\\ S\subseteq B}} u_B(x)$ and $\complement_{\mathbf{FM}(B)}$ D = $\bigcup_{\substack{y\in T\\ T\subseteq B}} u_B(y)$.

Thus **FM**(B) = D ∪ ($\complement_{\mathbf{FM}(B)}$D) = ($\bigcup_{\substack{x\in S\\ S\subseteq B}} u_B(x)$)∪ ($\bigcup_{\substack{y\in T\\ T\subseteq B}} u_B(y)$).

Since **FM**(B) is a compact set we can obtain a finite covering: **FM**(B) = $u_B(x_1) \cup ... \cup u_B(x_n) \cup u_B(y_1) \cup ... \cup u_B(y_m)$ (with $x_i \in S$, $y_j \in T$, $1 \le i \le n$, $1 \le j \le m$).

If n=0 then D = X = $u_B(\mathbf{1})$ and if m = 0, D = ∅ = $u_B(\mathbf{0})$.

In the case when m ≠ 0, n ≠ 0, D = $u_B(x_1 \vee ... \vee x_n \vee y_1 \vee ... \vee y_m)$.

Now let X ∈ $\tilde{B}$ and α : X → **FM**(T(X)), α(x) = {D ∈ T(X) : x ∈ D}. Clearly α is surjective.

Also α is injective, since if x ≠ y, there is D ∈ T(X) such that x ∈ D and y ∉ D (X is Boole space!).

The function α is bicontinuous because if D′ is a clopen set in **FM**(T(X)), e.g. D′ = u_B(D) with D ∈ T(X), then α^{-1}(D′) = D.

The rest are small calculus details which we let them for the reader. ∎

4.4. Representable functors. Adjoint functors

Let **C** be a category, F : **C** → **Set** a covariant functor, X ∈ **C** and (h^X, F) the class of functorial morphisms from the functor h^X to the functor F. Consider canonical function $\alpha = \alpha(F, X) : (h^X, F) \to F(X)$, $\alpha(\varphi) = \varphi(X)(1_X)$ for every $\varphi \in (h^X, F)$.

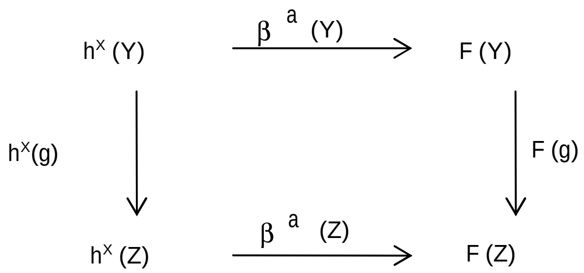
Lemma 4.4.1. (*Yoneda - Grothendieck*). The function a **is bijective and functorial with respect to F and X.**

Proof. We will construct $\beta : F(X) \to (h^X, F)$, the converse of α.

Indeed, for a ∈ F(X) and Y∈ **C** we consider the function $\beta^a(Y) : h^X(Y) \to F(Y)$, $\beta^a(Y)(f) = F(f)(a)$, for every $f \in \mathbf{C}(X,Y)$.

The morphisms $(\beta^a(Y))_{Y\in \mathbf{C}}$ are the components of a functorial morphism $\beta^a : h^X \to \mathbf{C}$.

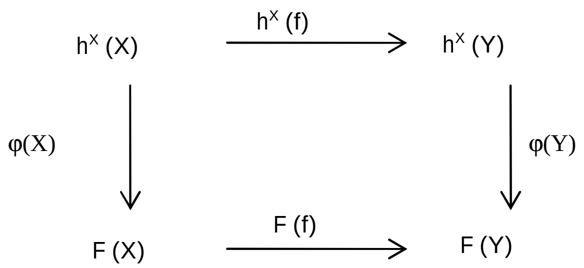
For this, let Z ∈ **C**, g ∈ **C**(Y, Z) and consider the diagram



If $f \in h^X(Y) = \mathbf{C}(X, Y)$, then $(F(g)\circ \beta^a(Y))(f) = (F(g)\circ F(f))(a) = F(g\circ f)(a) = \beta^a(Z)(g\circ f) = (\beta^a(Z)\circ h^X(g))(f)$, hence the above diagram is commutative, so β^a is a functorial morphism from h^X to F, that is, β is correctly defined.

To prove that β is the converse of α, we have to prove that $\beta\circ\alpha = 1_{(h^X,F)}$ and $\alpha\circ\beta = 1_{F(X)}$.

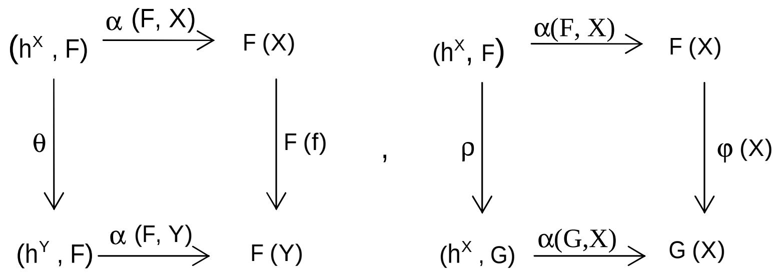
Indeed, let $\varphi \in (h^X, F)$. We have $(\beta\circ\alpha)(\varphi) = \beta(\alpha(\varphi)) = \beta(\varphi(X)(1_X)) = \beta^a$ (where $a = \varphi(X)(1_X) \in F(X)$). Since for every Y ∈ **C** and f ∈ **C**(X, Y), the diagram



is commutative, we deduce that $\beta^a(Y)(f) = F(f)(a) = F(f)(\varphi(X)(1_X)) = (\varphi(Y)\circ h^X(f))(1_X) = \varphi(Y)(f)$, hence $\beta^a = \varphi$, so $(\beta\circ\alpha)(\varphi) = \varphi \Rightarrow \beta\circ\alpha = 1_{(h^X,F)}$.

Conversely, if $a\in F(X)$, then $(\alpha\circ\beta)(a) = \alpha(\beta^a) = \beta^a(X)(1_X) = F(1_X)(a) = 1_{F(X)}(a) = a$, hence $\alpha\circ\beta = 1_{F(X)}$.

Since for every $f \in$ **C**(X,Y), $\varphi \in (h^X,F)$ and G : **C** → **Set** is a covariant functor it is easy to see that the diagrams



are commutative (θ, ρ are defined by the composition to the left, respective to the right, of $h^f : h^Y \to h^X$ with φ, where for $Z\in$ **C** and $g\in h^f(Z)(g)=gof$, we deduce the functoriality of α in F and X. **n**

Remark 4.4.2.

(i) If f : **C** ® **Set** is a contravariant, then for every $X \in$ **C,** the canonical function $\alpha(F, X) : (h_X, F) \to F(X)$ $(\varphi \to \varphi(X)(1_X))$ is bijective and functorial in F and X (the converse $\beta : F(X) \to (h_X, F)$, $a \to \beta_a$ will be defined analogously).

(ii) From the above lemma we deduce that (h^X, F) as (h_X, F) are sets.

Definition 4.4.3. **We say that the covariant functor F : C ® Set is *representable* if there is a pair (X, a) (with X Î C and a Î F(X)) such**

that the functorial morphism $\beta^{a} : h^{X} \to F$ (the corresponding of a by Yoneda-Grothendieck lemma) is a functorial isomorphism.

The pair (X, a) will be called the *pair of representation* for F.

Remark 4.4.4. In a dual way, the contravariant functor F : **C** → **Set** will be called *corepresentable* if there exist X∈ C, a∈ F(X) such that the functorial morphism $\beta_a : h_X \to F$ will be a functorial isomorphism.

Since every contravariant functor F : **C** → **Set** can be considered as a covariant functor from $\mathbf{C}^0$ to **Set**, in what follows we will consider only covariant functors.

Let **C**, **C′** be two categories and T : **C** → **C′**, S : **C′** → **C** two covariant functors. We will define two new covariant functors $\overline{T}$, $\overline{S}$: $\mathbf{C}^0 \times \mathbf{C}'$ → **Set** in the following way: if (X, X′) ∈ $\mathbf{C}^0 \times \mathbf{C}'$, then $\overline{T}$(X, X′) = **C′**(T(X), X′) and $\overline{S}$(X, X′) = **C**(X, S(X′)); (f, f ′) : (X, X′) → (Y, Y′) is a morphism in $\mathbf{C}^0 \times \mathbf{C}'$, then we define $\overline{T}$(f, f ′) : **C′**(T(X), X′) → **C′**(T(Y), Y′) by $\overline{T}$(f, f′)(α) = f′∘α∘T(f) for every α∈ **C′** (T(X), X′) and $\overline{S}$(f, f′) : **C**(X, S(X′)) → **C**(Y, S(Y′) by $\overline{S}$(f, f′)(α) = S(f′)∘α∘f, for every α ∈ **C**(X, S(X′)).

Lemma 4.4.5. $\overline{T}$, $\overline{S}$ **:** $\mathbf{C}^0 \times \mathbf{C}' \to$ **Set are covariant functors.**

Proof. We will only prove for $\overline{T}$ (for $\overline{S}$ will be analogous). We have $\overline{T}(1_{(X,X')}) = 1_{\overline{T}(X,X')} \Leftrightarrow \overline{T}(1_X, 1_{X'}) = 1_{\overline{T}(X,X')} \Leftrightarrow \overline{T}(1_X, 1_{X'})(a) = a$ for every α ∈ **C′**(T(X), X′) ⇔ $1_{X'} \circ \alpha \circ T(1_X) = \alpha \Leftrightarrow 1_{X'} \circ \alpha \circ 1_{T(X)} = \alpha$ which is clear.

Now let (f, f′) : (X, X′) → (Y, Y′) and (g, g′) : (Y, Y′) → (Z, Z′) be two morphisms in $\mathbf{C}^0 \times \mathbf{C}'$ (so, we have $Z \xrightarrow{g} Y \xrightarrow{f} X$ morphisms in **C** and $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z'$ morphisms in **C′**).

Then (g, g′)∘(f, f′) (in $\mathbf{C}^0 \times \mathbf{C}'$) = (g∘f (in $\mathbf{C}^0$), g′∘f′ (in **C′**)) = (f∘g (in **C**), g′∘f′(in **C′**)) = (f∘g, g′∘f′)(in **C**×**C′**) so, to prove $\overline{T}((g,g') \circ (f,f')) = \overline{T}(g,g') \circ \overline{T}(f,f') \Leftrightarrow \overline{T}(f \circ g, g' \circ f')(a) = \overline{T}(g,g')(\overline{T}(f,f')(a))$, for every α ∈ $\overline{T}$(X, X′) = **C′**(T(X), X′) ⇔ g′∘f ′∘α∘T(f∘g) = T(g, g′)(f ′∘α∘T(f)) ⇔ g′∘f ′∘ α∘T(f∘g) = g′∘ (f′∘α∘T(f))∘T(g) ⇔ g′ ∘f ′∘ α∘ T(f∘g) = g′∘f ′∘ α∘ T(f)∘T(g) which is clear (since T is a covariant functor, hence T(f∘g) = T(f)∘T(g)). ∎

Definition 4.4.6. Let T : C $\rightarrow$ C′ and S : C′ $\rightarrow$ C be two covariant functors. We say that T is a *left adjoint* of *S* (or that S is a *right adjoint* of T) if $\overline{T} \approx \overline{S}$ (i.e, there is a functorial isomorphism $\psi : \overline{T} \rightarrow \overline{S}$).

Now let $\psi : \overline{T} \rightarrow \overline{S}$ a functorial morphism of components ψ(X, X′): $\overline{T}$(X, X′) = **C**′(T(X), X′) → (X, X′) = **C**(X, S(X′)) with (X, X′)∈ $\mathbf{C}^{\mathbf{0}} \times \mathbf{C}'$ and we denote $\psi_X = \psi(X, T(X))\,(1_{T(X)}) : X \rightarrow (ST)(X)$.

Lemma 4.4.7. Relative to the above notations and hypothesis, the morphisms $(\psi_X)_{X \in C}$ are the components of the functorial morphism $\overline{\psi}$: $1_C \rightarrow$ ST. The assignment $\psi \rightarrow \overline{\psi}$ is a bijection between the functorial morphisms from $\overline{T}$ to $\overline{S}$ and functorial morphisms from 1_C to ST (e.g. from $(\overline{T}, \overline{S})$ to (1_C, ST), if we consider the notations from the above paragraph).

Proof. ([70]). To prove that $\overline{\psi}$ is a functorial morphism, it should be proved that for every X, Y ∈ **C** and f ∈ **C**(X, Y), the diagram

$$\begin{array}{ccc} X & \xrightarrow{\psi_X} & (ST)(X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle (ST)(f)} \\ Y & \xrightarrow{\psi_Y} & ST(Y) \end{array}$$

is commutative, that is, $(ST)(f) \circ \psi_X = \psi_Y \circ f$. (1)

Indeed, by hypothesis the following diagram:

$$\begin{array}{ccc} C'(T(X), T(X)) = \overline{T}(X, T(X)) & \xrightarrow{\psi(X,\, T(X))} & C(X, ST(X)) = \overline{S}(X, ST(X)) \\ {\scriptstyle \overline{T}(1_X,\ T(f))}\downarrow & & \downarrow{\scriptstyle \overline{S}(1_X,\ T(f))} \\ C'(T(X), T(Y)) = \overline{T}(X, T(Y)) & \xrightarrow{\psi(X,\, T(Y))} & C(X, ST(Y)) = \overline{S}(X, ST(Y)) \end{array}$$

is commutative hence $(\overline{S}(1_X, T(f)) \circ \psi(X, T(X)))(1_{T(X)}) =$
$= (\psi(X, T(Y)) \circ \overline{T}(1_X, T(f)))(1_{T(X)}) \Leftrightarrow$
$\overline{S}(1_X, T(f))(\psi(X, T(X))(1_{T(X)})) = \psi(X, T(Y))(\overline{T}(1_X, T(f))(1_{T(X)})) \Leftrightarrow$
$\overline{S}(1_X, T(f))(\psi_X) = \psi(X,T(Y)(T(f) \circ 1_{T(X)} \circ T(1_X)) \Leftrightarrow$
$S(T(f)) \circ \psi_X \circ 1_X = \psi(X,T(Y)(T(f)) \Leftrightarrow$
$(ST)(f) \circ \psi_X = \psi(X, T(Y)(T(f))$ (2)

Also, the diagram

$$\begin{array}{ccc} C'(T(Y), T(Y)) = \overline{T}(Y, T(Y)) & \xrightarrow{\psi(Y, T(Y))} & C(Y, ST(Y)) = \overline{S}(Y, ST(Y)) \\ \downarrow{\scriptstyle \overline{T}(f, 1_{T(Y)})} & & \downarrow{\scriptstyle \overline{S}(f, 1_{T(Y)})} \\ C'(T(X), T(Y)) = \overline{T}(X, T(Y)) & \xrightarrow{\psi(X, T(Y))} & C(X, ST(Y)) = \overline{S}(X, ST(Y)) \end{array}$$

is commutative, hence
$(\overline{S}(f, 1_{T(Y)}) \circ \psi(Y, T(Y)))(1_{T(Y)}) = (\psi(X, T(Y)) \circ \overline{T}(f, 1_{T(Y)}))(1_{T(Y)}) \Leftrightarrow$
$(\overline{S}(f, 1_{T(Y)})(\psi(Y, T(Y))(1_{T(Y)})) = (\psi(X, T(Y))(T(f, 1_{T(Y)})(1_{T(Y)})) \Leftrightarrow$
$(\overline{S}(f, 1_{T(Y)})(\psi_Y) = (\psi(X, T(Y))(1_{T(Y)} T(f)) \Leftrightarrow$
$S(1_{T(Y)}) \circ \psi_Y \circ f = (\psi(X, T(Y))(T(f)) \Leftrightarrow \psi_Y \circ f = (\psi(X, T(Y))(T(f))$ (3)

From (2) and (3) we deduce (1), hence $\overline{y}$ is a functorial morphism from $1_{\mathbf{C}}$ to ST.

Let $\alpha : (\overline{T}, \overline{S}) \to (1_C, ST)$, $a(y) = \overline{y}$ for every $y \in (\overline{T}, \overline{S})$.

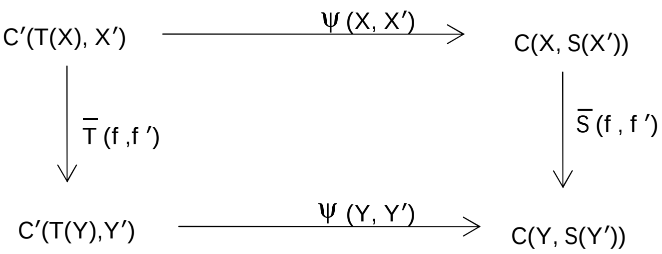
To prove that α is bijective, we will construct $\beta : (1_{\mathbf{C}}, ST) \to (\overline{T}, \overline{S})$ which will be the inverse of α.

So let $\overline{y} \in (1_C, ST)$ of components $(\overline{y_X})_{X \in C}$ with $\overline{y}_X : X \to (ST)(X)$, for every $X \in \mathbf{C}$.

For every $(X, X') \in \mathbf{C}^0 \times \mathbf{C}'$ we consider the mapping $y(X, X') : \overline{T}(X, X') \to \overline{S}(X, X')$ defined by $\psi(X, X')(\alpha) = S(a) \circ \overline{y}_X$ for every $\alpha \in \mathbf{C}'(T(X), X')$.

Lemma 4.4.8. The functions $(y\,(X, X'))_{(X, X')\in C^0\times C'}$ **are the components of a functorial morphism** $y : \overline{T} \to \overline{S}$ **.**

Proof. ([70]).It should be proved that for every morphism (f, f′) : (X, X′) → (Y, Y′) from $\mathbf{C^0} \times \mathbf{C'}$, the diagram



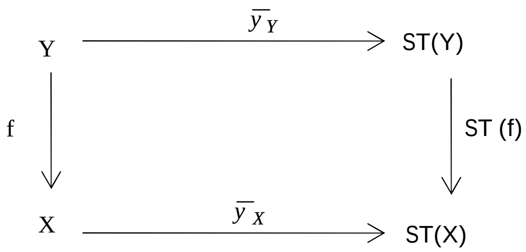
is commutative.

Indeed, if α ∈ **C′**(T(X), X′), then

($\overline{S}$ (f, f′)∘ ψ(X, X′))(α) = $\overline{S}$ (f, f′)(ψ(X, X′)(α)) = S(f′)∘ψ(X, X′)(α) ∘f = =S(f′) ∘ S(α) ∘$\overline{y}_X$ ∘f (4)

and (ψ(Y, Y′)∘$\overline{T}$ (f, f′))(α) = ψ(Y, Y′)($\overline{T}$ (f, f′)(α)) =
= ψ(Y, Y′)(f′∘α∘T(f)) = S(f′∘α∘T(f))∘ $\overline{y}_Y$ = S(f′)∘S(α)∘(ST)(f)∘ $\overline{y}_Y$.
(5)

Since the diagram



is commutative, we deduce that (ST)(f) ∘ $\overline{y}_Y$ = $\overline{y}_X$ ∘f. (6)

From (6), (4) and (5) we deduce that the diagram from the start of the proof is commutative, hence $y : \overline{T} \to \overline{S}$ is a functorial morphism.

We define $\beta : (1_{\mathbf{C}}, ST) \to (\bar{T}, \bar{S})$ with the help of the above lemma by $\beta(\bar{y}) = y$, for every $\bar{y} \in (1_{\mathbf{C}}, ST)$. ∎

Lemma 4.4.9. The functions α and β defined above are one the converse of the other (that is, $\alpha \circ \beta = 1_{(1_C, ST)}$ and $\beta \circ \alpha = 1_{(\bar{T}, \bar{S})}$).

Proof. ([70]). Let $\psi \in (\bar{T}, \bar{S})$; then $(\beta \circ \alpha)(\psi) = \beta(\alpha(\psi))$ and to prove that $\beta(\alpha(\psi)) = \psi$ is equivalent with $(\beta(\alpha(\psi)))(X, X') = \psi(X, X')$, for every $(X, X') \in \mathbf{C}^{0} \times \mathbf{C}'$.

We have that $(\beta(\alpha(\psi)))(X, X') : \mathbf{C}'(T(X), X') \to \mathbf{C}(X, S(X'))$ is defined such that $\beta(\alpha(\psi)))(X, X')(f) = S(f) \circ \psi(X, T(X))(1_{T(X)})$.

By the commutativity of the diagram

$$\begin{array}{ccc} C'(T(X), T(X)) & \xrightarrow{\psi(X, T(X))} & C(X, (ST)(X)) \\ \Big\downarrow{\scriptstyle \bar{T}(1_X, f)} & & \Big\downarrow{\scriptstyle \bar{S}(1_X, f)} \\ C'(T(X), X') & \xrightarrow{\psi(X, X')} & C(X, S(X')) \end{array}$$

we deduce that:

$(\bar{S}(1_X, f) \circ \psi(X, T(X)))(1_{T(X)}) = (\psi(X, X') \circ \bar{T}(1_X, f))(1_{T(X)}) \Leftrightarrow$

$\bar{S}(1_X, f)(\psi(X, T(X))(1_{T(X)})) = \psi(X, X')(\bar{T}(1_X, f)(1_{T(X)})) \Leftrightarrow$

$S(f) \circ \psi(X, T(X))(1_{T(X)}) \circ 1_X = \psi(X, X')(f \circ 1_{T(X)} \circ T(1_X)) \Leftrightarrow$

$S(f) \circ \psi(X, T(X))(1_{T(X)}) = \psi(X, X')(f) \Leftrightarrow \beta(\alpha(\psi))(X, X')(f) = \psi(X, X')(f)$,

so, we deduce that $\beta(\alpha(\psi)) = \psi$, hence $\beta \circ \alpha = 1_{(\bar{T}, \bar{S})}$.

Now let $\varphi \in (1_C, ST)$. For $X \in \mathbf{C}$ we have $((\alpha\, \beta)(\varphi))_X = (\alpha(\beta(\varphi)))_X =$ $(\varphi)(X, T(X))(1_{T(X)}) = \varphi(X, T(X))\,(1_{T(X)}) = S(1_{T(X)}) \circ \varphi_X = 1_{ST(X)} \circ \varphi_X = \varphi_X$, hence $(\alpha\, \beta)(\varphi) = \varphi$, that is, $\alpha \circ \beta = 1_{(1_C, ST)}$. ∎

Remark 4.4.10. Dually, if $T : \mathbf{C} \to \mathbf{C}'$ and $S : \mathbf{C}' \to \mathbf{C}$ are two covariant functors, then to every functorial morphism $\varphi : \bar{S} \to \bar{T}$ of components $\varphi(X, X')$: $\bar{S}(X, X') \to \bar{T}(X, X')$, with $(X, X') \in \mathbf{C}^{0} \times \mathbf{C}$ we obtain a family of morphisms $(\bar{\varphi}_{X'})_{X' \in C'}$ where $\bar{\varphi}_{X'} = \varphi(S(X'), X')(1_{S(X')})$,

$\bar{\varphi}_{X'}$:(TS)(X′) → X′ and the assignment X′ → $\bar{\varphi}_{X'}$, X′ ∈ **C′**, we define a functorial morphism $\bar{\varphi}$: TS → $1_{\mathbf{C}'}$. The assignment φ → $\bar{\varphi}$ is a bijection from $(\bar{S}, \bar{T})$ to (TS, $1_{\mathbf{C}'}$), its opposite assign to every functorial morphism $\bar{\varphi}$ ∈ (TS, $1_{\mathbf{C}}$) of components $(\bar{\varphi}_{X'})_{X' \in C'}$ the functorial morphism $\varphi : \bar{S} \to \bar{T}$ of components φ(X, X′): $\bar{S}(X, X') \to \bar{T}(X, X')$, φ(X, X′)(f) = $\bar{\varphi}_{X'}$ ∘T(f), for every f ∈ **C**(X, S(X′)).

Definition 4.4.11. Let $\psi : \bar{T} \to \bar{S}$ and $\varphi : \bar{S} \to \bar{T}$ be two functorial morphisms and $\bar{\psi} : 1_C \to ST$, $\bar{\varphi} : TS \to 1_C$ the functorial morphisms corresponding to the above lemmas.

If T is the left adjoint of S and φ is the converse isomorphism of ψ, we say that $\bar{\psi}$ and $\bar{\varphi}$ are the *adjoint arrows* (one *quasiconverse* for another).

Let S : **C′** → **C** be a covariant functor. For every X ∈ **C** we denote by X / (**C′**, S) (respective (**C′**, S) / X) the category whose objects are pairs (f, X′) (respective (X′, f)) with f ∈ **C**(X, S(X′)) (respective f ∈ **C**(S(X′), X)).

A morphism α : (f, X′) → (g,Y′) (respective α : (X′, f) → (Y′, g)) is by definition a morphism α : X′ → Y′ from **C′** such that S(α)∘f = g (respective g ∘ S(α) = f).

Proposition 4.4.12. If S : C′ → C is a covariant functor, then the following assertions are equivalent:

(i) There is a covariant functor T : C → C′ left adjoint for S;

(ii) For every X ∈ C, the functor h^X S : C′ → Set is representable;

(iii) For every X ∈ C the category X / (C′, S) has an initial object.

Proof. ([70]). (i) ⇒ (ii). Since T is the left adjoint for S, there is a functorial isomorphism $\psi : \bar{T} \to \bar{S}$, so, for every X ∈ **C** we have a functorial isomorphism in Y, ψ_Y = ψ(X, Y) : **C′**(T(X), Y) → **C**(X, S(Y)) ⇔ ψ_Y : $h^{T(X)}$(Y) → (h^XS) (Y), that is, F = h^XS is representable with (T(X), a) as pair of representation (where a = ψ(X, T(X)) ($1_{T(X)}$) ∈ **C**(X, (ST)(X)) = (h^XS) (T(X)) = F(T(X))).

(ii) ⇒ (iii). Suppose that for every X∈**C** the functor F = h^XS is representable and let (X′, a) a pair of representation with X′ ∈ **C′** and a ∈ F(X′) = (h^XS) (X′) = h^X(S(X′)) = **C**(X, S(X′)).

We have the functorial isomorphism $\beta^a : h^X \to h^X S$, that is, for every $Y' \in \mathbf{C}'$ we have a bijection $\beta^a(Y') : \mathbf{C}'(X', Y') \to \mathbf{C}(X, S(Y'))$ (with functorial properties).

Then (f, X'), with $f = \beta^a(X')(1_{X'}) \in \mathbf{C}(X, S(X'))$ is an initial object in the category $X / (\mathbf{C}', S)$.

Indeed, if (g, Y') is another object in $X / (\mathbf{C}', S)$, then $g \in \mathbf{C}(X, S(Y'))$ so, there is a unique $\alpha \in \mathbf{C}'(X', Y')$ such that $\beta^a(Y')(\alpha) = g$. We have to prove that α is a morphism in $X / (\mathbf{C}', S)$.

Indeed, from the commutative diagram

$$\begin{array}{ccc} C'(X', X') & \xrightarrow{\beta^a(X')} & C(X, S(X')) \\ {\scriptstyle h^{X'}(\alpha)}\downarrow & & \downarrow{\scriptstyle (h^X S)(\alpha)} \\ C'(X', Y') & \xrightarrow{\beta^a(Y')} & C(X, S(Y')) \end{array}$$

we deduce that:

$((h^X S)(\alpha) \circ \beta^a(X'))(1_{X'}) = (\beta^a(Y') \circ h^{X'}(\alpha))(1_{X'}) \Leftrightarrow$

$\Leftrightarrow (h^X S)(\alpha)(\beta^a(X')(1_{X'})) = \beta^a(Y')(h^{X'}(\alpha)(1_{X'})) \Leftrightarrow$

$\Leftrightarrow (h^X S)(\alpha)(f) = \beta^a(Y')(\alpha) \Leftrightarrow h^X(S(\alpha))(f) = g \Leftrightarrow S(\alpha) \circ f = g$, hence $\alpha : (f, X') \to (g, Y')$ is a morphism in $X / (\mathbf{C}', S)$.

(iii) $\Rightarrow$ (i). For every $X \in \mathbf{C}$, we denote by $(i_X, T(X))$ an initial object in the category $X / (\mathbf{C}', S)$ (with $T(X) \in \mathbf{C}'$ and $i_X \in \mathbf{C}(X, (ST)(X))$.

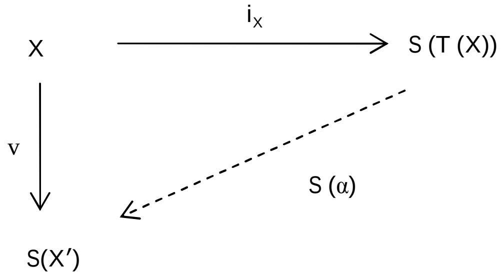
If we have $X, Y \in \mathbf{C}$ and $u \in \mathbf{C}(X, Y)$, and if we define $T(u) : T(X) \to T(Y)$ as the unique morphism with the property that the diagram

$$\begin{array}{ccc} X & \xrightarrow{i_X} & S(T(X)) \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle S(T(u))} \\ Y & \xrightarrow{i_Y} & S(T(Y)) \end{array}$$

is commutative, it is easy to see that the assignments X $\to$ T(X) and u $\to$ T(u) define a covariant functor T : **C** $\to$ **C**′.

To prove that T is a left adjoint of S, we should prove that there is a functorial isomorphism $j : \overline{S} \to \overline{T}$.

For this, if (X, X′) $\in$ $\mathbf{C}^{\mathbf{0}} \times \mathbf{C}'$ we define φ(X, X′): $\overline{S}(X, X') = C(X, S(X')) \to \overline{T}(X, X') = C'(T(X), X')$ in the following way: For v$\in$ **C**(X, S(X′)), φ(X, X′)(v) is the unique morphism α$\in$ **C**′(T(X), X′) such that the diagram



is commutative.

There results that φ(X, X′) is an injective function and since for every β $\in$ **C**′(T(X), X′), φ(X, X′)(S(β) $\circ$ i_X) = β we deduce that φ(X, X′) is surjective function, that is, φ(X, X′) is a bijective function.

Since it is easy to see that φ is a functorial morphism, the proof of this proposition is complete. ∎

The dual result is:

Proposition 4.4.13. Let T : C → C′ a covariant functor. The following assertions are equivalent:

(i) There is a right adjoint functor S : C′ → C for T;

(ii) For every X′ ∈ C′, the functor $h_{X'}T$ is corepresentable;

(iii) For every X′ ∈ C′, the category (C, T)/X′ has a final object.

Remark 4.4.14. The left (right) adjoint for a functor, if there is, is unique up to a functorial isomorphism.

Indeed, let S : **C**′$\to$ **C** be a covariant functor and T,T′ : **C** $\to$ **C**′ two left adjoints for S. By Proposition 4.4.12, for every X $\in$ **C**, the functor $h^X S$ is representable, hence there exist the functorial isomorphisms α : $h^{T(X)} \to h^X S$ and β : $h^{T'(X)} \to h^X S$. We deduce the existence of a functorial isomorphism $\alpha^{-1} \circ \beta : h^{T'(X)} \to h^{T(X)}$ which implies the existence of an isomorphism $\gamma(X) : T'(X) \to T(X)$ in **C′** such that $h^{\gamma(X)} = \alpha^{-1} \circ \beta$ (this is possible because for every $Y \in$ **C′**, h^Y is faithful and full).

Since $\alpha^{-1} \circ \beta$ is a functorial morphism, we deduce that the family of morphisms $(\gamma(X))_{X \in \mathbf{C}}$ are the components of a functorial isomorphism $\gamma : T' \to T$. Analogously we prove the dual result.

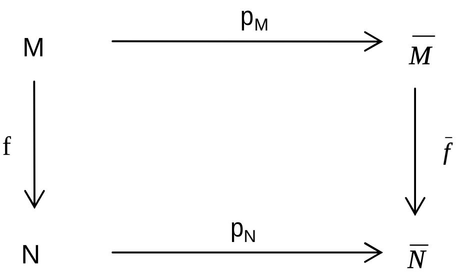
Examples

1. The inclusion functor i : **Ord** → **Pre** (see Chapter 2) has a left adjoint j : **Pre** → **Ord**.

Indeed, let (M, ≤) ∈ **Pre**. On M we consider the relation R : xRy ⇔ x ≤ y and y ≤ x; it is immediate to see that R is an equivalence relation on M compatible with ≤ (i.e, x R x′, y R y′ and x ≤ y imply x′ ≤ y′).

Let $\overline{M} = M / R$ be the quotient set equipped with *preorder quotient* (i.e, for $\bar{x}, \bar{y} \in \overline{M}$, $\bar{x} \le \bar{y}$ ⇔ x ≤ y) and $p_M : M \to \overline{M}$ the canonical isotone surjective function. Let N be an ordered set and g : M → N an isotone function. If R(g) is the equivalence relation on M associate with g (that is, xR(g)y ⇔ g(x) = g(y)), then R ≤ R(g), hence there is a unique isotone function $\bar{g} : \overline{M} \to N$ such that $\bar{g} \circ p = g$.

It is immediate that if f : M → N is an isotone function, then there is a unique isotone function $\bar{f} : \overline{M} \to \overline{N}$ such that the diagram



is commutative.

From the above property of uniqueness we deduce that the assignments $M \to \overline{M}$ and $f \to \bar{f}$ define a covariant functor j : **Pre** → **Ord**.

This, by Proposition 4.4.12, is the left adjoint functor for i (since from the above we deduce that for every M∈**Pre**, the object $(p_M, \overline{M})$ is the initial in the category M / (**Ord**, i)).

2.The subjacent functor S : **Top** → **Set** has a left adjoint functor D : **Set** → **Top** and a right adjoint functor G : **Set** → **Top**, defined in the following way:

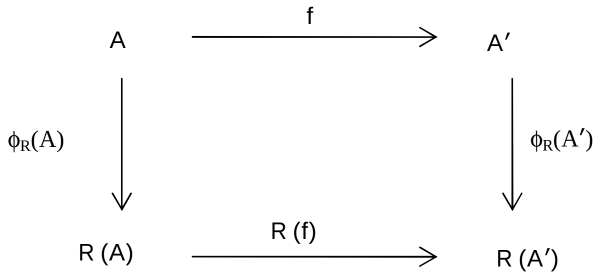
The functor D is the functor *discrete topology* which assigns to every set X the discrete topological space (X, P(X)) and to every function the same function (which is clearly a continuous function relative to discrete topologies).

The functor G is the functor *rough topology* which assign to every set X the rough topological space (X, {∅, X}) and to every function the same function which is clearly a continuous function relative to rough topologies.

4.5. Reflectors. Reflective subcategories

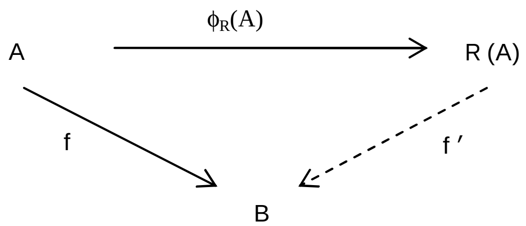
Definition 4.5.1. A subcategory $\mathcal{C}'$ of a category $\mathcal{C}$ is called *reflective* if there is a covariant functor $R : \mathcal{C} \to \mathcal{C}'$, called *reflector* such that for every $A \in \mathcal{C}$ there is a morphism $\phi_R(A) : A \to R(A)$ in $\mathcal{C}'$ with the properties:

(i) If $f \in \mathcal{C}(A, A')$, then the diagram



is commutative, that is, $\phi_R(A') \circ f = R(f) \circ \phi_R(A)$;

(ii) If $B \in \mathcal{C}'$ and $f \in \mathcal{C}(A, B)$, then there is a unique morphism $f' \in \mathcal{C}'(R(A), B)$ such that the diagram



is commutative (i.e, $f' \circ \phi_R(A) = f$).

Remark 4.5.2.

(i). In some books the reflectors are called *reflefunctors.*

(ii). Let $\mathbf{C}' \subseteq \mathbf{C}$ a subcategory of the category **C.** Then $\mathbf{C}'$ is a reflective subcategory of **C** iff there exists a function which assigns to $A \in \mathbf{C}$ an object $R(A) \in \mathbf{C}'$ and a function which assigns to every $A \in \mathbf{C}$ a morphism $\phi_R(A) : A \to R(A)$ of **C** such that for every $B \in \mathbf{C}'$ and $f \in \mathbf{C}(A, B)$ there is a unique morphism $f' \in \mathbf{C}'(R(A), B)$ such that $f' \circ \phi_R(A) = f$.

Indeed, the implication from left to right is immediate.

For another implication, we extend the above assignment from Ob(**C**) to Ob($\mathbf{C}'$) to a functor $R : \mathbf{C} \to \mathbf{C}'$.

For $f \in \mathbf{C}(A, A')$, we define $R(f) \in \mathbf{C}'(R(A), R(B))$ to be the unique morphism in $\mathbf{C}'$ for which $R(f)\circ \phi_R(A) = \phi_R(A')\circ f$. Then (i) and (ii) from Definition 4.5.1 are satisfied and it remains to show that R is a functor. Indeed, if $f\in \mathbf{C}(A, A')$, $g\in \mathbf{C}(A', A'')$, then $R(f) \circ \phi_R(A) = \phi_R(A')\circ f$ and $R(g)\circ\phi_R(A') = \phi_R(A'') \circ g$, so $R(g) \circ R(f) \circ \phi_R(A) = R(g)\circ \phi_R(A') \circ f = \phi_R(A'') \circ g\circ f$ and by uniqueness we deduce that $R(g) \circ R(f) = R(g\circ f)$.

For $1_A \in \mathbf{C}(A, A)$, we deduce that $R(1_A)\circ \phi_R(A) = \phi_R(A) \circ 1_A = \phi_R(A)$, so by uniqueness $1_{R(A)} = R(1_A)$.

(iii). If $R : \mathbf{C} \to \mathbf{C}'$, $S : \mathbf{C}' \to \mathbf{C}''$ are two reflectors, then $SR : \mathbf{C} \to \mathbf{C}''$ is a reflector.

Indeed, we check that the conditions of (ii) are satisfied. For $A \in \mathbf{C}$ let $\phi_{SR}(A) = \phi_S(\phi_R(A))\circ \phi_R(A)$.

If $C \in \mathbf{C}''$ and $f \in \mathbf{C}(A, C)$, then there exists a unique $f' \in \mathbf{C}'(R(A), C)$ such that $f'\circ\phi_R(A) = f$ and a unique $f'' \in \mathbf{C}''((SR)(A), C)$ such that $f''\circ\phi_S(R(A))=f'$. It easily follows that $f'' \circ \phi_{SR}(A) = f$. For uniqueness, let $g \in \mathbf{C}''((SR)(A),C)$ such that $g \circ \phi_{SR}(A) = f$; then $g \circ\phi_S(R(A))\circ \phi_R(A) = f$, hence $f' = g\circ\phi_S(R(A))$ and then $g = f''$.

(iv). If $\mathbf{C}'$ is a full subcategory of **C,** then to say that $R : \mathbf{C} \to \mathbf{C}'$ is reflector is equivalent with to say that R is a left adjoint for the inclusion functor from $\mathbf{C}'$ to **C**.

Lemma 4.5.3. Every reflector $R : \mathbf{C} \to \mathbf{C}'$ preserves epimorphisms.

Proof. Suppose that $f \in \mathbf{C}(A, A')$ is an epimorphism in **C** and let $g, h \in \mathbf{C}'(R(A'), B)$ such that $g \circ R(f) = h \circ R(f)$. Then $g \circ \phi_R(A') = g \circ R(f) \circ \phi_R(A) = h \circ R(f) \circ \phi_R(A) = h \circ \phi_R(A') \circ f$; since f is epimorphism in **C** we deduce that $g \circ \phi_R(A') = h \circ \phi_R(A')$, hence g = h, that is, R(f) is an epimorphism. ∎

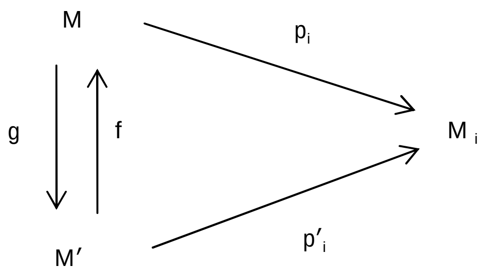
4.6. Products and coproducts of a family of objects

Let **C** be a category and $F = (M_i)_{i \in I}$ be a non-empty family of objects in **C**.

Definition 4.6.1. We call *direct product* of the family F a pair $(M, (p_i)_{i \in I})$ with $M \in \mathbf{C}$ and $p_i \in \mathbf{C}(M, M_i)$, for every $i \in I$ such that for every other pair $(M', (p'_i)_{i \in I})$ with $p'_i \in \mathbf{C}(M', M_i)$, $i \in I$, there is a unique $f \in \mathbf{C}(M', M)$ such that $p'_i = p_i \circ f$, for every $i \in I$.

Remark 4.6.2. In the case of existence, the direct product of a family F is unique up to an isomorphism.

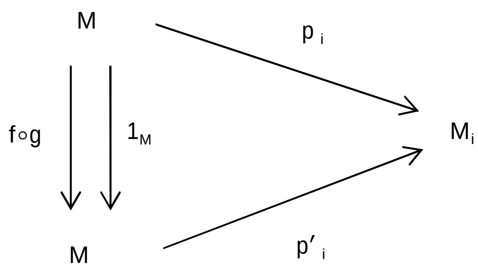
Indeed, suppose that we have two direct products $(M, (p_i)_{i \in I})$ and $(M', (p'_i)_{i \in I})$ for F. If we consider the diagram



then there is a unique $f \in \mathbf{C}(M', M)$ and a unique $g \in \mathbf{C}(M, M')$ such that $p'_i = p_i \circ f$ and $p_i = p'_i \circ g$, for every $i \in I$.

Then $p_i \circ (f \circ g) = p_i$ and $p'_i \circ (g \circ f) = p'_i$, for every $i \in I$.

If we consider now the diagram



from the uniqueness in the direct product definition we deduce that f∘g = 1_M.

Analogously we deduce that g∘f = $1_{M'}$, hence M ≈ M′.

The direct product of a family F if exists, will be denoted by $\prod_{i\in I} M_i$ and $p_j : \prod_{i\in I} M_i \to M_j$ will be called the *j-th canonical projection.*

Lemma 4.6.3. Let $\prod_{i\in I} M_i$ = (M, $(p_i)_{i\in I}$) a direct product of the family F. Then, for every i Î I the i-th projection p_i has a section (hence is epimorphism) Û C(M_i, M_j) ¹ Æ, for every j Î I.

Proof. Suppose that for every j ∈ I, **C**(M_i, M_j) ≠ ∅ and choose f_{ij} ∈ **C**(M_i, M_j) such that $f_{ii} = 1_{M_i}$. There is a unique morphism $f_i : M_i \to \prod_{j\in I} M_j$ such that $p_j \circ f_i = f_{ij}$, for every j ∈ I. In particular, for i = j we have $p_i \circ f_i = f_{ii} = 1_{M_i}$, hence p_i has a section, so is an epimorphism.

Conversely, if p_i has a retraction and s_i is a right inverse of p_i, then for every j ∈ I, $p_j \circ s_i$ ∈ **C**(M_i, M_j), hence **C**(M_i, M_j) ≠ ∅, for every j ∈ I. **n**

Corollary 4.6.4. If C is a category with a nullary object O, then the canonical projections of a direct product in C are epimorphisms with sections.

Proof. In the above lema it is suffice to consider for every j ∈ I, $f_{ij} = O_{M_i M_j}$ and $f_{ii} = 1_{M_i}$. **n**

Definition 4.6.5. We say that a category C is with *products*, if each family of objects in C has a direct product.

Examples

1. The category Set is a category with products (see §5 from Chapter 1).

2.Every equational category is a category with products (see Chapter 3). More general, every category of algebras of the same type τ is a category with products (see §3 from Chapter 3).

3. Gr is a category with products.

Indeed, if $F = (G_i)_{i\in I}$ is a family of groups, then if we consider in **Set** $\prod_{i\in I} G_i = (G, (p_i)_{i\in I})$ and if we define for two elements f, g $\in$ G, $f = (f_i)_{i\in I}$, $g = (g_i)_{i\in I}$, with $f_i, g_i \in G_i$ for every $i \in I$, $f\circ g = (f_i \circ g_i)_{i\in I}$, it is easy to see that relative to this multiplication G become a group and every projection p_i is a morphisms of groups. Then $(G, (p_i)_{i\in I}) = \prod_{i\in I} G_i$ in the category **Gr**.

4. The category **Fd** of fields is not a category with products (so, **Fd** is not an equational class).

Indeed, if K and K′ are two fields with different characteristics, then it is easy to see that it doesn't exist K Π K′ in **Fd** (since if between two fields K and K′ there is a morphism of fields, then K and K′ have the same characteristic).

The dual notions of the direct product is the notion of *direct coproducts*(also called *direct sum*). In fact we have the following definition:

Definition 4.6.6. We call *coproduct* in the category C for a family $F = (M_i)_{i\in I}$ of objects in C, a pair $((\alpha_i)_{i\in I}, M)$ where $M \in C$ and $\alpha_i \in C(M_i, M)$, for every $i \in I$ such that for every pair $((\alpha'_i)_{i\in I}, M')$ with $M' \in C$ and $\alpha'_i \in C(M_i, M')$, $i \in I$, there is a unique $f \in C(M, M')$ such that $f \circ \alpha_i = \alpha'_i$ for every $i \in I$.

Remark 4.6.7. As in the case of direct product, it is immediate to see that if a coproduct exists for a family F, then it is unique up to an isomorphism.

We denote the coproduct of the family F by $\coprod_{i\in I} M_i$.

For every $j \in I$, $\alpha_j : M_j \to \coprod_{i\in I} M_i$ will be called the *j-th canonical injection*.

From Lemma 4.6.3 and Corollary 4.6.4 we obtain dual results for the coproduct:

Lemma 4.6.8. For every $j \in I$, α_j is a morphism with retract (hence is monomorphism) iff $C(M_i, M_j) \neq \emptyset$, for every $i \in I$.

Corollary 4.6.9. If C has a nullary object, then the canonical injections of every coproduct in C are monomorphisms with retraction.

Definition 4.6.10. We say that a category C is with *coproducts* if each family of objects of C has a coproduct.

Examples

1. Set is a category with coproducts (see §5 from Chapter 1).

2. Let's see what is the situation of coproducts in an equational category **K**. For $A\in$**K** and $S\subseteq A$ we denote by [S] the subalgebra of A generated by S (see Chapter 3).

Proposition 4.6.11.([2]). Let C be an equational category, $(A_i)_{i\in I}$ a family of algebras in K and $(\alpha_i : A_i \to A)_{i\in I}$ a family of morphisms such that if $(f_i : A_i \to B)_{i\in I}$ $(B \in K)$ is another family of morphisms in K, then there is $f \in K(A, B)$ such that $f \circ \alpha_i = f_i$, for every $i \in I$.

Then $(A, (\alpha_i)_{i\in I}) = \coprod_{i\in I} A_i$ iff $\bigcup_{i\in I} \alpha_i(A_i)$ is a generating set for A (i.e, $[\bigcup_{i\in I} \alpha_i(A_i)] = A$).

Proof. ([2]). "⇐". We only have to prove the uniqueness of f. This follows from Lemma 3.1.11.

"⇒". Let $A' = [\bigcup_{i\in I} \alpha_i(A_i)]$. For every $i\in I$ we define $f'_i : A_i \to A'$ by $f'_i(x) = \alpha_i(x)$, for $x \in A_i$ and since **K** is an equational category we deduce that $f'_i \in$ **K**(A_i, A') for every $i \in I$. By hypothesis there is $f \in$ **K**(A, A') such that $f'_i = f \circ \alpha_i$ and $1_{A',A} \circ f'_i = \alpha_i$ for every $i \in I$. Since $1_A \circ \alpha_i = \alpha_i$ for every $i \in I$ we deduce that $1_{A',A} \circ f = 1_A$, hence $1_{A',A}$ is onto, so $A' = A$. ∎

Remark 4.6.12. In Chapter 3 we have defined the notion of free algebra over a class of algebras.

Now let's have a generalization of this notion:

Definition 4.6.13. An algebra A from an equational category K is called *free for K over a set S* if [S] = A and there is a function i : S → A such that for every function f : S → B (with B ∈ K) there is a unique g ∈ K(A, B) such that g∘i = f.

Clearly, if $S \subseteq A$ is non-empty and if we consider $i = 1_{S,A}$ we obtain the notion defined in Chapter 3.

The set S is called a set of *free generators*. We denote $A =$ $\mathbf{F_K}(S)$ (see Chapter 3).

Corollary 4.6.14. Let K be a nontrivial equational category and S be a non-empty set. If $\coprod_{s\in S} F_K(\{s\}) = (A, (\alpha_s)_{s\in S})$ with A ∈ K and $\alpha_s \in$ K($F_K(\{s\})$, A) for s ∈ S, then A ≈ $F_K(S)$.

Proof. Let $f : S \to A$, $f(s) = \alpha_s(s)$ for every $s \in S$, $B \in$ **K** and $g : S \to B$ a mapping. Then for every $s \in S$ there is a unique $g_s \in$ **K**$(\mathbf{F_K}(\{s\}), B)$ such that $g_s(s) = g(s)$ for every $s \in S$. There is a unique $h \in$ **K**(A, B) such that $h \circ \alpha_s = g_s$ and thus $h \circ f = g$. By the uniqueness of h and Proposition 6.11, we deduce that $A \approx \mathbf{F_K}(S)$. ∎

In the book [69,p.107] , it is proved the following result:

Proposition 4.6.15. Let K be an equational category and $(A_i)_{i\in I}$ a family of algebras in K. If every algebra A_i is a subalgebra of an algebra B_i ∈ K and for i ≠ j there is α_{ij} ∈ K(A_i, B_j), then there exists $\coprod_{i\in I} A_i$.

3. Coproducts in the categories **Mon** and **Gr** ([74]).

Let M be a set. The existence of the free monoid (group) generated by M is assured by Theorem 6.16 in Chapter 2 from [74]. Next we will have a description of those.

If $M^* = \coprod_{n\geq 0} M^n$ (in **Set**), then the elements of M^* are pairs (f, n) with $n \in \mathbf{N}$ and $f = (x_1, \ldots, x_n) \in M^n$.

If we denote by ( ) the empty sequence (of length 0), then $M^0 = \{(\ ), 0\}$.

On M^* we consider an operation of composition (by juxtaposition) in the following way: if $x = ((x_1, \ldots, x_n), n)$ and $x' = ((x'_1, \ldots, x'_{n'}), n') \in M^*$, then $xx' = ((x_1, \ldots, x_n, x'_1, \ldots, x'_{n'}), n+n') \in M^*$.

It is immediate to see that in this way M^* becomes a monoid (where the neutral element is the empty sequence $e_{M^*} = ((\ ), 0)$ and $i_M : M \to M^*$, $i_M(x) = ((x), 1)$ is an injective morphism of monoids. Since for every monoid M' and every function $f : M \to M'$, $\overline{f} : M^* \to M'$, $\overline{f}(((\ ),0)) = e_{M'}$ and $\overline{f}(((x_1,\ldots,x_n),n)) = f(x_1)\cdot\ldots\cdot f(x_n)$ (for $n \geq 1$) is the unique morphism of monoids with the property that $\overline{f} \circ i_M = f$ we deduce that M^* is the *free monoid generated by M* (i.e, $M^* = \mathbf{F_{Mon}}(M)$).

Let now $(M_i)_{i\in I}$ be a non-empty family of monoids, $M = \coprod_{i\in I} M_i$ (in **Set**) with canonical injections $\alpha_i : M_i \to M$ ($i \in I$) and M^* the free monoid generated by M (before described). The elements of M^* are pairs $((a_1, \ldots, a_n), n)$ with $(a_1, \ldots, a_n) \in M^n$, hence $a_j = (x_j, i_j)$ with $x_j \in M_{ij}$ and $i_j \in I$.

Let θ_M be the congruence of M^* generated by the elements $((x_j, i_j)\,(y_j, i_j),\ (x_j y_j, i_j))$, $((e_j, i_j), (\ ))$, with $x_j, y_j \in M_{ij}$ and e_j the neutral element of M_j ($j, i_j \in I$).

If by $p_{q_M} : M^* \to M^*/\theta_M$ we denote the canonical onto morphism of monoids and $\overline{a_i} = p_{q_M} \circ a_i$ ($i \in I$), then $(M^* / \theta_M, (\overline{a_i})_{i\in I}) = \coprod_{i\in I} M_i$ in **Mon**.

Following the above result and since every equational category is with products we obtain:

Proposition 4.6.16. The category Mon is a category with products and coproducts.

Now we consider the problem of coproducts in the category **Gr (**[74, p.130]).

Firstly, we will give a characterization for the *free group generated* by a set M. We denote by M′ an isomorphic image of M such that $M \cap M' = \emptyset$ (for $x \in M$ we denote by x′ the image of x by the above fixed isomorphism).

On a free monoid $(M \amalg M')^*$ (where $M \amalg M'$ is the coproduct of M with M′ in **Set**) we consider the congruence ρ_M generated by the elements ((x) (x′), ( )) and ((x′) (x), ( )) with $x \in M$. I suggest the reader to prove that the quotient monoid $(M \amalg M')^* / \rho_M$ is really the free group generated by the set M.

If $(G_i)_{i\in I}$ is a non-empty family of groups, then we have the same description of $\coprod_{i\in I} G_i$ in **Gr** as in the case of **Mon**.

So, we have:

Proposition 4.6.17. Gr is a category with products and coproducts.

4. The category **Fd** of fields is not a category with coproducts.

Indeed, if K, K′ are two fields with different characteristics, then it doesn't exist $K \amalg K'$ in **Fd** (the same argument as in the case of product).

If **C** is a category with products and coproducts then for every $M \in$ **C** and $I \neq \emptyset$ we denote $M^I = \prod_{i\in I} M_i$ and $M^{(I)} = \coprod_{i\in I} M_i$, where $M_i = M$, for every $i \in I$.

Remark 4.6.18.

The canonical injections (projections) of a coproduct (product) are not in the general monomorphisms (epimorphisms).

5. The category **Pre** is a category with products and coproducts

Indeed, let $((X_i, \leq))_{i\in I}$ be a family of elements in **Pre**, $(X', (p_i)_{i\in I}) = \prod_{i\in I} X_i$ and $((\alpha_i)_{i\in I}, X'') = \coprod_{i\in I} X_i$ in **Set.**

For $x, y \in X'$, $x = (x_i)_{i\in I}$, $y=(y_i)_{i\in I}$ we define $x \leq' y \Leftrightarrow x_i \leq y_i$, for every $i \in I$ and for $(x, i), (y, j) \in X''$ we define $(x, i) \leq'' (y, j) \Leftrightarrow i = j$ and $x \leq y$ in X_i.

Then

1) Relative to the order $\leq'$ the projections are isotone mappings and $((X', \leq'), (p_i)_{i\in I}) = \prod_{i\in I}(X_i, \leq)$ in **Pre**.

2) Relative to the order $\leq''$ the canonical injections are isotone and $((\alpha_i)_{i\in I}, (X'', \leq'')) = \coprod_{i\in I}(X_i, \leq)$ in **Pre**.

Analogously we prove that **Ord** is a category with products and coproducts.

For existence and characterization of coproducts in categories of lattices see Chapter 7 from [2].

6. Let $(G_i)_{i\in I}$ be a family of abelian aditive groups and $G = \prod_{i\in I} G_i$. We consider the subgroup G' of G whose elements are the elements $(x_i)_{i\in I}$ with the components equal with 0 excepting a finite numbers of them and for every $i\in I$, $\alpha_i:G_i \to G'$ $\alpha_i(x_i) = (y_i)_{i\in I}$, where $y_i = x_i$ and $y_j = 0$ for $j \neq i$.

Then for every $i \in I$, α_i is a morphism of groups and $((\alpha_i)_{i\in I}, G') = \coprod_{i\in I} G_i$ in **Ab**.

Remark 4.6.19. It is possible that $\coprod_{i\in I} G_i$ in **Gr** to be different from $\coprod_{i\in I} G_i$ in **Ab.**

7. In the category of cycle groups the product of the groups $\mathbf{Z_2}$ and $\mathbf{Z_3}$ does not exist.

8. In the category of abelian finite groups the product and coproduct of the family of aditive groups $(\mathbf{Z_n})_{n\in N}$ do not exist.

9. Let $(X_i, \tau_i)_{i\in I}$ be a family of topological spaces, $(X, (p_i)_{i\in I}) = \prod_{i\in I} X_i$ and $((\alpha_i)_{i\in I}, X') = \coprod_{i\in I} X_i$ in **Set**.

If we equip the set X with at finest topology τ such that all projections p_i are continuous functions and X' with the most fine topology τ' such that all injections α_i are continuous functions, then $((X, \tau), (p_i)_{i\in I}) = \prod_{i\in I}(X_i, t_i)$ and $((\alpha_i)_{i\in I}, (X', \tau')) = \coprod_{i\in I}(X_i, t_i)$ in **Top**.

Let $(X_i)_{i\in I}$ and $(X'_i)_{i\in I}$ be two families of objects in a category **C** with products and $(f_i)_{i\in I}$ a family of morphisms in **C** with $f_i \in$ **C**(X_i, X'_i), for every $i \in I$.

If we denote $\prod_{i\in I} X_i = (X, (p_i)_{i\in i})$ and $\prod_{i\in I} X'_i = (X', (p'_i)_{i\in i})$ following the universality property of product, there is a unique $f \in$ **C**(X, X') such that $f_i \circ p_i = p'_i \circ f$, for every $i \in I$, and if f_i is a monomorphism in **C**, then f is also a monomorphism in **C**.

We denote $f = \prod_{i\in I} f_i$ and we call f the *product* of the family of morphisms $(f_i)_{i\in I}$.

Let $(X_i)_{i\in I}$ and $(X'_i)_{i\in I}$ be two families of objects in a category **C** with coproducts and $(f_i)_{i\in I}$ a family of morphisms in **C** with $f_i \in$ **C**(X_i, X'_i), for every $i \in I$.

If $\coprod_{i\in I} X_i = ((\alpha_i)_{i\in I}, X)$ and $\coprod_{i\in I} X'_i = ((\alpha'_i)_{i\in I}, X')$, following the property of universality of coproduct, there is a unique $f \in$ **C**(X, X') such that $f \circ \alpha_i = \alpha'_i \circ f_i$, for every $i \in I$; if for every $i \in I$, f_i is an epimorphism in **C**, then f is also an epimorphism in **C**.

We denote $f = \coprod_{i\in I} f_i$ and we call f the *coproduct* of the family $(f_i)_{i\in I}$ of morphisms.

4.7. Limits and colimits for a partially ordered system

Let $(I, \leq)$ be a *directed* set (i.e, for every $i, j \in I$, there is $k \in I$, such that $i, j \leq k$), and **C** a category.

Definition 4.7.1. We call *inductive system* of objects in C with respect to directed index set I a pair $\mathfrak{I} = ((A_i)_{i\in I}, (\varphi_{ij})_{i,j\in I})$ with $(A_i)_{i\in I}$ a family of objects of C and $(\varphi_{ij})_{i,j\in I}$ a family of morphisms $\varphi_{ij} \in$ C(A_i, A_j), with $i \leq j$, such that

(i) $\varphi_{ii} = 1_{A_i}$ **, for every** $i \in I$**;**

(ii) If $i \leq j \leq k$**, then** $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$**.**

If there is no danger of confusion, the above inductive system $\Im$ will be denoted by $\Im$ **= (A_i, j $_{ij}$).**

Definition 4.7.2. Let $\Im$ = (A_i, j $_{ij}$) be an inductiv system of objests in C relative to a directed index set I.

A pair (A, $(e_i)_{i\in I}$) with A Î C and $(e_i)_{i\in I}$ a family of morphisms, with e_i Î C(A_i, A) for every i Î I, is called *inductive limit* of the inductive system $\Im$ = (A_i, j $_{ij}$), if:

(i) For every i £ j we have $e_j \circ$ j $_{ij}$ = e_i;

(ii) For every B Î C and every family $(f_i)_{i\in I}$ of morphisms with f_i Î C(A_i, B) for every i Î I such that $f_j \circ$ j $_{ij}$ = f_i for every i £ j, there is a unique morphism f Î C(A, B) such that f $\circ$ e_i = f_i, for every i Î I.

We will say that a category C is a *category with inductive limits* if every inductive system in C has an inductive limit.

Remark 4.7.3. As in the case of products or coproducts it is immediate to see that if (A, $(\varepsilon_i)_{i\in I}$) and (A′, $(\varepsilon'_i)_{i\in I}$) are two inductive limits for inductive system $\Im$ = (A_i, φ_{ij}), then there is a unique isomorphism f $\in$ C(A, A′) such that f $\circ$ $\varepsilon_i = \varepsilon'_i$, for every i $\in$ I.

If (A, $(\varepsilon_i)_{i\in I}$) is the inductive limit of inductive system $\Im$**,** we denote A= $\varinjlim_{i\in I} A_i$.

Examples

1. The category Set is a category with inductive limits.

Indeed, let $\Im$ = (A_i, φ_{ij}) be an inductive system of sets and $((a_i)_{i\in I}, \overline{A}) = \coprod_{i\in I} A_i$ in **Set;** then $\overline{A} = \bigcup_{i\in I} \overline{A_i}$, where $\overline{A_i} = A_i \times \{i\}$, for every i$\in$ I (see §8 from Chapter 1).

On a set $\overline{A}$ we consider the binary relation ρ: (x, i) ρ (y, j) ⇔ there is k $\in$ I such that i ≤ k, j ≤ k and $\varphi_{ik}(x) = \varphi_{jk}(y)$.

We have to prove that ρ is an equivalence on $\overline{A}$. Since the reflexivity and the symmetry of ρ are clear, to prove the transitivity of ρ, let (x, i), (y, j), (z, k) elements in $\overline{A}$ such that (x, i) ρ (y, j) and (y, j) ρ (z, k), hence there exist t, s ∈ I such that i, j ≤ t, j, k ≤ s, $\varphi_{it}(x) = \varphi_{jt}(y)$ and $\varphi_{js}(y) = \varphi_{ks}(z)$. We find r ∈ I such that t ≤ r, s ≤ r and since $\varphi_{ir}(x)= (\varphi_{tr} \circ \varphi_{it})(x) = \varphi_{tr}(\varphi_{it}(x)) = \varphi_{tr}(\varphi_{jt}(y)) = (\varphi_{tr} \circ \varphi_{jt})(y) = \varphi_{jr}(y) = (\varphi_{sr} \circ \varphi_{js})(y)= \varphi_{sr}(\varphi_{ks}(z)) = (\varphi_{sr} \circ \varphi_{ks})(z) = \varphi_{kr}(z)$ we deduce that (x, i) ρ (z, k), hence ρ is transitive, that is, an equivalence on $\overline{A}$.

Let $A = \overline{A}/r$, $p:\overline{A} \to A = \overline{A}/r$ be a canonical surjective function and for every i ∈ I, $\varepsilon_i = p \circ \alpha_i$, where $\alpha_i : A_i \to \overline{A}$ is the i-th canonical injection of coproduct in **Set**.

We have to prove then $\varinjlim_{i\in I} A_i = (A, (\varepsilon_i)_{i\in I})$.

Indeed, if i, j ∈ I, i ≤ j, $\varepsilon_j \circ \varphi_{ij} = \varepsilon_i \Leftrightarrow \varepsilon_j(\varphi_{ij}(x))= \varepsilon_i(x)$, for every x ∈ $A_i \Leftrightarrow p(\alpha_j(\varphi_{ij}(x))) = 0 \Leftrightarrow p((\varphi_{ij}(x), j)) = p((x, i))$ which is clear since if we choose k = j then i, j ≤ k, $\varphi_{ik}(x) = \varphi_{ij}(x)$ and $\varphi_{jk}(\varphi_{ij}(x)) =\varphi_{jj}(\varphi_{ij}(x)) = 1_{A_j}(j_{ij}(x)) = \varphi_{ij}(x)$, hence $\varphi_{ik}(x) = \varphi_{jk}(\varphi_{ij}(x))$.

Now let B be another set and $(f_i)_{i\in I}$ a family of functions with $f_i : A_i \to B$, for every i ∈ I and $f_j \circ \varphi_{ij} = f_i$, for every i ≤ j. Following the property of universality of coproduct, there is a unique function $g:\overline{A} = \coprod_{i\in I} A_i \to B$ such that $g \circ \alpha_i = f_i$, for every i ∈ I.

If (x, i), (y, i) ∈ $\overline{A}$ such that (x, i) ρ (y, j), then there is k ∈ I such that i, j ≤ k and $\varphi_{ik}(x) = \varphi_{jk}(y) \Rightarrow f_k(\varphi_{ik}(x)) = f_k(\varphi_{jk}(y)) \Rightarrow (f_k \circ \varphi_{ik})(x) = (f_k \circ \varphi_{jk})(y) \Rightarrow f_i(x)= f_j(y) \Rightarrow g((x, i)) = g((y, j))$, so we deduce that f : A → B, f((x, i) / ρ) = g(x, i) is correctly defined and it is immediate to verify that f is the unique function defined on A with values in B with the property that $f \circ \varepsilon_i = f_i$, for every i ∈ I, so the proof is complete. ∎

2. We have to prove, more general, that if $\mathcal{E}$ is an equational category and $\mathfrak{I} = (A_i, \varphi_{ij})$, i, j ∈ I, i ≤ j is an inductive system in $\mathcal{E}$, then in $\mathcal{E}$ exists $\varinjlim_{i\in I} A_i$.

Indeed, suppose that $\coprod_{i\in I} A_i = (A, (\alpha_i)_{i\in I})$ with $\alpha_i \in \mathcal{E}(A_i, A)$ (i∈I) and let $q = \Theta X$ the congruence on A generated by X = {$\alpha_i(x), \alpha_j(\varphi_{ij}(x))$: i, j ∈ I, i ≤ j and x ∈ A_i} (see Chapter 3). Since $\mathcal{E}$ is an equational category, B =

$A / \theta \in \mathcal{E}$ and $\overline{\alpha_i} = \pi_\theta \circ \alpha_i : A_i \to B$ is a family of morphisms in $\mathcal{E}$ (with $i \in I$ and $\pi_\theta : A \to B$ the canonical surjective function).

We have to prove that $(B, (\overline{\alpha_i})_{i\in I}) = \varinjlim_{i\in I} A_i$.

Firstley, we remark that for every $i, j \in I$, $i \le j$ and $x \in A_i$, following the definition of θ we have that $(\alpha_i(x), \alpha_j(\varphi_{ij}(x))) \in \theta$, so we deduce that $\pi_\theta(\alpha_i(x)) = \pi_\theta(\alpha_j(\varphi_{ij}(x))$, hence $\overline{\alpha_i} = \overline{\alpha_j} \circ \varphi_{ij}$.

Now let $B' \in \mathcal{E}$ and for every $i \in I$, $\alpha'_i \in \mathcal{E}(A_i, B')$ such that $\alpha'_j \circ \varphi_{ij} = \alpha'_i$ for $i \le j$. Since $A = \mathop{\mathbf{C}}_{i\in I} A_i$, there is an unique $u \in \mathcal{E}(A, B')$ such that $u \circ \alpha_i = \alpha'_i$ for every $i \in I$.

Since for every $i, j \in I$ with $i \le j$ and $x \in A_i$ we have $u(\alpha'_j(\varphi_{ij}(x))) = \alpha'_i(x) = u(\alpha_i(x))$, then we deduce that $((\alpha'_j(\varphi_{ij}(x)), \alpha_i(x)) \in$ **Ker**(u).

Since $\theta \subseteq$ **Ker**(u), there is a unique $v \in \mathcal{E}(B, B')$ such that $v \circ \pi_\theta = u$. Then for every $i \in I$, $v \circ \overline{\alpha_i} = v \circ (\pi_\theta \circ \alpha_i) = u \circ \alpha_i = \alpha'_i$.

To prove the unicity of v with the property that $v \circ \overline{\alpha_i} = \alpha'_i$ for every $i \in I$, let $w \in \mathcal{E}(B, B')$ such that $w \circ \overline{\alpha_i} = \alpha'_i$, for every $i \in I$.

From the uniqueness of u, we deduce that $u = w \circ \pi_\theta$, so, for $x \in A$, $v(x/\theta) = v(\pi_\theta(x)) = (v \circ \pi_\theta)(x) = u(x) = (w \circ \pi_\theta)(x) = w(x/\theta)$, that is, $w = v$.

■

Definition 4.7.3. Let $(I, \le)$ be a directed set . By *projective system* of objects in C we understand a pair $\wp = ((A_i)_{i\in I}, (\varphi_{ij})_{i,j\in I})$ with $(A_i)_{i\in I}$ a family of objects in C and $\varphi_{ij} \in C(A_j, A_i)$ for $i, j \in I$, $i \le j$ such that:

(i) $\varphi_{ii} = 1_{A_i}$, for every $i \in I$;

(ii) If $i \le j \le k$, then $\varphi_{ik} = \varphi_{ij} \circ \varphi_{jk}$.

If there is no danger of confusion, we denote the above projective system by $\wp = (A_i, \varphi_{ij})$.

Definition 4.7.4. Let $\wp = (A_i, \varphi_{ij})$ be a projective system in C.

A pair $(A, (q_i)_{i\in I})$, with $A \in C$ and $q_i \in C(A, A_i)$ is called *projective limit* of projective system $\wp$ if:

(i) j $_{ij}$○ **q_j = q_i for every i £ j;**

(ii) If (A′, (q'_i)$_{i\in I}$), is another pair with A¢ Î C and q¢$_i$ Î C(A′, A_i) with the property that for every i, j Î I with i £ j, j $_{ij}$○ **q'_j =q'_i , then there is a unique f Î C(A¢, A) such that q_i ∘ f = q¢$_i$ for every i Î I .**

As in the case of inductive limit of an inductive system, it is easy to see that the projective limit of a projective system, if exists,it is unique up to an isomorphism.

If (A, $(q_i)_{i\in I}$) is the projective limit of the projective system ℘ = (A_i, φ_{ij}), we denote A = $\underset{\overleftarrow{i\in I}}{\lim} A_i$.

We will say that a category C is a *category with projective limits* if every projective system in C has a projective limit.

Examples

1. In the category **Set**, let (A_i, φ_{ij}) be a projective system of sets, $\prod_{i\in I} A_i = (B, (p_i)_{i\in I})$ and suppose that A={a∈ B:($\varphi_{ij}\circ p_j$)(a)=p_i(a) for every i ≤ j} ≠ ∅.

If for every i ∈ I we denote by q_i the restriction of p_i to A, then it is immediate to prove that $\underset{\overleftarrow{i\in I}}{\lim} A_i$ = (A, $(q_i)_{i\in I}$)**.**

2. More generally, if **A** is an equational category, ℘= ($(A_i)_{i\in I}$, $(\varphi_{ij})_{i\le j}$) a projective system in **A** and A = {x ∈ $\underset{i\in I}{\times} A_i$: $p_i(x) = \varphi_{ij}(p_j(x))$ for i ≤ j} ≠ ∅, then $\underset{\overleftarrow{i\in I}}{\lim} A_i$ = (A, $(p_{i\,|A})_{i\in I}$) **.**

3. Following what we establish in the end of §6, the category **Top** is a category with products and coproducts.

Now let (X_i, τ_i)$_{i\in I}$ be a family of topological spaces and ℘ = (X_i, φ_{ij}) a projective system. If we consider (X, $(p_i)_{i\in I}$) = $\prod_{i\in I} X_i$ (in **Top**) and

Y = {y ∈ X : $\varphi_{ji}(p_i(y)) = p_j(y)$ for every i, j ∈ I with i ≤ j} then if we denote for every i∈ I, $p'_I = p_{i|Y}$ then it is immediate to see that (Y,$(p'_i)_{i\in I}$)= $\underset{\overleftarrow{i\in I}}{\lim} (X_i, t_i)$ (in **Top**).

4. The equational categories with nullary operations are categories with projective limits.

Remark 4.7.5. In the particular case when (I, ≤) is a chain then the inductive (projective) limit of an inductive (projective) where for every i ∈ I, $\varphi_{ij} = 1_{A_i}$, coincides with $\coprod_{i\in I} A_i$ (respective $\prod_{i\in I} A_i$).

So,the products and coproducts are particular cases of inductive (projective) limits.

Definition 4.7.6. Let $\Im = (A_i, \varphi_{ij})$ and $\Im' = (A'_i, \varphi'_{ij})$ be two inductive systems over the ordered set I (directed to right).

We call *inductive system of morphisms* from $\Im$ to $\Im'$ a family $(f_i)_{i\in I}$ of morphisms with $f_i \in$ C(A_i, A'_i) for every i ∈ I such that for every i, j ∈ I with i ≤ j, $f_j \circ \varphi_{ij} = \varphi'_{ij} \circ f_i$.

Remark 4.7.7. In the hypothesis of Definition 4.7.6, following the universality property of inductive limit, it is immediate to see that if we denote $\varinjlim_{i\in I} A_i$ = (A, $(\varepsilon_i)_{i\in I}$) and $\varinjlim_{i\in I} A'_i$ = (A′, $(\varepsilon'_i)_{i\in I}$), then there is a unique morphism f ∈ **C**(A, A′) such that $f \circ \varepsilon_i = \varepsilon'_i \circ f_i$, for every i ∈ I.

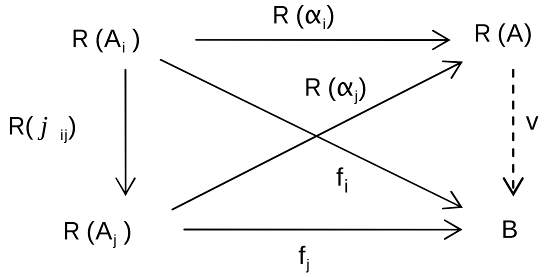
The morphism f will be called the *inductive limit* of the inductive system of morphisms $(f_i)_{i\in I}$ and we denote $f = \varinjlim_{i\in I} f_i$ (we have analogous notion for projective limits). Analogously for the case of projective limits.

Theorem 4.7.8. Every reflector preserve inductive limits (hence the coproducts).

Proof. Let **C′** ⊆ **C** be a reflexive subcategory of **C** and R:**C**→ **C′** be a reflector. Consider $\Im = (A_i, \varphi_{ij})$ an inductive system in **C** and $\varinjlim_{i\in I} A_i$ **=** (A, $(\alpha_i)_{i\in I}$), where (I, ≤) is an ordered set directed to right.

To prove that (R(A), $(R(\alpha_i))_{i\in I}$) = $\varinjlim_{i\in I} R(A_i)$ we remark that $R(\varphi_{ii}) = R(1_{A_i}) = 1_{R(A_i)}$ and for i ≤ j, $R(\alpha_j)\,R(\varphi_{ij}) = R(\alpha_j \circ \varphi_{ij}) = R(\alpha_i)$, hence (R(A), $R(\varphi_{ij})$) is an inductive system in **C′**.

Now let $(f_i : R(A_i) \to B)_{i\in I}$ be a family of morphisms in **C′** such that $f_j \circ R(\varphi_{ij}) = f_i$, for every $i \le j$. We should prove the existence of a unique $v \in$ **C′**$(R(A), B)$ such that $v \circ R(\alpha_i) = f_i$, for every $i \in I$.



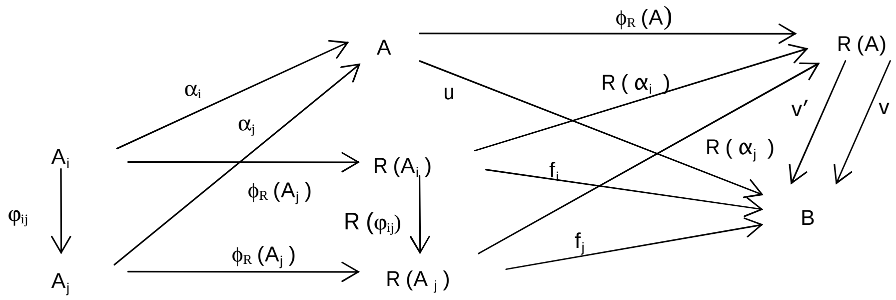
Thus, for every $i \le j$, $f_j \circ \phi_R(A_j) \circ \varphi_{ij} = f_j \circ R(\varphi_{ij}) \circ \phi_R(A_i) = f_i \circ \phi_R(A_i)$.

Since $A = \varinjlim_{i\in I} A_i$ we deduce the existence of a unique $u \in$ **C**(A, B) such that $u \circ \alpha_j = f_j \circ \phi_R(A_j)$, for every $j \in I$.

Then there is a unique $v \in$ **C′**$(R(A), B)$ such that $v \circ \phi_R(A) = u$.

We have $v \circ R(\alpha_i) \circ \phi_R(A_i) = v \circ \phi_R(A) \circ \alpha_i = u \circ \alpha_i = f_i \circ \phi_R(A_i)$, hence $v \circ R(\alpha_i) = f_i$, for every $i \in I$.

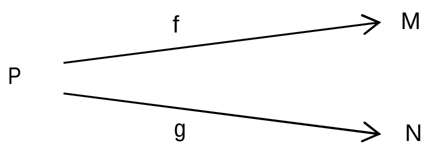
For the uniqueness of v, suppose that we have again $v' \in$ **C′**$(R(A), B)$ such that $v' \circ R(\alpha_i) = f_i$, for every $i \in I$.



Then $v' \circ R(\alpha_i) \circ \phi_R(A_i) = f_i \circ \phi_R(A_i)$, so $v' \circ \phi_R(A) \circ \alpha_i = f_i \circ \phi_R(A_i)$ and by the uniqueness of u we deduce that $v' \circ \phi_R(A) = u$. By the uniqueness from Definition 4.5.1 we deduce that $v = v'$. **n**

4.8. Fibred coproducts (poshout) and fibred product (pullback) of two objects

In the category **C** we consider the diagram



Definition 4.8.1. We call *fibred coproduct* of M with N over P a triple (i_M, i_N, L), where L ∈ C, i_M ∈ C(M, L), i_N ∈ C(N, L) such that:

(i) $i_M \circ f = i_N \circ g$;

(ii) If (i'_M, i'_N, L′) is another triple, with L′ ∈ C, i'_M ∈ C(M, L′), i'_N, ∈ C(N, L′) which verifies (i), then there is a unique u ∈ C(L, L′) such that $u \circ i_M = i'_M$ and $u \circ i_N = i'_N$.

Remark 4.8.2. For P ∈ **C** we define a new category P / **C** (respective **C** / P) in the following way:

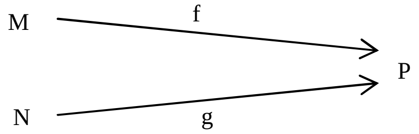
Ob(P / **C**) = {(P, f, X) : X ∈ **C** and f ∈ **C**(P, X)} (respective **Ob**(**C** / P) ={(X, f, P) : X ∈ **C** and f ∈ **C**(X, P)}).

For two objects (P, f, X), (P, g, Y) in P / **C**, we define a morphism α : (P, f, X) → (P, g, Y) as the morphism $\alpha \in$ **C**(X, Y) with the property that $\alpha \circ f = g$. The composition of morphism will be canonical and it is easy to see that in this way we obtain a category P / **C** (dual for **C** / P).

We remark that the fibred coproduct (i_M, i_N, L) of M with N over P above defined is really the coproduct of (P, f, M) and (P, g, N) in the category P / **C**. So, we deduce that if fibred coproduct of M with N over P exists, then it is unique up to an isomorphism.

We denote (i_M, i_N, L) = $M \underset{P}{\mathbf{C}} N$.

The dual notion of fibred coproduct is the notion of *fibred product.* More precisely consider the following diagram in **C:**



Definition 4.8.3. We call *fibred product* of M with N over P a triple (K, p_M, p_N), with $K \in$ C, $p_M \in$ C(K, M), $p_N \in$ C(K, N) such that:

(i) $f \circ p_M = g \circ p_N$;

(ii) If (K', p'_M, p'_N) is another triple, with $K' \in$ C, $p'_M \in$ C(K', M), $p'_N \in$ C(K', N) such that $f \circ p'_M = g \circ p'_N$, then there is a unique $u \in$ C(K', K) such that $p_M \circ u = p'_M$ and $p_N \circ u = p'_N$.

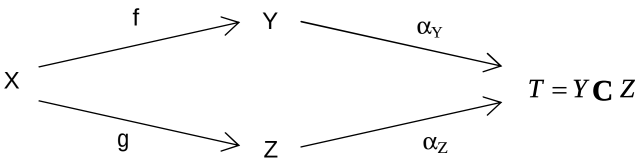
Dually we deduce that (K, p_M, p_N), the fibred product of M with N over P, is really the product of objects (M, f, P) and (N, g, P) in the category **C** / P and so, if the fibred product of M with N over P exists, then it is uniquely determined up to an isomorphism.

We denote $(K, p_M, p_N) = M\prod_P N$.

Examples

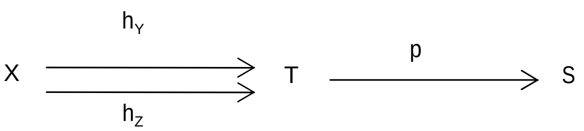
1. The category Set is a category with fibred coproducts and products.

Indeed, we consider the diagram:



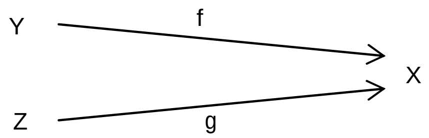
where α_Y, α_Z are the canonical injections of the coproduct.

If we consider $h_Y = \alpha_Y \circ f$ and $h_Z = \alpha_Z \circ g$, let (S, p) = **Coker** (h_Y, h_Z) (see Theorem 1.4.5).



Following the universality property of cokernel of a pair of morphisms, we deduce that if we consider $i_Z = p \circ \alpha_Z$ and $i_Y = p \circ \alpha_Y$, then $(i_Y, i_Z, S) = Y \underset{X}{\mathbf{C}} Z$.

For the existence of fibred product in **Set** we consider the diagram



$K = \{(y, z) \in Y \times Z : f(y) = g(z)\}$, $p_Y : K \to Y$ and $p_Z : K \to Z$ the restrictions of cartesian product $Y \times Z$ to K. It is immediate to see that $(K, p_Y, p_Z) = Y \prod_X Z$.

2. The category Top is a category with fibred coproducts and products

From the above remark **Set** is a category with fibred coproducts and products. Preserve the notations from Example 1 and consider in **Set** $T = Y \underset{X}{\mathbf{C}} Z$ with the topology of coproduct (that is, the less fine topology on T for which α_Y and α_Z are continuous mappings) and $S = Y \prod_X Z$ with quotient topology (since S is T factorized by the equivalence relation $\overline{r}$ generated by $\rho = \{(h_Y(x), h_Z(x)) : x \in X\}$); then the functions i_Y and i_Z are continuous and we continue from here as in the case of **Set**.

For the existence of fibred product in **Top** we do as in the case of **Set**, equipping $K = \{(y, z) \in Y \times Z : f(y) = g(z)\}$ with the restriction of product topology from $Y \times Z$ to K (where p_Y and p_Z are continuous functions).

3. The category Ab is a category with fibred coproducts and products.

Let $G, G', G'' \in$ **Ab**, $f \in$ **Ab**(G, G''), $g \in$ **Ab**(G', G''), $K = \{(x, x') : f(x) = g(x')\}$ and $\overline{p}_G, \overline{p}_{G'}$ the restrictions to K of the projections p_G, $p_{G'}$ of G Π G′ on G, respective G′. Then (K, $\bar{p}_G, \bar{p}_{G'}$) = $G\prod_{G''} G'$ in **Ab**.

For the case of the fibred coproduct let G, G′, G″ ∈ **Ab**, f ∈ **Ab**(G″, G′), g ∈ **Ab**(G″, G′) and $(\alpha_G, \alpha_{G'}, G \coprod G')$ a coproduct of G and G′ in **Ab**.

If we denote H = {α_G (f(x)) - $\alpha_{G'}$ (g(x)) : x ∈ G″}, then $H \le G \coprod G'$ and $(G\coprod G'/H, p \circ \alpha_G, p \circ \alpha_{G'}) = G \coprod_{G''} G'$ in **Ab**, where $p : G \coprod G' \to G\coprod G'/H$ is the canonical surjective function.

4. The category Gr is a category with fibred coproducts and products.

The fibred product in **Gr** is as in the case of **Ab**.

Now let G, G′, G″ ∈ **Gr**, f ∈ **Gr**(G″, G), g ∈ **Gr**(G″, G′), $(\alpha_G, \alpha_{G'}, G \coprod G')$ a coproduct of G and G′ in **Gr**, H = {α_G (f(x)) · ($\alpha_{G'}$ (g(x)))$^{-1}$: x ∈ G″} and N(H) the normal subgroup of $G \coprod G'$ generated by H.

If we denote $K = (G\coprod G')/N(H)$ and $p : G\coprod G' \to K$ is the canonical onto morphism of groups, then $(K, p \circ \alpha_G, p \circ \alpha_{G'}) = G \coprod_{G''} G'$ in **Gr**.

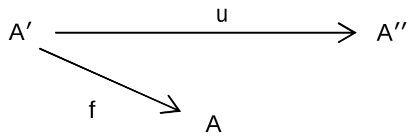
Remark 4.8.4. In general, in a category **C**, the notions of inductive (projective) limits of an inductive (projective) system, coproduct (product), fibred coproduct (product) and kernel (cokernel) of a pair of morphisms appear in the theory of categories in a unitar context as inductive and projective limits of some functors F : **I** → **C** where **I** is a small category.

This particular ,case of inductive limits or projective ones of some particular functors, are suffices for what we need now (in this case I have abandoned this point of view).

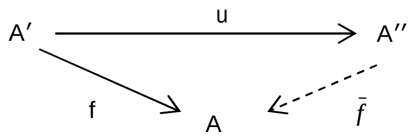
I recommend the reader to study the book [70].

4.9. Injective (projective) objects. Injective (projective) envelopes

Definition 4.9.1. Let C be a category. An object A Î C is called *injective* in C if for every diagram in C



with u a monomorphism, there is a morphism $\bar{f}$: A′′ ® **A such that the diagram**



is commutative (i.e, $\bar{f}$ **ou = f**).

We say that a category C is with *enough injectives* (or a category with *injective embedding*) if for any object A Î C there is an injective object B and a monomorphism u : A ® B (that is, A is subobject of an injective object).

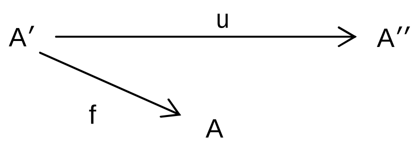
Examples

1. Every final object is injective.
2. In **Set** every non-empty set is injective.
3. In **Top**, every rough topological space is injective.

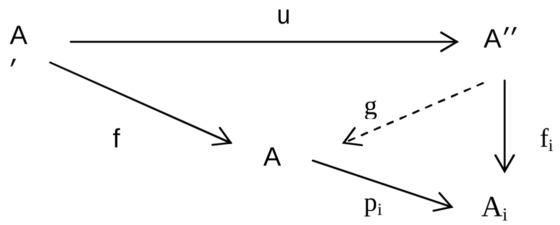
Proposition 4.9.2. Let $(A_i)_{i\in I}$ **be a family of injective objects in C for which their product** $(A, (p_i)_{i\in I}) = \prod_{i\in I} A_i$ **exists.**

Then A is also an injective object.

Proof. Consider in **C** the following diagram:



with u a monomorphism and for every $i \in I$ the diagram:



For every $i \in I$ there is $f_i \in$ **C**(A'', A_i) such that $f_i \circ u = p_i \circ f$.

By the universality property of product we deduce the existence of a unique morphism $g \in$ **C**(A'', A) such that $p_i \circ g = f_i$, for every $i \in I$.

If $v = g \circ u$, we have $p_i \circ v = p_i \circ (g \circ u) = (p_i \circ g) \circ u = f_i \circ u = p_i \circ f$, for every $i \in I$, so, by the uniqueness property of product we obtain that $v = f$, hence A is injective object in **C**. **n**

Remark 4.9.3. (i). In some categories, as for example the categories with nullary objects, the converse of Proposition 4.9.2 is true.

(ii). Every monomorphism with the domain injective object has a retraction.

(iii). Let R : **C** ® **C′** be a reflector which preserves the monomorphisms. If B is an injective object in **C′,** then B is also an injective object in **C.**

Indeed, if we suppose that $f : A \to C$ is a monomorphism in **C** and $g \in$ **C**(A, B), then there is $h \in$ **C′**$(R(A), B)$ such that $h \circ \phi_R(A) = g$. Since $R(f)$ is a monomorphism in **C′** and B is injective in **C′**, there is $k \in$ **C′**$(R(C), B)$ such that $k \circ R(f) = h$. If we consider the morphism $k \circ \phi_R(C) : C \to B$ then $k \circ \phi_R(C) \circ f = k \circ R(f) \circ \phi_R(A) = h \circ \phi_R(A) = g$.

Definition 4.9.4. A monomorphism i Î C(X, Y) is called *essential* if for every morphism f Î C(Y, Z) with the property that f ∘ i is a monomorphism, then f is an monomorphism.

A pair (i, Q) is called *injective envelope* for an object X if Q is injective and $i \in C(X, Q)$ is an essential monomorphism.

Remark 4.9.5. If u and v are composable monomorphisms in **C**, then if u and v are essentials, then $u \circ v$ is also essential.

Definition 4.9.6. We say that a category C has the *property* $\mathcal{E}$ if for every two composable monomorphisms u and v in C, if u∘v is essential, then u and v are essentials.

Lemma 4.9.7. In a category C with the property $\mathcal{E}$, if the injective envelope of an object exists , then it is unique up to an isomorphism.

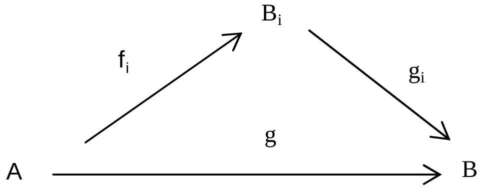
Proof. Let $A \in \mathbf{C}$ and $(i, Q), (i', Q')$ two injective envelopes for A. So, $i \in \mathbf{C}(A, Q)$, $i' \in \mathbf{C}(A, Q')$ are essential monomorphisms and Q, Q′ are injective objects.

Since Q′ is injective, there is $f \in \mathbf{C}(Q, Q')$ such that $f \circ i = i'$ and since i is essential, we deduce that f is a monomorphism.

Since Q is injective, f has a retraction, so, there is $f' \in \mathbf{C}(Q', Q)$ such that $f' \circ f = 1_Q$.

Since 1_Q is essential monomorphism and **C** has the property $\mathcal{E}$, then f′ is monomorphism, and from $f' \circ f = 1_Q$ we deduce that $(f' \circ f) \circ f' = f' \Rightarrow f' \circ (f \circ f') = f' \circ 1_{Q'} \Rightarrow f \circ f' = 1_{Q'}$, hence f is isomorphism , so $Q \approx Q'$. **n**

Definition 4.9.8. We say that a category C has the *amalgamation property* provided that if $(f_i)_{i \in I}$ is a family of monomorphisms with f_i Î C (A, B_i) for every i Î I, then there exists B Î C, a family of monomorphisms g_i Î C (B_i, B) and a monomorphism g Î C (A, B) such that the diagram



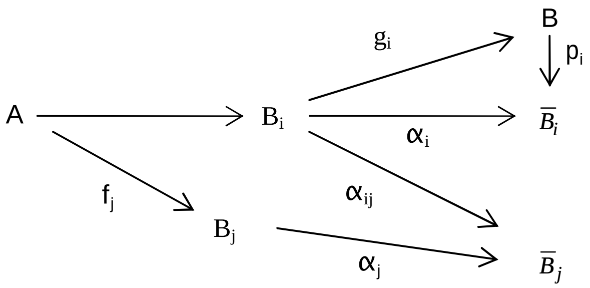
is commutative (i.e. $g_i \circ f_i = g$, for every i Î I).

Theorem 4.9.9. (*Pierce R. S.*) Every equational category A with enough injectives has the amalgamation property.

Proof. ([58]). Let $(f_i : A \to B_i)_{i\in I}$ a non-empty family of monomorphisms and for every $i \in I$ let $a_i : B_i \to \overline{B}_i$ be a monomorphism with $\overline{B}_i$ injective object in **A**.

Since in every equational category there exist products, let $B = \prod_{i\in I} \overline{B}_i \in$ **A** (which by Proposition 4.9.2 is injective object) and $p_i : B \to \overline{B}_i$ the i-th projection $(i \in I)$.

Since we can suppose $|I| \geq 2$, for $i, j \in I$, $i \neq j$, by the injectivity of $\overline{B}_j$ there is $\alpha_{ij} \in \mathbf{A}(B_i, \overline{B}_j)$ such that $\alpha_{ij} \circ f_i = \alpha_j \circ f_j$. By the universality property of product for every $i \in I$ we find $g_i \in \mathbf{A}(B_i, B)$ such that for every $j \in I$, $j \neq i$ we have $p_i \circ g_i = \alpha_i$ and $p_j \circ g_i = \alpha_{ij}$:



So, for $i \neq j$, $p_i \circ g_i \circ f_i = \alpha_i \circ f_i = \alpha_{ji} \circ f_j = p_i \circ g_j \circ f_j$ hence $g_i \circ f_i = g_j \circ f_j = g$.

Since for every $i \in I$, α_i is a monomorphism and $\alpha_i = p_i \circ g_i$ we deduce that g_i is a monomorphism, hence $g_i \circ f_i$ is a monomorphism.

So, we have obtained a monomorphism $g \in \mathbf{A}(A, B)$ and a family of monomorphisms $g_i \in \mathbf{A}(B_i, B)$ such that $g_i \circ f_i = g$, for every $i \in I$, hence **A** has the amalgamation property. ∎

Remark 4.9.10. The above result of Pierce is true in every category **C** with products(with the canonical projections epimorphisms)and enough injectives.

In what follows we shall present some results from the paper [42].

Theorem 4.9.11. Let C, C¢ two categories , S : C ® C¢, T : C¢® C two covariant functors such that S is the right adjoint of T.

If

a) S is faithful and full , T is faithful ;

b) T preserves monomorphisms ;

c) In C every object has an injective envelope,

then the following assertions are equivalent :

(i) A is an injective object in C¢ ;

(ii) A is the retract of all his extensions ;

(iii) A doesn't have proper essential extensions ;

(iv) Adjoint morphism y_A : A ® (ST)(A) is an isomorphism and T(A) is injective object in C.

Proof. (i) ⇒ (ii). Let $i : A \to B$ a monomorphism in **C′**. If A is injective then there is $f : B \to A$ such that $f \circ i = 1_A$.

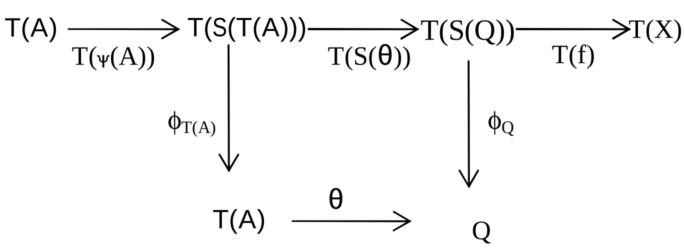
(ii) ⇒ (iii). If $f : A \to A'$ is an essential monomorphism, then there is a monomorphism $g : A' \to A$ such that $g \circ f = 1_A$.

Then $g\circ(f\circ g) = (g\circ f)\circ g = 1_A\circ g = g = g \circ 1_{A'}$; since g is a monomorphism, then $f\circ g = 1_{A'}$, hence $A \approx A'$.

(iii) ⇒ (iv). For $T(A) \in$ **C** , there is an essential monomorphism $\theta : T(A) \to Q$, with Q injective. Since T is supposed faithful, $\psi_A : A \to S(T(A))$ is a monomorphism in **C′**. We shall prove that $S(\theta) \circ \psi_A$ is an essential monomorphism.

For this, let $f \in$ **C′** (S (Q), X) such that $f\circ S(\theta)\circ\psi_A$ is a monomorphism. Since S is faithful and full, the another adjoint morphism $\phi : TS \to 1_{\mathbf{C}}$ is a functorial isomorphism.

Consider now in **C** the following commutative diagram:

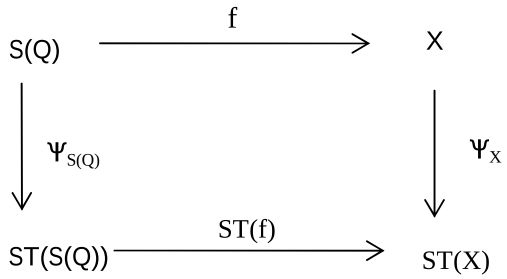


We have $T(f)\circ TS(\theta)\circ T(\psi_A)=T(f)\circ\phi^{-1}_Q\circ\ \theta\circ\phi_{T(A)}\circ T(\psi_A\ )$.

Since $\phi_{T(A)}\circ T(\psi_A) = 1_{T(A)}$, then $T(f\circ S(\theta)\circ\psi_A) = T(f)\circ TS(\theta)\circ T(\psi_A) = =T(f)\circ\phi^{-1}_Q\circ\theta$.

Since the functor T preserves monomorphisms, we deduce that $T(f)\circ\phi^{-1}_Q\circ\theta$ is a monomorphism in **C**. Since θ essential, then $T(f)\circ\phi^{-1}_Q$ is a monomorphism. We deduce that $T(f)$ is a monomorphism (since ϕ^{-1}_Q is an isomorphism).

In **C′** we have the following commutative diagram :

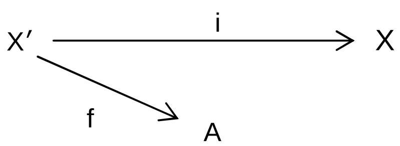


Since S is the right adjoint of T , then S also preserves monomorphisms , hence ST(f) is a monomorphism. But $S(\phi_Q)\circ\psi_{S(Q)} = 1_{S(Q)}$ and $S(\phi_Q) = S(1_Q) = 1_{S(Q)}$, hence $\psi_{S(Q)} = 1_{S(Q)}$.

From the above diagram, there results that $\psi_X\circ f = ST(f)$, hence f is a monomorphism in **C′** and S(Q) is an essential extension of A.

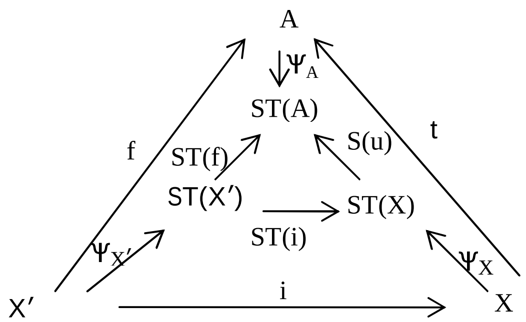
Since by hypothesis A doesn't have proper essential extensions, then $S(\theta)\circ\psi_A$ is an isomorphism and since ψ_A and $S(\theta)$ are monomorphisms, there results that they are isomorphisms, hence T(A) is an injective object (since $\theta\approx TS(\theta)$).

(iv) $\Rightarrow$ (i). Consider in **C′** the diagram



where i is a monomorphism. Since T preserves monomorphisms and T(A) is injective, there is a monomorphism $u : T(X) \to T(A)$ such that $u\circ T(i) = f$.

We have the following commutative diagram in **C′** :

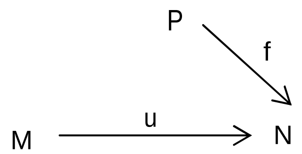


where $t = \psi^{-1}_A \circ S(u) \circ \psi_X$. We have $t \circ i = \psi^{-1}_A \circ S(u) \circ \psi_X \circ i = \psi^{-1}_A \circ S(u) \circ \circ ST(i) \text{ o } \psi_{X'} = \psi^{-1}_A \circ ST(f) \text{ o } \psi_{X'} = f$ (since $\psi_A \circ f = ST(f) \circ \psi_{X'}$). There result that A is injective in **C′**. **n**

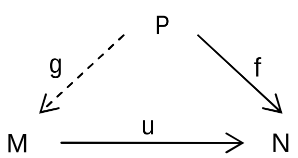
Corollary 4.9.12. Let M Î C¢ . If Q is the injective envelope of T(M) in C, then S(Q) is the injective envelope of M in C¢.

The dual notion for injective object is the notion of *projective* object .

Definition 4.9.13. An object P in a category C is called *projective*, if for any diagram in C



with u an epimorphism, there is a morphism g:P ® M such that the following diagram is commutative :



(i.e, uog = f).

Examples

1. In **Set** every object is projective.

2. In **Top** every discrete space is projective.

Proposition 4.9.14. If $(A_i)_{i\in I}$ a family of projective objects in a category C and A is its coproduct , then A is also a projective object in C.

Proof. It is the dual of Proposition 4.9.2. ■

Remark 4.9.15. (i) Every epimorphism with codomain a projective object has a section;

(ii) In some categories (for example in categories which have a nullary object) the converse of Proposition 4.9.14 is true .

Definition 4.9.16. An epimorphism p $\in$ C (X, Y) is called *superfluous* if every f $\in$ C (Z, X) with the property that p$\circ$f is an epimorphism , then f is an epimorphism.

A pair (P, p) is called *projective envelope* of X if P is a projective object and p : P $\to$ X is a superflous epimorphism .

Theorem 4.9.17. Let C be a category , S : C $\to$ Sets a covariant functor and T : Sets $\to$ C a left adjoint of S. If

a) S is faithful ;

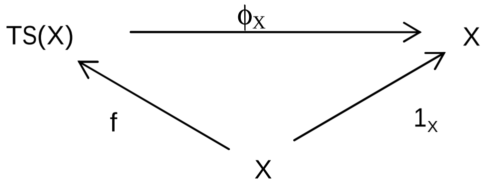
b) S preserves epimorphisms ,

then the following assertions are equivalent :

(i) X is a projective object in C ;

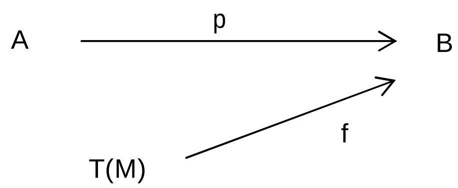
(ii) There is a set M and morphisms $X \xrightarrow{f} T(M) \xrightarrow{g} X$ **such that g$\circ$f = 1_X.**

Proof. (i) $\Rightarrow$ (ii). Since S is faithful, the adjoint morphism ϕ_X : TS(X) $\to$ X is epimorphism. Since X is projective, we have in **C** the following diagram:

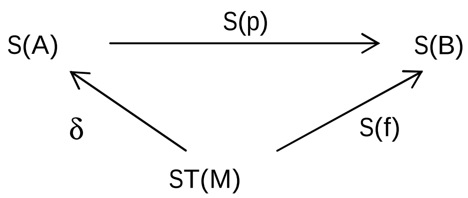


We choose M = S(X).

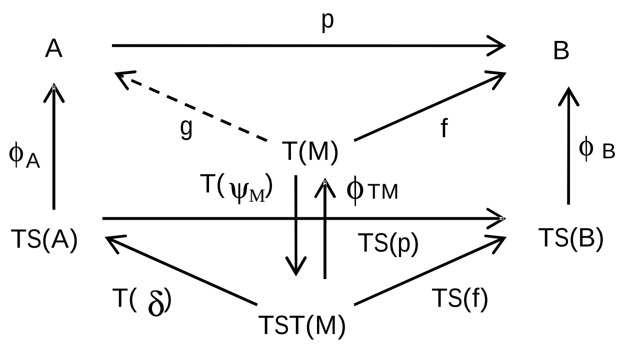
(ii) $\Rightarrow$ (i). Firstly, we shall prove that every element in **C** of the form T(M) is projective. For this, we consider in **C** the diagram



with p an epimorphism. Since S preserves epimorphisms and every object in **Set** is projective, there is a mapping $\delta : ST(M) \rightarrow S(A)$ such that the following diagram is commutative in **Set** :



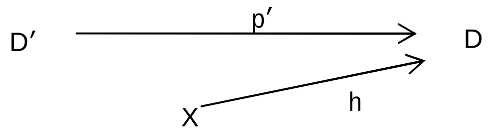
In **C** we have the following commutative diagram:



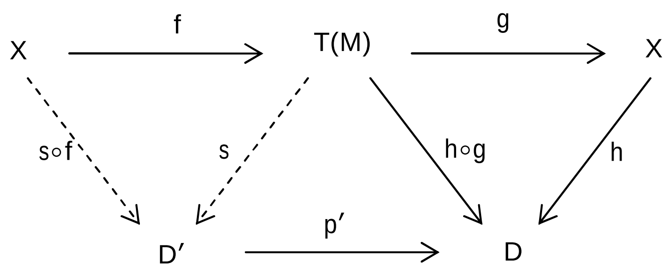
where $g = \phi_A \circ T(\delta) \circ T(\psi_M)$ is the canonical morphism of adjunction.

But $p \circ g = p \circ \phi_A \circ T(\delta) \circ T(\psi_M) = \phi_B \circ TS(p) \circ T(\delta) \circ T(\psi_M) =$ $= \phi_B \circ TS(f) \circ T(\psi_M) = f \circ \phi_{T(M)} \circ T(\psi_M) = f \circ 1_{T(M)} = f$, hence T(M) is projective.

To prove that X is projective , we consider the following diagram in **C**:



with p' an epimorphism. Since T(M) is projective, we have in **C** the following commutative diagram:



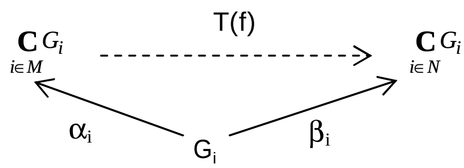
There is $s : T(M) \to D'$ such that $p' \circ s = h \circ g$, hence $p' \circ s \circ f =$ $h \circ g \circ f =$ $= h \circ 1_X = h$, that is , X is projective. ∎

Corollary 4.9.18. Let C be a category with coproducts and G∈C a projective generator . Then for X∈C the following assertions are equivalent:

(i) X is projective in C ;

(ii) There is a set M and morphisms $X \xrightarrow{f} G^{(M)} \xrightarrow{g} X$ **such that** $g \circ f = 1_X$**.**

Proof. We consider the functor $T : \mathbf{Set} \to \mathbf{C}$ defined by $T(M) = G^{(M)}$, for every $M \in$ **Set** and for $M, N \in$ **Set** and $f : M \to N$ a function $T(f) : G^{(M)} \to G^{(N)}$ is the unique morphism in **C** such that the diagram



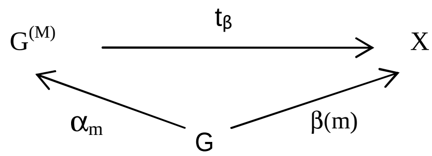
is commutative (where $G_i = G$, for every $i \in M$, $(\alpha_i)_{i\in I}$ are the canonical morphisms of coproduct and $b_i : G_i \to \coprod_{i\in N} G_i$, $\beta_i = \alpha_{f(i)}$, for every $i \in M$).

Then it is immediate to prove that T becomes a covariant functor.

We have to prove that T is left adjoint for the functor h^G : **C** $\to$ **Set**. For this, for X $\in$ **C** and M $\in$ **Set**, we have to prove the existence of an isomorphism (functorial in X and M): **C**(T(M), X) $\approx$ **Set** (M, h^G (X)).

Indeed, if f $\in$ **C**((T(M), X), we consider s_f : M $\to$ h^G(X) defined by $s_f(m) = f \circ \alpha_m$, for every m $\in$ M.

If β : M $\to$ h^G(X) is a function, by the universality property of coproduct, there is a unique morphism in **C** t_β : $G^{(M)} \to X$ such that the following diagram is commutative:



It is immediate to see that the assignments f $\to$ s_f and $\beta \to t_\beta$ are one the converse of the other, hence in this way we obtain the desired isomorphism. Since the projectivity of G is assured by b) from Theorem 4.9.17, the proof of this theorem is complete. **n**

4.10. Injective Boolean algebras. Injective (bounded) distributive lattices

We start this paragraph with the characterization of the injective objects in the category **B** of Boolean algebras (see Chapter 2).

Following the categorial equivalence between Boolean algebras and Boolean rings, we will work (relative to context) with Boolean algebras (using the operations $\wedge$, $\vee$ and $'$) or with the corresponding Boolean rings (using the operations + and $\cdot$) - see §7 from Chapter 2.

We don't have special problems since if B_1, B_2 are two Boolean algebras, $\overline{B}_1, \overline{B}_2$ the corresponding Boolean rings, f $\in$ **B**(B_1, B_2) and $\bar{f} : \overline{B}_1 \to \overline{B}_2$ the corresponding morphism of Boolean rings, then f is a morphism in **B** iff $\bar{f}$ is a morphism of Boolean rings.

Definition 4.10.1. We say that a Boolean ring is *complete* if the corresponding Boolean algebra is complete.

Lemma 4.10.2. Let A be a Boolean ring, A¢ Ì A a subring, a Î A \ A¢ and A¢(a) the subring of A generated by A¢ È {a}.

If C is a complete Boolean ring then for every morphism of Boolean rings $f : A' \to C$ there is a morphism of Boolean rings $f' : A'(a) \to C$ such that $f'_{|A} = f$.

Proof. Clearly, $A'(a) = \{x + ay : x, y \in A'\}$. Since C is supposed complete there exist $m_a = \bigvee_{\substack{x \in A' \\ x \le a}} f(x)$ and $M_a = \bigwedge_{\substack{x \in A' \\ a \le x}} f(x)$ in C. We remark that $m_a \le M_a$, so we can choose $m_a \le m \le M_a$.

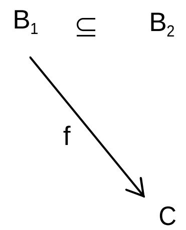
Now let $z \in A'(a)$ and suppose that for z we have two representations $z = x_1 + ay_1 = x_2 + ay_2$ with $x_1, x_2, y_1, y_2 \in A'$.

Then $x_1 + x_2 = a(y_1 + y_2)$, hence $x_1 + x_2 \le a$, so we deduce that $a(x_1 + x_2 + y_1 + y_2 + 1) = a(x_1 + x_2) + a(y_1 + y_2) + a = a(y_1 + y_2) + a(y_1 + y_2) + a = a$, that is, $a \le x_1 + x_2 + y_1 + y_2 + 1$. Following these last two relations we deduce that $f(x_1) + f(x_2) \le m \le f(x_1) + f(x_2) + f(y_1) + f(y_2) + 1$, so $m = m[f(x_1) + f(x_2) + f(y_1) + f(y_2) + 1] = m[f(x_1) + f(x_2)] + m[f(y_1) + f(y_2)] + m = f(x_1) + f(x_2) + m f(y_1) + m f(y_2) + m$ hence $f(x_1) + f(x_2) + m f(y_1) + m f(y_2) = 0 \iff f(x_1) + m f(y_1) = f(x_2) + m f(y_2)$. Thus we can define for $z = x + ay \in A'(a)$, $f'(z) = f(x) + m f(y)$ and it is immediate to prove that this is the desired morphism. ∎

Theorem 4.10.3. (*Sikorski*). Complete Boolean algebras are injective objects in the category B.

Proof. Let C be a complete Boolean algebra and B_1, B_2 Boolean algebras such that B_1 is subalgebra of B_2.

To prove that C is injective object in **B** we consider in **B** the diagram



Let $\mathbf{M} = \{(B', f') : B_1 \subseteq B' \subseteq B_2$, B' is a Boolean subalgebra of B_2 and $f' : B' \to C$ is a morphism in **B** such that $f'_{|B_1} = f\}$. Since $(B_1, f) \in \mathbf{M}$, then $\mathbf{M} \ne \emptyset$ and it is immediate to prove that relative to the ordering $(B', f') \le (B'', f'') \iff B' \subseteq B''$ and $f''_{|B'} = f'$, $(\mathbf{M}, \le)$ is inductive, hence by Zorn's lemma there is a maximal element $(B_0, f_0) \in \mathbf{M}$. If we prove that $B_0 = B_2$, the proof is ended.

Indeed, if by contrary $B_0 \neq B_2$, then there is $a \in B_2 \setminus B_0$. By Lemma 4.10.2 (following the equivalence between Boolean algebras and Boolean rings) we deduce that there is $f' : B_0(a) \to C$ morphism in **B** such that $f'_{|B_0} = f_0$, hence $(B_0(a), f') \in$ **M** which is contradictory with the maximality of (B_0, f_0) (here by $B_0(a)$ we have denoted the Boolean algebra generated by $B_0 \cup \{a\}$). ∎

Remark 4.10.4. (i). In the above we have identified the subobjects of a Boolean algebra with his subalgebras (this is possible since the category **B** is equational).

(ii). In particular we deduce that every finite Boolean algebra is injective (as **2** = {0, 1} for example).

Corollary 4.10.5. 2 = { 0, 1} is injective cogenerator in B.

Proof. From Theorem 4.10.3 we deduce that **2 = {0, 1}** is an injective object in **B**.

Now let two distinct morphisms $f, g : B_1 \to B_2$ in **B**. Thus there is $a \in B_1$ such that $f(a) \neq g(a)$.

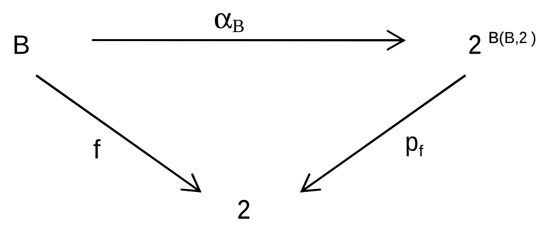
By Corollary 2.8.2 there is a maximal filter F_a in B_2 such that $f(a) \in F_a$ and $g(a) \notin F_a$. If we consider $h_a : B_2 \to$ **2** the morphism in **B** induced by F_a (that is, $h_a(x) = 1$ if $x \in F_a$ and 0 if $x \notin F_a$ - see Proposition 2.6.20) it is immediate to say that $h_a \circ f \neq h_a \circ g$, that is, **2** is cogenerator in **B**. ∎

Lemma 4.10.6. Let A and B be two ordered sets, f : A → B a morphism in Ord such that there is g : B → A a morphism in Ord such that g∘f = 1$_A$. If B is complete, then A is also complete.

Proof. It is immediate to prove that for $S \subseteq A$, then $\sup(S) = g(\sup(f(S)))$ and $\inf(S) = g(\inf(f(S)))$. ∎

Theorem 4.10.7. (*Halmos*). Every injective Boolean algebra is complete.

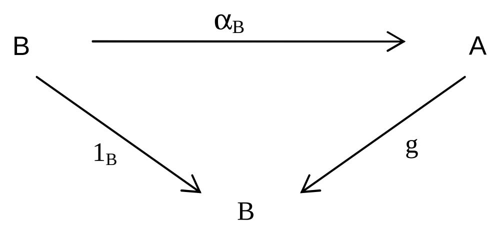
Proof. Let B be a Boolean algebra. By the universality property of product there is a morphism in **B**, $\alpha_B : B \to \mathbf{2}^{\mathbf{B}(B,2)}$ such that the following diagram



is commutative, where $(p_f)_{f\in \mathbf{B}(B,2)}$ are the canonical projections. To prove α_B is a monomorphism in **B** let $\beta, \gamma : A \to B$ be a morphism in **B** such that $\alpha_B \circ \beta = \alpha_B \circ \gamma$.

There result that $f \circ \beta = f\circ \gamma$, for every $f \in \mathbf{B}(B, \mathbf{2})$ and since we have proved that **2** is injective cogenerator in **B** (Corollary 4.10.5), we deduce that $\beta = \gamma$, that is, α_B is a monomorphism in **B**.

Clearly $A = \mathbf{2}^{\mathbf{B}(B,2)}$ is a complete Boolean algebra. By hypothesis, B is an injective Boolean algebra. Since $\alpha_B : B \to A$ is a monomorphism in **B**, there is a morphism $g : A \to B$ in **B** such that the diagram



is commutative, hence $g \circ \alpha_B = 1_B$. By Lemma 4.10.6 we deduce that B is complete. ∎

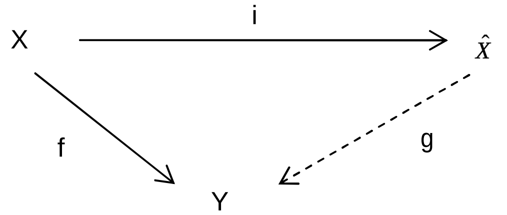
Corollary 4.10.8. In the category B of Boolean algebras the injective objects are exactly the complete Boolean algebras.

Corollary 4.10.9. The category B of Boolean algebras is a category with enough injectives.

Proof. Since **2** is an injective Boolean algebra, by Proposition 4.9.2, we deduce that $A = \mathbf{2}^{\mathbf{B}(B,2)}$ is an injective Boolean algebra for every Boolean algebra B. Since $\alpha_B : B \to A$ is a monomorphism in **B** we obtain the desired conclusion. ∎

Let's pass to the characterization of injective objects in the category **Ld**(**0**, **1**) (for this we need some notions introduced in §3).

Lemma 4.10.10. Let X be a Stone space. Then there exist a unique Boole space $\hat{X}$ and a strong continuous function i : X $\to$ $\hat{X}$ with the following universality property: for every Boole space Y and every strong continuous function f : X $\to$ Y there is a unique continuous function g : $\hat{X}$ $\to$ Y such that the following diagram is commutative:



(i.e, g$\circ$i = f).

Proof. The subjacent set for $\hat{X}$ will be X and a basis of open sets will be: **D** = D(X) $\cup$ {V : $\complement$V $\in$ D(X)}, where D(X) is the set of opens in X.

Clearly $\hat{X}$ is a Hansdorff space and his clopen sets determine a basis.

We have to prove that X is compact and for this we consider a family of closed sets from basis with the empty intersection :

$\left(\bigcap_{i\in I} V_i\right) \cap \left(\bigcap_{j\in J} CW_j\right) = \varnothing$ with V_i, $W_j \in$ D(X) for i $\in$ I and j $\in$ J. If we consider the close set F = $\bigcap_{j\in J} CW_j$ (from $\hat{X}$) then F $\cap$($\bigcap_{i\in I} V_i$) = $\varnothing$.

If $\{V_i\}_{i\in I}$ doesn't have the property of finite intersection, then the proof is clear. If $(V_i)_{i\in I}$ has the finite intersection property, then there is i $\in$ I such that F $\cap$ $V_i = \varnothing$. Since V_i is a compact set and $(\complement W_j \cap V_i)_{j\in J}$ are closed in V_i, we deduce that there exist $W_1, ..., W_n$ such that $\complement W_1 \cap ... \cap \complement W_n \cap V_i = \varnothing$; thus X is a compact set, hence $\hat{X}$ is a Boole space.

Thus we choose i = 1_X and g = f and the proof is complete. ∎

Consider now S : **B** $\to$ **Ld**(**0**,**1**) the subjacent functor which assigns to every Boolean algebra his subjacent bounded distributive lattice.

Proposition 4.10.11. The functor S has a left adjoint functor

T : Ld (0,1) ® B.

Proof. Since the dual category of **B** is equivalent with the category of Boole spaces $\tilde{B}$ (see Theorem 4.3.26), by Lemma 4.10.10 we deduce that for every L ∈ **Ld**(**0**, **1**) there is a unique Boolean algebra $\hat{L}$ and a unique morphism of lattices i : L → $\hat{L}$ such that for every Boolean algebra A and every morphism of lattices f : L → A there is a unique morphism of Boolean algebras g : $\hat{L}$ → A such that g ∘ i = f (see Corollary 2.8.10).

The functor T will be assigned to every **L** ∈ **Ld**(**0**, **1**), $\hat{L}$ ∈ **B** and the definition of T on morphisms is immediate, following Lemma 4.10.10.

By Proposition 4.4.9 we deduce that T is a left adjoint of S. ∎

Theorem 4.10.12. (*Banaschewski,Bruns*). In the category Ld(0,1) the injective objects are exactly the complete Boolean algebras.

Proof. 1. Follows from Corollary 4.10.8 since the functors S and T verify the conditions a), b) and c) from Theorem 4.9.11.

Proof. 2. Suppose that L is injective in **Ld**(**0**, **1**). In §4 from Chapter 2 we have defined φ_L : L → **Spec**(L) by φ_L(x) = {P ∈ **Spec**(L) : x ∉ P} and we have proved that φ_L is a monomorphism in **Ld**(**0**, **1**).

Then we can consider φ_L: L → **P**(**Spec**(L)) and since L is injective there is s : **P**(**Spec**(L)) → L a morphism in **Ld**(**0**, **1**) such that s ∘ φ_L = 1_L.

By Lemma 4.10.6 we deduce that L is complete.

Since s is surjective and **P**(**Spec**(L)) is a Boolean algebra, we deduce that L = s(**P**(**Spec**(L))) hence L is a complete Boolean algebra.

Now let B be a complete Boolean algebra.

In [30], to §7 from Chapter 3 it is proved that **B** is a reflexive subcategory of **Ld**(**0**,**1**) and the reflector R_{01} : **Ld**(**0**,**1**) → **B** preserves monomorphisms.

By Remark 4.9.3 (iii), we deduce that the injective objects in **B** are also injective and in **Ld**(**0**, **1**).

By Corollary 4.10.8, B is injective in **B** (since is complete) hence B will also be injective in **Ld** (**0**,**1**).

Corollary4.10.13. In the category Ld the injective objects are exactly complete Boolean algebras.

Proof. If L ∈ **Ld** is injective, as in the second proof of Theorem 4.10.12, we deduce that L is a complete Boolean algebra. For the converse we use that **Ld**(**0**,**1**) is a reflexive subcategory of **Ld** and the reflector U : **Ld** → **Ld**(**0**,**1**) preserves monomorphisms (see [30], Proposition 7.2, Chapter 3), so, since a complete Boolean algebra is injective object in **Ld**(**0**,**1**) (by Theorem 4.10.12) it will be injective also in **Ld**. ∎

Chapter 5

ALGEBRAS OF LOGIC

The origin of many algebras is in Mathematical Logic.
The first paragraph of this chapter contains some notions about Heyting algebras, which have their origins in mathematical logic ,too.
It was A. Heyting who in [48] formalized the propositional and predicate calculus for the intuitionist view of mathematics.
In 1923, David Hilbert was the first who remarked the possibility of studying a very interesting part of the classical propositional calculus taking as axioms only the ones verified by logical implication (this field is known as *positive implicative propositional calculus*) and it is interesting because his *theorems* are those theorems of intuitionist propositional calculus which contains only logical implication and which is called *intuitionist implicative calculus*. The study of this fragment was started by D. Hilbert and P. Bernays in [49].
We can study this fragment with the help of specific algebraic technique because we have an algebraic structure: the notion of *implicative model* introduced by Henkin in 1950.
The dual algebras of implicative models were called by A. Monteiro *Hilbert algebras.* In some papers Hilbert algebras are called *positive implicative algebras* ([73],[75]).
In this chapter are also studied Hertz algebras (which in some papers are called *implicative semilattices* see -[57-60]) and residuated lattices.
The origin of residuated lattices is in Mathematical Logic without contraction .

The last paragraph of this chapter is dedicated to Wajsberg algebras and to their connections with residual lattices .

For more information about Wajsberg algebras, I recommend the reader the paper [39].

About the connection of these algebras with fuzzy logic algebras (MV-algebras) I recommend the reader the book [81].
Though the origin of all these algebras is in Mathematical Logic, in this chapter we are interested only by the study of these algebras from the Universal algebra (see Chapter 3) and Theory of categories (see Chapter 4) view points.
In this chapter we have included classical results and all my original results relative to these algebras (more of these results are included in my Ph.Thesis : **Contributions to the study of Hilbert algebras** - see [18]).
The guide-line in the study of localization of Hilbert and Hertz algebras is the case of rings (see [71]).

a. Heyting algebras

5.1. Definitions. Examples. Rules of calculus

Definition 5.1.1. Let L be a lattice and a, b Î L. The *pseudocomplement of a relative to b* is the element of L denoted by a → b such that a → b = sup({x Î L : a Ù x £ b}).

Therefore, $a \wedge x \leq b \Leftrightarrow x \leq a \rightarrow b$.

Definition 5.1.2. A *Heyting algebra* is a lattice L with L such that a → b there exists for any a, b Î L.

Examples

1. If $(B, \wedge, \vee, ', 0, 1)$ is a Boolean algebra, then $(B, \wedge, \vee, \rightarrow, 0)$ is a Heyting algebra, where for $a, b \in B$, $a \rightarrow b = a' \vee b$.
2. If L is a chain with 0 and 1, then L becomes a Heyting algebra, where for $a, b \in L$, $a \rightarrow b = 1$ if $a \leq b$ and b if $a > b$.
3. If (X, τ) is a topological space, then $(\tau, \rightarrow, \emptyset)$ becomes a Heyting algebra, where for $D_1, D_2 \in \tau$, $D_1 \rightarrow D_2 = \mathrm{int}[(X \setminus D_1) \cup D_2]$.
In [75, p.58], Heyting algebras are called *pseudo-boolean* algebras.

Heyting algebras in which we ignore $\vee$ (which is not necessary for the definition of implication $\rightarrow$) are called, by Nemitz,*implicative semilattices* in [63]–[65]; in [45] (Chapter 4, p.61), these are called *meet-semilattices relatively pseudocomplemented* (in the above mentioned papers, the element $x \rightarrow y$ is denoted by $x * y$).

Therefore, in the case of Heyting algebras or implicative semilattices, for two elements x, y, $x \to y = \sup\{z : x \wedge z \leq y\}$.
In what follows, by H we denote a Heyting algebra (unless otherwise specified).

Proposition 5.1.3. If for every S Í H there is sup(S), then for every a Î H there is sup({a Ù s : s Î S}) and sup({a Ù s :s Î S}) = a Ù sup(S).

Proof. Let b = sup (S). Then $a \wedge s \leq a \wedge b$ for every $s \in S$; if we have $x \in H$ such that $a \wedge s \leq x$ for every $s \in S$, then $s \leq a \to x$, hence $b \leq a \to x \Rightarrow a \wedge b \leq x$, that is, we obtain the equality from the enounce. **n**

Corollary 5.1.4. H Î Ld(0, 1).

Proof. By Proposition 5.1.3 we deduce that H ∈ **Ld**(0). Clearly, $1 = a \to a$ for some $a \in H$. **n**

For $x \in H$ we denote $x^* = x \to 0$.

Remark 5.1.5. Since x^* is the pseudocomplement of x, we deduce that Heyting algebras are pseudocomplemented lattices.

In §8 from Chapter 2 we have defined for a distributive lattice L and I, J∈**I**(L), $I \to J = \{x \in L: [x) \cap I \subseteq J\} = \{x \in L: x \wedge i \in J$, for every $i \in I\}$ (see Lemma 2.8.2).

Theorem 5.1.6. For every distributive lattice L with 0, (I(L), →, 0={0}) is a Heyting algebra.

Proof. We will prove that for I, J∈**I**(L), then $I \to J \in$ **I**(L). If $x, y \in L$, $x \leq y$ and $y \in I \to J$, then for every $i \in I$, $y \wedge i \in J$. Since $x \wedge i \leq y \wedge i$ we deduce that $x \wedge i \in J$, hence $x \in I \to J$. If $x, y \in I \to J$ and $i \in I$, since $x \wedge i$, $y \wedge i \in J$ and $(x \vee y) \wedge i = (x \wedge i) \vee (y \wedge i) \in J$ we deduce that $x \vee y \in I \to J$, hence $I \to J \in$ **I**(L).

Now we will prove that if K∈**I**(L), then $I \cap K \subseteq J \Leftrightarrow K \subseteq I \to J$.

Indeed, if $x \in K$ then, since for every $i \in I$ we have $x \wedge i \in K \cap I \subseteq J$ we deduce that $x \wedge i \in J$, hence $x \in I \to J$, so $K \subseteq I \to J$.

Now let $x \in I \cap K$. Then $x \in I$ and $x \in K \subseteq I \to J$, hence $x \in I \to J$, so $x \wedge x = x \in J$, therefore $I \cap K \subseteq J$. ■

Corollary 5.1.7. Let L be a distributive lattice. Then, for every $I \in I(L)$, $I^* = I \to \{0\} = \{x \in L : x \wedge i = 0$, for every $i \in I\}$.

Proposition 5.1.8. Let H be a Heyting algebra and $x, y \in H$.
Then $(x] \to (y] = (x \to y]$.

Proof. If $z \in (x] \to (y]$, then for every $i \in (x]$ (that is, $i \le x$) we have $z \wedge i \in (y]$, hence $z \wedge i \le y \Leftrightarrow z \le i \to y$.

In particular, for $i = x$ we deduce that $z \le x \to y \Leftrightarrow z \in (x \to y]$, hence $(x] \to (y] \subseteq (x \to y]$. If $z \in (x \to y]$, then $z \le x \to y \Leftrightarrow z \wedge x \le y$, so if $i \in (x]$, $i \le x$ and $z \wedge i \le z \wedge x \le y$, hence $z \wedge i \in (y] \Leftrightarrow z \in (x] \to (y]$. Therefore we also have the inclusion $(x \to y] \subseteq (x] \to (y]$, that is, $(x \to y] = (x] \to (y]$. ■

Theorem 5.1.9. For every elements $x, y, z \in H$ we have
h_1: $x \wedge (x \to y) \le y$;
h_2: $x \wedge y \le z \Leftrightarrow y \le x \to z$;
h_3: $x \le y \Leftrightarrow x \to y = 1$;
h_4: $y \le x \to y$;
h_5: $x \le y \Rightarrow z \to x \le z \to y$ and $y \to z \le x \to z$;
h_6: $x \to (y \to z) = (x \wedge y) \to z$;
h_7: $x \wedge (y \to z) = x \wedge [(x \wedge y) \to (x \wedge z)]$;
h_8: $x \wedge (x \to y) = x \wedge y$;
h_9: $(x \vee y) \to z = (x \to z) \wedge (y \to z)$;
h_{10}: $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$;
h_{11}: $(x \to y)^* = x^{**} \wedge y^*$;
h_{12}: $x \wedge x^* = 0$;
h_{13}: $x \le y \Rightarrow y^* \le x^*$;
h_{14}: $(x \vee y)^* = x^* \wedge y^*$.

Proof. **h_1** and **h_2** follows from Definition 5.1.1.
h_3. We have $x \to y = \mathbf{1} \Leftrightarrow \mathbf{1} \le x \to y \Leftrightarrow x \wedge \mathbf{1} \le y \Leftrightarrow x \le y$.
h_4. We have $x \wedge y \le y \Rightarrow y \le x \to y$.
h_5. We have $z \wedge (z \to x) \le x \le y$, hence $z \to x \le z \to y$ .Since $x \wedge (y \to z) \le y \wedge (y \to z) \le z$ we deduce that $y \to z \le x \to z$.

h_6. We have $(x \wedge y) \wedge [x \rightarrow (y \rightarrow z)] = y \wedge \{x \wedge [x \rightarrow (y \rightarrow z)]\} \leq y \wedge (y \rightarrow z) \leq z$, hence $x \rightarrow (y \rightarrow z) \leq (x \wedge y) \rightarrow z$.
Conversely, $(x \wedge y) \wedge [(x \wedge y) \rightarrow z] \leq z \Rightarrow x \wedge [(x \wedge y) \rightarrow z] \leq y \rightarrow z \Rightarrow (x \wedge y) \rightarrow z \leq x \rightarrow (y \rightarrow z)$.
h_7. From $x \wedge (x \rightarrow y) \leq x$ and $(x \wedge y) \wedge x \wedge (y \rightarrow z) \leq x \wedge z \Rightarrow x \wedge (y \rightarrow z) \leq (x \wedge y) \rightarrow (x \wedge z)$, hence $x \wedge (y \rightarrow z) \leq x \wedge [(x \wedge y) \rightarrow (x \wedge z)]$.
Conversely, $x \wedge [(x \wedge y) \rightarrow (x \wedge z)] \leq x$ and $y \wedge x \wedge [(x \wedge y) \rightarrow (x \wedge z)] \leq x \wedge z \leq z$, hence $x \wedge [(x \wedge y) \rightarrow (x \wedge z)] \leq y \rightarrow z$, therefore $x \wedge [(x \wedge y) \rightarrow (x \wedge z)] \leq x \wedge (y \rightarrow z)$.
h_8. Clearly, $x \wedge (x \rightarrow y) \leq x, y$ and $x \wedge y \leq x, \ x \rightarrow y$.
h_9. From $x, y \leq x \vee y \Rightarrow (x \vee y) \rightarrow z \leq x \rightarrow z, y \rightarrow z$. Conversely, $(x \vee y) \wedge (x \rightarrow z) \wedge (y \rightarrow z) \leq [x \wedge (x \rightarrow z)] \vee [y \wedge (y \rightarrow z)] \leq z \vee z = z$, therefore $(x \rightarrow z) \wedge (y \rightarrow z) \leq (x \vee y) \rightarrow z$.
h_{10}. From $y \wedge z \leq y, z \Rightarrow x \rightarrow (y \wedge z) \leq x \rightarrow y, x \rightarrow z \Rightarrow x \rightarrow (y \wedge z) \leq (x \rightarrow y) \wedge (x \rightarrow z)$. Since $x \wedge (x \rightarrow y) \wedge (x \rightarrow z) \leq x \wedge y \wedge (x \rightarrow z) \leq y \wedge z \Rightarrow (x \rightarrow y) \wedge (x \rightarrow z) \leq x \rightarrow (y \wedge z)$.
h_{11}. From $y \leq x \rightarrow y \Rightarrow (x \rightarrow y)^* \leq y^*$ and $x^* = x \rightarrow 0 \leq x \rightarrow y \Rightarrow (x \rightarrow y)^* \leq x^{**} \Rightarrow (x \rightarrow y)^* \leq x^{**} \wedge y^*$.
Conversely, $x^{**} \wedge y^* \wedge (x \rightarrow y) \leq x^{**} \wedge y^* \wedge [(x \wedge y^*) \rightarrow (y \wedge y^*)] = x^{**} \wedge y^* \wedge [(x \wedge y^*) \rightarrow 0] = x^{**} \wedge y^* \wedge [(x \wedge y^*) \rightarrow (0 \wedge y^*)] =$
$x^{**} \wedge y^* \wedge (x \rightarrow 0) = x^{**} \wedge y^* \wedge x^* = \mathbf{0}$, hence $x^{**} \wedge y^* \leq (x \rightarrow y)^*$.
h_{12}. Follows from h_1 or h_8;
h_{13}. Follows from h_5;
h_{14}. Follows from h_9. ∎

Corollary 5.1.10. If for $x_1, \ldots, x_n \in H$, we define $[x_1] = x_1$ and $[x_1, \ldots, x_{n+1}] = [x_1, \ldots, x_n] \rightarrow x_{n+1}$, then for every $x \in H$ and $1 \leq i \leq n$ we have
h_{15}: $x \wedge [x_1, \ldots, x_n] = x \wedge [x_1, \ldots, x_{i-1}, x \wedge x_i, x_{i+1}, \ldots, x_n]$.

We denote by **H** the class of Heyting algebras.

Corollary 5.1.11. The class H of Heyting algebras is equational.

Proof. It is immediate that $(L, \wedge, \vee, \rightarrow, 0) \in \mathbf{H}$ iff $(L, \wedge, \vee, \mathbf{0}) \in \mathbf{L}(\mathbf{0})$ and verifies the identities
H_1: $x \wedge (x \rightarrow y) = x \wedge y$;
H_2: $x \wedge (y \rightarrow z) = x \wedge [(x \wedge y) \rightarrow (x \wedge z)]$;
H_3: $z \wedge ((x \wedge y) \rightarrow x) = z$. ∎

For $H_1, H_2 \in$ **H**, a function $f : H_1 \to H_2$ is called a *morphism of Heyting algebras* if f is a morphism in **Ld(0)** and $f(x \to y) = f(x) \to f(y)$ for every $x, y \in H_1$.

Theorem 5.1.12. Let H Î Ld(0, 1). The following assertions are equivalent:
(i) H is a Heyting algebra ;
(ii) Every interval [a, b] in H is pseudocomplemented.

Proof. (i)⇒ (ii). Let $a, b \in H$ with $a \le b$; we shall prove that $[a, b] \in$ **H ,** so let $c, d \in [a, b]$.
We remark that $a \le (c \to d) \wedge b \le b$, hence $(c \to d) \wedge b \in [a, b]$.
Also, $c \wedge ((c \to d) \wedge b) = c \wedge (c \to d) \wedge b = c \wedge d \wedge b \le d$, so if $x \in [a, b]$ and $c \wedge x \le d$, then $x \le c \to d$. Since $x \le b$ we deduce that $x \le (c \to d) \wedge b$.
From the above we deduce that $c \wedge d = (c \to d) \wedge b$.
(ii)⇒(i). Let $a, b \in H$; we will prove that $a \to b = a^{*[a\wedge b)}$ (where by $a^{*[a\wedge b)}$ we denote the pseudocomplement of a in the filter $[a \wedge b)$, that is, a is the great element $x \in [a \wedge b)$ with the property $a \wedge x = a \wedge b$).
So, $a \wedge a^{*[a\wedge b)} = a \wedge b \le b$. Suppose that $a \wedge x \le b$.
Since $a \wedge b \le x \vee (a \wedge b) \le 1$ and $a \wedge [x \vee (a \wedge b)] = (a \wedge x) \vee (a \wedge b) = a \wedge b$, we deduce that $x \vee (a \wedge b) \le a^{*[a\wedge b)}$, hence $x \le a^{*[a\wedge b)}$. **n**

Corollary 5.1.13. If H is a Heyting algebra, then every closed interval in H is a Heyting algebra .

Corollary 5.1.14. If H is a Heyting algebra and (x → y) Ú (y → x) = 1 is an identity in H, then this is an identity in every interval in H.

Proof. By Theorem 5.1.12, if $c, d \in H$, $c \le d$ and $a, b \in [c, d]$, then
$(a \to b) \vee (b \to a) = [(a \to b) \wedge d] \vee [(b \to a) \wedge d] = ((a \to b) \vee (b \to a)) \wedge d = \ =1 \wedge d = d$. **n**

Theorem 5.1.15. Let H be a Heyting algebra and φ_H: H → I(H), φ_H (x) = (x] for every x Î H. Then φ_H is an embedding of H in the complete Heyting algebra I(H).

Proof. By Corollary 2.3.11 we deduce that φ_H is a morphism of lattices with 0. It is immediate that φ_H is injective.
By Proposition 5.1.8 we deduce that φ_H is a morphism of Heyting algebras. **n**

For $F \in$ **F** (H), we consider the binary relation on H:
$\theta_F = \{(x, y) \in H^2 : x \wedge i = y \wedge i$ for some $i \in F\}$ (see Proposition 2.5.3).

We denote by **Con(**H**)** the congruence lattice of H (see Chapter 2).

Theorem 5.1.16. If F Î F(H), then q_F Î Con(H) and the assignment F ® q_F is an isomorphism of ordered set between F(H) and Con(H).

Proof. Since $H \in$ **Ld(0, 1**), then $\theta_H \in$ **Con(H)** (in **Ld**). So, we only have to prove that θ_H is compatible with $\rightarrow$. Let $(x, x'), (y, y') \in \theta_F$.
Then there are $i, j \in F$ such that $x \wedge i = x' \wedge i$ and $y \wedge j = y' \wedge j$.
We deduce that $i \wedge j \wedge (x \rightarrow y) = i \wedge j \wedge [(x \wedge i \wedge j) \rightarrow (y \wedge i \wedge j)] = i \wedge j \wedge [(x \wedge i) \rightarrow (y \wedge j)] = i \wedge j \wedge [(x' \wedge i) \rightarrow (y' \wedge j)] = i \wedge j \wedge (x' \rightarrow y')$.
Since $i \wedge j \in F$ we deduce that $(x \rightarrow y, x' \rightarrow y') \in \theta_F$.
Clearly, if $F, G \in$ **F**(H) and $F \subseteq G \Rightarrow \theta_F \subseteq \theta_G$.
Suppose that $\theta_F \subseteq \theta_G$ and let $x \in F$.
Then $(x, \mathbf{1}) \in \theta_F$ (because $x \wedge x = \mathbf{1} \wedge x$), hence $(x, \mathbf{1}) \in \theta_G$, therefore there is $i \in G$ such that $x \wedge i = i$, hence $i \leq x$. Then $x \in G$, hence $F \subseteq G$.
To prove the surjectivity of the function $F \rightarrow \theta_F$ let $\theta \in$ **Con(**H**)** in **H** and denote $F_\theta = \{x \in H : (x, 1) \in \theta\}$. Then $F_\theta \in$ **F**(H) and we will prove that $\theta(F_\theta) = \theta$. If $(x, y) \in \theta(F_\theta)$, then $x \wedge i = y \wedge i$ for some $i \in F_\theta$, hence $(i, 1) \in \theta$ and $(i \wedge x, x), (i \wedge y, y) \in \theta$. Since $x \wedge i = y \wedge i$ we deduce that $(x, y) \in \theta$, hence $\theta(F) \subseteq \theta$.
Conversely, let $(x, y) \in \theta$. Then $(x \rightarrow y, 1) = (x \rightarrow y, y \rightarrow y) \in \theta$, hence $x \rightarrow y \in F_\theta$. Analogously $y \rightarrow x \in F_\theta$; since $x \wedge [(x \rightarrow y) \wedge (y \rightarrow x)] = x \wedge y = y \wedge [(x \rightarrow y) \wedge (y \rightarrow x)]$ (and $(x \rightarrow y) \wedge (y \rightarrow x) \in F_\theta$) we deduce that $(x, y) \in \theta(F_\theta)$, hence we have the equality $\theta(F_\theta) = \theta$. **n**

Proposition 5.1.17. If H is a Heyting algebra and $F \subseteq H$ is a non-empty set then the following are equivalent:

(i) $F \in F(H)$;

(ii) $1 \in F$ and if $x, y \in H$ such that $x, x \rightarrow y \in F$, then $y \in F$.

Proof. (i)⇒(ii). Clearly 1∈F and if x, y∈H such that x, x→y∈F then by Theorem 5.1.9, h_8, x∧(x→y) = x∧y∈F. Since x∧y ≤ y we deduce that y∈F.

(ii)⇒(i). If x, y∈H, x ≤ y and x∈F, since c = 1∈F we deduce that y∈F. Suppose that x, x→y ∈F. Since y ≤ x→y (by Theorem 5.1.9, h_4) we deduce that x→y∈F, so x∧y = x∧(x→y)∈F. ■

Remark 5.1.18. Following Proposition 5.1.17, the filters in a Heyting algebra are also called *deductive systems*.

For a Heyting algebra H we denote **D**(H) = {x∈H: x* = 0} (these elements will be called *dense*) .

Proposition 5.1.19. D(H)∈F(H).

Proof. By Proposition 5.1.9 it it will suffice to prove that **D**(H) is a deductive system. Since 1* = 1 → 0 = 0 we deduce that 1∈ **D**(H). Now let now x, y∈H such that x, x→y ∈ **D**(H), that is, x* = (x→y)* = 0. By Theorem 5.1.12, h_{11}, we deduce that (x→y)* = x** ∧ y* ⇔ 0 = 1∧y* ⇔ y* = 0, hence y∈**D**(H). ■

Corollary 5.1.20. A Heyting algebra H is a Boolean algebra iff D(H)={1}.

Proof. "⇒" Clearly (because if we have x∈H such that x* = x′∨0 = 0 ⇒x′ = 0 ⇒x = 1).
"⇐". Let x∈H. We have x∧x* = 0 and (x∨x*)* = x*∧x** = 0, hence x∨x* ∈**D**(H). By hypothesis x∨x* = 1, hence x* is the complement of x , so H is a Boolean algebra. ■

b.Hilbert and Hertz algebras

5.2. Definitions. Notations. Examples.Rules of calculus

Following Diego (see [37, p. 4]), by *Hilbert algebra* we mean the following concept:

Definition 5.2.1. We call *Hilbert algebra* an algebra (A,→, 1), of type (2,0) satisfying the following conditions:

a_1: x → (y → x) = 1;
a_2: (x → (y → z)) → ((x → y) → (x → z)) = 1;
a_3: If x → y = y → x = 1, then x = y.

In the same paper it is proved that Definition 5.2.1 is equivalent with

Definition 5.2.2. A *Hilbert algebra* is an algebra (A,→), where A is a nonempty set and → a binary operation on A such that the following identities are verified :
a_4: (x → x) → x = x;
a_5: x → x = y → y;
a_6: x → (y → z) = (x → y) → (x → z);
a_7: (x → y) → ((y → x) → x) = (y → x) → ((x → y) → y).

We deduce that the class of Hilbert algebras is equational. In [73]and [75], Hilbert algebras are called *positive implicative algebras.*

Examples

1. If (A, ≤) is a poset with 1, then (A,→, 1) is a Hilbert algebra, where for x, y ∈ A,

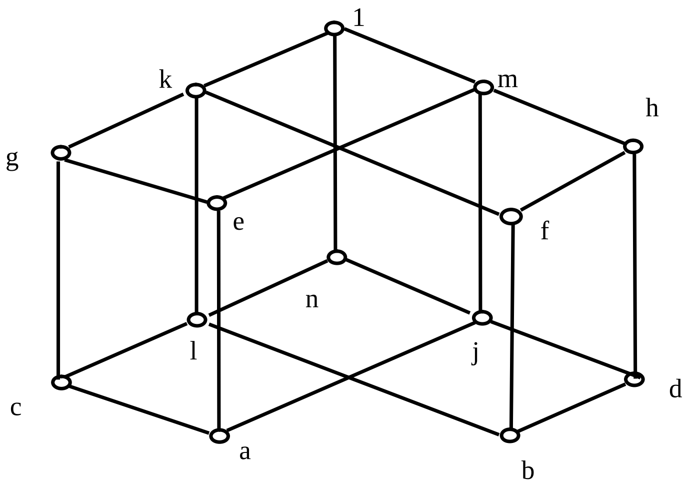
$$x \rightarrow y = \begin{cases} 1, & \text{if } x \leq y \\ y, & \text{if } x \nleq y . \end{cases}$$

2. If X is a nonempty set and τ a topology on X, then (τ,→, X) becomes a Hilbert algebra if for D_1, D_2 ∈ τ, we define

$$D_1 \rightarrow D_2 = \text{int}\, [(X \setminus D_1) \cup D_2].$$

3. If (A,∨,∧,0) is a Heyting algebra then for every x, y ∈ A there is an element denoted by x → y ∈ A such that if z ∈ A, then x ∧ z ≤ y iff z ≤ x → y; so, (A,→, 1) become a Hilbert algebra (where 1 = a → a, for an element a ∈ A).
4. If (A,∨,∧,′,0,1) is a Boolean algebra, then (A,→,1) is a Hilbert algebra, where for x, y ∈ A, x → y = x′ ∨ y.

5. There are Hilbert algebras which are not Heyting or Boolean algebras. Such an example is offered by the following diagram (see [37, p.9]):



The table of composition of this diagram is given by Skolem and it is mentioned in [37], at page 10.
If (A,→) is a Hilbert algebra in the sense of Definition 5.1.2, then we denote $1 = a \to a$ for some element $a \in A$ (this is possible by the axiom a_5).
On A we define a relation of order: $x \leq y$ iff $x \to y = 1$ (see [37, p.5]). This order will be called the *natural ordering* on A. Relative to the natural ordering on A, 1 is the greatest element. If relative to natural ordering A has the smallest element 0, we say that A is *bounded*; in this case, for $x \in A$ we denote $x^* = x \to 0$. If A is a Boolean algebra, then $x^* = x'$.

Definition 5.2.3. If A is a Hilbert algebra, we call *deductive system* in A every non-empty subset D of A which verifies the following axioms:

a_8: $1 \in D$;

a_9: If $x, y \in A$ and $x, x \to y \in D$, then $y \in D$.

It is immediate that {1} and A are trivial examples of deductive systems of A; every deductive system different from A will be called *proper*.
We denote by **Ds**(A) the set of all deductive systems of A. If A is bounded, then $D \in$ **Ds**(A) is proper iff $0 \notin D$.
In the case of Heyting or Boolean algebras, the deductive systems are in facts the filters of respective algebras.
For two elements x, y of a bounded Hilbert algebra A we denote:

$$x \sqcup y = (x \to y) \to y;$$

$x \veebar y = x^* \to y$;
$x \,\Delta\, y = (x \to y) \to ((y \to x) \to x)$.

As it follows $x \sqcup y$, $x \veebar y$, $x \,\Delta\, y$ are by natural order on A, majorants for x and y.
It is shown that in general, this majorants are different for a pair (x, y) of elements in A; it is also shown what they becames in an Heyting or Boole algebra and in what case one of them is the supremum of x and y.

Definition 5.2.4. If A is a Hilbert algebra, we call *Hilbert subalgebra* of A every nonempty subset $S \subseteq A$ which verifies the axiom
a_{10}: If $x, y \in S$, then $x \to y \in S$.

If A is bounded, we add, to a_9,the condition that $0 \in S$.
In the case of unbounded Hilbert algebras, their deductive systems are also Hilbert subalgebras. We denote by **Alg**(A) the set of all *subalgebras* of A (see Chapter 3).

Definition 5.2.5. If A_1 and A_2 are two Hilbert algebras, a function $f : A_1 \to A_2$, will be called *morphism of Hilbert algebras* if for every x, y $\in A_1$ we have:
a_{11}: $f(x \to y) = f(x) \to f(y)$.

If A_1 and A_2 are bounded Hilbert algebras, f will be called *morphism of bounded Hilbert algebras* if verifies a_{11} and the condition $f(0) = 0$.
We note that the morphisms of Hilbert algebras map 1 into 1 (this follows immediate from a_{11} if we consider $x = y = 1$).
In what follows by $\mathbf{H_i}$ (respective, $\overline{H_i}$) we denote the *category of Hilbert algebras* (respective, *bounded Hilbert algebras*).
Since the class of Hilbert algebras is equational, in $\mathbf{H_i}$ the monomorphisms are exactly injective morphisms; the same thing is also valid for $\overline{H_i}$ (see Proposition 4.2.9).

Definition 5.2.6. If A is a Hilbert algebra and $S \subseteq A$ is a non-empty subset, we denote by <S> the lowest deductive system of A (relative to inclusion) which contains S; we call <S> *deductive system generated by S*.

In [80] Tarski proves that $\langle S\rangle = \bigcup_{\substack{F\subseteq S,\\ F \text{ finite}}} \langle F\rangle$.

If $F = \{a_1, a_2, ..., a_n\} \subseteq A$ is a finite set, we denote by $<a_1,a_2,..., a_n>=<F>$; if $F = \{a\} \subseteq A$, then we denote $<a> = <F>$ which will be called the *principal deductive system* generated by a.

In [75,p. 27] it is proved that

$<a_1, a_2, ..., a_n> = \{x \in A : a_1 \to (a_2 \to ... \to (a_n \to x) ...) = 1\}$.

In particular, we deduce that $<a> = \{x \in A : a \le x\} = [a)$.

It is immediate that relative to inclusion **Ds**(A) becomes a bounded lattice, where for $D_1, D_2 \in Ds(A)$, $D_1 \wedge D_2 = D_1 \cap D_2$, $D_1 \vee D_2 = < D_1 \cup D_2 >$, **0** = {1} and **1** = A.

Definition 5.2.7. An element x of a Hilbert algebra A is called *regular* if x = x and *dense* if x* = 0.**

We denote by **D**(A), respective **R**(A), the set of all dens, respective regular elements of A.

If A is a Hilbert algebra and D∈**Ds**(A), then the relation $(x, y)\in \theta(D)$ iff $x \to y, y \to x \in D$ is a congruence on A (see [18], [37]); for an element $x \in A$ we denote by x/D the equivalence class of a relative to θ(D) and by A/D the quotient Hilbert algebra, where for x, y∈ A, $(x/D) \to (y/D) = (x \to y)/D$ and **1** = 1/D = D.

Definition 5.2.8. If (A, ≤) is a poset with 1, we say that p∈A is the *penultimate element* of A, if p ≠ 1 and for every x ∈ A, x ≠ 1, we have x ≤ p.

Remark 5.2.9. If A and A′ are Hilbert algebras and $f : A \to A'$ is a morphism of Hilbert algebras, we denote by **Ker**(f) = {x ∈ A: f(x) = 1}. It is immediate that **Ker**(f) ∈ **Ds**(A) and f is injective iff **Ker**(f) = {1}, (see [18],[37]).

Now let some rules of calculus in a Hilbert algebra.

Theorem 5.2.10. If A is a Hilbert algebra and x, y, z ∈ A, then :

c_1: $1 \rightarrow x = x$, $x \rightarrow 1 = 1$;
c_2: $x \leq y \rightarrow x$, $x \leq (x \rightarrow y) \rightarrow y$;
c_3: $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$;
c_4: $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$;
c_5: If $x \leq y$, then $z \rightarrow x \leq z \rightarrow y$ and $y \rightarrow z \leq x \rightarrow z$;
c_6: $((x \rightarrow y) \rightarrow y) \rightarrow y = x \rightarrow y$.

Proof. Excepting c_6, all is proved in [37, p.5].
To prove c_6, we deduce from c_2 and c_5 that $((x \rightarrow y) \rightarrow y) \rightarrow y \leq x \rightarrow y$ and by c_2 that $x \rightarrow y \leq ((x \rightarrow y) \rightarrow y) \rightarrow y$, hence $((x \rightarrow y) \rightarrow y) \rightarrow y = x \rightarrow y$ ■

Corollary 5.2.11. If A is a bounded Hilbert algebra and $x, y \in A$, then
c_7: $0^* = 1$, $1^* = 0$;
c_8: $x \rightarrow y^* = y \rightarrow x^*$;
c_9: $x \rightarrow x^* = x^*$, $x^* \rightarrow x = x^{**}$, $x \leq x^{**}$, $x \leq x^* \rightarrow y$;
c_{10}: $x \rightarrow y \leq y^* \rightarrow x^*$;
c_{11}: If $x \leq y$, then $y^* \leq x^*$;
c_{12}: $x^{***} = x^*$.

Proof. c_7 follows from c_1 for $x = 0$. c_8 follows from c_3 for $z = 0$. The first relation of c_9 follows from a_6 for $y = x$ and $z = 0$, the third relation follows from c_2 for $y = 0$ and for the last relation we use c_6. ■

Remark 5.2.12. If (X, τ) is a topological space and $D \in \tau$, then $D^* = \text{int}(X - D)$, $D^{**} = \text{int}(\overline{D})$, where $\overline{D}$ is the *aderence* of D.
For n elements $x_1, x_2, ..., x_n$ of a Hilbert algebra A we define:

$$(x_1, x_2, ..., x_{n-1}; x_n) = \begin{cases} x_n, & \text{if } n = 1 \\ x_1 \rightarrow (x_2, ..., x_{n-1}; x_n), & \text{if } n \geq 2 \end{cases}$$

Theorem 5.2.13. Let A be a Hilbert algebra and $x, y, x_1, x_2, ..., x_n \in A$ ($n \geq 2$).
Then:
c_{13}: If σ is a permutation of elements of the set {1, 2, ..., n}, we have $(x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(n)}; x) = (x_1, ..., x_n; x)$;
c_{14}: $x \rightarrow (x_1, x_2, ..., x_{n-1}; x_n) = (x, x_1, ..., x_{n-1}; x_n) = \ldots = (x_1, x_2, ..., x_{n-1}, x; x_n)$;
c_{15}: $(x_1, x_2, ..., x_n; x \rightarrow y) = (x_1, x_2, ..., x_n; x) \rightarrow (x_1, x_2, ..., x_n; y)$.

Proof. c_{13} and c_{14} follow using mathematical induction relative to n and c_{15} follow from a_6. ■

Remark 5.2.14. If A is a Hilbert algebra without 0, then by adding a new element $0 \notin A$ and define in $A' = A \cup \{0\}$ the implication as in table

→	0	x
	1	
0	1	1
x	1	
1	0	1
	1	
	0	x
	1	

(where $x \in A$), then $(A', \to, 0, 1)$ becomes a bounded Hilbert algebra.
We verify the axioms $a_4 – a_7$.
a_4: $(0 \to 0) \to 0 = 1 \to 0 = 0$;

a_5: $0 \to 0 = 1 = x \to x$ for every $x \in A$;

a_6: If $x = 0$ and $y, z \in A$, then $x \to (y \to z) = 0 \to (y \to z) = 1$ and $(x \to y) \to (x \to z) = (0 \to y) \to (0 \to z) = 1 \to 1 = 1$, so a_6 is verified.
If $y = 0$, then $x \to (y \to z) = x \to (0 \to z) = x \to 1 = 1$ and $(x \to y) \to (x \to z) = (x \to 0) \to (x \to z) = 0 \to (x \to z) = 1$, hence a_6 is verified.
If $z = 0$, then $x \to (y \to z) = x \to (y \to 0) = x \to 0 = 0$ and $(x \to y) \to (x \to z) = (x \to y) \to (x \to 0) = (x \to y) \to 0 = 0$, so a_6 is also verified.

If $x = y = 0$ and $z \in A$, then $x \to (y \to z) = 0 \to (0 \to z) = 1$ and $(x \to y) \to (x \to z) = (0 \to 0) \to (0 \to z) = 1 \to 1 = 1$, hence a_6 is verified.
If $y = z = 0$, then $x \to (y \to z) = x \to (0 \to 0) = x \to 1 = 1$ and $(x \to y) \to (x \to z) = (x \to 0) \to (x \to 0) = 0 \to 0 = 1$, hence a_6 is also verified.
If $x = z = 0$, then $x \to (y \to z) = 0 \to (y \to 0) = 0 \to 0 = 1$ and $(x \to y) \to (x \to z) = (0 \to y) \to (0 \to 0) = 1 \to 1 = 1$, hence a_6 is verified.
Since we have verified all possibilities, we deduce that a_6 is verified.
a_7: If $x = 0$, then $(x \to y) \to ((y \to x) \to x) = (0 \to y) \to ((y \to 0) \to 0) = 1 \to (0 \to 0) = 1 \to 1 = 1$ and $(y \to x) \to ((x \to y) \to y) =$

$(y \to 0) \to ((0 \to y) \to y) = 0 \to (1 \to y) = 0 \to y = 1$, hence a_7 is also verified.
If $y = 0$, analogously we deduce that a_7 is verified.

If $x = y = 0$, then $(x \to y) \to ((y \to x) \to x) = (0 \to 0) \to ((0 \to 0) \to 0) = (y \to x) \to ((x \to y) \to y)$ hence a_7 is true.

Remark 5.2.15. In general, in a bounded Hilbert algebra, for two elements x, y, the elements $x \sqcup y$, $x \veebar y$ and $x \Delta y$ are different two by two.
Indeed, if $A = \{0, a, b, c, d, e, f, g, h, i, j, k, m, n, 1\}$ is the Skolem's example, then the table of composition is the following:

→	0	a	b	c	d	e	f		h	i	j	k	m	n	1
→			b	c		e	f	g	h	i		k	m		
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
a	0	1	h	1	h	1	h	1	h	1	1	1	1	1	1
b	0	g	1	g	1	g	1	g	1	1	1	1	1	1	1
c	0	m	h	1	h	m	h	1	h	1	m	1	m	1	1
d	0	g	k	g	1	g	k	g	1	k	1	k	1	1	1
e	0	n	d	n	d	1	h	1	h	n	n	1	1	n	1
f	0	c	n	c	n	g	1	g	1	n	n	1	1	n	1
g	0	j	d	n	d	m	h	1	h	n	j	1	m	n	1
h	0	c	i	c	n	g	k	g	1	i	n	k	1	n	1
i	0	e	h	g	n	e	h	g	h	1	m	1	m	1	1
j	0	g	f	g	h	g	f	g	h	k	1	k	1	1	1
k	0	a	d	c	d	e	h	g	h	n	j	1	m	n	1
m	0	c	b	c	d	g	f	g	h	i	n	k	1	n	1
n	0	e	f	g	h	e	f	g	h	k	m	k	m	1	1
1	0	a	b	c	d	e	f	g	h	i	j	k	m	n	1

For the elements a, b we have:

$a \sqcup b = (a \to b) \to b = h \to b = i$

$a \veebar b = (a \to 0) \to b = 0 \to b = 1$
$a \Delta b = (a \to b) \to ((b \to a) \to a) = h \to (g \to a) = h \to j = n$,

so the elements $a \sqcup b$, $a \veebar b$ and $a \Delta b$ in general are different two by two.

If A is a Heyting, it doesn't result that for $x, y \in A$, $x \sqcup y$, $x \veebar y$ or $x \Delta y$ is the supremum of x and y; indeed, if A is the chain $\{0, x, y, 1\}$, this become in canonical way Hilbert (Heyting) algebras.

In this algebra we have: $x \Delta y = (x \to y) \to ((y \to x) \to x) = 1 \to (x \to x) = 1 \to 1 = 1$, but $x \vee y = y$; also $x \veebar y = (x \to 0) \to y = 0 \to y = 1 \neq x, y$.

If A is a Boolean algebra, then for $x, y \in A$ we have $x \sqcup y = x \veebar y = x \vee y$.

Indeed, $x \sqcup y = (x \to y) \to y = (x \to y)' \vee y = (x' \vee y)' \vee y = (x \wedge y') \vee y = (x \vee y) \wedge (y' \vee y) = x \vee y$, $x \veebar y = x' \to y = x'' \vee y = x \vee y$, $x \Delta y = (x \to y) \to ((y \to x) \to x) = (x' \vee y)' \vee (x \vee y) = (x \wedge y') \vee (x \vee y) = x \vee y$.

Theorem 5.2.16. If A is a bounded Hilbert algebra and x, y, z $\in$A, then
c_{16}: $x \leq x \sqcup y$, $y \leq x \sqcup y$, $x \sqcup x = x$, $x \sqcup 0 = x^{**}$, $x \sqcup 1 = 1$, $x \sqcup x^* = 1$, $x \leq y$ iff $x \sqcup y = y$, $(x \to y) \wedge (x \sqcup y) = y$;
c_{17}: $x \leq x \veebar y$, $y \leq x \veebar y$, $x \veebar x = x^{**}$, $x \veebar 0 = x^{**}$, $x \veebar 1 = 1$,
$x \veebar x^* = 1$.

Proof. **c_{16}.** From c_2 we deduce that $x \sqcup y$ is a majorant for x and y. We have $x \sqcup x = (x \to x) \to x = 1 \to x = x$, $x \sqcup 0 = (x \to 0) \to 0 = x^{**}$, $x \sqcup 1 = (x \to 1) \to 1 = 1 \to 1 = 1$, $x \sqcup x^* = (x \to x^*) \to x^* = x^* \to x^* = 1$. If $x \leq y$, then $x \sqcup y = (x \to y) \to y = 1 \to y = y$.

If $x \sqcup y = y$, since $x \leq x \sqcup y$ we deduce that $x \leq y$.

We have that $x \leq y \to x$ and $x \leq ((y \to x) \to x$ and let $t \in A$ such that $t \leq y \to x$ and $t \leq ((y \to x) \to x$.
Then $((y \to x) \to x) \to x \leq t \to x$, hence $y \to x \leq t \to x$; since $t \leq y \to x$, by transitivity we deduce that $t \leq t \to x$, hence $t \leq x$, from where the last equality results.
c_{17}. From $0 \leq y$ we deduce that $x \to 0 \leq x \to y$, hence $x^* \leq x \to y$ and $x \leq x^* \to y$, therefore $x \leq x \veebar y$. Also, $x \veebar x = x^* \to x$; now let's prove that $x^* \to x = x^{**}$. For this, if in a_7 we consider $y = 0$, we obtain $(x \to 0) \to ((0 \to x) \to x) = (0 \to x) \to ((x \to 0) \to 0)$, hence $x^* \to x = x^{**}$. We also have $x \veebar 0 = x^* \to 0 = x^{**}$, $x \veebar 1 = x^* \to 1 = 1$ and $x \veebar x^* = x^* \to x^* = 1$. ■

Theorem 5.2.17. If A is a bounded Hilbert algebra and x, y, z $\in$A, then
c_{18}: $x \Delta y = y \Delta x$, $x \leq x \Delta y$, $y \leq x \Delta y$;
c_{19}: $x \Delta x = x$, $x \Delta 0 = x^{**}$, $x \Delta 1 = 1$;

c_{20}: $x \Delta (x \to y) = 1$, $x \Delta x^* = 1$;
c_{21}: $z \to (x \Delta y) = (z \to x) \Delta (z \to y)$;
c_{22}: $(x \to y) \Delta z = x \to (y \Delta z)$, $(x \to y)^{**} = x^{**} \to y^{**}$;
c_{23}: $(x \to y) \Delta (y \to x) = 1$.

Proof. **c_{18}.** Follows from a_7 and c_2.
c_{19}. We have $x \Delta x = (x \to x) \to ((x \to x) \to x) = 1 \to (1 \to x) = 1 \to x = x$, $x \Delta 0 = (x \to 0) \to ((0 \to x) \to x) = x^* \to (1 \to x) = x^* \to x = x^{**}$, and $x \Delta 1 = (x \to 1) \to ((1 \to x) \to x) = 1 \to (x \to x) = 1 \to 1 = 1$.
c_{20}. We have $x \Delta (x \to y) = (x \to (x \to y)) \to (((x \to y) \to x) \to x) = = (x \to y) \to (((x \to y) \to x) \to x) = 1$ (by c_2).
For $y = 0$, we obtain that $x \Delta x^* = 1$.
c_{21}. Using a_6 we have $z \to(x \Delta y) = z \to ((x \to y) \to((y \to x) \to x)) = (z \to (x \to y))\to (z \to ((y \to x) \to x)) = ((z \to x) \to (z \to y)) \to (((z \to y) \to (z \to x)) \to (z \to x)) = (z \to x) \Delta (z \to y)$.
c_{22}. We have $(x \to y) \Delta z = ((x \to y) \to z) \to ((z \to (x \to y)) \to (x \to y))$.
But $(z \to (x \to y)) \to (x \to y) = (x \to (z \to y)) \to (x \to y) = x \to ((z \to y) \to y)$ hence $(x \to y) \Delta z = ((x \to y)\to z) \to (x \to ((z \to y) \to y)) = x \to (((x \to y) \to z) \to ((z \to y) \to y)) = ((x \to y) \to (x \to z)) \to (x \to ((z \to y) \to y)) = x \to ((y \to z) \to ((z \to y) \to y)) = x \to (y \Delta z)$, that is, the desired relation.
For $z = 0$ we obtain that $(x \to y) \Delta 0 = x \to (y \Delta 0)$. By c_{19} we have $(x \to y) \Delta 0 = (x \to y)^{**}$ and $y \Delta 0 = y^{**}$, so $(x \to y)^{**} = x \to y^{**}$.
By c_8 we have $x \to y^{**} = y^* \to x^*$ and $x^{**} \to y^{**} = y^* \to x^{***} = y^* \to x^*$, hence $(x \to y)^{**} = x^{**} \to y^{**}$.
c_{23}. We have $(x \to y) \Delta (y \to x) = ((x \to y) \to (y \to x)) \to (((y \to x) \to (x \to y)) \to (x \to y))$. But $(x \to y) \to (y \to x) = y \to ((x \to y) \to x) = ((y \to (x \to y)) \to (y \to x) = 1 \to (y \to x) = y \to x$, hence $(x \to y) \Delta (y \to x) = (y \to x) \to ((x \to y) \to (x \to y)) = (y \to x) \to 1 = 1$. ■

In the following paragraphs we will put in evidence some rules of calculus relative to $\veebar$ and Δ.
We recall that for two deductive systems $D_1, D_2 \in$ **Ds**(A), in the lattice (**Ds**(A), $\subseteq$) we have $D_1 \vee D_2 = \langle D_1 \cup D_2 \rangle$.

Theorem 5.2.18. If A is a Hilbert algebra and $D_1, D_2 \in$ Ds(A), then $D_1 \vee D_2 = \{x \in A$: there are $x_1, x_2, ..., x_n \in D_1$ such that $(x_1, ... , x_n; x) \in D_2\}$.

Proof. Let $D = \{x \in A$: there are $x_1, x_2, ..., x_n \in D_1$ such that $(x_1, ..., x_n; x) \in D_2\}$.

Firstly we will prove that $D \in$ **Ds**(A). Clearly $1 \in D$ and let $x, x \to y \in D$; then there are $x_1, x_2, ..., x_n, y_1, y_2, ..., y_m \in D_1$ such that $(x_1, x_2, ..., x_n; x)$, $(y_1, y_2, ..., y_m; x \to y) \in D_2$.

By c_{15} we deduce that $(y_1, y_2, ..., y_m; x) \to (y_1, y_2, ..., y_m; y) \in D_2$, therefore $x_n \to ((y_1, y_2, ..., y_m; x) \to (y_1, y_2, ..., y_m; y)) \in D_2$.

By c_{14} and c_{15} we can write the last relation as
$(y_1, y_2, ..., y_m ; x_n \to x) \to (y_1, y_2, ..., y_m; x_n \to y) \in D_2$.

By inductively reasoning relative to n we deduce that $(y_1, y_2, ..., y_m, x_1, x_2, ..., x_n; x) \to (y_1, y_2, ..., y_m, x_1, x_2, ..., x_n; y) \in D_2$.

Since $(y_1, y_2, ..., y_m, x_1, x_2, ..., x_n; x) = (y_1, y_2, ..., y_m; (x_1, x_2, ..., x_n; x)) \in D_2$ we deduce that $(y_1, y_2, ..., y_m, x_1, x_2, ..., x_n; y) \in D_2$, hence $y \in D$.

We will prove that $D_1 \vee D_2 \subseteq D$. If $x \in D_1$, then, since $x \to x = 1$, we deduce that $x \in D$, hence $D_1 \subseteq D$. Since for $x \in D_2$, $1 \to x = x \in D_2$ we deduce that $D_2 \subseteq D$, hence $D_1 \vee D_2 \subseteq D$.

To prove $D \subseteq D_1 \vee D_2$, let $x \in D$; then there are $x_1, x_2, ..., x_n \in D_1$ such that $(x_1, x_2, ..., x_n; x) \in D_2 \subseteq D_1 \vee D_2$.

Since $(x_1, x_2, ..., x_n; x) = (x_2, ..., x_n; x) \in D_2 \subseteq D_1 \vee D_2$ and $x \in D_1$ we deduce that $(x_2, ..., x_n; x) \in D_1 \vee D_2$; reasoning inductively relative to n, we deduce that $x \in D_1 \vee D_2$, hence $D \subseteq D_1 \vee D_2$. ■

Corollary 5.2.19. If A is a Hilbert algebra, $D \in$ Ds(A), $a, x_1, x_2, ..., x_n \in A$, then

c_{24}: $[a) \vee D = \{x \in A : a \to x \in D\}$;

c_{25}: $<x_1, x_2, ..., x_n> = \{x \in A: (x_1, x_2, ..., x_n; x) = 1\}$.

Proof. **c_{24}.** Let $x \in [a) \vee D$; by Theorem 5.2.18 there are $x_1, x_2,..., x_n \in D$ such that $(x_1, x_2, ..., x_n; x) \in [a)$, hence
$a \leq (x_1, x_2, ..., x_n; x) \Leftrightarrow (x_1, x_2, ..., x_n; a \to x) = 1$.

Since $x_1, x_2, ..., x_n \in D$ and $x_1 \to (x_2, x_3, ..., x_n; a \to x) = 1 \in D$, we deduce that $(x_2, x_3, ..., x_n; a \to x) \in D$; successively we deduce that $a \to x \in D$, hence $[a) \vee D \subseteq \{x \in A: a \to x \in D\}$. Since the other inclusion is clear (since $a \in [a)$) we obtain the equality from the enounce.

c_{25}. We write $<x_1, x_2, ..., x_n> = <x_1, x_2, ..., x_{n-1}> \vee [x_n)$ and use c_{24}. ■

In what follows we will establish the condition that a Hilbert algebra is a Boolean algebra relative to natural ordering.

Theorem 5.2.20. For a bounded Hilbert algebra A, the following assertions are equivalent:
(i) A is a Boolean algebra relative to natural ordering;
(ii) For every $x \in A$, $x^{**} = x$.

Proof. (i)⇒(ii). If A is a Boolean algebra relative to natural ordering, then for every $x \in A$ we have $x^{**} = x'' = x$.

(ii)⇒(i). Firstly we shall prove that for every $x, y \in A$ there are $x \wedge y \in A$ and $x \wedge y = (x \to y^*)^*$.

Indeed, from $0 \le y^*$ we deduce successively $x \to 0 = x^* \le x \to y^*$, $(x \to y^*)^* \le x^{**} = x$ and by $y^* \le x \to y^*$ we deduce that $(x \to y^*)^* \le y^{**} = y$. Now let $t \in A$ such that $t \le x$, $t \le y$. Then $y^* \le t^*$, hence $x \to y^* \le x \to t^* \le t \to t^* = t^*$, therefore $t = t^{**} \le (x \to y^*)^*$.

We have to prove now that for every $x, y \in A$ there are $x \vee y \in A$ and $x \vee y = x^* \to y = x \veebar y$.

Indeed, by c_{17} $x, y \le x \veebar y$. Now let $t \in A$ such that $x \le t$ and $y \le t$. From $x \le t$ we deduce that $t^* \le x^*$, hence $x^* \to y \le t^* \to y \le t^* \to t = t^{**} = t$.

Therefore we have proved that $(A, \vee, \wedge, 0, 1)$ is a bounded lattice.

We have to prove now that A is a Heyting algebra.

Indeed, if $x, y, z \in A$, then $x \wedge z \le y \Leftrightarrow (x \to z^*)^* \le y$, hence we deduce that $y^* \le x \to z^* \Rightarrow x \le y^* \to z^* \le z^{**} \to y^{**} = z \to y \Rightarrow z \le x \to y$. Since the proof of converse implication is analogous, we deduce that $x \wedge z \le y$ iff $z \le x \to y$, hence $(A, \vee, \wedge, \to, 0)$ is a Heyting algebra.

Following Corollary 5.1.20, to prove that A is a Boolean algebra it will suffice to prove that $\mathbf{D}(A) = \{1\}$.

Indeed, if $x \in \mathbf{D}(A)$, then $x^* = 0$, hence $x = x^{**} = 0^* = 1$. ■

Corollary 5.2.21. A bounded Hilbert algebra A is a Boolean algebra (relative to natural ordering) iff for every $x, y \in A$ we have $(x \to y) \to x = x$.

Proof. "⇒" If A is a Boolean algebra, then for every $x, y \in A$, we have $(x \to y) \to x = (x \to y)' \vee x = (x' \vee y)' \vee x = (x'' \wedge y') \vee x = (x \wedge y') \vee x = x$.

"⇐" If $(x \to y) \to x = x$ for every $x, y \in A$, then for $y = 0$ we obtain that for every $x \in A$, $x^* \to x = x$, hence $x^{**} = x$. By Theorem 5.2.20, A is a Boolean algebra. ■

Corollay 5.2.22. For a bounded Hilbert algebra A, the following assertions are equivalent:

(i) A is Boolean algebra (relative to natural ordering);

(ii) For every $x, y \in A$, $x \sqcup y = y \sqcup x$;

(iii) For every $x, y \in A$, $x \veebar y = y \veebar x$;

(iv) For every $x, y \in A$, $x \sqcup y = x \vee y$;

(v) For every $x, y \in A$, $x \veebar y = x \vee y$.

Proof. The implications (i)⇒(ii), (iii), (iv), (v) are immediate, since in a Boolean algebra A for any elements $x, y \in A$ we have $x \sqcup y = x \veebar y = x \vee y$.

(ii) ⇒ (i). If in the equality $(x \to y) \to y = (y \to x) \to x$ we put $y = 0$, we obtain that $(x \to 0) \to 0 = (0 \to x) \to x$, hence $x^{**} = x$ and apply Theorem 5.2.20.

(iii)⇒(i). If in the equality $x^* \to y = y^* \to x$ we put $y = 0$ we obtain that $x^* \to 0 = 0^* \to x$, hence $x^{**} = x$ and if we apply Theorem 5.2.20.

(iv)⇒(i). Also if in the equality $(x \to y) \to y = x \vee y$ if we put $y = 0$, then we obtain that $x^{**} = x$ and apply Theorem 5.2.20.

(v)⇒(i). If in the equality $x^* \to y = x \vee y$ we put $y = 0$, then we obtain that $x^{**} = x$ and we apply Theorem 5.2.20. ■

Corollary 5.2.23. If A is a bounded Hilbert algebra, then the following assertions are equivalent:
(i) A is Boolean algebra (relative to natural ordering);
(ii) For every x, y $\in$ A, x Δ y = x $\vee$ y;
(iii) For every x, y, z $\in$ A, if x $\le$ y, then x Δ z $\le$ y Δ z.

Proof. The implications (i)$\Rightarrow$ (ii), (iii) are true because A is a Boolean algebra, hence if x, y $\in$ A, then x Δ y = x $\vee$ y.
(ii)$\Rightarrow$(i). For y = 0, we obtain that for every x $\in$ A, x Δ 0 = x$\vee$0 $\Leftrightarrow$ x** = x and applying Theorem 5.1.20.
(iii)$\Rightarrow$(i). Since 0 $\le$ x, then x Δ 0 $\le$ x Δ x, hence x**$\le$ x, therefore x** = x and applying Theorem 5.2.20. ■

The following result shows that and for Hilbert algebras we have a theorem of *Glivenko type* (see and Proposition 2.6.17):

Theorem 5.2.24. If A is a bounded Hilbert algebra, then R(A) becomes a Boolean algebra, where for x, y $\in$ R (A),

$$\mathbf{x \wedge y = (x \to y^*)^* \in R(A)}$$
$$\mathbf{x \vee y = (x^* \wedge y^*)^* \in R(A)}$$
$$\mathbf{x' = x^* \in R(A).}$$

The function φ_A : A $\to$ R(A), φ_A (x) = x for every x $\in$ A is a surjective morphism of bounded Hilbert algebras.**

Proof. Firstly we remark that if x, y $\in$ **R**(A), then x** = x and y** = y, hence x $\to$ y $\in$ **R**(A), because by c_{22} we have (x $\to$ y)** = x $\to$ t. The proof continues as in the case of Theorem 5.2.20, because in fact x $\vee$ y = (x*$\to$ y**)** = (x*$\to$ y)** = x* $\to$ y = x $\veebar$ y.
The fact that φ_A is a surjective morphism of bounded Hilbert algebras follows from c_{12} and c_{22}. ■

Lemma 5.2.25. If A is a bounded Hilbert algebra, then D(A) $\in$ Ds(A).

Proof. Since 1* = 1 $\to$ 0 = 0 we deduce that 1 $\in$ **D**(A).

Now let's suppose that $x, x \to y \in \mathbf{D}(A)$; then $x^* = (x \to y)^* = 0$ and we will prove that $y^* = 0$.
By $(x \to y) \to 0 = 0$ we deduce that $x \to ((x \to y) \to 0) = x \to 0 = 0$, hence $(x \to y) \to (x \to 0) = 0 \Leftrightarrow x \to (y \to 0) = 0 \Leftrightarrow x \to y^* = 0$. On the other hand, by $x \to 0 = 0$ we deduce that $y \to (x \to 0) = y \to 0 \Leftrightarrow x \to y^* = y^*$. Since $x \to y^* = 0$, we deduce that $y^* = 0$, hence $y \in \mathbf{D}(A)$. ■

Lemma 5.2.26. If A is a bounded Hilbert algebra, then:
(i) For every $x \in A$, $x^{**} \to x \in D(A)$; $x \in D(A)$ iff $x = y^{**} \to y$ for some $y \in A$;
(ii) For every $x \in A$ there exists $x^{**} \wedge (x^{**} \to x)$ and $x^{**} \wedge (x^{**} \to x) = x$;
(iii) For every $x \in A$, $(x^{**} \to x) \to x = x^{**}$.

Proof. (i). Let $d = x^{**} \to x$; from c_2 and c_{17} we deduce that $x \le d$ and $x^* \le d$, hence $d^* \le x^* \le d$, therefore $d^* \le d \Leftrightarrow d^* \to d = 1 \Leftrightarrow d^{**} = 1 \Leftrightarrow d^* = 0 \Leftrightarrow d \in \mathbf{D}(A)$.
If $x \in \mathbf{D}(A)$, then for $y = x$, we obtain that $y^{**} \to y = 0^* \to x = 1 \to x = x$.
(ii). Clearly $x \le x^{**}$ and $x \le x^{**} \to x$; now let $t \in A$ such that $t \le x^{**}$ and $t \le x^{**} \to x$. We deduce that $x^{**} \le t \to x$, hence $t \le x^{**} \le t \to x$, so $t \le x$, that is, $x = x^{**} \wedge (x^{**} \to x)$.
(iii). If in a_7 we consider $y = x^{**}$ we obtain $(x \to x^{**}) \to ((x^{**} \to x) \to x) = (x^{**} \to x) \to ((x \to x^{**}) \to x^{**})$.
Since $x \to x^{**} = 1$ we obtain that $(x^{**} \to x) \to x = (x^{**} \to x) \to x^{**} = (x^{**} \to x) \to (x^*)^* = x^* \to (x^{**} \to x)^* = x^* \to 0 = x^{**}$. ■

Corollary 5.2.27. If A is a bounded Hilbert algebra, then for every $x \in A$ we have $x = y \wedge z$, with $y = x^{**} \in R(A)$ and $z = x^{**} \to x \in D(A)$.

Remark 5.2.28. In [66, p. 133], Nemitz proves an analogous result for implicative semilattices.

Theorem 5.2.29. Let A be a bounded Hilbert algebra and $x, y \in A$. Then $x^{**} = y^{**}$ iff there are $d_1, d_2 \in D(A)$ such that $d_1 \to x = d_2 \to y$.

Proof. "⇒". Suppose that $d_1 \to x = d_2 \to y$, with $d_1, d_2 \in$ **D**(A).

From c_{22} we deduce that $(d_1 \to x)^{**} = (d_2 \to y)^{**} \Leftrightarrow d_1^{**} \to x^{**} = d_2^{**} \to y^{**}$. Since $d_1^{**} = d_2^{**} = 1$, we deduce that $x^{**} = y^{**}$.

"⇐". If $x^{**} = y^{**}$, then by Lemma 5.2.26, (iii) we deduce that $(x^{**} \to x) \to x = x^{**} = y^{**} = (y^{**} \to y) \to y$ hence we can consider

$d_1 = x^{**} \to x \in$ **D**(A) and $d_2 = y^{**} \to y \in$ **D**(A). ■

Remark 5.2.30. If A is an implicative semilattice, in [63] Nemitz prove that $x^{**} = y^{**}$ iff there is $d \in$ D(A) such that $d \wedge x = d \wedge y$.

In what follows we will extend the notions of dense and regular elements in the case of unbounded Hilbert algebras.

Definition 5.2.31. If A is a Hilbert algebra and $x, y \in A$, we say that y is *fixed* by x if $x \to y = y$. If $S \subseteq A$, we say that S is *fixed* by x iff every element of S is fixed by x. If $T \subseteq A$, we say that S is *fixed* by T if every element of S is fixed by T.

We denote for $S \subseteq A$, **Fix**(S) = $\{x \in A$: S is fixed by x$\}$ = $\{x \in A : x \to s = s$ for every $s \in S\}$ and

Fixat(S) = $\{x \in A$: x is fixed by S$\}$=$\{x \in A: s \to x = x$ for every $s \in S\}$.

Lemma 5.2.32. If A is a Hilbert algebra, then for every $S \subseteq A$, Fix(S) $\in$ Ds(A) and Fixat(S) $\in$ Alg(A).

Proof. Firstly we will prove that **Fix**(S) $\in$ **Ds**(A).

Since for every $s \in S$, $1 \to s = s$ we deduce that $1 \in$ **Fix**(S).

Suppose that $x, x \to y \in$ **Fix**(S), that is, $x \to s = (x \to y) \to s = s$ for every $s \in S$.

We deduce that for every $s \in S$, $(x \to y) \to (x \to s) = s$, hence successively we obtain $x \to (y \to s) = s$, $y \to (x \to s) = s$, $y \to s = s$, therefore $y \in$ Fix(S).

To prove Fixat(S) $\in$ **Alg**(A), let x, y $\in$ **Fixat**(S); then $s \to x = x$ and $s \to y = y$ for every s$\in$ S. Then for every s $\in$ S we have $s \to (x \to y) = (s \to x) \to (s \to y) = x \to y$, hence $x \to y \in$ **Fixat**(S). ■

Lemma 5.2.33. Let A be a Hilbert algebra and x$\in$A. Then Fixat({x}) = $x \to A$, where $x \to A = \{x \to y: y \in A\}$.

Proof. By definition, **Fixat**({x}) = $\{z \in A: x \to z = z\}$. If $y = x \to z \in x \to A$ (hence $z \in A$), then $x \to y = x \to (x \to z) = x \to z = y$, hence y $\in$ Fixat({x}), so, we obtain the inclusion $x \to A \subseteq$ **Fixat**({x}). If y $\in$ **Fixat**({x}), then $y = x \to y \in x \to A$, hence **Fixat**({x}) $\subseteq x \to A$, that is, **Fixat**({x}) **=** $x \to A$. ■

Lemma 5.2.34. If A is a Hilbert algebra and x, y $\in$ A, then $x\to A= y\to A$ iff x = y.

Proof. It is suffice to prove the implication: if $x \to A \subseteq y \to A$, then $x \le y$.

From $x \to A \subseteq y \to A$ we deduce that for every z $\in$ A there is t $\in$ A such that $x \to z = y \to t$. In particular, for z = y we find t$\in$A such that $x \to y = y \to t$.

Since $y \le x \to y$ we deduce that $y \le y \to t \Leftrightarrow y \to t = 1$, hence $x \to y = 1 \Leftrightarrow x \le y$. ■

The dual notion of implicative semilattice is the notion of *difference semilattice.* If (A,$\vee$, 0) is a join–semilattice with 0, we say that A is a *difference semilattice* if for any elements x, y $\in$ A there is an element of A denoted by x - y such that $x - y = \sup\{z \in A: x \le y \vee z\}$.

It is immediate that if A is a difference semilattice and x,y,z $\in$ A, then we have the following rules of calculus:

c_{26}: $x - y \le x$;

c_{27}: $x - (y \vee z) = (x - y) - z = (x - z) - y$;

c_{28}: $(x - z) - (y - z) \le (x - z) - z$;

$\mathbf{c_{29}}$: $(x \vee y) - z = (x - z) \vee (y - z)$;
$\mathbf{c_{30}}$: $x = x - 0$;
$\mathbf{c_{31}}$: $x \leq y$ iff $x - y = 0$;
$\mathbf{c_{32}}$: If there exists $y \wedge z$, then $x - (y \wedge z) = (x - y) \vee (x - z)$;
$\mathbf{c_{33}}$: If there exists $y \wedge z$, then $x = (x \wedge y) \vee (x - z)$.

Lemma 5.2.35. If A is a Hilbert algebra and x, y $\in$ A, then in the join-semilattice (Ds(A), $\vee$) (where 0 = {1}) there exists [x) – [y) and [x) – [y) = [y $\to$ x).

Proof. Firstly we will prove that $[x) \subseteq [y) \vee [y \to x)$.
By c_{24} we have $[y) \vee [y \to x) = \{z \in A: y \to z \in [y \to x)\} =$
$\{z \in A: y \to x \leq y \to z\}$, so if $z \in [x)$ then $x \leq z$, hence $y \to x \leq y \to z$, so $z \in [y) \vee [y \to x)$; we deduce that $[x) \subseteq [y) \vee [y \to x)$.
Now let $D \in$ **Ds**(A) such that $[x) \subseteq [y) \vee D$ and we will prove that $[y \to x) \subseteq D$; since $x \in [x)$ we deduce that $x \in [y) \vee D$, hence $y \to x \in D$, that is, $[y \to x) \subseteq D$. ■

Theorem 5.2.36. If A is a Hilbert algebra and x $\in$ A, then $<x \to A> = A - [x)$.

Proof. Firstly we will prove that $A \subseteq < x \to A > \vee [x)$, that is, $< x \to A > \vee [x) = A$.
By Theorem 5.1.18 it must be proved that for every $a \in A$ there exist
$a_1, a_2, ..., a_n \in <x \to A>$ such that $(a_1, a_2, ..., a_n; a) \in [x)$.
Clearly $a_1 = x \to a \in <x \to A>$; since $x \leq (x \to a) \to a = a_1 \to a$, we deduce that $(a_1; a) = a_1 \to a \in [x)$, hence $<x \to A> \vee [x) = A$.
Now let $D \in$ **Ds**(A) such that $[x) \vee D = A$; then for every $a \in A$, $a \in [x) \vee D$, hence $x \to a \in D$. Then $x \to A \subseteq D$, hence $<x \to A> \subseteq D$. ■

After this training we can extend the notions of regular and dense element to the case of unbounded Hilbert algebras.

Definition 5.2.37. If A is an unbounded Hilbert algebra, we say that an element x∈A is *regular* if for every y ∈ A we have (x → y) → x =x. We denote by $\overline{R}(A)$ the set of all regular elements of A.

Theorem 5.2.38. If A is a bounded Hilbert algebra, then $\overline{R}(A)$ = R(A).

Proof. If x∈ $\overline{R}(A)$ then for every y ∈ A we have (x → y) → x = x; in particular for x = 0 we obtain (x → 0) → x = x ⇔ x** = x ⇔ x ∈ **R**(A), hence $\overline{R}(A)$ ⊆ **R**(A).

Now let x ∈**R**(A) and y∈ A .
Since 0 ≤ y we deduce that x* ≤ x → y, hence (x → y) → x ≤ x* → x = x** = x; since x ≤ (x → y) → x we deduce that (x → y) → x = x, hence x ∈ $\overline{R}(A)$, so **R**(A) ⊆ $\overline{R}(A)$, that is, $\overline{R}(A)$ = **R**(A). ■

Definition 5.2.39. If A is an unbounded Hilbert algebra, we define $\overline{D}(A) = Fix(\overline{R}(A))$; an element x ∈ A will be called *dense* if x ∈ $\overline{D}(A)$ (that is, x ∈ A is dense iff for every r ∈ $\overline{R}(A)$ we have x → r = r).

Theorem 5.2.40. If A is a bounded Hilbert algebra, then $\overline{D}(A)$ = D (A).

Proof. Since (0 → y) → 0 = 1 → 0 = 0, for every y ∈ A, we deduce that 0 ∈ $\overline{R}(A)$.

Let now x ∈ **D**(A); since 0 ∈ **R**(A), in particular we obtain x → 0 = = 0, hence x* = 0, that is, x ∈ **D**(A).
Let now x ∈ **D**(A), (hence x* = 0) and r ∈ $\overline{R}(A)$ = **R**(A), (hence r** = r). Then x → r = x → r**= x → (r*→ 0) = r*→ (x → 0) = r*→ 0 = r** = r, hence x ∈ $\overline{D}(A)$, that is, $\overline{D}(A)$ = **D**(A). ■

5.3. The lattice of deductive systems of a Hilbert algebra

According to the notations from Chapters 1 and 3, for a Hilbert algebra A we denote by **Echiv**(A) (respective, **Con**(A)) the set of all equivalence relations (respective, congruence relations) on A.

For $D \in$ **Ds**(A) we consider the equivalence relation $\theta(D)$ on A defined in § 1: $(x, y) \in \theta(D)$ iff $x \to y, y \to x \in D$.

Lemma 5.3.1. $\theta(D) \in$ Con (A).

Proof. Let $x, x', y, y' \in A$ such that $(x, y), (x', y') \in \theta(D)$, that is, $x \to y$, $y \to x$, $x' \to y'$, $y' \to x' \in D$.

We deduce that $x \to (x' \to y')$, $x \to (y' \to x') \in D$, hence $(x \to x') \to (x \to y')$, $(x \to y') \to (x \to x') \in D$, that is, $(x \to x', x \to y') \in \theta(D)$.

Analogously we deduce that $(x \to y', y \to y') \in \theta(D)$ (since by c_4, $x \to y \le (y \to y') \to (x \to y')$ and $y \to x \le (x \to y') \to (y \to y')$). By the transitivity of $\theta(D)$ we deduce that $(x \to x', y \to y') \in \theta(D)$, hence $\theta(D) \in$ Con (A). ■

Lemma 5.3.2. If $\theta \in$ Con (A), then $D(\theta) = \{ x \in A: (x, 1) \in \theta \} \in$ Ds(A).

Proof. Clearly $1 \in D(\theta)$; let $x, x \to y \in D(\theta)$ and we shall prove that $y \in D(\theta)$. From $(x, 1) \in \theta$ we deduce that $(x \to y, 1 \to y) \in \theta$, hence $(x \to y, y) \in \theta$. Then $(y,1) \in \theta$ (by the transitivity of θ), hence $y \in D(\theta)$, that is, $D(\theta) \in$ **Ds**(A). ■

Lemma 5.3.3. If $D \in$ Ds(A) and $\theta \in$ Con(A), then $\theta(D(\theta)) = \theta$ and $D(\theta(D)) = D$.

Proof. We shall firstly prove by double inclusion that $\theta(D(\theta)) = \theta$.

If $(x, y) \in \theta$, to prove $(x, y) \in \theta(D(\theta))$ it must be proved that $x \to y, y \to x \in D(\theta) \Leftrightarrow (x \to y, 1), (y \to x, 1) \in \theta$, which is immediate because from $(x, y) \in \theta$ we deduce that $(x \to y, y \to y), (y \to x, y \to y) \in \theta$, that is, $(x \to y, 1)$, $(y \to x, 1) \in \theta$. Hence $\theta \subseteq \theta(D(\theta))$. For the other inclusion, let (x, y) ∈ θ(D(θ)) ⇔(x → y, 1), (y → x, 1) ∈ θ. Since θ is a congruence on A we deduce that:

(1) ((x → y) → y, y), ((y → x) →x, x) ∈ θ.
From (1) we deduce that:

(2) $\begin{cases} ((y \to x) \to ((x \to y) \to y), (y \to x) \to y) \in q, \\ ((x \to y) \to ((y \to x) \to x), (x \to y) \to x) \in q. \end{cases}$

But (y → x) → ((x → y) → y) = (x → y) → ((y → x) → x) = xΔy, hence from (2) we deduce that:

(3) ((y → x) → y, (x → y) → x) ∈ θ.

On the other hand, from (x → y, 1), (y → x, 1) ∈ θ we deduce that ((x → y) → x, x), ((y → x) → y, y) ∈ θ, so, if we use (3), we deduce that (x, y) ∈ θ, hence we have the equality θ(D(θ)) = θ.

To prove the equality D(θ(D)) = D, we use the equivalence x ∈ D(θ(D)) iff (x, 1) ∈ θ(D). ■

Theorem 5.3.4. If A is a Hilbert algebra, then there is a bijective isotone function between Con(A) and Ds(A).

Proof. We define f : **Con**(A) → **Ds**(A) by f (θ) = D(θ) for every θ∈**Con**(A) and g : **Ds**(A) → **Con**(A) by g (D) = θ(D) for every D ∈ **Ds**(A); it is immediate to see that f is isotone. Following Lemma 5.3.3 we deduce that $f \circ g = 1_{Ds(A)}$ and $g \circ f = 1_{Con(A)}$, hence f is bijective function and g is its converse ■

If A is a Hilbert algebra, then (**Ds**(A),∨,∧) becomes a bounded lattice where for D_1, D_2 ∈ **Ds**(A), $D_1 \wedge D_2 = D_1 \cap D_2$, $D_1 \vee D_2 = < D_1 \cup D_2 >$, **0** = {1} and **1** = A.

In fact this is a complete lattice, where for a family $\{D_i\}_{i\in I}$ of deductive systems of A, then $\bigwedge_{i\in I} D_i = \bigcap_{i\in I} D_i$ and $\bigvee_{i\in I} D_i = \left\langle \bigcup_{i\in I} D_i \right\rangle$.

For D_1, D_2 ∈ **Ds**(A) we define $D_1 \to D_2$ = {x ∈ A : [x) ∩ $D_1 \subseteq D_2$}.

Lemma 5.3.5. If A is a Hilbert algebra and D, D_1, D_2 ∈ Ds(A), then

(i) $\mathbf{D_1 \to D_2 \in Ds(A)}$ **;**

(ii) $\mathbf{D_1 \cap D \subseteq D_2}$ **iff** $\mathbf{D \subseteq D_1 \to D_2}$**.**

Proof. (i). Since $[1) = \{1\}$, $[1) \cap D_1 = \{1\} \subseteq D_2$, hence $1 \in D_1 \to D_2$.
Now let x, $x \to y \in D_1 \to D_2$; to prove that $y \in D_1 \to D_2$ let $t \in [y) \cap D_1 \Leftrightarrow t \in D_1$ and $y \le t$.
Since $x \le (x \to t) \to t$ we deduce that $(x \to t) \to t \in [x)$; since $t \le (x \to t) \to t$, then $(x \to t) \to t \in [x) \cap D_1 \subseteq D_2$, hence

(1) $(x \to t) \to t \in D_2$.

Analogously we deduce

(2) $((x \to y) \to t) \to t \in D_2$.

Since $y \le t$, from c_5 we deduce successively: $x \to y \le x \to t$, $(x \to t) \to t \le (x \to y) \to t$, $((x \to y) \to t) \to t \le (x \to t) \to t$. By the last inequality and c_6, we deduce that:

(3) $((x \to y) \to t) \to t \le x \to t$.

From (2) and (3) we deduce that $x \to t \in D_2$, hence by (1) follows that $t \in D_2$ (since D_2 is a deductive system).
Therefore $[y) \cap D_1 \subseteq D_2$, so, $y \in D_1 \to D_2$.
(ii) "$\Rightarrow$" If $D_1 \to D \subseteq D_2$, let $a \in D$ and $t \in [a) \cap D_1$; then $a \le t$ and $t \in D_1$ implies $t \in D$, hence $t \in D_1 \cap D \subseteq D_2$.
Thus $[a) \cap D_1 \subseteq D_2$, hence $D \cap D_1 \subseteq D_2$.
"$\Leftarrow$". Suppose that $D \subseteq D_1 \cap D_2$ and consider $x \in D_1 \cap D$; then $x \in D \subseteq D_1 \to D_2$, hence $[x) \cap D_1 \subseteq D_2$. Since $x \in [x) \cap D_1$, we deduce that $x \in D_2$
■

Remark 5.3.6. From Lemma 5.3.5 we deduce that (**Ds**(A),$\vee$, $\wedge$, $\{1\}$, A) is a Heyting algebra, where for $D \in Ds(A)$, $D^* = D \to 0 = D \to \{1\} = \{x \in A : [x) \cap D = \{1\}\}$, so, for $a \in A$, $[a)^* = \{x \in A: [x) \cap [a) = \{1\}\}$.
If A is a Heyting algebra, then $D^* = \{x \in A: x \vee y = 1$, for every $y \in D\}$ and $[a)^* = \{x \in A: x \vee a = 1\}$.

To prove the last assertion we remark that if $x \in D^*$ and $y \in D$, then since $x \vee y \in [x)$ and $x \vee y \in D$, we deduce that $x \vee y \in [x) \cap D = \{1\}$, hence $x \vee y = 1$.

If $x \vee y = 1$ for every $y \in D$, then $[x) \cap D = \{1\}$, since if $y \in [x) \cap D$ then from $x \leq y$, we deduce that $y = x \vee y = 1$, hence $[x) \cap D = \{1\}$, that is, $x \in D$.

We want to see in what conditions **Ds**(A) is a Boolean algebra; in this way we will prove:

Theorem 5.3.7. If A is a bounded Hilbert algebra, then the following assertions are equivalent:

(i) (Ds(A),$\vee$, $\wedge$, *, {1}, A) is a Boolean algebra ;

(ii) A is a finite Boolean algebra (relative to natural ordering).

Proof. (i)$\Rightarrow$(ii). Let $x \in A$; since **Ds**(A) is supposed Boolean algebra, then $[x) \vee [x)^* = A$. By c_{24}, $[x) \vee [x)^* = \{y \in A: x \to y \in [x)^*\} = \{y \in A: [x \to y) \cap [x) = \{1\}\}$, so, for every $y \in A$ we have

(1) $[x \to y) \cap [x) = \{1\}$.

Since $x \to y \leq ((x \to y) \to x) \to x$ and $x \leq ((x \to y) \to x) \to x$, we deduce that $((x \to y) \to x) \to x \in [x \to y) \cap [x) = \{1\}$, hence

(2) $((x \to y) \to x) \to x = 1$.

Since $x \leq (x \to y) \to x$, from (2) we deduce that $(x \to y) \to x = x$, so, by Corollary 5.2.21 we deduce that A is a Boolean algebra .

We shall prove that every filter of A is principal, hence A will be finite (see [45]).

Let now $D \in$ **Ds**(A); since we have supposed that **Ds**(A) is a Boolean algebra, we have that $D \vee D^* = A$, hence $0 \in D \vee D^*$.

By Theorem 5.2.18 there exist $x_1, x_2, ..., x_n \in D$ such that $(x_1, x_2, ..., x_n; 0) \in D^*$,so, by the above remark we deduce that for every $y \in D$, $(x_1, x_2, ..., x_n; 0) \vee y = 1$.

Since in a Heyting algebra A for $x_1, x_2, ..., x_n \in A$ we have :

c_{34} : $(x_1, x_2, ..., x_{n-1}; x_n) = (x_1 \wedge x_2 \wedge ... \wedge x_{n-1}) \to x_n$, then the relation $(x_1, x_2, ..., x_n; 0) \vee y = 1$ it is successively equivalent with

$$((x_1 \wedge ... \wedge x_n) \to 0) \vee y = 1$$
$$(x_1 \wedge ... \wedge x_n)^* \vee y = 1$$
$$(x_1 \wedge ... \wedge x_n) \wedge y^* = 0$$
$$x_1 \wedge ... \wedge x_n \leq y,$$

hence $D = [a)$, where $a = x_1 \wedge ... \wedge x_n \in D$.

(ii)⇒(i). Suppose that A is a finite Boolean algebra; then every filter of A is principal.
By Remark 5.3.6, **Ds**(A) is a Heyting algebra, hence to prove that **Ds**(A) is a Boolean algebra it will suffice to prove that if $D = [a) \in$ **Ds**(A), with $a \in A$ and $D^* = \{1\}$, then $D = A$ (see Corollary 5.1.20).

Also, $D^* = \{x \in A: x \vee y = 1$ for every $y \geq a\}$. Since for every $y \geq a$, $a^*\vee y \geq a^*\vee a = 1$,so we deduce that $a^*\vee y = 1$, hence $a^* \in [a)^* = 1$. We obtain $a^* = 1$, hence $a = 0$, therefore $D = [0) = A$, that is **Ds**(A) is a Boolean algebra. ■

In what follows we want to see in what conditions a lattice L can be the lattice of deductive systems for a Hilbert algebra.
For this we will prove:

Theorem 5.3.8. A lattice L is the lattice of deductive systems of a Hilbert algebra iff it is complete and algebraic (with a base of compacts B ⊆ L which verify the condition : if x, y ∈ B, then x ∨ y, x - y ∈ B).
In this case L will be isomorphic with Ds(A), where A is the dual of B (which is an implicative semilattice, hence a Hilbert algebra).

Proof. "⇒". Suppose that L = **Ds**(A), with A a Hilbert algebra. Then L is complete and consider B = {<F>: F ⊆ A is finite}⊆ L.
We know that if A is an algebra of some type, then the lattice **Con**(A) is algebraic, where the principal congruence are compact elements(see Chapter 3 or [45]). Since L is the lattice of congruence of A (by Theorem 5.3.4), then L is algebraic, and the principal deductive systems of A are compact elements in L. Since if F ⊆ A is a finite set, then <F> = ∨ {[x): x ∈ F}, we deduce that the elements of B are compacts.

Since for $D \in L = \mathbf{Ds}(A)$ we have $D = \sup \{<F>: F \subseteq D, F \text{ is finite}\}$, we deduce that B is a compact base for L.

Let now $X = <F_1>$, $Y = <F_2>$, with $F_1, F_2 \subseteq A$ finite; then $X \vee Y =$
$= <F_1 \cup F_2> \in B$.

We shall prove that $X - Y \in B$; we recall that following Lemma 5.2.35,for every $a, b \in A$, then there exists $[a) - [b)$ in $L = \mathbf{Ds}(A)$, and $[a) - [b) = [b \to a)$; also, we use the rules of calculus $c_{26} - c_{33}$.

Let $X = <x_1, x_2, ..., x_n>$, $Y = <y_1, y_2, ..., y_m>$, with $x_i, y_j \in A$, $i = 1, 2, ..., n$, $j = 1, 2, ..., m$, m and n naturals numbers .
We have :

$X - Y = X - ([y_1) \vee [y_2) \vee ... \vee [y_m)) = (... ((X - [y_1) - [y_2) - ... - [y_m))$ and

$X - [y_1) = ([x_1) \vee [x_2) \vee ... \vee [x_n)) - [y_1) = ([x_1) - [y_1)) \vee ... \vee ([x_n) - [y_1))$
$= [y_1 \to x_1) \vee [y_1 \to x_2) \vee \ldots \vee [y_1 \to x_n) = <y_1 \to x_1, y_1 \to x_2, ..., y_1 \to x_n> \in B$, so, recursively we deduce that $X - Y = <F>$, where $F = < y_j \to x_i : i = 1, 2, ..., n$ and $j = 1, 2, ..., m>$, hence $X - Y \in B$.

In [63], we have an analogous result for implicative semilattices, where the principal filters are considered basis. In our case we cannot consider as basis for L the principal deductive systems of A, since in this case, if $X = [a)$, $Y = [b)$, then $X \vee Y = <a, b>$, which is not principal.

"$\Leftarrow$". The dual of B, $B^0 = (B, \geq)$ will be a implicative semilattice (hence a Hilbert algebra). A deductive system of B^0 will be a filter of B^0, hence an ideal of B; so $\mathbf{Ds}(B^0)$ is in fact $\mathbf{I}$ (B) (the set of ideals of B).
So, I must prove that $\mathbf{I}(B)$ and L are isomorphic as lattices.

For $I \in \mathbf{I}(B)$, if we denote $f(I) = \sup(I)$, we obtain a function $f : \mathbf{I}(B) \to L$, which we will prove that is an isomorphism of lattices; clearly f is morphism of lattices.

Since B is a compact base, f will be surjective isotone function (for $x \in L$, if we take $I = (x] \in \mathbf{I}(B)$, then $f(I) = x$).

To prove the injectivity of f , let $I_1, I_2 \in \mathbf{I}(B)$ such that $f(I_1) \leq f(I_2)$ and we shall prove that $I_1 \subseteq I_2$. Let $y \in I_1$; then $y \leq \sup(I_1) \leq \sup(I_2)$. Since y is compact, there is $I' \subseteq I_2$ such that $y \leq \sup(I')$, hence $y \in I_2$ (since $\sup(I') \in I_2$). We deduce that $I_1 \subseteq I_2$, that is, f is injective ■

Remark 5.3.9. In [45, p. 94]), Grätzer proves that a lattice L is algebraic iff it is isomorphic with the lattice of ideals of a meet-semilattice with 0.

Definition 5.3.10. For a Hilbert algebra A, we say that D∈Ds(A) is *irreducible(completely irreducibile)* if, as an element of the complete lattice Ds(A), is a meet- irreducibile (completely meet-irreducibile) element .

Clearly, every completely irreducible deductive system is irreducible; if A is a Heyting algebra, D ∈ **Ds** (A) is irreducibile iff it is *prime filter* .

In [73,p.34], it is proved that in the case of *implication algebras* (that is, Hilbert algebras with the property that for every two elements x, y, then (x → y) → x = x), a deductive system (called in [75] *implicative filter*) is irreducible iff it is prime iff it is maximal.

In [37, pp. 21-22] it is proved the following results:

Theorem 5.3.11. D ∈ Ds(A) is irreducibile iff for any x, y ∉ D there is z ∉ D such that x ≤ z and y ≤ z.

Theorem 5.3.12. D ∈ Ds(A) is completely irreducibile iff there is a ∉ D such that D is a *maximal relative to a* (that is, D is maximal in Ds(A) with the property a ∉ D) .

Theorem 5.3.13. D ∈ Ds(A) is maximal relative to a iff a ∉ D and (x ∉ D implies x → a ∈ D).

In what follows we will present other criteria for meet-irreductibility (complete irreductibility) relative to a deductive system.

Theorem 5.3.14. For D ∈ Ds(A) the following are equivalent :
(i) D is meet-irreducible ;
(ii) For every H ∈ Ds (A), H → D = D or H ⊆ D ;
(iii) If x, y ∈ A and [x) ∩ [y) ⊆ D, then x ∈ D or y ∈ D ;

(iv) For $\alpha, \beta \in A / D$, $\alpha \neq 1$, $\beta \neq 1$, there is $\gamma \in A / D$ such that $\gamma \neq 1$ and $\alpha, \beta \leq \gamma$.

Proof. (i)⇒(ii). Suppose that D is meet-irreducible and let $H \in$ **Ds**(A); since **Ds**(A) is a Heyting algebra, by c_{16}, we have $D = (H \to D) \cap ((H \to D) \to D)$.Since D is meet-irreducible, we have $D = D \to H$ or $D= (H \to D) \to D$; in the second case, since $H \subseteq (H \to D) \to D$ we deduce that $H \subseteq D$.

(ii)⇒(i). Let $D_1, D_2 \in$ **Ds**(A) such that $D = D_1 \cap D_2$; then $D_1 \subseteq D_2 \to D$, so, if $D_2 \subseteq D$, then $D_2 = D$ and if $D_2 \to D = D$, then $D_1 = D$.

(i)⇒(iii). Let $x, y \in A$ such that $[x) \cap [y) \subseteq D$ and suppose that $x \notin D$, $y \notin D$; by Theorem 5.3.11 there is $z \notin D$ such that $x \leq z$ and $y \leq z$. Then $z \in [x) \cap [y) \subseteq D$, hence $z \in D$, a contradiction !

(iii)⇒(ii). Let $H \in$ **Ds**(A) such that $H \nsubseteq D$ and we shall prove that $H \to D = D$. Let $x \in H \to D$; then $[x) \cap H \subseteq D$ and if $y \in H \setminus D$, then $[y) \subseteq H$, hence $[x) \cap [y) \subseteq [x) \cap H \subseteq D$. Since $y \notin D$, we deduce that $x \in D$, hence $H \to D = D$.

(i)⇒(iv). Let $\alpha, \beta \in A / D$, $\alpha \neq 1$, $\beta \neq 1$; then $\alpha = x / D$, $\beta = y / D$ with $x, y \notin D$. By Theorem 5.3.11 there is D such that $x \leq z$ and $y \leq z$. If we take $\gamma = z / D \in A / D$, $\gamma \neq 1$ and $\alpha, \beta \leq \gamma$, since $x \to z = y \to z = 1 \in D$.

(iv) ⇒ (i). Let $x, y \notin D$; if we take $\alpha = x / D$, $\beta = y / D$, $\alpha, \beta \in A / D$, $\alpha \neq 1$, $\beta \neq 1$, hence there is $\gamma = z / D$, $\gamma \neq 1$, (hence $z \notin D$) such that $\alpha, \beta \leq \gamma$.

Thus $x \to z, y \to z \in D$.

We compute $z' = ((x \to z) \to z) \Delta ((y \to z) \to z)$ (clearly $(x \to z) \to z$, $(y \to z) \to z \notin D$ since if we suppose by contrary, we deduce that $z \in D$ - a contradiction !).

We have $((x \to z) \to z) \to ((y \to z) \to z) = (y \to z) \to (((x \to z) \to z) \to z) = (y \to z) \to (x \to z)$ and $((y \to z) \to z) \to ((x \to z) \to z) = (x \to z) \to (y \to z)$ so $z' = ((x \to z) \to z) \Delta ((y \to z) \to z) = ((y \to z) \to (x \to z)) \to (((x \to z) \to (y \to z)) \to ((x \to z) \to z)) = ((y \to z) \to (x \to z)) \to ((x \to z) \to ((y \to z) \to z)) = ((y \to z) \to (x \to z)) \to ((y \to z) \to ((x \to z) \to z)) =$

$= (y \to z) \to ((x \to z) \to ((x \to z) \to z)) = (y \to z) \to ((x \to z) \to z)$.

We will prove that $z' \notin D$; if suppose $z' \in D$, then since $y \to z \in D$, then follows that $((x \to z) \to z) \in D$; since $x \to z \in D$ we obtain that $z \in D$, a contradiction, hence $z' \notin D$.

Clearly, $x \le (x \to z) \to z \le z'$ and $y \le (y \to z) \to z \le z'$, hence by Theorem 5.3.11, D is meet-irreducible . ■

Corollary 5.3.15. If D ∈ Ds(A) is irreducible, then in Heyting algebra Ds(A), D is dense or regular.

Proof. If $H = D^* \in$ **Ds**(A), by Theorem 5.3.14, (ii) we have $D^* \subseteq D$ or $D^* \to D = D$; in the first case we obtain that $D^* \to D = 1$ or $D^{**} = 1$, hence $D^* = 0$, so D is dense element in **Ds**(A); in the second case we deduce that $D^* \to D = D \Leftrightarrow D^{**} = D$, hence D is a regular element in **Ds**(A). ■

Theorem 5.3.16. For D ∈ Ds(A) the following are equivalent :

(i) D is completely meet-irreducible;

(ii) If $\bigcap_{x \in I \subseteq A} [x) \subseteq D$**, then I ∩ D ≠ ∅ ;**

(iii) A/D have a penultimate element.

Proof. (i)⇒(ii) clearly .

(ii)⇒(i). Let $D = \bigcap_{i \in I} D_i$ with $D_i \in$ **Ds**(A) for every $i \in I$, and suppose that for every $i \in I$ there exists $x_i \in D_i \setminus D$. Since $[x_i) \subseteq D_i$ for every $i \in I$, we deduce that $\bigcap_{i \in I} [x_i) \subseteq \bigcap_{i \in I} D_i = D$, so, by hypothesis there is $i \in I$ such that $x_i \in D$, a contradiction ! .

(i) ⇒ (iii). By Theorem 5.3.12, D is maximal relative to an element $a \notin D$. We shall prove that $\alpha = a / D$ is a penultimate element of A / D. Let $\beta = x / D \in A / D$ with $\beta \neq 1$ (hence $x \notin D$).

By Theorem 5.3.13, $x \to a \in D$, hence $\beta = x/D \le a / D = \alpha$.

(iii) ⇒ (i). Suppose that A / D has a penultimate element $\alpha = a / D$.

We deduce that a $\notin$ D and for $\beta = x / D \neq 1$ (hence x $\notin$ D), x /D $\leq$ a / D. There results that for every x $\notin$ D, x $\to$ A $\in$ D, hence D is maximal relative to a , hence by Theorem 5.3.13, D is completely meet-irreducible . ■

In [37, p. 22], it is proved:

Theorem 5.3.17. If D$\in$Ds(A) and a$\notin$D, there is a complete meet-irreducible deductive system M such that D $\subseteq$ M and a $\notin$ M.
If a, b$\in$M, a$\neq$b, then there is a completely meet-irreducible deductive system M such that a $\notin$ M and b $\in$ M.

In what follows, for a Hilbert algebra A, we denote by **Ir**(A) (**Irc**(A)) the set of all meet-irreducible (completely meet-irreducible) deductive systems of A.

Theorem 5.3.18. If A is a Hilbert algebra and D$\in$Ds(A), then

D = { M $\in$ Irc (A): D $\subseteq$ M}.

Proof. Let D′ = {M $\in$ **Irc**(A): D $\subseteq$ M}; clearly D $\subseteq$ D′.
To prove another inclusion we shall prove the inclusion of the complementaries.
If a $\notin$ D, then by Theorem 5.3.17, there is M$\in$ **Irc**(A) such that D$\subseteq$M and a $\notin$ M. There results that a$\notin${M $\in$ **Irc** (A): D $\subseteq$ M} = D′, so a $\notin$ D′, hence D′$\subseteq$ D, that is, D = D′. ■

Theorem 5.3.19. If A is a bounded Hilbert algebra, then the following are equivalent:

(i) Every D $\in$ Ds(A) has a unique representation as an intersection of elements from Irc(A);
(ii) A is a finite Boolean algebra (relative to natural ordering).

Proof. (i)$\Rightarrow$(ii). To prove **Ds**(A) is a Boolean algebra, let D $\in$ **Ds**(A) and consider D′ **=** {M $\in$ **Irc**(A): D $\nsubseteq$ M}$\in$ **Ds**(A).

We have to prove that D' is the complement of D in Heyting algebra **Ds**(A).

Clearly $D \cap D' = \{1\}$; if $D \vee D' \neq A$, then by Theorem 5.3.17, there is $D'' \in$ **Irc**(A) such that $D \vee D' \subseteq D''$, $D'' \neq A$, hence D has two distinct representations as intersection of elements from Irc(A):

$D' = \cap \{M \in$ **Irc**(A): $D \not\subseteq M\}$ and

$D' = D'' \cap (\cap \{M \in$ **Irc**(A): $D \not\subseteq M\})$, a contradiction, hence

$D \vee D' = A$, that is, **Ds**(A) is a Boolean algebra.
By Theorem 5.3.7, A is a finite Boolean algebra.

(ii)⇒(i). This implication is straightforward (see [35], Chapter 4, page 77).

■

Remark 5.3.20. For the case of lattices with 0 and 1 we have an analogous result of Hashimoto(see [47]).

Definition 5.3.21. We say that M ∈ Ds(A), M ≠ A, is *maximal* if it is a maximal element in the lattice (Ds(A), ⊆).

Let us denote by **Max(A)** the set of maximal deductive systems of A.

Definition 5.3.22. We say that a Hilbert algebra is *semisimple* if the intersection of all maximal deductive systems of A is {1}.

Theorem 5.3.23. If A is a bounded Hilbert algebra and there is a deductive system D ≠ A, then there is a maximal deductive system M of A such that D ⊆ M.

Proof. It is an immediate consequence of Theorem 5.3.17, since for the case of a = 0, a deductive system is maximal iff it is maximal relative to 0. ■

Theorem 5.3.24. For M ∈ Ds(A), with A an bounded Hilbert algebra, the following are equivalent:
(i) M is maximal;
(ii) If x ∉ M, then x* ∈ M.

Proof. (i)⇒(ii). Suppose M maximal and consider $x \notin M$; then $[x) \vee M = A$. By c_{24}, $[x) \vee M = \{y \in A: x \to y \in M\}$; in particular $0 \in [x) \vee M$, hence $x \to 0 = x^* \in M$.

(ii)⇒(i). Suppose by contrary that M is not maximal, that is, there is N $\in Ds(A)$ such that $M \subset N \subset A$; then there is $x \in N$ such that $x \notin M$. Since $x \notin M$, then $x^* \in M$, hence $x^* \in N$; since $x \in N$ we deduce that $0 \in N \Leftrightarrow N = A$, a contradiction since N is proper. ■

Theorem 5.3.25. For $M \in Ds(A)$, with A an bounded Hilbert algebra, the following are equivalent:

(i) M is maximal;

(ii) For any $x, y \in A$, if $x \veebar y \in M$, then $x \in M$ or $y \in M$.

Proof. (i)⇒(ii). Let $x, y \in A$ such that $x \veebar y \in M$ and suppose that $x \notin M$, $y \notin M$. By Theorem 5.3.24, we deduce that $x^* \in M$, $y^* \in M$. From $x \veebar y = x^* \to y \in M$ and $x^* \in M$, we deduce that $y \in M$. But $y^* \in M$, hence $0 \in M$, that is, $M = A$, which is a contradiction!.

(ii)⇒(i). If $x \in A$, since $x \veebar x^* = x^* \to x^* = 1 \in M$, then if $x \notin M$, we deduce that $x^* \in M$ hence, by Theorem 5.3.24, M is maximal. ■

Theorem 5.3.26. If A is a bounded Hilbert algebra and $M \in Ds(A)$, $M \neq A$, then the following are equivalent:

(i) $M \in Max(A)$;

(ii) For any $x, y \in A$, if $x \Delta y \in M$, then $x \in M$ or $y \in M$.

Proof.(i)⇒(ii). Suppose by contrary that there are $x, y \in A$ such that $x \Delta y \in M$, $x \notin M$, $y \notin M$; by Theorem 5.3.24, $x^* \in M$, $y^* \in M$.

From $x^* \le x \to y$, $y^* \le y \to x$ we deduce that $x \to y, y \to x \in M$.

On the other hand, from $x^* \le x \to y$ we deduce that $(x \to y) \to ((y \to x) \to x) \le x^* \to ((y \to x) \to x) = (y \to x) \to (x^* \to x) = (y \to x) \to x^{**} = x^* \to (y \to x)^*$.

Then $x \Delta y \leq x^* \to (y \to x)^*$, hence $x^* \to (y \to x)^* \in M$; since $x^* \in M$ we deduce that $(y \to x)^* \in M$, which is contradictory since $y \to x \in M$.

(ii) ⇒ (i). If $x \in A$, by c_{20}, $x \veebar x^* = 1 \in M$, so, if $x \notin M$, then $x^* \in M$, that is, M is maximal (by Theorem 5.3.24). ■

Clearly, if A is a Hilbert algebra, then **Max**(A) ⊆ **Irc**(A).

We want to see in what conditions **Max(**A) = **Irc**(A), with A a bounded Hilbert algebra.

The answer is given by:

Theorem 5.3.27. If A is a bounded Hilbert algebra, then the following are equivalent:

(i) Max(A) = Irc(A);

(ii) A is Boolean algebra (relative to natural ordering).

Proof. (i)⇒(ii).If by contrary A is not a Boolean algebra, then by Theorem 5.3.24, there is $a \in A$ such that $a^{**} \neq a \Leftrightarrow a^{**} \nleq a$.

By Theorem 5.3.17, there is a deductive system D ∈ **Irc**(A) such that $a^{**} \in D$ and $a \notin D$.

But **Max**(A) = **Irc**(A), hence D ∈ **Max**(A).

Since $a \notin D$, we deduce that $a^* \in D$; from $a^* \in D$ and $a^{**} \in D$ we deduce that that $0 \in D$, hence D = A , which is a contradiction!.

(ii)⇒(i). See [45], Chapter 4. ■

Theorem 5.3.28. If A is a bounded Hilbert algebra, then D(A) is irreducible iff it is maximal.

Proof. ”⇒”. We will prove that for any $a \in A$, then $[a) \cap [a^*) \subseteq$ **D**(A).

Indeed, let $z \in [a) \cap [a^*)$, that is, $a \leq z$ and $a^* \leq z$.

We deduce that $z^* \leq a^* \leq z$, hence $z^* \to z = 1$; then $z^{**} = 1 \Leftrightarrow z^* = 0$, hence $z \in$ **D**(A). Thus $[a) \cap [a^*) \subseteq$ **D**(A).

Since D is supposed irreducible and $[a) \cap [a^*) \subseteq D$, by Theorem 5.3.14 we deduce that $a \in$ **D**(A) or $a^* \in$ **D**(A), hence **D**(A) is maximal (by Theorem 5.3.24).

"$\Leftarrow$". This implication is straightforward. ■

In what follows we will continue with the study of **Max**(A), with A a bounded Hilbert algebra; the main result will be that **Max**(A) can be organized as a Boole space. As an immediate consequence we can define a contravariant functor from the category of bounded Hilbert algebras to the category of Boole spaces.
We recall that a *Boolean space* (see Definition 4.3.23) is a compact Haussdorf topological space generated by his clopen sets.

For $a \in A$, we denote $\sigma_A(a) = \{M \in \mathbf{Max}(A) : a \in M\}$.
Let τ_A be the topology of **Max**(A) generated by the family $\{\sigma_A(a)\}_{a\in A}$ of subsets of **Max**(A); an element of τ_A is a union of finite intersections of sets of the form $\sigma_A(a)$, with $a\in A$.

Lemma 5.3.29. For any x, y $\in$ A we have:
(i) $\sigma_A(0) = \emptyset$, $\sigma_A(1) = \mathbf{Max}(A)$, $\sigma_A(x^{**}) = \sigma_A(x)$;
(ii) $\sigma_A(x \to y) = \sigma_A(x) \to \sigma_A(y)$, $\sigma_A(x^*) = \mathbf{Max}(A) \setminus \sigma_A(x)$;
(iii) $\sigma_A(x) \cap \sigma_A(y) = \sigma_A((x \to y^*)^*)$;
(iv) $\sigma_A(x) \cup \sigma_A(y) = \sigma_A(x^* \to y)$.

Proof. (i). Since all deductive systems of **Max**(A) are proper (hence it doesn't contain 0) there results that $\sigma_A(0) = \emptyset$; since all deductive systems from **Max**(A) contain 1 there results that $\sigma_A(1) = \mathbf{Max}(A)$.
If $M \in \sigma_A(x)$, then $x \in M$, hence $x^* \notin M$; then $x^{**} \in M$, hence $\sigma_A(x) \subseteq \sigma_A(x^{**})$. Analogously we prove another inclusion, hence $\sigma_A(x) = \sigma_A(x^{**})$.
(ii). We recall that $\sigma_A(x) \to \sigma_A(y) = \mathrm{int}((\mathbf{Max}(A) \setminus \sigma_A(x)) \cup \sigma_A(y))$.
Firstly we will prove that

$$(1)\ (\mathbf{Max}(A) \setminus \sigma_A(x)) \cup \sigma_A(y) \subseteq \sigma_A(x \to y).$$

Indeed, let $M \in (\mathbf{Max}(A) \setminus \sigma_A(x)) \cup \sigma_A(y)$, that is, $x \notin M$ or $y \in M$; if $y \in M$, then $x \to y \in M$, hence $M \in \sigma_A(x \to y)$.
If $x \notin M$, then $[x)\vee M = A$, hence $x \to y \in M$ (by c_{24}), so $M\in \sigma_A(x \to y)$.
If we consider the interior in both members of (1) we deduce that

$$(2)\ \sigma_A(x) \to \sigma_A(y) \subseteq \sigma_A(x \to y).$$

Now we will prove that:

$$(3)\ \sigma_A(x \to y) \subseteq (\mathrm{Max}(A) \setminus \sigma_A(x)) \cup \sigma_A(y).$$

Indeed, if $M \in \sigma_A(x \to y)$, then $x \to y \in M$; if $x \in M$, then $y \in A$, hence $M \in \sigma_A(y)$ and in this case (3) is verified.

If $x \notin M$, then $M \notin \sigma_A(x)$, hence $M \in ((\mathbf{Max}(A) \setminus \sigma_A(x))$ and (3) is also verified. If we consider the interior in both members of (3) we deduce that $\sigma_A(x \to y) \subseteq \sigma_A(x) \to \sigma_A(y)$ which together with (2) imply the equality $\sigma_A(x \to y) = \sigma_A(x) \to \sigma_A(y)$.
In particular, if $y = 0$, we obtain that $\sigma_A(x \to 0) = \sigma_A(x) \to \sigma_A(0)$ hence $\sigma_A(x^*) = \mathbf{Max}(A) \setminus \sigma_A(x)$ (we can also obtain this equality and from the equivalence $M \in \sigma_A(x^*)$ iff $x \notin M$).
(iii). We will prove the equality from the enounce by double inclusion.
Since $y^* \leq x \to y^*$ we deduce that $(x \to y^*)^* \leq y^{**}$, hence $\sigma_A((x \to y^*)^*) \subseteq \sigma_A(y^{**}) = =\sigma_A(y)$.
Since $x \to y^* = y \to x^*$ (by c_8) we change x with y and we obtain $\sigma_A((x \to y^*)^*) \subseteq \sigma_A(x)$, hence $\sigma_A((x \to y^*)^*) \subseteq \sigma_A(x) \cap \sigma_A(y)$.
Now let $M \in \sigma_A(x) \cap \sigma_A(y)$, that is, $x,y \in M$; we will prove that $M \in \sigma_A((x \to y^*)^*) \Leftrightarrow (x \to y^*)^* \in M$.
Since M is maximal, if $(x \to y^*)^* \notin M$, then $x \to y^* \in M$; since $x \in M$ we deduce that $y^* \in M$, which is a contradiction (since $y \in M$).
So, we also obtain the inclusion $\sigma_A(x) \cap \sigma_A(y) \subseteq \sigma_A((x \to y^*)^*)$, hence $\sigma_A(x) \cap \sigma_A(y) = \sigma_A((x \to y^*)^*)$.
(iv). Since $x \leq x^* \to y$ and $y \leq x^* \to y$ we deduce that $\sigma_A(x) \cup \sigma_A(y) \subseteq \sigma_A(x^* \to y)$.
Since $x^* \to y = x \veebar y$, if $M \in \sigma_A(x \veebar y)$, then $x \veebar y \in M$; by Theorem 5.3.25, $x \in M$ or $y \in M$, hence $M \in \sigma_A(x) \cup \sigma_A(y)$, that is, we obtain the desired equality. ■

Corollary 5.3.30. An element of τ_A has the form $\bigcup_{i \in I} s_A(x_i)$ with x_i elements from A ($i \in I$) .

Proof. This follows from Lemma 5.3.29, (iii) and since an element of τ_A is a union of finite intersections of elements of the form $\sigma_A(a)$, with $a \in A$. ■

Since for every $x \in A$ we have $\sigma_A(x) = \sigma_A(x^{**})$ and $x^{**} \in \mathbf{R}(A)$, it is a natural idea to see if **Max**(A) coincides with **Max** (**R**(A)) (in the sense that between the two sets we have a bijection). This thing proves to be true, as we will prove in what follows.

Lemma 5.3.31. If D∈ Ds(A), then D ∩ R (A) = {x : x ∈ D} and D ∩ R(A) is a deductive system in R(A) (that is, a filter in R(A), since by Theorem 1.24, R(A) is a Boolean algebra).**

Proof. If $x \in D \cap \mathbf{R}(A)$, then $x^{**} = x \in D$, hence we have an inclusion; if $x \in D$, since $x \le x^{**}$ we deduce that $x^{**} \in D$, hence $x^{**} \in D \cap \mathbf{R}(A)$ (since $x^{**} \in \mathbf{R}(A)$), so we have another inclusion, that is, the equality from the enounce.

Clearly $1 \in D \cap \mathbf{R}(A)$; if $x, y \in \mathbf{R}(A)$ such that $x, x \to y \in D \cap \mathbf{R}(A)$, then $y \in D$, hence $y \in D \cap \mathbf{R}(A)$, so $D \cap \mathbf{R}(A)$ is a deductive system in **R**(A).■

Lemma 5.3.32. If F is a filter in R(A), then

$$\overline{F} = \{x \in A : x^{**} \in F\} \in Ds(A).$$

Proof. Since $1^{**} = 1 \in F$, we deduce that $1 \in \overline{F}$; now let $x, y \in A$ such that $x, x \to y \in \overline{F}$, that is, $x^{**}, (x \to y)^{**} \in F$. By c_{22}, $(x \to y)^{**} = x^{**} \to y^{**}$; since $x^{**} \in F$ then $y^{**} \in F$, hence $y \in \overline{F}$, that is, $\overline{F}$ is a deductive system of A. ■

Lemma 5.3.33. If M ∈ Ds(A), then M ∈ Max(A) iff M ∩ R(A) is maximal in R(A).

Proof. Suppose that M is maximal in A and we have to prove that M ∩ **R**(A) is maximal in **R**(A). Now let $x \in \mathbf{R}(A)$ such that $x \notin M$; then $x^* \in M$ and since $x^* \in \mathbf{R}(A)$ (by c_{12}) we deduce that $x^* \in M \cap \mathbf{R}(A)$, that is, $M \cap \mathbf{R}(A)$ is maximal in **R**(A) (by Theorem 5.3.24).

Suppose now that M ∩ **R**(A) is maximal in **R**(A) and we will prove that M is maximal in A. Now let x ∉ M; if x* (which is in **R**(A)) is not in M, then x* ∉ M ∩ **R**(A); since we have supposed that M ∩ **R**(A) is maximal in **R**(A) then x ∈ M ∩ **R**(A), a contradiction, hence x* ∈ M, that is, M is maximal in A. ■

Lemma 5.3.34. If F ∈ Ds(R (A)), then F is a deductive system (that is , a filter) maximal in R(A) iff $\overline{F}$ is a maximal deductive system in A.

Proof. Firstly suppose that F is a maximal deductive systems in **R**(A) and we shall prove that $\overline{F}$ is maximal deductive system in A; let now x ∈ A such that x ∉ $\overline{F}$. Then x** ∉ F, hence x* ∈ F, so x* ∈ $\overline{F}$ (since (x*)** = x* ∈ F).
Suppose now $\overline{F}$ is a maximal deductive system in A and we shall prove that F is maximal in **R**(A); let now x ∈ **R**(A) such that x ∉ F. Since x ∈ **R**(A), then x = y*, with y ∈ A. If suppose that x* = y** ∉ F, then y ∉ $\overline{F}$; since $\overline{F}$ is maximal, we deduce that y* = x ∈ $\overline{F}$, hence x** = x ∈ F, a contradiction, since x ∉ F. Hence x* ∈ F, that is, F is maximal in **R**(A). ■

Lemma 5.3.35. If D ∈ Ds (A) and F ∈ Ds(R (A)), then $\overline{D \cap R(A)}$ = D and $\overline{F}$ ∩ R (A) = F.

Proof. We have $\overline{D \cap R(A)}$ = {x ∈ A : x** ∈ D∩**R**(A)}; since x ∈ **R**(A) we deduce that x** = x ∈ D, hence we have the inclusion $\overline{D \cap R(A)}$ ⊆ D. If x ∈ D, since x ≤ x** we deduce that x** ∈ D, hence x** ∈ D∩**R**(A), that is , D ∈ $\overline{D \cap R(A)}$, so we obtain the equality $\overline{D \cap R(A)}$ = D.
For the second equality we remark that $\overline{F}$ ∩ **R**(A) = {x∈**R**(A) : x** ∈ F}, hence if x ∈ $\overline{F}$ ∩ R(A), then x** = x ∈ F, so $\overline{F}$ ∩ **R**(A) ⊆ F.
Now let x ∈ F. Since F ⊆ **R**(A), x** = x ∈ F, hence x ∈ $\overline{F}$ ∩ **R**(A), so we have another inclusion F ⊆ $\overline{F}$ ∩**R**(A), that is, $\overline{F}$ ∩ **R**(A) = F. ■

Theorem 5.3.36. There is a bijection between Max(A) and Max(R(A)).

Proof. We define f : **Max**(A) → **Max**(**R**(A)) by f(M) = M ∩ **R**(A) for every M ∈ **Max**(A) and g : **Max**(**R**(A)) → **Max**(A) by g(F) = $\overline{F}$ for every F ∈ **Max**(**R**(A)). By Lemma 5.3.34, the functions f and g are correctly defined. By Lemma 5.3.35, we have $f \circ g = 1_{Max(R(A))}$ and $g \circ f = 1_{Max(A)}$, hence we deduce that f is a bijection and g is the converse of f. ■

Thorem 5.3.37. Topological space (Max(A), τ_A) is a Boolean space.

Proof. Since **R**(A) is a Boolean algebra (by Theorem 5.2.24), then **Max**(**R**(A)) is a Boolean space and all follows from Theorem 5.3.36. ■

Lemma 5.3.38. If A, A′ are two bounded Hilbert algebras and f: A → A′ is a morphism of bounded Hilbert algebras, then for every M ∈ Max(A′) we have that f^{-1}(M) ∈ Max(A).

Proof. Since f(1) = 1 ∈ M we deduce that 1 ∈ f^{-1}(M).
Suppose now that x, x → y ∈ f^{-1}(M), that is, f(x), f(x → y) = f(x) → f(y) ∈ M; then f(y) ∈ M, hence y ∈ f^{-1}(M), that is, f^{-1}(M) ∈ Ds(A).
We will prove that f^{-1}(M) ∈ **Max**(A); if x ∉ f^{-1}(M), then f(x) ∉ M, hence (f(x))* = f(x*) ∈ M, so x* ∈ f^{-1}(M). Clearly, f^{-1}(M) is proper because if we suppose that f^{-1}(M) = A, then we obtain that 0 ∈ f^{-1}(M), hence f(0) = 0 ∈ M and M = A′, which is a contradiction!. ■

Corollary 5.3.39. The assignments A → Max(A) and f → Max(f) (where Max(f) is defined by Lemma 5.3.38) defines a contravariant functor from the category of bounded Hilbert algebras to the category of Boolean spaces.

Proof. If we prove that for every f : A → A′, **Max**(f):**Max**(A′)→**Max**(A), **Max**(f)(M) = f^{-1}(M) for every M ∈ **Max**(A′) is a continuous function, then we apply Theorem 5.3.37 and Lemma 5.3.38.

Since **Max**(f) commutes with ∪ and ∩ , to prove that the function **Max**(f) is continuous it will suffice to prove that for every x ∈ A′, **Max**(f)(σ_A(x)) is open in **Max**(A).

We have **Max**(f) (σ_A(x)) = {M∈**Max**(A′) : **Max**(f)(M) ∈ σ_A(x)} = {M ∈ **Max**(A′) : f^{-1}(M) ∈ σ_A(x)} = {M ∈ **Max**(A′) : x ∈ f^{-1}(M)} = {M ∈ **Max**(A′) : f(x) ∈ M} = σ_A(f (x)) ∈ τ_A. ■

For M ∈ **Max**(A) we consider the function f_M : A → {0, 1} defined by

$$f_M(x) = \begin{cases} 0, & for\ x \notin M, \\ 1, & for\ x \in M. \end{cases}$$

Lemma 5.3.40. The function f_M : A → {0, 1} is a morphism of bounded Hilbert algebras.

Proof. We must prove that for any x,y∈A, then f_M(x →y) = f_M(x) → f_M(y) and f_M(0) = 0.

If x → y ∉ M, then y ∉ M (because if by contrary y∈ M, then x → y ∈ M). We will prove that x ∈ M. If x ∉ M, then x* ∈ M (since M is maximal); since x* ≤ x → y, so we deduce again x → y ∈ M, which is a contradiction!. So, in this case, f_M(x → y) = 0, f_M(x) = 1, f_M(y) = 0 and we have the equality f_M(x → y) = f_M(x) → f_M(y), because 1 = 0 → 0.

Suppose that x → y ∈ M; if x ∈ M, then y ∈ M and we have again the equality f_M(x → y) = f_M(x) → f_M(y) because 1 = 1 → 1.

If x ∉ M, then either y is or not in M we have the equality f_M(x → y) = f_M(x) → f_M(y), because 1 = 0 → 1 = 0 → 0.

Since 0 ∉ M, we deduce that f_M(0) = 0.

Since f_M(x) = 1 iff x ∈ M, we deduce that **Ker**(f_M) = M. ■

Theorem 5.3.41. If A is a bounded Hilbert algebra, then there is a bijection between Max(A) and H_i(A,{0, 1}) = { f: A → {0,1} : f is a morphism of Hilbert algebras}.

Proof. We define F : **Max**(A) → $\mathbf{H_i}$ (A,{0,1}) by F(M) = f_M for every M ∈ **Max**(A) and G : H_i(A,{0,1}) → **Max**(A) by G(f) = **Ker**(f) for every f ∈ $\mathbf{H_i}$(A, {0,1}) (clearly **Ker**(f) ∈ **Max**(A), since if x ∉ **Ker**(f), then f(x) = 0, so x*∈Ker(f) since f(x*) = (f(x))* = 0* = 1).
If M ∈ **Max**(A), then (G∘F)(M) = G(F(M)) = Ker(f_M) = M, that is, $G \circ F = 1_{Max(A)}$.
If f ∈ $\mathbf{H_i}$ (A,{0,1}), then (F∘G)(f) = F (G (f)) = $f_{Ker(f)}$.
We will prove that $f_{Ker(f)}$ = f; if x ∈ **Ker**(f), then f(x) = 1, so $f_{Ker(f)}$(x) = 1, and if x ∉ **Ker**(f), then f(x) = 0 and $f_{Ker(f)}$(x) = 0.
We deduce that $F \circ G = 1_{H_i(A,\{0,1\})}$, that is, F and G are bijections. ■
In [37, p. 24], it is considered a fixed family X of deductive systems which contains **Irc**(A) and to every element a ∈ A it is assigned φ(a) = {D ∈ X : a ∈ D}; if we consider π_A the topology of X generated by the sets of the form $\{\varphi(a)\}_{a\in A}$, then it is proved the following theorem of representation:

Theorem 5.3.42. The function $\varphi_A : A \to \pi_A$, defined by $\varphi_A(a) = \varphi(a)$, for every a ∈ A, is a monomorphism of Hilbert algebras and the space (X, π_A) is T_0.
If X = Irc(A), in general, this space is not quasi-compact.

In [37, p. 27], it is proved that if we denote $\mathbf{Ds}^2$(A) = **Ds**(**Ds**(A)), then we have:

Theorem 5.3.43. There is a monomorphism of Hilbert algebras $\psi_A : A \to \mathbf{Ds}^2(A)$.

The two representation theorems are still valid in the case when A is bounded (that is, we have a bounded monomorphism of Hilbert algebras). Let's see in what conditions we obtain a representation theorem by the same type as Theorem 5.3.42, for a bounded Hilbert algebra A, when instead of X we consider **Max**(A).

For this we will prove:

Theorem 5.3.44. If A is a bounded Hilbert algebra, then $\sigma_A : A \to \tau_A$ is a morphism of bounded Hilbert algebras.

σ_A is a monomorphism of bounded Hilbert algebras iff A is semisimple (see Definition 5.3.22).

Proof. From Lemma 5.3.29 we deduce that σ_A is a morphism of bounded Hilbert algebras.
Seeing in what case σ_A is a morphism of bounded Hilbert algebras we come to see in what conditions **Ker**(σ_A) = {1}.

We have a ∈ **Ker**(σ_A) iff σ_A(a) = 1 iff σ_A(a) = **Max**(A) iff a ∈ M, for every M ∈ **Max**(A) iff a ∈ $\bigcap_{M \in Max(A)} M$ hence **Ker**(σ_A) = $\bigcap_{M \in Max(A)} M$, so σ_A is a monomorphism of bounded Hilbert algebras iff A is semisimple. ■

Lemma 5.3.45. If A is a bounded Hilbert algebra, then D(A) = $\bigcap_{M \in Max(A)} M$.

Proof. If x ∈ D(A) and M ∈ **Max**(A), then x* = 0 ∉ M, hence x ∈ M, that is, x ∈ $\bigcap_{M \in Max(A)} M$, so we have the inclusion **D**(A) ⊆ $\bigcap_{M \in Max(A)} M$.

If x ∉ **D**(A), then x* ≠ 0, hence there is a maximal deductive system M such x* ∈ M; then x ∉ $\bigcap_{M \in Max(A)} M$, hence we deduce other inclusion, that is, we have the equality **D**(A) **=** $\bigcap_{M \in Max(A)} M$. ■

Theorem 5.3.46. A bounded Hilbert algebra is semisimple iff it is Boolean algebra.

Proof. "⇐". By Lemma 5.3.45 we have $\bigcap_{M \in Max(A)} M$ = **D**(A), hence if A is a Boolean algebra, then **D**(A) = {1}, that is, A is semisimple.
" ⇒ ". Suppose that $\bigcap_{M \in Max(A)} M$ = {1}; by Lemma 5.2.26, if x ∈ A, then x** → x ∈ D(A) **=** $\bigcap_{M \in Max(A)} M$ = {1}, hence x** → x = 1, so x** = x and by applying Theorem 5.2.20 we obtain that A is a Boolean algebra . ■

5.4. Hertz algebras. Definitions. Examples. Rules of calculus. The category of Hertz algebras

Definition 5.4.1. We call *Hertz algebra* a Hilbert algebra A with the property that for any elements x, y $\in$ A there exists $x \wedge y \in A$ (relative to natural ordering).

Heyting algebras and Boolean algebras are examples of Hertz algebras (later we will put in evidence a Hertz algebra which is not a Heyting algebra).
It is immediate that Definition 5.4.1 is equivalent with:

Definition 5.4.2. A *Hertz algebra* is an algebra $(A,\wedge,\rightarrow)$ of type (2,2) such that the followings identities are verifyed:
a_{12}: $x \rightarrow x = y \rightarrow y$;
a_{13}: $(x \rightarrow y) \wedge y = y$;
a_{14}: $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$;
a_{15}: $x \wedge (x \rightarrow y) = x \wedge y$.

Corollary 5.4.3. The class H_z of Hertz algebras is equational.

Lemma 5.4.4. If A is a Hertz algebra and x, y, z$\in$ A, then $x \wedge z \leq y$ iff $z \leq x \rightarrow y$.

Proof. "$\Rightarrow$". Suppose that $x \wedge z \leq y$; then $x \rightarrow (x \wedge z) \leq x \rightarrow y$, hence $(x \rightarrow x) \wedge (x \rightarrow z) \leq x \rightarrow y$ (by a_{14}), so $x \rightarrow z \leq x \rightarrow y$. Since $z \leq x \rightarrow z$, we deduce that $z \leq x \rightarrow y$.
"$\Leftarrow$". Conversely, if $z \leq x \rightarrow y$; then $x \wedge z \leq x \wedge (x \rightarrow y) = x \wedge y$ (by a_{15}), hence $x \wedge z \leq y$. ■

Remark 5.4.5. Following this lemma we can conclude that Hertz algebras are *implicative semilattices* (see [63]–[66]).

Lemma 5.4.6. If A is a Hertz algebra and x, y, z $\in$ A, then
c_{34}: $x \rightarrow (y \rightarrow z) = (x \wedge y) \rightarrow z$;
c_{35}: $(x \rightarrow y) \wedge (y \rightarrow z) \leq x \rightarrow z$;
c_{36}: If $x \leq y$, then $x \wedge (y \rightarrow z) = x \wedge z$;

$\mathbf{c_{37}}$**:** $\mathbf{x \wedge (y \rightarrow z) = x \wedge ((x \wedge y) \rightarrow (x \rightarrow z))}$**;**

$\mathbf{c_{38}}$**:** $\mathbf{(x \rightarrow y)^* = x^{**} \wedge y^*}$**.**

Proof. We use that if $(A,\vee,\wedge,\rightarrow,0)$ is a Heyting algebra, then $(A, \wedge, \rightarrow)$ is a Hertz algebra; so the equalities $c_{34} - c_{38}$ are true in a Hertz algebra because these are true in a Heyting algebra (see §1). ■

Lemma 5.4.7. If A is a bounded Hilbert algebra and x, y, z ∈ A, then
$\mathbf{c_{39}}$**:** $\mathbf{x \Delta (y \Delta z) = y \Delta (x \Delta z)}$**;**

$\mathbf{c_{40}}$**:** $\mathbf{x \veebar (y \veebar z) = (x \veebar y) \veebar z = y \veebar (x \veebar z)}$**.**

Proof. By Theorem 5.3.43, we can suppose that A is a Heyting algebra (or Hertz algebra). In consequence we can use the rules of calculus $c_{34} - c_{38}$.

$\mathbf{c_{39}}$**.** We have: $x \Delta (y \Delta z) = x \Delta ((z \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow z))$ = (by c_{22}) $= (z \rightarrow y) \rightarrow (x \Delta ((y \rightarrow z) \rightarrow z)) = (z \rightarrow y) \rightarrow ((y \rightarrow z)\rightarrow(x \Delta z)) = (z \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow ((z \rightarrow x) \rightarrow ((x \rightarrow z) \rightarrow z)))$ = (by c_{34}) $= ((z \rightarrow y) \wedge (y \rightarrow z) \wedge (z \rightarrow x) \wedge (x \rightarrow z)) \rightarrow z$ and $y \Delta (x \Delta z) = y \Delta ((z \rightarrow x) \rightarrow((x \rightarrow z)\rightarrow z)$ = (by c_{22}) $= (z \rightarrow x) \rightarrow (y \Delta ((x \rightarrow z) \rightarrow z)) = (z \rightarrow x) \rightarrow ((x \rightarrow z) \rightarrow (y \Delta z)) = (z \rightarrow x) \rightarrow ((x \rightarrow z) \rightarrow ((z \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow z)))$ = (by c_{34}) $= ((z \rightarrow x) \wedge (x \rightarrow z) \wedge (y \rightarrow z) \wedge (z \rightarrow y)) \rightarrow z$, hence $x \Delta (y \Delta z) = y \Delta (x \Delta z)$.

$\mathbf{c_{40}}$**.** We have $x \veebar (y \veebar z) = x^* \rightarrow (y^* \rightarrow z)$ = (by c_{34}) $=(x^* \wedge y^*) \rightarrow z$, and $(x \veebar y) \veebar z = (x^* \rightarrow y)^* \rightarrow z$ = (by c_{38}) $= (x^{***} \wedge y^*) \rightarrow z = (x^* \wedge y^*) \rightarrow z$, hence we deduce that $x \veebar (y \veebar z) = (x \veebar y) \veebar z$; since $x \veebar (y \veebar z) = y \veebar (x \veebar z)$ we obtain the required equalities. ■

In the case of bounded Hertz algebras, the notions of *dense* and *regular* element will be defined as in the case of bounded Hilbert algebras; consequently, Theorem 5.1.24 is true for the case A is Hertz algebra.

We remark that if $x, y \in \mathbf{R}(A)$, hence $x^{**} = x$ and $y^{**} = y$, then the meet between x and y in $\mathbf{R}(A)$ (that is, $(x \rightarrow y^*)^*$) doesn't coincide with the meet between x and y in A (that is, with $x\wedge y$).
We want to establish in what conditions these two infimums coincide.

Suppose $x\wedge y = (x \rightarrow y^*)^*$, for any $x, y \in A$; in particular we have $x\wedge x = (x \rightarrow x^*)^* \Leftrightarrow x = x^{**}$, that is, A is a Boolean algebra (by Theorem 5.2.20).

Definition 5.4.8. If A_1, A_2 are two Hertz algebras, we call *morphism of Hertz algebras* a function $f : A_1 \to A_2$ such that for every $x, y \in A_1$ we have
a_{16}: $f(x \to y) = f(x) \to f(y)$;
a_{17}: $f(x \wedge y) = f(x) \wedge f(y)$.
If A_1 and A_2 are bounded, we add the condition $f(0) = 0$.

We denote by $\mathbf{H_z}$ ($\overline{H}_z$) the category of Hertz algebras (bounded Hertz algebras). Since these categories are equational, the monomorphisms are exactly the injective morphisms (by Proposition 4.2.9).

Lemma 5.4.9. If A is a bounded Hertz algebra and $x, y \in A$, then
c_{41}: $(x \wedge y)^{**} = x^{**} \wedge y^{**}$ (the meet between x^{**} and y^{**} is in R(A)).

Proof. If in c_{34} we consider $z = 0$, we obtain that $(x \wedge y)^* = x \to y^*$, so $(x\wedge y)^{**} = (x \to y^*)^*$.
On the other hand, in **R**(A) we have $x^{**}\wedge y^{**} = (x^{**} \to y^{***})^* =$
$= (x^{**} \to y^*)^* =$ (by c_8) $= (y \to x^*)^* = (x \to y^*)^*$, so we obtain the desired equality. ■

Corollary 5.4.10. If A is a bounded Hertz algebra, then the function $\varphi_A:A \to R(A)$, defined by $\varphi_A(x) = x^{**}$ for every $x \in A$, is an surjective morphism of bounded Hertz algebras.

Theorem 5.4.11. The category $\overline{H}_z$ is a reflexive subcategory of $\overline{H}_i$.

Proof. ([18],[73]).We have to define a reflector $R : \overline{H}_i \to \overline{H}_z$.
For $A\in \overline{H}_i$ we denote by F(A) the family of finite and non-empty subsets of A, and I = {1}.
If $X,Y \in F(A)$, $X = \{x_1, x_2, ..., x_n\}$, $Y = \{y_1, y_2, ..., y_m\}$ we define $X \to Y = \bigcup_{j=1}^{m} \{(x_1, x_2, ..., x_n; y_j)\}$ and $X \wedge Y = X \cap Y$.

On F(A) we define a binary relation ρ_A; $(X, Y) \in \rho_A$ iff $X \to Y = Y \to X = I$.

Clearly, ρ_A is an equivalence on F(A); we will prove the compatibility of ρ_A with the operations $\rightarrow$ and $\wedge$ defined above on F(A).

Let $Z = \{z_1, z_2, ..., z_p\} \in F(A)$.

To prove that $(Z \rightarrow X, Z \rightarrow Y) \in \rho_A$, we denote $t_i = (z_1, ..., z_p; x_i)$ and $q_j = (z_1, ..., z_p; y_j)$, i = 1, 2, ..., n, j = 1, 2, ..., m.

Then $Z \rightarrow X = \{t_1, t_2, ..., t_n\}$ and $Z \rightarrow Y = \{q_1, q_2, ..., q_m\}$, so $(Z \rightarrow X) \rightarrow (Z \rightarrow Y) = \bigcup_{j=1}^{m} \{(t_1, t_2, ..., t_n; q_j)\}$ and

$(Z \rightarrow Y) \rightarrow (Z \rightarrow X) = \bigcup_{i=1}^{n} \{(q_1, q_2, ..., q_m; t_i)\}$.

But for $j \in \{1,2,...,m\}$ we have $(t_1, t_2, ..., t_n; q_j) = ((z_1, ..., z_p; x_1), ..., (z_1, ..., z_p; x_n); (z_1, ..., z_p; y_j))$ = (by $c_{13} – c_{15}$) = $(z_1, z_2, ..., z_p; (x_1, .., x_n; y_j)) = (z_1, ..., z_p; 1) = 1$, hence $(Z \rightarrow X) \rightarrow (Z \rightarrow Y) = I$.

Analogously we deduce that $(Z \rightarrow Y) \rightarrow (Z \rightarrow X) = I$,

$(X \rightarrow Z) \rightarrow (Y \rightarrow Z) = (Y \rightarrow Z) \rightarrow (X \rightarrow Z) = I$, hence $(Z \rightarrow X, Z \rightarrow Y) \in \rho_A$ and $(X \rightarrow Z, Y \rightarrow Z) \in \rho_A$.

To prove the compatibility of ρ_A with $\wedge$ we remark that $(X,Y) \in \rho_A$ iff $\langle X \rangle = \langle Y \rangle$ (by c_{25}).

So $(X \wedge Z, Y \wedge Z) \in \rho_A \Leftrightarrow \langle X \cup Z \rangle = \langle Y \cup Z \rangle \Leftrightarrow \langle X \rangle \vee \langle Z \rangle = \langle Y \rangle \vee \langle Z \rangle$, which is true since we have supposed that $\langle X \rangle = \langle Y \rangle$.

For $X \in F(A)$ we denote by X / ρ_A the equivalence class of X relative to ρ_A and $H_A = F(A) / \rho_A$.

For $X/\rho_A, Y/\rho_A \in H_A$ we define $X/\rho_A \rightarrow Y/\rho_A = (X \rightarrow Y)/ \rho_A$ and $X /\rho_A \wedge Y / \rho_A = (X \cup Y)/ \rho_A$.

Since ρ_A is compatible with $\rightarrow$ and $\wedge$, the operations on H_A are correctly defined.

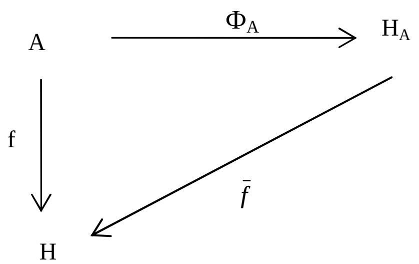
Also, $X / \rho_A \leq Y / \rho_A$ iff $X / \rho_A \wedge Y / \rho_A = X / \rho_A$ iff $(X \cup Y) / \rho_A = X / \rho_A$ iff $X \rightarrow (X \rightarrow Y) = I$ iff $\langle Y \rangle \subseteq \langle X \rangle$.

In [73] it is proved that $(H_A, \rightarrow, \wedge)$ become a bounded Hertz algebra where $0 = \{0\} / \rho_A$ and $1 = \{1\} / \rho_A$ (see also §2 from Chapter 3).

We will prove that $\Phi_A : A \rightarrow H_A$, $\Phi_A (a) = \{a\} / \rho_A$, for every $a \in A$, is a monomorphism of bounded Hilbert algebras; indeed, if $a, b \in A$, then

$\Phi_A(a \to b) = \{a \to b\}/\rho_A$,$\Phi_A(a) \to \Phi_A(b) = \{a\}/\rho_A \to \{b\}/\rho_A = (\{a\} \to \{b\})/\rho_A = \{a \to b\}/\rho_A = \Phi_A(a \to b)$ and $\Phi_A(0) = \{0\}/\rho_A = 0$.
If $\Phi_A(a) = \Phi_A(b)$, then [a) = [b), hence a = b.
If we put **R**(A) = H_A we obtain the definition of the reflector R on objects.
To define R on morphisms we will prove that the pair (H_A,Φ_A) verifies the property:

For every bounded Hertz algebra H and every morphism of bounded Hilbert algebras $f : A \to H$, there is a unique morphism of bounded Hertz algebras $\bar{f} : H_A \to H$ such that the diagram



is commutative (i.e, $\bar{f} \circ \Phi_A = f$).

Indeed, for X = $\{x_1, ..., x_n\} \in$ F(A) we define $\bar{f}(X / r_A) = \overset{n}{\underset{i=1}{\wedge}} f(x_i)$.

To prove $\bar{f}$ is correctly defined, let Y = $\{y_1, y_2, ..., y_m\} \in$ F(A) such that

$X/\rho_A = Y/\rho_A \Leftrightarrow X \to Y = Y \to X = \{1\}$, hence

(1) $$\begin{cases}(x_1,...,x_n;y_j)=1, & j=1,2,...,m,\\(y_1,...,y_m;x_i)=1, & i=1,2,...,n.\end{cases}$$

Since f is a morphism of bounded lattices, by c_{34} and (1) we deduce

(2) $$\begin{cases}f(x_1)\wedge ... \wedge f(x_n) \le f(y_j), & j=1,2,...,m,\\f(y_1)\wedge ... \wedge f(y_m) \le f(x_i), & i=1,2,...,n,\end{cases}$$

hence $\overset{n}{\underset{i=1}{\wedge}} f(x_i) = \overset{m}{\underset{j=1}{\wedge}} f(y_j) \Leftrightarrow \bar{f}(X/\rho_A) = \bar{f}(Y/\rho_A)$, that is, $\bar{f}$ is correct defined.
We will prove that $\bar{f}$ is a morphism of bounded Hertz algebras.

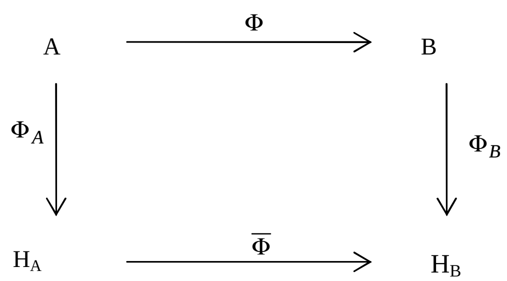
We have $\bar{f}$ (X / ρ_A $\to$Y / ρ_A) = $\bar{f}$ ((X $\to$ Y) /ρ_A) =
$\bar{f}((\bigcup_{j=1}^{m}\{(x_1,...,x_n;y_j)\})/r_A)$ = $\wedge_{j=1}^{m}(f(x_1),...,f(x_n);f(y_j))$,and
$\bar{f}$ (X / ρ_A) $\to$ $\bar{f}$ (Y / ρ_A) = ($\wedge_{i=1}^{n} f(x_i)$) $\to$ ($\wedge_{j=1}^{m} f(y_j)$) (by a_{14}) =
$\wedge_{j=1}^{m}$ (($\wedge_{i=1}^{n} f(x_i)$) $\to$ $f(y_j)$) = $\wedge_{j=1}^{m}$ (f (x_1), f (x_2), ..., f (x_n); f (y_j)) =
= $\bar{f}$ (X/ρ_A $\to$ Y/ρ_A); also, $\bar{f}$ (0) = $\bar{f}$ ({0} / ρ_A) = f(0) = 0 and
$\bar{f}$ (X/ρ_A $\wedge$ Y/ρ_A) = $\bar{f}$ ((X$\wedge$Y)/ ρ_A) = $\bar{f}$ ((X $\cup$ Y)/ ρ_A) = ($\wedge_{i=1}^{n} f(x_i)$)$\wedge$
($\wedge_{j=1}^{m} f(y_j)$) = $\bar{f}$ (X / ρ_A) $\wedge$ $\bar{f}$ (Y / ρ_A).

If a $\in$ A, then ($\bar{f}$ $\circ$ Φ_A)(a) = $\bar{f}$ (Φ_A (a)) = $\bar{f}$ ({a} / ρ_A) = f (a), that is, $\bar{f}$ $\circ$ Φ_A = f.

To prove the uniqueness of $\bar{f}$, let $\bar{\bar{f}}$: H_A $\to$ H be another morphism of bounded Hertz algebras such that $\bar{\bar{f}}$ $\circ$ Φ_A = f and X = {x_1, x_2, ..., x_n}$\in$F(A).

Since X = {x_1} $\cup$ {x_2} $\cup$ ... $\cup$ {x_n} we have X / ρ_A = ({x_1}$\cup${x_2}$\cup$ ... $\cup$ {x_n}) / ρ_A = ({x_1}/ρ_A) $\wedge$ ({x_2}/ρ_A) $\wedge$ ... $\wedge$ ({x_n}/ρ_A); but $\bar{f}$ $\circ$ Φ_A = f, so we obtain that $\bar{\bar{f}}$ (X / ρ_A) = $\bar{\bar{f}}$ (({x_1}/ρ_A) $\wedge$ ... $\wedge$ ({x_n}/ρ_A)) = $\bar{\bar{f}}$ ({x_1}/ ρ_A) $\wedge$...$\wedge$ $\bar{\bar{f}}$ ({x_n}/ ρ_A) = f (x_1) $\wedge$ f (x_2) $\wedge$...$\wedge$ f (x_n) = $\bar{f}$ (X / ρ_A), hence $\bar{f}$ = $\bar{\bar{f}}$.

It is immediate that if A and B are two bounded Hilbert algebras and Φ : A $\to$ B is a morphism of bounded Hilbert algebras, then there is a unique morphism of bounded Hertz algebras $\bar{\Phi}$: H_A $\to$ H_B such that the diagram

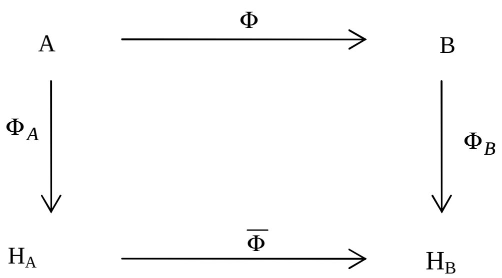


is commutative (i.e, $\Phi_B \circ \Phi = \overline{\Phi} \circ \Phi_A$).
Clearly, if $X = \{x_1, \ldots, x_n\}$, then $\overline{\Phi}(X / \rho_A) = \{\Phi(x_1), \ldots, \Phi(x_n)\} / \rho_B$. If we put $R(\Phi) = \overline{\Phi}$ we obtain the definition of $R : \overline{H}_i \to \overline{H}_z$ by morphisms. Now, the proof that $\overline{H}_z$ is a reflexive subcategory of $\overline{H}_i$ is a routine. ■

Remark 5.4.12. For a Hilbert algebra A, $(H_A, \to, \wedge)$ is an example of Hertz algebra which is not a Heyting algebra; indeed, it is suffice to take X, $Y \in F(A)$ such that $X \cap Y = \emptyset \notin F(A)$ and thus in H_A it doesn't exist $X / \rho_A \vee Y / \rho_A$ since $X/\rho_A \vee Y/\rho_A = (X \vee Y) / \rho_A = (X \cap Y)/\rho_A = \emptyset/\rho_A \notin H_A$.

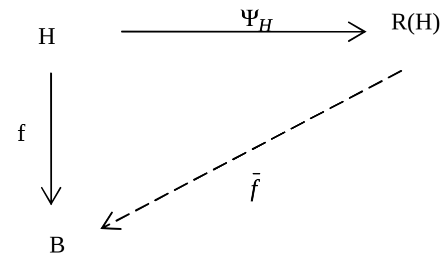
Theorem 5.4.13. The reflector R : $\overline{H}_i \to \overline{H}_z$ (defined in Theorem 5.4.11) preserves monomorphisms.

Proof. Let A, B be two bounded Hilbert algebras, $\Phi: A \to B$ a monomorphism of bounded Hilbert algebras; we will prove that the morphism $R(\Phi) = \overline{\Phi} : H_A \to H_B$ (defined in Theorem 5.3.11) such that the diagram



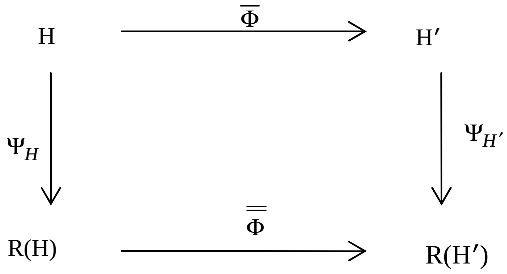
is commutative, is also a monomorphism of bounded Hertz algebras, that is, **Ker**$(\overline{\Phi}) = \{1\}$.
Indeed, if $X = \{x_1, x_2, ..., x_n\} \in F(A)$ and $\overline{\Phi}(X / \rho_A) = 1$, then $\{\Phi(x_1), \ldots, \Phi(x_n)\} / \rho_B = \{1\} / \rho_B \Leftrightarrow \Phi(x_1) = \ldots = \Phi(x_n) = 1$, hence $x_i = 1$, $i = 1, 2, ..., n$, that is, $X / \rho_A = I / \rho_A = 1$. ■

Theorem 5.4.14. Let H be a bounded Hertz algebra, B a Boolean algebra and f : H → B be a morphism of bounded Hertz algebras. Then:
(i) There is a unique morphism of Boolean algebras $\bar{f}$: R(H) → B such that the diagram



is commutative (i.e, $\bar{f} \circ \Psi_H = f$);

(ii) If H′ is another bounded Hertz algebra and $\overline{\Phi}$: H → H′ a morphism (monomorphism) of bounded Hertz algebras, then there is a unique morphism (monomorphism) of Boolean algebras $\overline{\overline{\Phi}}$: R(H) → R(H′) such that the diagram



is commutative (i.e, $\Psi_{H'} \circ \overline{\Phi} = \overline{\overline{\Phi}} \circ \Psi_H$).

Proof. (i). If x ∈ **R**(H) then x** = x; since f is a morphism of bounded Hertz algebras, we deduce that f(x**) = (f(x))** = f(x) (since f(x) ∈ B and B is a Boolean algebra). We can consider $\bar{f} = f_{|R(H)}$.

If x, y ∈ **R**(H), then x → y, x ∧ y ∈ **R**(H) since by c_{38} we have (x → y)** = ((x → y)*)* = (x** ∧ y*)* = (x** ∧ y*) → 0 = (by c_{34}) = x** → (y* → 0) = x** → y** = x → y and (x ∧ y)** = x** ∧ y** = x ∧ y (by c_{41}). So, if we consider $\bar{f} = f_{|R(H)}$ and x, y∈ **R**(H), then $\bar{f}$ (x → y) = f(x → y) = f(x) → f(y) = $\bar{f}$ (x) → $\bar{f}$ (y), $\bar{f}$ (x ∧ y) = f(x ∧ y) = f(x) ∧ f(y) = $\bar{f}$ (x) ∧ $\bar{f}$ (y) and $\bar{f}$ (0) = f(0) = 0.

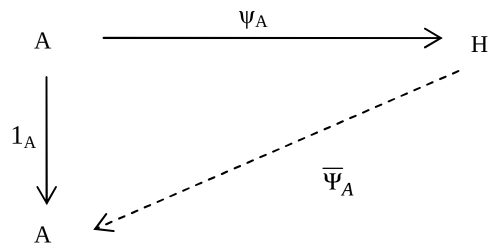
As in the case of Theorem 5.2.24, for x, y ∈ **R**(H), x ∨ y ∈ **R**(H) and x ∨ y = (x* ∧ y*)*, hence $\bar{f}$ (x ∨ y) = f(x ∨ y) = f((x* ∧ y*)*) = (f(x* ∧ y*))* = ((f(x))* ∧ (f(y))*)* = f(x) ∨ f(y) = $\bar{f}$ (x) ∨ $\bar{f}$ (y).
Since $\bar{f}$ (1) = f(1) = 1, from the above we deduce that $\bar{f}$ is a morphism of Boolean algebras.
(ii). From (i) we deduce the existence of $\bar{\bar{\Phi}}$ for f = $\Psi_{H'} \circ \bar{\Phi}$.
We have to prove that if $\bar{\Phi}$ is a monomorphism of bounded Hertz algebras, then $\bar{\bar{\Phi}}$ is a monomorphism of Boolean algebras.
Indeed, let x ∈ **R**(H) such that $\bar{\bar{\Phi}}$(x) = 1; since x = x** = ψ_H(x), there result that $(\bar{\bar{\Phi}} \circ \Psi_H)(x) = 1$, hence $(\Psi_{H'} \circ \bar{\Phi})(x) = 1 \Leftrightarrow$ ($\bar{\Phi}$ (x))** = 1.
But ($\bar{\Phi}$ (x))** = $\bar{\Phi}$ (x**) = $\bar{\Phi}$ (x), so we obtain that $\bar{\Phi}$ (x) = 1, hence x = 1 (since $\bar{\Phi}$ is supposed a monomorphism of bounded Hertz algebras). ■

Remark 5.4.15. Since by Theorem 4.2.24, in **R**(H) (with H a bounded Hilbert algebra), ∧, ∨ and ′ could be done only with the help of implication →, we deduce that a theorem as Theorem 5.3.15 is true in the case of bounded Hilbert algebras, too.
Indeed, if x, y ∈ **R**(H), then $\bar{f}$ (x ∧ y) = $\bar{f}$ ((x → y*)*) = f ((x → y*)*) = (f(x) → (f(y))*)* = ((f(x))* ∨ (f(y))*)* = f(x) ∧ f(y) = $\bar{f}$ (x) ∧ $\bar{f}$ (y), and $\bar{f}$ (x ∨ y) = $\bar{f}$ (x* → y) = f(x* →y) = (f(x))* → f(y) = f(x) ∨ f(y) = $\bar{f}$ (x) ∨ $\bar{f}$ (y), $\bar{f}$ (x′) = $\bar{f}$ (x*) = f(x*) = (f(x))* . ■

5.5. Injective objects in the categories of bounded Hilbert and Hertz algebras

Theorem 5.5.1. In the category $\overline{H}_i$ any injective object is a complete Boolean algebra.

Proof. Let A be an injective object in $\overline{H}_i$. By Theorem 5.3.43, there is a complete Heyting algebra H = **Ds**^{2}(A) and a monomorphism of bounded Hilbert algebras ψ_A : A → H.
Since A is injective, if we consider in $\overline{H}_i$ the diagram



there results the existence of a morphism of bounded Hilbert algebras $\overline{\Psi}_A : H \to A$ such that $\overline{\Psi}_A \circ \psi_A = 1_A$.

Since H is complete, and ψ_A, $\overline{\Psi}_A$ are in particular isotone functions, by Lemma 4.10.6, A is complete (by Lemma 5.4.4 we deduce that A is complete Heyting algebra); by Corollary 5.1.20, to prove A is a Boolean algebra it is suffice to prove that **D**(A) = {1} (where **D**(A) is the deductive system of the dense elements of A).

Clearly **D**(A) is a Hilbert subalgebra of A. Then by Remark 5.2.14, A′= **D**(A) ∪ {0} become a bounded Hilbert algebra.

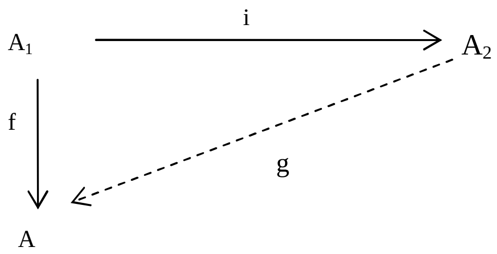
Let B = A′∪{α} with α∉A; B becomes a bounded Hilbert algebra if we define α → α = 1, 0 → α = 1, α → 0 = 0, a → α = 1 and α → a = 1, for every a∈ **D**(A) (see [44]). So A′ becomes a Hilbert subalgebra both for A and B. By the injectivity of A there is a morphism of bounded Hilbert algebras f : B → A such that $f_{|A'} = i_{A'}$ ($i_{A'}$ is the inclusion of A′ in A). Since α* = 0 we have (f(α))* = 0, hence f(α) ∈ **D**(A), so there is x ∈ **D**(A) such that x = f(α).

Then x → f (α) = 1; since x → f(α) = f(x) → f(α) = f(x → α) = f(α), we deduce that f(α) = 1.

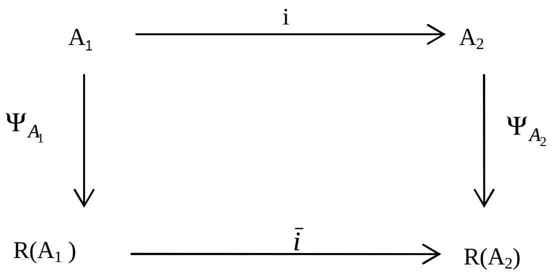
Since α → a = 1, for every a ∈ **D**(A), we obtain that f(α) → a = 1, so 1 → a = 1 ⇔ a = 1, hence **D**(A) = {1}. ■

Theorem 5.5.2. In the category $\overline{H}_i$ the complete Boolean algebras are injective objects.

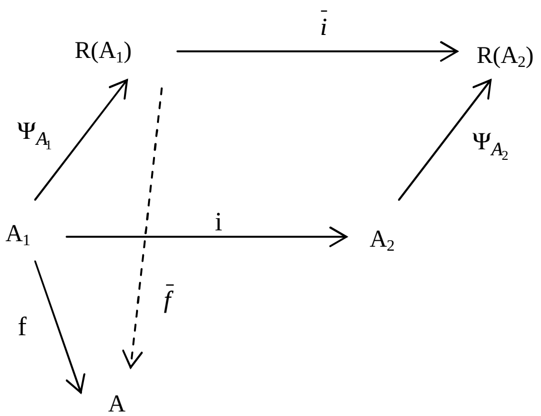
Proof. Let A be a complete Boolean algebra. In $\overline{H}_i$ we consider the diagram



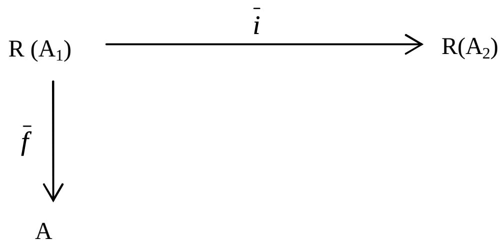
with A_1, A_2 bounded Hilbert algebras, $i : A_1 \to A_2$ a monomorphism of bounded Hilbert algebras and $f : A_1 \to A$ a morphism of bounded Hilbert algebras.
So, we have to prove the existence of a morphism of bounded Hilbert algebras $g : A_2 \to A$ such that $g \circ i = f$.
By Theorem 5.4.14 (which is true and for the case of bounded Hilbert algebras), we have the commutative diagram:



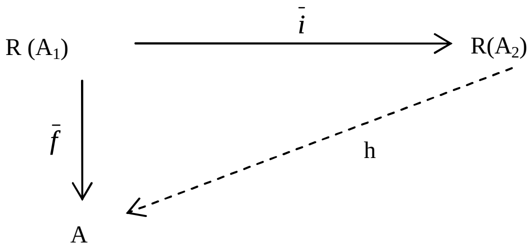
We obtain the following diagram



with $\bar{f} \circ \Psi_{A_1} = f$ (the existence of $\bar{f}$ is assured by Theorem 5.4.11).
We consider now the diagram



in the category **B** of Boolean algebras with $\bar{i}$ monomorphism in **B** (by Theorem 5.4.11). Then by a theorem of Sikorski (see Theorem 4.10.3), in the category **B** the injective objects are exactly complete Boolean algebras, hence there is a morphism of Boolean algebras h : **R**(A_2) → A such that the diagram



is commutative (i.e, $h \circ \bar{i} = \bar{f}$).
The desired morphism will be $g = h \circ \Psi_{A_2} : A_2 \rightarrow A$, (which is a morphism of bounded Hilbert algebras). Indeed, $g \circ i = (h \circ \Psi_{A_2}) \circ i = h \circ (\bar{i} \circ \Psi_{A_1}) = (h \circ \bar{i}) \circ \Psi_{A_1} = \bar{f} \circ \Psi_{A_1} = f$. ■

Corollary 5.5.3. In the category $\overline{H}_z$ injective objects are exactly complete Boolean algebras.

Proof. By Theorem 5.4.13, the reflector R : $\overline{H}_i \rightarrow \overline{H}_z$ preserves monomorphisms.
Now let B be an injective bounded Hertz algebra; by Remark 4.9.3, B is injective as bounded Hilbert algebra, hence B has to be complete Boolean algebra (by Theorem 5.5.1).
The fact, that a complete Boolean algebras is injective Hertz algebras is proved as in the case of bounded Hilbert algebras (see Theorem 5.5.2) by using Theorem 5.4.14. ■

The problems of injective envelopes in the category $\overline{H}_i$ follows from the following theorem:

Theorem 5.5.4. Let A be a Hilbert algebra and B be a Boolean algebra. If there is a monomorphism of bounded Hilbert algebras i : A → B, then A becomes a Boolean algebra.

Proof. Let $x \in A$; since $i(x^{**}) = (i(x))^{**} = i(x)$ and i is supposed to be a monomorphism, we deduce that $x^{**} = x$, hence A is a Boolean algebra (by Theorem 5.3.20). ■

5.6. Localization in the categories of bounded Hilbert and Hertz algebras

In this paragraph we consider only bounded Hilbert and Hertz algebras.
We recall that if A is a Hilbert algebra, then for $x, y \in A$, $x \veebar y = x^* \to y$.

Definition 5.6.1. If A is a Hilbert algebra, a non-empty subset $S \subseteq A$ is called $\veebar$ - *closed system* of A if it contains with elements x, y and the element $x \veebar y$,too ($x, y \in A$).

For example, the deductive systems of A are $\veebar$ - closed systems of A.
For a Hilbert algebra A and a $\veebar$ - closed system S of A we define on A the binary relation θ_S by:
$(x, y) \in \theta_S$ iff there is $t \in S$ such that $t \veebar x = t \veebar y$.

Lemma 5.6.2. θ_S is a congruence on A.

Proof. Firstly we have to prove that θ_S is an equivalence on A; clearly θ_S is reflexive and symmetric.
Now let $(x, y), (y, z) \in \theta_S$; then there are $t, t' \in S$ such that $t \veebar x = t \veebar y$ and $t' \veebar y = t' \veebar z$. By c_{40} we have $t' \veebar (t \veebar x) = t' \veebar (t \veebar y) \Leftrightarrow t \veebar (t' \veebar y) = t \veebar (t' \veebar z) \Leftrightarrow (t' \veebar t) \veebar x = (t' \veebar t) \veebar z$, so, if we denote $t'' = t' \veebar t$

$\in S$ we obtain that $t'' \veebar x = t'' \veebar z$, hence $(x, z) \in \theta_S$, that is, θ_S is transitive. Hence θ_S is an equivalence on A.

To prove the compatibility of θ_S with $\rightarrow$, let $(x, y) \in \theta_S$, hence there is t $\in S$ such that $t \veebar x = t \veebar y$.

If $z \in A$, then $z \rightarrow (t \veebar x) = z \rightarrow (t \veebar y) \Leftrightarrow t \veebar (z \rightarrow x) = t \veebar (z \rightarrow y)$ (by c_3), hence $(z \rightarrow x, z \rightarrow y) \in \theta_S$.

Also, $t \veebar (x \rightarrow z) = t \veebar (y \rightarrow z)$ (by a_6), hence $(x \rightarrow z, y \rightarrow z) \in \theta_S$. ■

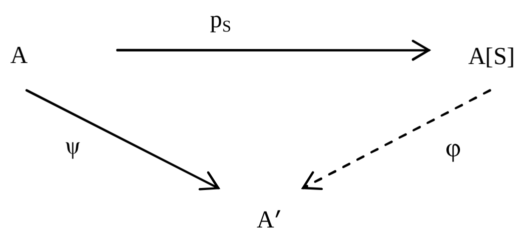
We denote $A[S] = A/\theta_S$ and by $p_S : A \rightarrow A[S]$ the canonical surjective function (which is a morphism of bounded Hilbert algebras).

If there is no danger of confusion, for $x \in A$, we denote $\hat{x} = p_S(x)$.

In A[S] the role of 0 is played by $\hat{0} = \{x \in A : (x, 0) \in \theta_S\} = \{x \in A : \text{there is } t \in S \text{ such that } t^* \leq x^*\}$ and the role of 1 by $\hat{1} = \{x \in A : (x, 1) \in \theta_S\} = \{x \in A : \text{there is } t \in S \text{ such that } t^* \leq x\}$.

Remark 5.6.3. If $s \in S$, since $s^* \rightarrow s = s^{**} = s^* \rightarrow 0$ (by c_9) we deduce that $(s, 0) \in \theta_S \Leftrightarrow p_S(s) = 0$, hence $p_S(S) = \{0\}$.

Lemma 5.6.4. If A′ is a Hilbert algebra and $\psi : A \rightarrow A'$ is a morphism of Hilbert algebras such that $\psi(S) = \{0\}$, then there is a unique morphism of Hilbert algebras $\varphi : A[S] \rightarrow A'$ such that the diagram



is commutative (i.e, $\varphi \circ p_S = \psi$).

Proof. For $\hat{x} \in A[S]$, with $x \in A$, we define $\varphi(\hat{x}) = \psi(x)$.

If $\hat{x} = \hat{y}$, then there is $t \in S$ such that $t^* \rightarrow x = t^* \rightarrow y$; since ψ is an morphism of Hilbert algebras we successively deduce $\psi(t^* \rightarrow x) = \psi(t^* \rightarrow$

y), $(\psi(t))^* \to \psi(x) = (\psi(t))^* \to \psi(y)$, $0^* \to \psi(x) = 0^* \to \psi(y)$, $1 \to \psi(x) = 1 \to \psi(y)$, $\psi(x) = \psi(y)$, hence φ is correctly defined. Clearly φ is a morphism of Hilbert algebras.

Since p_S is surjective we deduce the uniqueness of φ. ■

Definition 5.6.5. Following the above lemma, A[S] is called Hilbert algebra of fractions of A relative to the $\veebar$ - closed system S.

In what follows by A we denote a bounded Hilbert algebra.

Definition 5.6.6. A nonempty subset S $\subseteq$ A is called $\veebar$ - *subset* of A if for any a $\in$ A and x $\in$ S $\Rightarrow$ a $\veebar$ x $\in$ S.

We denote by S(A) the set of all $\veebar$ - subsets of A; clearly Ds(A) $\subseteq$ S(A) and if $D_1, D_2 \in S(A) \Rightarrow D_1 \cap D_2 \in S(A)$.

Lemma 5.6.7. If D $\in$ S(A), then

(i) 1 $\in$ D;

(ii) x $\in$ D $\Rightarrow$ x $\in$ D.**

Proof. (i). If $x \in D$, since $1 \in A \Rightarrow 1 \veebar x \in D \Leftrightarrow 0 \to x = 1 \in D$.

(ii). If $x \in S$, then $x \veebar x = x^{**} \in D$. ■

Definition 5.6.8. By *partial multiplier* on A we understand a function f:D $\to$ A with D $\in$ S(A) such that for any x, y $\in$ D and a $\in$ A we have

a_{18}: f(a $\veebar$ x) = a $\veebar$ f(x);

a_{19}: f(x) = f(x);**

a_{20}: x $\veebar$ f(y) = y $\veebar$ f(x).

By dom(f) $\in$ S(A) we denote the domain of f. If dom(f) = A, we say that f is *total*.

To simplify the language, we will use *multiplier* instead of *partial multiplier*, using *total* to indicate that the domain of a certain multiplier is A.

Examples

1. The function $1 : A \to A$, $1(x) = 1$ for every $x \in A$ is a total multiplier.

Indeed, if $x, a \in A$, then $a \veebar 1(x) = a^* \to 1 = 1 = 1(a \veebar x)$, $1(x^{**}) = 1 = 1(x)$ and for $x, y \in A$, $x \veebar 1(y) = x^* \to 1 = 1$ and $y \veebar 1(x) = y^* \to 1 = 1$ hence $x \veebar 1(y) = y \veebar 1(x)$.

2. The function $0 : A \to A$, $0(x) = x^{**}$ for every $x \in A$ is also a total multiplier.

Indeed, if $x, a \in A$, then $0(a \veebar x) = (a \veebar x)^{**} = (a^* \to x)^{**} = a^{***} \to x^{**}$ (by c_{22}) $= a^* \to x^{**} = a \veebar 0(x)$, $0(x^{**}) = x^{****} = x^{**} = 0(x)$. For $x, y \in A$, $x \veebar 0(y) = x \veebar y^{**} = x^* \to y^{**} = y^* \to x^{**}$ (by c_8) $= y^* \to 0(x) = y \veebar 0(x)$.

3. For $a \in A$ and $D \in S(A)$, the function $f_a : D \to A$, $f_a(x) = x \veebar a$ for any $x \in D$ is a multiplier on A (called *principal*).

Indeed, for $b \in A$, $x, y \in D$ we have $f_a(b \veebar x) = (b \veebar x) \veebar a = b \veebar (x \veebar a)$ (by c_{40}) $= b \veebar f_a(x)$, $f_a(x^{**}) = (x^{**}) \veebar a = x^{***} \to a = x^* \to a = f_a(x)$ and $x \veebar f_a(y) = x \veebar (y \veebar a) = y \veebar (x \veebar a) = y \veebar f_a(x)$.

Remark 5.6.9. If $\mathrm{dom}(f_a) = A$ we denote f_a by $\bar{f}_a$.

Lemma 5.6.10. If $f : D \to A$ is a multiplier on A ($D \in S(A)$), then
(i) $f(1) = 1$;
(ii) For every $x \in D$, $x^{**} \le f(x)$.

Proof. (i). If in a_{18} we put $a = 1$, then we obtain that for every $x \in D$, $f(1 \veebar x) = 1 \veebar f(x) \Leftrightarrow f(1) = 1$.

(ii). If in a_{18} we put $a = x$ we obtain that for every $x \in D$, $f(x \veebar x) = x \veebar f(x) \Leftrightarrow f(x^{**}) = x^* \to f(x) \Leftrightarrow f(x) = x^* \to f(x)$ (by a_{19})) $\Rightarrow x \le f(x) \Rightarrow x^* \to x \le x^* \to f(x) \Rightarrow x^{**} \le f(x)$. ■

For $D \in S(A)$ we denote $M(D, A) = \{f : D \to A : f$ is a multiplier on A$\}$ and $M(A) = \bigcup_{D \in S(A)} M(D, A)$.

If $D_1, D_2 \in S(A)$ and $f_i \in M(D_i, A)$, $i = 1, 2$, we define:
$f_1 \to f_2 : D_1 \cap D_2 \to A$ by $(f_1 \to f_2)(x) = f_1(x) \to f_2(x)$, for every $x \in D_1 \to D_2$.

Lemma 5.6.11. $f_1 \to f_2 \in M(D_1 \cap D_2, A)$.

Proof. If $a \in A$ and $x, y \in D_1 \cap D_2$, then $(f_1 \to f_2)(a \veebar x) = f_1(a \veebar x) \to f_2(a \veebar x) = (a \veebar f_1(x)) \to (a \veebar f_2(x)) = a \veebar (f_1(x) \to f_2(x)) = a \veebar (f_1 \to f_2)(x)$, $(f_1 \to f_2)(x^{**}) = f_1(x^{**}) \to f_2(x^{**}) = f_1(x) \to f_2(x) = (f_1 \to f_2)(x)$ and $x \veebar (f_1 \to f_2)(y) = x \veebar (f_1(y) \to f_2(y)) = (x \veebar f_1(y)) \to (x \veebar f_2(y)) = (y \veebar f_1(x)) \to (y \veebar f_2(x)) = y \veebar (f_1(x) \to f_2(x)) = y \veebar (f_1 \to f_2)(x)$. ■

Lemma 5.6.12. $(M(A), \to, 0, 1)$ **is a bounded Hilbert algebra.**

Proof. From Lemma 5.6.11 we deduce immediately that M(A) is a Hilbert algebra. If $D \in S(A)$, $f \in M(D, A)$ and $x \in D$, then $0(x) \le x^{**} \le f(x) \le 1 = 1(x)$. ■

Lemma 5.6.13. The function $v_A : A \to M(A)$, $v_A(a) = \bar{f}_a$ **for every** $a \in A$ **is a morphism in** $\overline{H}_i$.

Proof. If $a, b, x \in A$, then $(\bar{f}_a \to \bar{f}_b)(x) = \bar{f}_a(x) \to \bar{f}_b(x) = (x \veebar a) \to (x \veebar b) = x \veebar (a \to b) = \bar{f}_{a\to b}(x)$, hence $v_A(a) \to v_A(b) = v_A(a \to b)$.

Also, $v_A(0) = 0$ (since $\bar{f}_0(x) = x \veebar 0 = x^* \to 0 = x^{**} = 0(x)$ for every $x \in A$). ■

Definition 5.6.14. A nonempty subset $D \subseteq A$ **is called *regular* if for any** $x, y \in A$ **such that** $t \veebar x = t \veebar y$ **for any** $t \in D$**, then** $x = y$.

Example

Clearly, A is a regular subset of A since if $x, y \in A$ and $t \veebar x = t \veebar y$ for any $t \in A$, then in particular for $t = 0$ we obtain that $0 \veebar x = 0 \veebar y \Leftrightarrow 1 \to x = 1 \to y \Leftrightarrow x = y$.
More generally, every subset of A which contains 0 is a regular subset of A.

We denote by $\mathcal{R}(A) = \{D \subseteq A : D \text{ is a regular subset of } A\}$.

Lemma 5.6.15. If $D_1, D_2 \in S(A) \cap \mathcal{R}(A)$, then $D_1 \cap D_2 \in S(A) \cap \mathcal{R}(A)$.

Proof. Let $x, y \in A$ such that $t \veebar x = t \veebar y$ for every $t \in D_1 \cap D_2$.
Since for every $t_1, t_2 \in A$ we have $(t_1 \veebar t_2) \veebar 0 = (t_2 \veebar t_1) \veebar 0 = t_1 \veebar (t_2 \veebar 0)$, then if we consider $t = (t_1 \veebar t_2) \veebar 0 = (t_2 \veebar t_1) \veebar 0$ we have that $t \in D_1 \cap D_2$ (since $t_1 \veebar t_2 \in D_2$, so by Lemma 5.7, $t = (t_1 \veebar t_2) \veebar 0 = (t_1 \veebar t_2)^{**} \in D_2$).
Since $t \veebar x = t \veebar y$ we obtain that $((t_1 \veebar t_2) \veebar 0) \veebar x = ((t_1 \veebar t_2) \veebar 0) \veebar y \Leftrightarrow t_1 \veebar ((t_2 \veebar 0) \veebar x) = t_1 \veebar ((t_2 \veebar 0) \veebar y)$.
Since $t_1 \in D_1$ is arbitrary and $D_1 \in S(A) \cap \mathcal{R}(A)$, we obtain that $(t_2 \veebar 0) \veebar x = (t_2 \veebar 0) \veebar y \Leftrightarrow t_2 \veebar (0 \veebar x) = t_2 \veebar (0 \veebar y) \Leftrightarrow t_2 \veebar x = t_2 \veebar y$ (since $0 \veebar x = 1 \to x$ and $0 \veebar y = 1 \to y = y$). Since $t_2 \in D_2$ is arbitrary and $D_2 \in S(A) \cap \mathcal{R}(A)$, we obtain that $x = y$, hence $D_1 \cap D_2 \in S(A) \cap \mathcal{R}(A)$.
■

Remark 5.6.16. From Lemma 5.6.15 we deduce that
$M_r(A) = \{f \in M(A) : dom(f) \in S(A) \cap \mathcal{R}(A)\}$ is a Hilbert subalgebra of $M(A)$.

Definition 5.6.17. Given two multipliers f_1 and f_2 on A, we say that f_1 *extends* f_2 if $dom(f_2) \subseteq dom(f_1)$ and $f_1(x) = f_2(x)$ for every $x \in dom(f_2)$; in this case we write $f_2 \leq f_1$.
A multiplier f is called *maximal* if f can not be extended to a strictly larger domain which contain dom(f).

Lemma 5.6.18. (i) If $f_1, f_2 \in M(A)$ and $f \leq f_1$, $f \leq f_2$, then f_1 and f_2 agree on the $dom(f_1) \cap dom(f_2)$;
(ii) Every multiplier $f \in M_r(A)$ can be extended to a maximal multiplier. More precisely, every principal multiplier f_a with $dom(f_a) \in S(A) \cap \mathcal{R}(A)$ can be uniquely extended to a total multiplier $\bar{f}_a$ and each non-principal multiplier can be extended to a maximal non-principal one.

Proof. (i). If by contrary there is $t \in dom(f_1) \cap dom(f_2)$ such that $f_1(t) \neq f_2(t)$, since $dom(f) \in \mathcal{R}(A)$, there is $t' \in dom(f)$ such that $t' \veebar f_1(t) \neq t' \veebar f_2(t) \Leftrightarrow f_1(t' \veebar t) \neq f_2(t' \veebar t) \Leftrightarrow f_1((t' \veebar t)^{**}) \neq f_2((t' \veebar t)^{**})$, which is contradictory since $t_0 = (t' \veebar t)^{**} = ((t')^* \to t)^{**} = (t')^* \to t^{**} = t^* \to (t')^{**} = t \veebar (t')^{**} \in dom(f)$.

(ii). We first prove that f_a can not be extended to a non-principal multiplier. Let $D = dom(f_a) \in S(A) \cap \mathcal{R}(A)$, $f_a : D \to A$ and suppose by contrary that there is $D' \in S(A)$, $D \subseteq D'$ (hence $D' \in \mathcal{R}(A)$) and a non-principal multiplier $f \in M(D', A')$ which extends f_a.

Since f is non-principal there is $x_0 \in D'$, $x_0 \notin D$, such that $f(x_0) \neq x_0 \veebar a$. Since $D \in \mathcal{R}(A)$, then there is $t \in D$ such that $t \veebar f(x_0) \neq t \veebar (x_0 \veebar a) \Leftrightarrow f(t \veebar x_0) \neq (t \veebar x_0) \veebar a \Leftrightarrow f((t \veebar x_0)^{**}) \neq (t \veebar x_0) \veebar a = ((t \veebar x_0)^{**} \veebar a$.

Denoting $t_0 = (t \veebar x_0)^{**} = (t^* \to x_0)^{**} = t^{***} \to x_0^{**} = t^* \to x_0^{**} = x_0^* \to t^{**} = x_0 \veebar t^{**} \in D$ (since $t^{**} \in D$).

We obtain that $f(t_0) \neq t_0 \veebar a$, which is a contradictory since $f_a \leq f$.
Hence f_a is uniquely extended by $\bar{f}_a$.

Now, let $f \in M_r(A)$ non-principal and $M_f = \{(D,g) : D \in S(A),$ $g \in M(D,A)$, $dom(f) \subseteq D$ and $g_{|dom(f)} = f\}$ (clearly, if $(D,g) \in M_f$, then $D \in S(A) \cap \mathcal{R}(A)$).

The set M_f is ordered by $(D_1, g_1) \leq (D_2, g_2) \Leftrightarrow D_1 \subseteq D_2$ and $g_{2|D_1} = g_1$.

Let $(D_i, g_i)_{i \in I}$ be a chain in M_f.
Then $D' = \bigcup_{i \in I} D_i \in S(A) \cap \mathcal{R}(A)$ and $dom(f) \subseteq D'$.

So, $g' : D' \to A$ defined by $g'(x) = g_i(x)$ if $x \in D_i$ is correctly defined (since for $x \in D_i \cap D_j$ we have that $g_i(x) = g_j(x)$).
Clearly $g' \in M(D', A)$ and $g'_{|\,dom(f)} = f$ (since for $x \in dom(f) \subseteq D'$, then $x \in D'$ and so there is $i \in I$ such that $x \in D_i$, hence $g'(x) = g_i(x) = f(x)$).

So, (D', g') is an upper bound for the family $(D_i, g_i)_{i \in I}$, hence by Zorn's lemma M_f contains at least one maximal multiplier h which extends f. Since f is non-principal and h extends f, we deduce that h is non-principal.

■

On Hilbert algebra $M_r(A)$ we consider the relation ρ_A defined by:

$(f_1, f_2) \in \rho_A \Leftrightarrow f_1$ and f_2 agree on $dom(f_1) \cap dom(f_2)$.

Lemma 5.6.19. $\boldsymbol{\rho_A \in}$ **Con(M_r (A)) (in** $\overline{H}_i$ **).**

Proof. The reflexivity and the symmetry of ρ_A are immediate; to prove the transitivity of ρ_A, let $(f_1, f_2), (f_2, f_3) \in \rho_A$.
If by contrary there is $x_0 \in dom(f_1) \cap dom(f_3)$ such that $f_1(x_0) \neq f_3(x_0)$, since $dom(f_2) \in \mathcal{R}(A)$ there is $t \in dom(f_2)$ such that $t \veebar f_1(x_0) \neq t \veebar f_3(x_0)$ $\Leftrightarrow$ $f_1(t \veebar x_0) \neq f_3(t \veebar x_0) \Leftrightarrow f_1((t \veebar x_0)^{**}) \neq f_3((t \veebar x_0)^{**})$, which is a contradictory since $(t \veebar x_0)^{**} \in dom(f_1) \cap dom(f_2) \cap dom(f_3)$ (see the proof of Lemma 5.5.15). Hence $\rho_A \in$ **Echiv**($M_r(A)$).
Since the compatibility of ρ_A with $\rightarrow$ is immediate, we deduce that $\rho_A \in$ **Con**($M_r(A)$). ■

For $f \in M_r(A)$ we denote $[f] = f / \rho_A$ and $A'' = M_r(A) / \rho_A$.

Lemma 5.6.20. The function $\bar{v}_A$ **: A → A′′ defined by** $\bar{v}_A(a) = [\bar{f}_a]$**, for every a ∈ A is a monomorphism in** $\overline{H}_i$ **and** $\bar{v}_A$ **(A) ∈** $\mathcal{R}$**(A′′).**

Proof. The fact that $\bar{v}_A \in \overline{H}_i(A, A'')$ follows from Lemma 5.5.13.
To prove the injectivity of $\bar{v}_A$, let $a, b \in A$ such that $\bar{v}_A(a) = \bar{v}_A(b)$. Then $[\bar{f}_a] = [\bar{f}_b] \Leftrightarrow (\bar{f}_a, \bar{f}_b) \in \rho_A \Leftrightarrow \bar{f}_a(x) = \bar{f}_b(x)$, for every $x \in A \Leftrightarrow x \veebar a = x \veebar b$, for every $x \in A \Leftrightarrow a = b$.
To prove $\bar{v}_A(A) \in \mathcal{R}(A'')$, if by contrary there exist $f_1, f_2 \in M_r(A)$ such that $[f_1] \neq [f_2]$ (that is, there is $x_0 \in dom(f_1) \cap dom(f_2)$ such that $f_1(x_0) \neq f_2(x_0)$) and $[\bar{f}_a] \veebar [f_1] = [\bar{f}_a] \veebar [f_2] \Leftrightarrow [\bar{f}_a \veebar f_1] = [\bar{f}_a \veebar f_2]$, for every $a \in A$.
In particular for $a = x_0$, we obtain that $x \in dom(f_1) \cap dom(f_2)$,
$(\bar{f}_{x_0} \veebar f_1)(x) = (\bar{f}_{x_0} \veebar f_2)(x) \Leftrightarrow (\bar{f}^*_{x_0} \rightarrow f_1)(x) = (\bar{f}^*_{x_0} \rightarrow f_2)(x) \Leftrightarrow$
$(\bar{f}_{x_0}(x) \rightarrow 0(x)) \rightarrow f_1(x) = (\bar{f}_{x_0}(x) \rightarrow 0(x)) \rightarrow f_2(x) \Leftrightarrow ((x^* \rightarrow x_0) \rightarrow x^{**})$
$\rightarrow f_1(x) = ((x^* \rightarrow x_0) \rightarrow x^{**}) \rightarrow f_2(x) \Leftrightarrow (x^* \rightarrow x_0^*) \rightarrow f_1(x) = (x^* \rightarrow$

x_0*) → f_2(x); in particular for x = x_0 we obtain that 1 → $f_1(x_0)$ = 1 → $f_2(x_0)$ ⇔ $f_1(x_0) = f_2(x_0)$, which is contradictory. ■

Remark 5.6.21. (i). Since for every a ∈ A, $\bar{f}_a$ is the unique maximal multiplier on $[\bar{f}_a]$ (by Lemma 5.6.18), we can identify $[\bar{f}_a]$ with $\bar{f}_a$;
(ii). So, since $\bar{v}_A$ is a monomorphism in $\overline{H}_i$, the elements of A can be identified with the elements of the set { $\bar{f}_a$: a ∈ A}.

Lemma 5.6.22. In view of the identifications made above, if [f] ∈ A′′ (with f ∈ M_r (A) and D = dom(f) ∈ S(A) ∩ $\mathcal{R}$(A)), then D ⊆ {a ∈ A : $\bar{f}_a$ ⊻ [f] ∈ A}.

Proof. Let a ∈ D. If by contrary $\bar{f}_a$ ⊻ [f] ∉ A (that is, $[\bar{f}_a$ ⊻ f] ∉ $\bar{v}_A(A)$), then $\bar{f}_a$ ⊻ f is a non-principal multiplier on A.
By Lemma 5.6.18, $\bar{f}_a$ ⊻ f can be extended to a non-principal maximal multiplier $\bar{f}:\overline{D}\to A$ with $\overline{D}$ ∈ S(A).
Thus, D ⊆ $\overline{D}$ and for every x ∈ D, $\bar{f}$ (x) = ($\bar{f}_a$ ⊻ f)(x) = ($\bar{f}^*{}_a$ → f)(x) = (($\bar{f}_a$ → 0)→ f)(x) = ($\bar{f}_a$(x) → 0(x)) → f (x) = ((x* → a) → (x* → 0)) → f (x) = =(x* → a*) → f (x).
Thus, for every x ∈ D, x* → $\bar{f}$ (x) = x* → ((x* → a*) → f(x) ⇔ $\bar{f}$ (x* → x) = (x* → a*) → (x* → f (x)) ⇔ $\bar{f}$ (x**) = x* → (a* → f(x)) ⇔ $\bar{f}$ (x) = a* → (x* → f(x)) = a* → f(x) = a ⊻ f(x).
Since a ∈ D, then by a_{20} we deduce that for every x ∈ D, $\bar{f}$ (x) = a ⊻ f(x) = x ⊻ f(a), that is, $\bar{f}_{|D}$ is principal which is contradictory with the assumption that $\bar{f}$ is non-principal. ■

Definition 5.6.23. A Hilbert algebra A′ is called *Hilbert algebra of fractions* of A if

a_{21}: A is a Hilbert subalgebra of A′;

a_{22}: For every a′, b′, c′∈ A′, a′ ≠ b′, there is a ∈ A such that a ⊻ a′ ≠ a ⊻ b′ and a ⊻ c′ ∈ A.

As a notational convenience, we write A ≤ A′ to indicate that A′ is a Hilbert algebra of fractions for A (clearly, A ≤ A).

Definition 5.6.24. M is the *maximal Hilbert algebra of quotients* of A if $A \le M$ and for every A' with $A \le A'$ there is a monomorphism of Hilbert algebras $i : A' \to M$ in $\overline{H}_i$.

Lemma 5.6.25. Let $A \le A'$. Then for every $a', b' \in A'$, $a' \neq b'$ and any finite sequences $c'_1,...,c'_n \in A'$, there is $a \in A$ such that $a \veebar a' \neq a \veebar b'$ and $a \veebar c'_i \in A$ for i = 1, 2, ..., n.

Proof. For n = 1 the lemma is true since $A \le A'$.
Assume lemma hold true for n-1 (that is, there is $b \in A$ such that $b \veebar a' \neq b \veebar b'$ and $b \veebar c'_i \in A$ for i = 1, 2, ..., n-1).
Since $A \le A'$ we find $c \in A$ such that $c \veebar (b \veebar a') \neq c \veebar (b \veebar b')$ and $c \veebar c'_n \in A$. Then the element $a = b \veebar c \in A$ has the required properties. ■

Lemma 5.6.26. Let $A \le A'$ and $a' \in A'$. Then
$$D_{a'} = \{a \in A: a \veebar a' \in A\} \in S(A) \cap \mathcal{R}(A).$$

Proof. If $a \in A$ and $x \in D_{a'}$, then $x \veebar a' \in A$ and since $(a \veebar x) \veebar a' = a \veebar (x \veebar a') \in A$ it follows that $a \veebar x \in D_{a'}$, hence $D_{a'} \in S(A)$.
To prove $D_{a'} \in \mathcal{R}(A)$, let $x, y \in A$ such that $a \veebar x = a \veebar y$ for every $a \in D_{a'}$.
If by contrary $x \neq y$, since $A \le A'$, there is $a_0 \in A$ such that $a_0 \veebar a' \in A$ (hence $a_0 \in D_{a'}$) and $a_0 \veebar x \neq a_0 \veebar y$, which is contradictory. ■

Theorem 5.6.27. $A'' = M_r(A) / \rho_A$ is the maximal Hilbert algebra of quotients of A.

Proof. The fact that A is Hilbert subalgebra of A'' follows from Lemma 5.6.19.
To prove that $A \le A''$, let $[f], [g], [h] \in A''$ (with $f, g, h \in M_r(A)$) such that $[g] \neq [h]$ (that is, there is $x_0 \in dom(g) \cap dom(h)$ such that $g(x_0) \neq h(x_0)$).
Put $D = dom(f) \in S(A) \cap \mathcal{R}(A)$ and $D_{[f]} = \{ a \in A : \bar{f}_a \veebar [f] \in A\}$.

Then by Lemma 5.6.22, $D \subseteq D_{[f]}$.
If we suppose that for every $a \in D$, $\bar{f}_a \veebar [g] = \bar{f}_a \veebar [h]$, then $[\bar{f}_a \veebar g] = [\bar{f}_a \veebar h]$, hence for every $x \in dom(g) \cap dom(h)$ we have $(\bar{f}_a \veebar g)(x) = (\bar{f}_a \veebar h)(x) \Leftrightarrow (x^* \to a^*) \to g(x) = (x^* \to a^*) \to h(x)$.

We deduce that for every $x \in dom(g) \cap dom(h)$, $x^* \to ((x^* \to a^*) \to g(x)) = x^* \to ((x^* \to a^*) \to h(x)) \Rightarrow (x^* \to a^*) \to (x^* \to g(x)) = (x^* \to a^*) \to (x^* \to h(x)) \Rightarrow x^* \to (a^* \to g(x)) = x^* \to (a^* \to h(x)) \Rightarrow a^* \to (x^* \to g(x)) = a^* \to (x^* \to h(x)) \Rightarrow a^* \to g(x) = a^* \to h(x) \Leftrightarrow a \veebar g(x) = a \veebar h(x)$.
Since $D \in \mathcal{R}(A)$, we deduce that $g(x) = h(x)$ for every $x \in dom(g) \cap dom(h) \Leftrightarrow [g] = [h]$ which is contradictory.
Hence, if $[g] \neq [h]$, then there is $a \in D$ such that $\bar{f}_a \veebar [g] \neq \bar{f}_a \veebar [h]$; but for this $a \in D$, we clearly have, $\bar{f}_a \veebar [f] \in A$ (since $D \subseteq D_{[f]}$), hence $A \leq A''$.
To prove the maximality of A'', let A' such that $A \leq A'$. Then we have $i : A' \to A''$, $i(a') = [f_{a'}]$, for every $a' \in A'$ (with $dom(f_{a'}) = D_{a'}$).
Clearly, $f_{a'} \in M_r(A)$ and i is a morphism in $\overline{H}_i$.
To prove the injectivity of i, let $a', b' \in A'$ such that $[f_{a'}] = [f_{b'}] \Leftrightarrow f_{a'}(x) = f_{b'}(x)$ for every $x \in D_{a'} \cap D_{b'}$. If $a' \neq b'$, since $A \leq A'$, there is $a \in A$ such that $a \veebar a', a \veebar b' \in A$ and $a \veebar a' \neq a \veebar b'$, which is contradictory (since $a \veebar a', a \veebar b' \in A \Rightarrow a \in D_{a'} \cap D_{b'}$). ■

Definition 5.6.28. A non-empty subset F of S(A) is called a *Gabriel filter* on A if
a_{23}: $D_1 \in F$, $D_2 \in S(A)$ and $D_1 \subseteq D_2$, then $D_2 \in F$ (hence $A \in F$);
a_{24}: $D_1, D_2 \in F$, then $D_1 \cap D_2 \in F$.

We denote by G(A) the set of all Gabriel filters on A .

Examples

1. If $D \in S(A)$, then $F(D) = \{D' \in S(A) : D \subseteq D'\} \in G(A)$.
2. $\mathcal{R}(A) \cap S(A) \in T(A)$.

Indeed, a_{23} it is clearly verified. To verify a_{24}, let $D_1, D_2 \in \mathcal{R}(A) \cap S(A)$ and $x, y \in A$ such that $t \veebar x = t \veebar y$ for every $t \in D_1 \cap D_2$. Then for any $t_1, t_2 \in A$ we have $(t_1 \veebar t_2) \veebar 0 = (t_2 \veebar t_1) \veebar 0 = t_1 \veebar (t_2 \veebar 0)$ so , if $t_i \in D_i$, $i = 1, 2$, and if we take $t = (t_1 \veebar t_2) \veebar 0$, $t \in D_1 \cap D_2$ (since $t_1 \veebar t_2 \in D_2 \Rightarrow (t_1 \veebar t_2)^{**} \in D_2$ by Lemma 5.6.7).

Since $t \veebar x = t \veebar y \Rightarrow ((t_1 \veebar t_2) \veebar 0) \veebar x = ((t_1 \veebar t_2) \veebar 0) \veebar y \Leftrightarrow t_1 \veebar ((t_2 \veebar 0) \veebar x) = t_1 \veebar ((t_2 \veebar 0) \veebar y)$.

Since $t_1 \in D_1$ is arbitrary and $D_1 \in S(A) \cap \mathcal{R}(A)$ we obtain that $(t_2 \veebar 0) \veebar x = (t_2 \veebar 0) \veebar y \Leftrightarrow t_2 \veebar (0 \veebar x) = t_2 \veebar (0 \veebar y) \Leftrightarrow t_2 \veebar x = t_2 \veebar y$ (since $0 \veebar x = 1 \to x = x$).

Since $t_2 \in D_2$ is arbitrary and $D_2 \in \mathcal{R}(A)$ we deduce that $x = y$, hence $D_1 \cap D_2 \in \mathcal{R}(A)$, that is $\mathcal{R}(A) \cap S(A) \in G(A)$.

3. We recall that $S \subseteq A$ is called $\veebar$ - closed system if $x, y \in S \Rightarrow x \veebar y \in S$.

If we denote $F_S = \{D \cap S \neq \emptyset\}$, then $F_S \in G(A)$.

Indeed, the axiom a_{23} is verified since if $D_1, D_2 \in S(A)$, $D_1 \subseteq D_2$ and $D_1 \cap S \neq \emptyset$, then $D_1 \cap S \subseteq D_2 \cap S$, hence $D_2 \cap S \neq \emptyset$.

To prove the axiom a_{24}, let $D_1, D_2 \in F_S$, that is, there is $s_i \in D_i \cap S$, $i = 1, 2$.

If we denote $s = s_1 \veebar s_2$ and $s' = s \veebar 0$, then $s \in S$ and $s' = s^{**} = s \veebar s \in S$; since $s' \in D_1 \cap D_2$, then $s' \in (D_1 \cap D_2) \cap S$, that is, $D_1 \cap D_2 \in F_S$.

For $F \in G(A)$ we consider the binary relation on A defined by :

$(x, y) \in \theta_F \Leftrightarrow$ there is $D \in F$ such that $t \veebar x = t \veebar y$ for every $t \in D$

Lemma 5.6.29. $\boldsymbol{\theta_F \in}$ **Con (A).**

Proof. The reflexivity and symmetry of θ_F are immediate. To prove the transitivity of θ_F, let $(x, y), (y, z) \in \theta_F$. Then there are $D_1, D_2 \in F$ such that $t \veebar x = t \veebar y$ for every $t \in D_1$ and $t' \veebar x = t' \veebar y$ for every $t' \in D_2$.

If we consider $D = D_1 \cap D_2 \in F$, then for every $t \in D$, $t \veebar x = t \veebar z$, hence $(x, z) \in \theta_F$.

To prove the compatibility of θ_F with $\to$, let $x, y, z \in A$ such that $(x, y) \in \theta_F$, hence there is $D \in F$ such that $t \veebar x = t \veebar y$ for every $t \in D$.

Since $t \veebar (x \to z) = (t \veebar x) \to (t \veebar z) = (t \veebar y) \to (t \veebar z) = t \veebar (y \to z)$ and $t \veebar (z \to x) = (t \veebar z) \to (t \veebar x) = (t \veebar z) \to (t \veebar y) = t \veebar (z \to y)$ we deduce that $(x \to z, y \to z), (z \to x, z \to y) \in \theta_F$. ■

For $x \in A$ we denote by x/θ_F the equivalence class of x modulo θ_F and by $\pi_F : A \to A / \theta_F$ canonical surjective function defined for $a \in A$ by $\pi_F(a) = a / \theta_F$ (clearly π_F is an epimorphism in $\overline{H}_i$).

Definition 5.6.30. Let $F \in G(A)$. An F–*multiplier* is a function $f : D \to A / \theta_F$, where $D \in F$ and for any $x, y \in D$ and $a \in A$ the following axioms are fulfilled:

a_{25}: $f(a \veebar x) = (a / \theta_F) \veebar f(x)$;
a_{26}: $f(x^{**}) = f(x)$;
a_{27}: $(x / \theta_F) \veebar f(y) = (y / \theta_F) \veebar f(x)$.

Examples

1. If $F = \{A\}$ then θ_F is the identity, then an F– multiplier is in fact a total multiplier on A (in the sense of Definition 5.6.8).
2. The functions $0, 1 : A \to A / \theta_F$ defined by $0(x) = (x / \theta_F)^{**}$ and $1(x) = 1/\theta_F$ for every $x \in A$ are F – multipliers.
3. For $a \in A$, $f_a : A \to A / \theta_F$, $f_a(x) = (x / \theta_F) \veebar (a / \theta_F)$, for every $x \in A$ is a F – multiplier.

We denote by $M(D, A / \theta_F)$ the set of all F – multipliers having as domain $D \in F$. If $D_1, D_2 \in F$, $D_1 \subseteq D_2$ then we have a canonical function $j_{D_2}^{D_1} : M(D_2, A / \theta_F) \to M(D_1, A / \theta_F)$ defined by $j_{D_2}^{D_1}(f) = f_{|D_1}$, for $f \in M(D_2, A / \theta_F)$.
Let us consider the directed system of sets $(\{M(D, A / \theta_F)\}_{D\in F}, (j_{D_2}^{D_1})_{D_1 \subseteq D_2})$ and denote by A_F the inductive limit $A_F = \varinjlim_{D\in F} M(D, A / \theta_F)$ (in the category **Set** of sets; see Chapter 4).
For any F – multiplier $f : D \to A / \theta_F$ we will denote by $\overline{(D, f)}$ the equivalence class of f in A_F.

Remark 5.6.31. If $f_i : D_i \to A / \theta_F$, i = 1, 2 are two multipliers, then $\overline{(D_1, f_1)} = \overline{(D_2, f_2)}$ (in A_F) $\Leftrightarrow$ there is $D \in F$, $D \subseteq D_1 \cap D_2$ such that $f_{1|D} = f_{2|D}$.

For $f_i : D_i \to A / \theta_F$, i = 1, 2, F – multipliers let us consider the function $f_1 \to f_2 : D_1 \cap D_2 \to A / \theta_F$ defined by $(f_1 \to f_2)(x) = f_1(x) \to f_2(x)$, for any $x \in D_1 \cap D_2$ and $\overline{(D_1, f_1)} \to \overline{(D_2, f_2)} = \overline{(D_1 \cap D_2, f_1 \to f_2)}$

This last definition is correct .

Indeed, let $f'_i : D'_i \to A / \theta_F$ with $D'_i \in F$ such that $\overline{(D_i, f_i)} = \overline{(D'_i, f'_i)}$, i = 1, 2. Then there are D''_1, $D''_2 \in F$ such that $D''_1 \subseteq D_1 \cap D'_1$, $D''_2 \subseteq D_2 \cap D'_2$ and $f_{1|D''_1} = f'_{1|D''_1}$, $f_{2|D''_2} = f'_{2|D''_2}$.

If we set $D'' \subseteq D_1 \cap D_2 \cap D'_1 \cap D'_2$, then $D'' \in F$ and clearly $(f_1 \to f_2)_{|D''} = (f'_1 \to f'_2)_{|D''}$, hence $\overline{(D_1 \cap D_2, f_1 \to f_2)} = \overline{(D'_1 \cap D'_2, f'_1 \to f'_2)}$.

Lemma 5.6.32. $f_1 \to f_2 \in M(D_1 \cap D_2, A / \theta_F)$.

Proof. As in the case of Lemma 5.6.11. ■

Corollary 5.6.33. $(A_F, \to, \overline{0}, \overline{1}) \in \overline{\overline{H}}_i$, where $\overline{0} = \overline{(A,0)}$ and $\overline{1} = \overline{(A,1)}$.

Proof. As in the case of Lemma 5.6.12. ■

Definition 5.6.34. The bounded Hilbert algebra A_F will be called the *localization Hilbert algebra* of A with respect to the Gabriel filter F.

Lemma 5.6.35. The function $v_F : A \to A_F$ defined by $v_F(a) = \overline{(A, f_a)}$, for $a \in A$ is a morphism in $\overline{\overline{H}}_i$ and $v_F(A) \in \mathcal{R}(A_F)$.

Proof. If a, b ∈ A, then $v_F(a) \to v_F(b) = \overline{(A, f_a)} \to \overline{(A, f_b)} = = \overline{(A, f_a \to f_b)} = \overline{(A, f_{a \to b})} = v_F(a \to b)$ (by Lemma 5.6.13). Since $f_0(x) = (x / \theta_F)^* \to (0 / \theta_F) = (x / \theta_F)^{**} = \overline{0}(x)$, for any $x \in A$, we deduce that $v_F(0) = \overline{(A, f_0)} = \overline{(A,0)} = \overline{0}$.

To prove that $v_F(A) \in \mathcal{R}(A_F)$, let $\overline{(D_i, f_i)} \in A_F$ with $D_i \in F$, i = 1, 2 such that $\overline{(A, f_a)} \veebar \overline{(D_1, f_1)} = \overline{(A, f_a)} \veebar \overline{(D_2, f_2)}$, for any $a \in A$.

Then: $(\overline{(A, f_a)} \to \overline{0}) \to \overline{(D_1, f_1)} = (\overline{(A, f_a)} \to \overline{0}) \to \overline{(D_2, f_2)} \Leftrightarrow$

$\left(\overline{(A, f_a)} \to \overline{(A,0)}\right) \to \overline{(D_1, f_1)} = \left(\overline{(A, f_a)} \to \overline{(A,0)}\right) \to \overline{(D_2, f_2)} \Leftrightarrow$
$\overline{(A, f_a \to 0)} \to \overline{(D_1, f_1)} = \overline{(A, f_a \to 0)} \to \overline{(D_2, f_2)} \Leftrightarrow$
$\overline{(D_1, (f_a \to 0) \to f_1)} = \overline{(D_2, (f_a \to 0) \to f_2)}$, for any $a \in A$.

So, there is $D \subseteq D_1 \cap D_2$, $D \in F$ such that
$((f_a \to 0) \to f_1)_{|D} = ((f_a \to 0) \to f_2)_{|D}$

$((f_a \to 0) \to f_1)(x) = ((f_a \to 0) \to f_2)(x)$, for every $x \in D$ and $a \in A$ $\Leftrightarrow$ $((x / \theta_F)^* \to (a / \theta_F)^*) \to f_1(x) = ((x / \theta_F)^* \to (a / \theta_F)^*) \to f_2(x)$ for every $x \in D$ and $a \in A$.

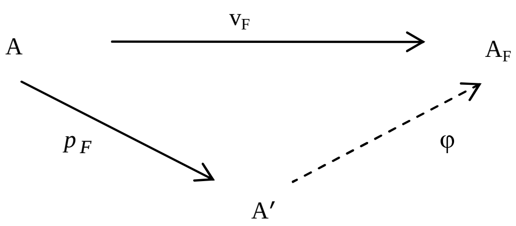
If we take $a = x \in D$, then we obtain $1 \to f_1(x) = 1 \to f_2(x) \Leftrightarrow$
$f_1(x) = f_2(x)$, hence $\overline{(D_1, f_1)} = \overline{(D_2, f_2)}$, that is, $v_F(A) \in \mathcal{R}(A_F)$. ■

In what follows we describe the localization of Hilbert algebra A_F in some special instances.

Applications

1. If $D \in S(A)$ and F is the Gabriel filter $F(D) = \{S' \in S(A) : D \subseteq D'\}$, then $A_F \subseteq M(D, A / \theta_F)$ and $v_F(a) = \overline{(D, f_{a|D})}$, for every $a \in A$.

For $x, y \in A$ we have: $(x, y) \in \theta_F \Leftrightarrow$ for any $t \in D$, $t \veebar x = t \veebar y$ $\Leftrightarrow$
$f_{x|D} = f_{y|D}$ $\Leftrightarrow v_F(x) = v_F(y)$ and then there exists a monomorphism
$\varphi : A / \theta_F \to A_F$ in $\overline{H}_i$ such that the diagram



is commutative (e.g. $\varphi \circ \pi_F = v_F$).
2. If $F = \mathcal{R}(A) \cap S(A)$ is the Gabriel filter of the sets from S(A) which are regular subsets of A, then $\theta_F = \Delta_A$ (hence $A / \theta_F = A$), so, an F –

multiplier on A in the sense of Definition 5.6.30 coincide with the notion of multiplier in the sense of Definition 5.6.8.
In this case $A_F = \varinjlim_{D\in F} M(D, A)$, where M(D, A) = {f : D → A : f is multiplier on A}, v_F is a monomorphism (coincides with $\bar{v}_A$ from Lemma 5.6.20) in $\overline{H}_i$ and $A_F = A''$. So, in the case F = $\mathcal{R}$(A) ∩ S(A), A_F is exactly the maximal Hilbert algebra of quotients of A (see Theorem 5.6.27).
(iii). Let S ⊆ A a $\veebar$ - closed subset of A.
Consider the congruence θ_S on A: (x, y) ∈ θ_S ⇔ there is s ∈ S such that s $\veebar$ x = s$\veebar$ y.
By Lemma 5.6.2 we deduce that θ_S ∈ Con(A) and A / θ_S = A[S] (see Definition 5.6.5).

Theorem 5.6.36. Let S ⊆ A a $\veebar$ - closed system of A and F_S = {D ∈ S (A) : D ∩ S ≠∅ } ∈ G(A). Then $A_{F_S} \approx$ A[S] (in $\overline{H}_i$).

Proof. For x, y ∈ A we have: (x, y) ∈ q_{F_S} ⇔ there is D ∈ F_S such that s $\veebar$ x = s $\veebar$ y, for every s ∈ D.
Since D ∩ S ≠ ∅, there is s_0 ∈ D∩S; in particular we obtain that $s_0 \veebar x = s_0 \veebar y$, hence (x, y) ∈ θ_S (see Lemma 5.6.2).
We consider $D_0 = [s_0) = \{a \in A : s_0 \le a\} \in Ds(A)$.
Since s_0 ∈ D ∩ S we deduce that $D_0 \in F_S$.
From $s_0 \veebar x = s_0 \veebar y \Rightarrow s^*_0 \to x = s^*_0 \to y \Rightarrow s^*_0 \le x \to y$ and $s^*_0 \le y \to x$. If s ∈ D_0, then $s_0 \le s \Rightarrow s^* \le s^*_0 \Rightarrow s^* \le x \to y$ and $s^* \le y \to x \Rightarrow s^* \to x = s^* \to y \Rightarrow s \veebar x = s \veebar y \Rightarrow$ (x, y) ∈ $q_{F_S} \Rightarrow q_{F_S} = \theta_S$, so A / q_{F_S} = A[S].
Therefore, an F_S–multiplier can be considered in this case a function f : D → A[S] (D ∈ F_S) having the properties: f(a $\veebar$ x) = $\hat{a} \veebar$ f(x), f(x**) = f(x) and $\hat{x} \veebar$ f (y) = $\hat{y} \veebar$ f (x), for any x, y ∈ D and a ∈A (we denoted $\hat{x}$ = x / θ_S).
If $\overline{(D_1, f_1)}, \overline{(D_2, f_2)} \in A_{F_S} = \varinjlim_{D\in F_S} M(D, A[S])$ and $\overline{(D_1, f_1)} = \overline{(D_2, f_2)}$, then there is D ∈ F_S such that D ⊆ $D_1 \cap D_2$ and $f_{1|D} = f_{2|D}$.

Since $D, D_1, D_2 \in F_S$ then $D \cap S$, $D_1 \cap S$, $D_2 \cap S \neq \emptyset$ and choose $s \in D \cap S$, $s_i \in D_i \cap S$, $i = 1, 2$.
We will prove that $f_1(s_1) = f_2(s_2)$.
Indeed, since for any $x, y \in A$ we have $x^* \to y^{**} = y^* \to x^{**}$ we deduce that $s_1 \veebar (s_2 \veebar s_1) = s_2 \veebar (s_1 \veebar s_1) = s^*_2 \to s^{**}_1 = s^*_1 \to s^{**}_2 = s_2 \veebar (s_1 \veebar s_2) \in S$. Hence, for $t = s \veebar (s_1 \veebar (s_2 \veebar s_1)) = s \veebar (s_2 \veebar (s_1 \veebar s_2))$ we obtain $t = s \veebar (s_2 \veebar (s_1 \veebar s_1)) = s_2 \veebar (s \veebar (s_1 \veebar s_1)) = s_2 \veebar (s^* \to s^{**}_1)) = s_2 \veebar (s^*_1 \to s^{**})) = s_2 \veebar (s_1 \veebar (s \veebar s)) \in D \cap S$.
Since $f_{1|D} = f_{2|D}$ and $t \in D$ we have $f_1(t) = f_2(t)$. Since f_1 and f_2 are F_S – multipliers, we obtain $f_1 (s \veebar (s_1 \veebar (s_2 \veebar s_1))) = f_2 (s \veebar (s_2 \veebar (s_1 \veebar s_2)))$
$\Leftrightarrow \quad (\hat{s})^* \to ((\hat{s}_2)^* \to ((\hat{s}_2)^* \to f_1(s_1))) = (\hat{s})^* \to ((\hat{s}_2)^* \to ((\hat{s}_1)^* \to f_2(s_2)))$.
But $s, s_1, s_2 \in S$, hence $\hat{s} = \hat{s}_1 = \hat{s}_2 = 0$, so $0^* \to (0^* \to (0^* \to f_1(s_1))) = 0^* \to (0^* \to (0^* \to f_2(s_2))) \Leftrightarrow 1 \to (1 \to (1 \to f_1(s_1))) = 1 \to (1 \to (1 \to f_2(s_2))) \Leftrightarrow f_1(s_1) = f_2(s_2)$.
Analogously we prove that $f_1(s_1) = f_2(s_2)$, for any $s_1, s_2 \in D \cap S$.
In accordance with these considerations we can consider the function $\alpha : A_{F_S} = \varinjlim_{D \in F_S} M(D, A[S]) \to A[S]$, $\alpha(\overline{(D, f)}) = f(s)$, where $s \in D \cap S$.
Clearly, α is a morphism in $\overline{H}_i$ (since if $\overline{(D_i, f_i)} \in A_{F_S}$, with $D_i \in F_S$, $i = 1, 2$, then $\alpha(\overline{(D_1, f_1)} \to \overline{(D_2, f_2)}) = \alpha(\overline{(D_1 \cap D_2, f_1 \to f_2)}) = (f_1 \to f_2)(s) = f_1(s) \to f_2(s) = \alpha(\overline{(D_1, f_1)}) \to \alpha(\overline{(D_2, f_2)})$, where $s \in (D_1 \cap D_2) \cap S$ and $\alpha(\overline{0}) = \alpha(\overline{(A,0)}) = 0(s) = (\hat{s})^{**} = 0^{**} = 0)$.
We will prove that α is bijective.
To prove the injectivity of α, let $\overline{(D_1, f_1)}, \overline{(D_2, f_2)} \in A_{F_S}$ such that $\alpha(\overline{(D_1, f_1)}) = \alpha(\overline{(D_2, f_2)})$.
Then for $s_1 \in D_1 \cap S$ and $s_2 \in D_2 \cap D$ we have $f_1(s_1) = f_2(s_2)$. We consider the element $s = s_1 \veebar (s_2 \veebar s_1) = s_2 \veebar (s_1 \veebar s_2) \in (D_1 \cap D_2) \cap S$.
We have $f_1(s) = \hat{s}_1 \veebar (\hat{s}_2 \veebar f_1(s_1)) = 0 \veebar (0 \veebar f_1(s_1)) = 1 \to (1 \to f_1(s_1)) = f_1(s_1)$ and analogously $f_2(s) = \hat{s}_2 \veebar (\hat{s}_1 \veebar f_2(s_2)) = f_2(s_2)$, hence $f_1(s) = f_2(s)$.
Now let $D_s = \{s' \in D_1 \cap D_2 : s' = s' \veebar s\}$.

Since $s^{**} = s \veebar s \in D_1 \cap D_2$ and $(s^{**}) \veebar s = s^{***} \to s = s^* \to s = s^{**}$ we deduce that $D_s \neq \varnothing$.

If $a \in A$ and $s' \in D_s$, then $a \veebar s' = a \veebar (s' \veebar s) = (a \veebar s') \veebar s$, hence $a \veebar s' \in D_s$, that is, $D_s \in S(A)$. Since $a^{**} \in D_s \cap S$, we deduce that $D_s \in F_S$.

If $s' \in D_s$, then $f_1(s') = f_1(s' \veebar s) = \hat{s}' \veebar f_1(s)$, $f_2(s') = f_2(s' \veebar s) = \hat{s}' \veebar f_2(s)$, hence $f_1(s') = f_2(s') \Rightarrow f_{1|D_s} = f_{2|D_s} \Rightarrow \overline{(D_1, f_1)} = \overline{(D_2, f_2)}$, that is, α is injective.

To prove the surjectivity of α, let $\hat{a} \in A[S]$ $(a \in A)$.

For $s \in S$, we consider $D = [s)$.

Then $D \in F_S$ and we define $f_a : D \to A[S]$, $f_a(x) = (x \veebar a) / \theta_S$ for any $x \in D$.

Clearly f_a is a F_S – multiplier and we shall prove that $\alpha(\overline{(D, f_a)}) = \hat{a}$.

Indeed, since $s^* \to (s^* \to a) = s^* \to a$, then $(s^* \to a, a) \in \theta_S$, hence $s \hat{\to} a = \hat{a} \Leftrightarrow f_a(s) = \hat{a} \Leftrightarrow \alpha(\overline{(D, f_a)}) = \hat{a}$. ■

We consider now the case of Hertz algebras .

Lemma 5.6.37. If $(H, \to, \wedge)$ is a Hertz algebra , then $D \in$ Ds(H) iff D is a filter of the meet–semilattice $(H, \wedge)$.

Proof. Suppose that $D \in$ **Ds**(H) and let $x, y \in D$; we will prove that $x \wedge y \in D$.

By a_{14} we have: $x \to (x \wedge y) = (x \to x) \wedge (x \to y) = 1 \wedge (x \to y) = x \to y \in D$; since $x \in D$ we deduce that $x \wedge y \in D$.

If $x \in D$, $y \in H$ and $x \leq y$, then $x \to y = 1 \in D$, hence $y \in D$.

Conversely, suppose that D is a filter of H and we will prove that D is a deductive system of H; clearly $1 \in D$ since $x \leq 1$ for every $x \in D$.

Suppose that $x, x \to y \in D$; then by a_{15}, $x \wedge y = x \wedge (x \to y) \in D$ and since $x \wedge y \leq y$ we deduce that $y \in D$. ■

The notion of $\veebar$ - closed system for Hertz algebras will be defineed as in the case of Hilbert algebras (see Definition 5.6.1).

We will define the notion of Hertz algebra of fractions relative to an ⊻ - closed system as for Hilbert algebras.

So, let (H, →, ∧) be a Hertz algebra and S⊆H an ⊻- closed system in H. By Lemma 5.6.2, the relation θ_S defined on H by (x, y)∈ θ_S iff there is t ∈ S such that t ⊻ x = t ⊻ y, is compatible with →. We will prove that θ_S is compatibile with ∧, too.

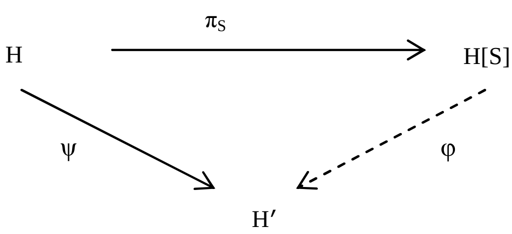
Let x, y, z ∈ A such that (x, y) ∈ θ_S; then there is t ∈ S such that t ⊻ x = = t ⊻ y ⇔ t* → x = t* → y.

By a_{14} we deduce t* → (x∧ z) = (t* → x)∧(t* → z) = (t* → y)∧(t* → z) = t* → (y ∧ z), hence (x ∧ z, y ∧ z) ∈ θ_S.

We denote H[S] = H / θ_S and by π_S : H → H[S] the canonical epimorphism of Hertz algebras which map an element in its equivalence class.

As in the case of Hilbert algebras, π_S(S) = {0}.

We will prove that Lemma 5.6.4 is valid and in the case of Hertz algebras; so, let H′ another Hertz algebra and ψ : H → H′ a morphism of Hertz algebras such that ψ(S) = {0}.



To prove the existence of unique morphism of Hertz algebras φ : H [S] → H′ for which the above diagram is commutative, it will suffice to prove that φ (defined as in the case of Lemma 5.5.4) is morphism of Hertz algebras, that is, for x, y ∈ H, we have $j(\hat{x}\wedge\hat{y}) = j(\hat{x})\wedge j(\hat{y})$.

Indeed, $j(\hat{x}\wedge\hat{y}) = j(\widehat{x\wedge y})$ = ψ (x∧y) = ψ (x)∧ ψ (y) = $j(\hat{x})\wedge j(\hat{y})$.

We call H [S] *Hertz algebra of fractions* of H relative to ⊻-closed system S.

If H, H′ are two Hertz algebras with H Hertz subalgebra of H′ (hence H contains two elements x′,y′of H′ and the elements x′ → y′ and x′∧y′,too), we say that H′ *is a Hertz algebra of fractions of H* if for any x′,y′, z′∈ H′ with x′ ≠ y′, there is a ∈ H such that $a \veebar x' \neq a \veebar y'$ and $a \veebar z' \in$ H.

The notions of $\veebar$ - *closed subset* of H and *multiplier* on H are defined as in the case of Hilbert algebras (see Definitions 5.6.6 and 5.6.8).

We will prove that if $f_i : D_i \to H$, $D_i \in$ **Ds**(H), i = 1,2 are multipliers on H, then $f_1 \wedge f_2 : D_1 \cap D_2 \to H$, defined for $x \in D_1 \cap D_2$ by $(f_1 \wedge f_2)(x) = f_1(x) \wedge f_2(x)$ is also a multiplier on H.

Indeed, if a ∈ H and $x \in D_1 \cap D_2$ we have $(f_1 \wedge f_2)(a \veebar x) = f_1(a \veebar x) \wedge f_2(a \veebar x) = (a \veebar f_1(x)) \wedge (a \veebar f_2(x)) = (a^* \to f_1(x)) \wedge (a^* \to f_2(x))$ = (by a_{14}) = $a^* \to (f_1(x) \wedge f_2(x)) = a \veebar (f_1 \wedge f_2)(x)$ and from $x^{**} \leq f_1(x)$ and $x^{**} \leq f_2(x)$ we deduce that $x^{**} \leq f_1(x) \wedge f_2(x) = (f_1 \wedge f_2)(x)$.

Therefore $H''=M_r(H)/\rho_H$ is a Hertz algebra (it is immediate the compatibility of ρ_H with ∧); to prove that H′′ is the maximal Hertz algebra of quotients of H it is suffice to prove that $\bar{v}_H : H \to H''$ defined in Lemma 5.6.20 is a morphism of Hertz algebras.

If a, b ∈ H, then for every x ∈ H, we have $x^* \to (a \wedge b) = (x^* \to a) \wedge (x^* \to b)$, hence $f_{a \wedge b}(x) = f_a(x) \wedge f_b(x) \Leftrightarrow \bar{v}_H(a \wedge b) = \bar{v}_H(a) \wedge \bar{v}_H(b)$, so $\bar{v}_H$ is morphism of Hertz algebras.

The notions of *Gabriel filter* F on a Hertz algebra H and F - ***multiplier*** are define as for Hilbert algebras; also the relation θ_F on H.

If H is a Hertz algebra, S an $\veebar$-closed system of H, then the compatibility of θ_F with ∧ on H is as in the case of compatibility of θ_S with ∧ (by using a_{14}).

By preserving the notations from Hilbert algebras, there results that H_F (see Definition 5.6.34) becomes in a canonical way bounded Hertz algebra, where for $\overline{(D_i, f_i)} \in H_F$ (i = 1, 2): $\overline{(D_1, f_1)} \to \overline{(D_2, f_2)} = \overline{(D_1 \cap D_2, f_1 \to f_2)}$, $\overline{(D_1, f_1)} \wedge \overline{(D_2, f_2)} = \overline{(D_1 \cap D_2, f_1 \wedge f_2)}$.

We call H_F *the Hertz algebra of localization* of H with respect to the Gabriel filter F.

Theorem 5.6.38. Let A, A′ be Hilbert algebras; then A ≤ A′ iff $H_A \leq H_{A'}$**.**

Proof. We recall that $\varphi_A : A \to H_A$ defined by $\varphi_A(a) = \hat{a}$, for $a \in A$ is a monomorphism of bounded Hilbert algebras (we denoted $\hat{a} = \{a\}/\rho_A$; see the notations from the proof of Theorem 5.4.11).

Firstly, suppose that $A \leq A'$; we will prove that $H_A \leq H_{A'}$. Clearly H_A is a Hetz subalgebra of $H_{A'}$ since if $\alpha = \hat{X}$, $\beta = \hat{Y} \in H_A$, where $X = \{x_1, x_2, ..., x_n\}$, $Y = \{y_1, y_2, ..., y_m\} \in F(A)$, then $\alpha \to \beta = \hat{X}'$ and $\alpha \wedge \beta = \hat{Y}'$ where $Y' = X \cup Y \subset A$ and $X' = \{x'_1, x'_2, ..., x'_m\} \subset A$ with $x'_j = (x_1, x_2, ..., x_n; y_j) \in A$, $j = 1, 2, ..., m$, hence $\alpha \to \beta$, $\alpha \wedge \beta \in H_A$.

Let $\alpha' = \hat{X}'$, $\beta' = \hat{Y}'$, $\gamma' = \hat{Z}' \in H_{A'}$, where $X' = \{a'_1, ..., a'_m\}$, $Y' = \{b'_1, ..., b'_n\}$, $Z' = \{c'_1, ..., c'_p\}$ are finite subsets of A' such that $\alpha' \neq \beta'$.

From $\alpha' \neq \beta'$ we deduce that there are $i_0 \in \{1, 2, \dots, n\}$, $j_0 \in \{1, 2, \dots, m\}$ such that $(a'_1, ..., a'_m; b'_{i_0}) \neq 1$ or $(b'_1, ..., b'_n; a'_{j_0}) \neq 1$.

Suppose that $(a'_1, ..., a'_m; b'_{i_0}) \neq 1$ with $i_0 \in \{1, 2, \dots, n\}$ (another case will be analogously).

By Lemma 5.6.25 there is $a \in A$ such that $a \veebar (a'_1, ..., a'_m; b'_{i_0}) \neq a \veebar 1 = 1$ and $a \veebar c'_k \in A$, for every $k \in \{1, 2, ..., p\}$.

Then, if denote $\alpha = \hat{a} \in H_A$, $c''_k = a \veebar c'_k$, $k = 1, 2, ..., p$ and $X'' = \{c''_1, ..., c''_p\} \subseteq A$, we immediately deduce that $\alpha \vee \gamma' = \hat{X}'' \in H_A$.

If we prove that $\alpha \veebar \alpha' \neq \alpha \veebar \beta'$, then the proof of this implication is complete.

We denote $a''_i = a \veebar a'_i$, $b''_j = a \veebar b'_j$, $i = 1, 2, ..., m$, $j = 1, 2, ..., n$.

If by contrary $\alpha \veebar \alpha' = \alpha \veebar \beta'$, then $(a''_1, ..., a''_m; b''_j) = (b''_1, ..., b''_n; a''_i) = 1$, for every $i \in \{1, 2, \dots, m\}$ and $j \in \{1, 2, \dots, n\}$.

But, using the rules of calculus from Theorem 5.2.13, from $(a''_1, ..., a''_m; b''_j) = 1$, for every $j \in \{1, 2, ..., n\}$, we deduce that $a \veebar (a''_1, ..., a''_m; b''_j) = 1$, for every $j \in \{1, 2, ..., n\}$, which is a contradiction!.

Now suppose that $H_A \leq H_{A'}$ and we will prove that $A \leq A'$.

To prove that A is a Hilbert subalgebra of A′, let a, b ∈ A. Since $\hat{a}$, $\hat{b}$ ∈ H_A and H_A is a Hertz subalgebra (hence in particular Hilbert subalgebra) of $H_{A'}$ we have that $\hat{a} \to \hat{b} = \widehat{a \to b} \in H_A$, hence a → b ∈ A.

Now let a′, b′, c′ ∈ A′, with a′ ≠ b′; if we consider the elements $\hat{a}'$, $\hat{b}'$, $\hat{c}' \in H_{A'}$ we have that $\hat{a}' \neq \hat{b}'$ (see the proof of Theorem 5.3.11); since we supposed that $H_A \leq H_{A'}$ there is X = {x_1,..., x_n} a finite subset of A such that $\hat{X} \veebar \hat{a}' \neq \hat{X} \veebar \hat{b}'$ and $\hat{X} \veebar \hat{c}' \in H_A$.

Since in $H_{A'}$, $\hat{X}^* = \hat{X} \to \hat{0} = \widehat{(x_1,..., x_n;0)}$ we obtain that (denoting a = (x_1, ..., x_n; 0) ∈ A) $\hat{a} \to \hat{a}' \neq \hat{a} \to \hat{b}'$ and $\hat{a} \to \hat{c}' \in H_A$, hence a → a′ ≠ a → b′ and a → c′ ∈ A.

We will prove that a ∈ **R**(A), that is, a** = a; indeed, since for every x, y ∈ A by c_{22} we have (x → y)** = x** → y** = (by c_8) = y* → x*** = y* → x* = x → y**, then a** = (x_1, x_2, ..., x_{n-1}; (x_n → 0)**) = (x_1, x_2, ..., x_{n-1}; x_n***) = (x_1, x_2, ..., x_{n-1}; x_n*) = (x_1, x_2, ..., x_n; 0) = a.

So, the relations a → a′ ≠ a → b′ and a → c′ ∈ A becomes a**→ a′ ≠ a** → b′ and a** → c′ ∈ A or (a*)* → a′ ≠ (a*)* → b′ and (a*)* → c′ ∈ A; if we denote b = a* ∈ A we have b ⊻ a′ ≠ b ⊻ b′ and b ⊻ c′ ∈ A, hence A ≤ A′.■

Corollary 5.6.39. If A is a bounded Hilbert algebra, then $H_{A''}$ is a Hertz subalgebra of (H_A)′′ (where by A′′ we denoted the maximal Hilbert algebra of quotients of A).

Proof. Preserving the notations from Definition 5.6.23, by Theorem 5.6.27, A ≤ A′′. By Theorem 5.6.38, $H_A \leq H_{A''}$, and by the maximality of (H_A)′′ we deduce that $H_{A''}$ is a Hertz subalgebra of (H_A)′′. ■

Let’s study now the case of Boole algebras.

If (B, ∨, ∧, ′, 0, 1) is a Boolean algebra, then as in the case of Hilbert or Hertz algebras it is immediate that the deductive systems of B are in fact the filters of B.

Since for x, y ∈ B, x* → y = (x*)′ ∨ y = x′′ ∨ y = x ∨ y, a multiplier on B will be a function f : D → B (with D filter in B) such that for every a ∈ B

and $x \in D$ we have $f(a \vee x) = a \vee f(x)$ (if we take $a = x$ we deduce that $f(x) = x \vee f(x)$, hence $x^{**} = x \leq f(x)$ and the axiom a_{21} follows from a_{22}.
If D_1, D_2 are filters of B and $f_i : D_i \rightarrow B$ are multipliers on B, then $f_1 \vee f_2 : D_1 \cap D_2 \rightarrow B$ defined by $(f_1 \vee f_2)(x) = f_1(x) \vee f_2(x)$ for every $x \in D_1 \cap D_2$, is a multiplier on B since for every $a \in A$ and $x \in D_1 \cap D_2$ we have:
$(f_1 \vee f_2)(a \vee x) = f_1(a \vee x) \vee f_2(a \vee x) = (a \vee f_1(x)) \vee (a \vee f_2(x)) =$
$a \vee (f_1(x) \vee f_2(x)) = a \vee (f_1 \vee f_2)(x)$.
Also, if $f : D \rightarrow B$ is a multiplier on B, then $f' : D \rightarrow B$ defined by $f'(x) = f(x) \rightarrow 0(x) = f(x) \rightarrow x$ is also a multiplier on B (as in the case of Hilbert algebras).
If B, B′ are two Boolean algebras, we say that B′ is a *Boolean algebra of fractions of B* if B is a Boolean subalgebra of B′ and if $a', b', c' \in B'$ then there is $a \in B$ such that $a \vee a' \neq a \vee b'$ and $a \vee c' \in B$.
A filter D of B will be called *regular* if for any $a, b \in B$ such that $t \vee a = t \vee b$ for every $t \in D$, then $a = b$.
It is immediate that, as in the case of Hilbert and Hertz algebras, that $(B'', \vee, \wedge, ', 0, 1)$ is *Boolean maximal algebra of fractions of* B.
In fact, B′′ is the *Dedekind – Mac Neille completion* of B (see [77], [78]).
The notion of *Boolean algebra of localization with respect to a Gabriel filter on B* will be introduced now in canonical way as in the case of Hilbert and Hertz algebras.

Theorem 5.6.40. If A, A′ are Hilbert algebras such that $A \leq A'$, then $\mathbf{R}(A) \leq \mathbf{R}(A')$ (as Boolean algebras).

Proof. Clearly, **R**(A) is a Boolean subalgebra of **R**(A′); let now $a', b', c' \in \mathbf{R}(A')$ such that $a' \neq b'$. Since $A \leq A'$, then there is $a \in A$ such that $a^* \rightarrow a' \neq a^* \rightarrow b'$ and $a^* \rightarrow c' \in A$.
Since $a^* = a^{***}$ we deduce that $a^{***} \rightarrow a' \neq a^{***} \rightarrow b'$ and $a^{***} \rightarrow c \in A$, hence if we denote $b = a^{**} \in \mathbf{R}(A)$, then $b^* \rightarrow a' \neq b^* \rightarrow b'$ and $b^* \rightarrow c' \in A$.
But by c_{22}, $(b^* \rightarrow c')^{**} = b^{***} \rightarrow c'^{**} = b^* \rightarrow c'$, hence $b^* \rightarrow c' \in \mathbf{R}(A)$.

Finally we obtain that b ∨ a′ ≠ b ∨ b′ and b ∨ c′ ∈ **R**(A) (see Theorem 5.2.24), hence **R**(A) ≤ **R**(A′). ■

Lemma 5.6.41. If A is a Hilbert algebra and D ∈ $\mathcal{R}$(A), then we have D ∩ R(A) ∈ $\mathcal{R}$(R (A)).

Proof. Let a, b ∈ **R**(A) such that t ⊻ a = t ⊻ b for every t ∈ $\overline{D}$ = D∩**R**(A); since for d ∈ D, d** ∈ D ∩ **R**(A) = $\overline{D}$, we deduce that (d**)* → a = (d**)* → b ⇔ d ⊻ a = d ⊻ b, hence a = b, since we supposed that D is regular in A. ■

We recall that if A is a Hilbert (Hertz or Boole) algebra, then by A′′ we denoted the maximal Hilbert (Hertz or Boole) algebra of quotients.

Theorem 5.6.42. If A is a Hilbert algebra, then R(A′′) is a Boolean subalgebra of (R(A))′′.

Proof. If $\overline{(D, f)}$ ∈ R(A′′), then D ∈ $\mathcal{R}$(A) and f : D → A is a multiplier on A such that f** = f; hence for every x ∈ D we have (f(x) → x**) → x** = =f(x) ⇔ (x* → (f(x))*) → (x*)* = f(x) ⇔ x* → (f(x))** = f(x), we deduce (by c_2) that (f(x))** ≤ f(x), hence (f(x))** = f(x), so f(D) ⊆ **R**(A). By Lemma 5.6.41, $\overline{D}$ =D∩R(A) is regular in **R**(A), hence $\bar{f} = f_{|\overline{D}} : \overline{D} \to R(A)$ is a multiplier on **R**(A), so $\overline{(\overline{D}, \bar{f})}$ ∈ (R (A))′′ (since f is a multiplier on A). Clearly the assignment $\overline{(D, f)} \to \overline{(\overline{D}, \bar{f})}$ defines a morphism of Hilbert algebras and Boolean algebras (since in a Boolean algebra the operations ∨, ∧, ′ can be defined with the aid of →); we will prove that this assignment is injective.

If (D_1, f_1), (D_2, f_2) ∈ **R**(A′′) such that $\overline{(\overline{D}_1, \bar{f}_1)} = \overline{(\overline{D}_2, \bar{f}_2)}$ then $\bar{f}_1 = \bar{f}_2$ on $\overline{D}_1 \cap \overline{D}_2$ = (D_1 ∩ D_2) ∩ **R**(A), hence f_1 = f_2 on (D_1 ∩ D_2) ∩ **R**(A), so $\overline{(D_1, f_1)} = \overline{(D_2, f_2)}$. ■

5.7. Valuations on Hilbert algebras

In this paragraph by A we denote a Hilbert algebra and by $\mathbb{R}$ the set of real nmbers.

Definition 5.7.1. A function v : A → $\mathbb{R}$ is called a *pseudo-valuation* on A if:

a_{28}: v(1) = 0;

a_{29}: v(x → y) ≥ v(y) - v(x), for any x, y ∈ A.

The pseudo- valuation v is said to be a *valuation* if

a_{30}: x ∈ A and v(x) = 0 ⇒ x = 1.

Remark 5.7.2. If we interpret A as an implicational calculus**,** x → y as the proposition "x ⇒ y" and **1** as truth, a pseudo-valuation on A can be interpreted as *falsity – valuation*.

Examples

1. v : A → $\mathbb{R}$, v(x) = 0 for any x ∈A is a pseudo-valuation on A (called *trivial*).

2. If D ∈ **Ds**(A) and 0 ≤ r ∈ $\mathbb{R}$, then v_D : A → $\mathbb{R}$,

$$v_D(x) = \begin{cases} 0, & for\ x \in D \\ r, & for\ x \notin D \end{cases} .$$

is a pseudo-valuation on A and a valuation iff D = {1} and r > 0.

3. If M is a finite set with n elements and A = (P(M), ∪, ∩, C_M, ∅, M) is the Boolean algebra of power set of M, then v : P(M) → $\mathbb{R}$, v(X) = n - | X | is a valuation on A (where by | X | we denote the cardinal of X, that is, the numbers of elements of X).

Remark 5.7.3. If v : A → $\mathbb{R}$ is a pseudo-valuation on A and $x, x_1, \ldots, x_n \in A$ such that $(x_1, \ldots, x_n ; x) = 1$ (that is, $x \in \langle x_1, \ldots, x_n \rangle$ – see Corollary 5.2.19), then

$$c_{42}:\ v(x) \le \sum_{i=1}^{n} v(x_i)\ .$$

Lemma 5.7.4. ([26], [28]) If v : A → ℝ is a pseudo-valuation on A, then

(i) D_v = {x ∈ A : v(x) = 0} ∈ Ds (A).

Conversely, if D ∈ Ds(A), then there is a pseudo-valuation v_D : A → ℝ (see Example 2) such that $D_{v_D} = D$**;**

(ii) The pseudo-valuation v on A, is a decreasing positive function satisfying

c_{43}: v(x → y) + v(y → z) ≥ v(x →z) for any x, y, z ∈ A.

We recall that by a pseudo-metric space we mean an ordered pair (M, d) where M is a non-empty set and d : M × M → ℝ is a positive function such that the following properties are satisfied: *d(x, x) = 0, d(x, y) = d(y, x)* and *d(x, z)≤ d(x, y) + d(y, z)* for every x,y,z ∈M. If in the pseudo-metric space (M,d) the implication *d(x, y) = 0 ⇒ x = y* hold, then (M, d) is called *metric space.*

Lemma 5.7.5. ([26], [28]) Let v : A → ℝ be a pseudo-valuation on A. If we define d_v : A × A → ℝ, d_v(x, y) = v(x → y) + v(y → x), for every x, y ∈ A, then (A, d_v) is a pseudo-metric space satisfying

c_{44}: max{ d_v(x → z, y → z), d_v(z → x, z → y) } ≤ d_v(x, y),

for any x, y, z ∈ A.

So, the operation → it is a uniformly continuous function in both variables. (A, d_v) is a metric space iff v is a valuation on A.

Definition 5.7.6. A pseudo-valuation v : A → ℝ is called *bounded* if there is a real positive number M_v such that 0 ≤ v(x) ≤ M_v for every x ∈ A.

Remark 5.7.7. All pseudo-valuations from examples 1-3 are bounded; if A is a bounded Hilbert algebra, then every pseudo-valuation on A is bounded (we can consider M_v = v(0)).

Theorem 5.7.8. ([26], [28]) (i) If D ∈ Ds(A), a ∈ A \ D and v : D → ℝ is a bounded pseudo-valuation on A, then there is a bounded pseudo-valuation on A v′ : D(a) → ℝ such that $v'_{|D}$ = v, where D(a) = <D ∪ {a}> = { x ∈ A : a → x ∈ A} (see Corollary 5.2.19);

(ii) If B is another Hilbert algebra such that A ⊆ B (as subalgebra) and v : A → ℝ is a pseudo-valuation (valuation) on A, then there is a

pseudo-valuation (valuation) v′ : A → ℝ such that $v'_{|A}$ = v, where by <A> we denoted the deductive system of B generated by A (see Definition 5.2.6).

Proof. We recall only the definition of an extension of v to v′ in both cases:

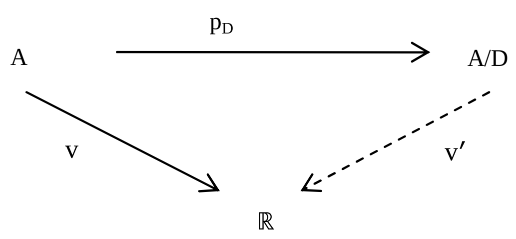
(i) For x ∈ D(a) we define v′ : D(a) → ℝ by

$v'(x) = \begin{cases} v(x), & for\ x \in D \\ M_v, & for\ x \notin D \end{cases}$ (where M_v has the property that 0 ≤ v(x) ≤ M_v for any x ∈ D).

(ii) For x ∈ <A> we define v′ : <A> → ℝ by $v'(x) = \inf\{\sum_{i=1}^{n} v(x_i) : x_1, ..., x_n \in A, (x_1, ..., x_n; x) = 1\}$. ■

Theorem 5.7.9. If D ∈ Ds (A) and v : A → ℝ is a pseudo-valuation (valuation) on A, then the following are equivalent:

(i) There is a pseudo-valuation (valuation) v′ : A / D → ℝ such that the diagram



is commutative (i.e, v′ ∘ p_D = v, where p_D (x) = x / D, for every x ∈ A);

(ii) v(a) = 0 for every a ∈ D.

Proof. (i) ⇒ (ii). If there is a pseudo-valuation v′ : A / D → ℝ such that v′ ∘ p_D = v, then for every a ∈ D, v(a) = (v′ ∘ p_D) (a) = v′ (p_D (a)) = v′(**1**) = 0.

(ii) ⇒(i). For x ∈ A we define v′ : A / D → ℝ by v′(x / D) = v(x).

If x, y ∈ A and x / D = y / D, then x → y, y → x ∈ D. We obtain 0 = v(x → y) ≥ v(y) – v(x) and 0 = v(y → x) ≥ v(x) – v(y), so v(x) = v(y), hence v′ is well defined.

We have $v'(\mathbf{1}) = v'(1 / D) = v(1) = 0$ and $v'(x / D \rightarrow y / D) = v'((x \rightarrow y) / D) = v(x \rightarrow y) \geq v(y) - v(x) = v'(y / D) \rightarrow v'(x / D)$, that is, v' is a pseudo-valuation on A / D. Clearly $v' \circ p_D = v$.

If v is a valuation on A and $x \in A$ such that $v'(x / D) = 0$, then $v(x) = 0$, hence $x = 1$, and $x / D = 1 / D = \mathbf{1}$, that is, v' is a valuation on A / D. ■

Definition 5.7.10. In [29], for a Hilbert algebra A, by A^0 it is denoted the Heyting algebra Ds(A) with the order $D_1 \leq D_2 \Leftrightarrow D_2 \subseteq D_1$.

In $(A^0, \leq)$, $0 = A$, $1 = \{1\}$ and for $D_1, D_2 \in A^0$, $D_1 \sqcap D_2 = \langle D_1 \cup D_2 \rangle = D_1 \vee D_2$, $D_1 \sqcup D_2 = D_1 \cap D_2$ and $D_1 \rightarrow D_2 = \sqcup \{D \in A^0 : D_2 \subseteq D_1 \vee D\}$.

Also, $j_A : A \rightarrow A^0$, $j_A(a) = \langle a \rangle$ for every $a \in A$ is a monomorphism of Hilbert algebras.

Definition 5.7.11. We say that a Hilbert algebra A has the property $\mathcal{F}$ if for every $D \in A^0$ there are $x_1, \ldots, x_n \in A$ such that $D \subseteq \langle x_1, \ldots, x_n \rangle$.

As examples of Hilbert algebras with property $\mathcal{F}$ we remark the bounded Hilbert algebras (since in this case $A = \langle 0 \rangle$) and finite Hilbert algebras.

Theorem 5.7.12. Let A be a Hilbert algebra with the property $\mathcal{F}$ and $v : A \rightarrow \mathbb{R}$ a pseudo-valuation on A. Then there is $v' : A^0 \rightarrow \mathbb{R}$ a pseudo-valuation on A^0 such that $v' \circ j_A = v$.

Proof. For $D \in A^0$ we define

$$v'(D) = \inf\{\sum_{i=1}^{n} v(x_i) : x_1, \ldots, x_n \in A, D \subseteq \langle x_1, \ldots, x_n \rangle\}.$$

Clearly $v'(1) = \inf\{\sum_{i=1}^{n} v(x_i) : x_1, \ldots, x_n \in A, \{1\} \subseteq \langle x_1, \ldots, x_n \rangle\} = v(1) = 0.$

To prove that v' verify a_{29} let $D_1, D_2 \in A^0$, $x_1, \ldots, x_n, z_1, \ldots, z_m \in A$ such that $D_1 \subseteq \langle x_1, \ldots, x_n \rangle$ and $D_1 \rightarrow D_2 \subseteq \langle z_1, \ldots, z_m \rangle$.

Then $D_2 \subseteq D_1 \vee (D_1 \rightarrow D_2) \subseteq \langle x_1, \ldots, x_n \rangle \vee \langle z_1, \ldots, z_m \rangle = \langle x_1, \ldots, x_n, z_1, \ldots, z_m \rangle$, hence $v'(D_2) \leq \sum_{i=1}^{n} v(x_i) + \sum_{j=1}^{m} v(z_j)$, so

$$v'(D_2) \leq \inf\{\sum_{i=1}^{n} v(x_i) : x_1, \ldots, x_n \in A, D_1 \subseteq \langle x_1, \ldots, x_n \rangle\} +$$

$$+ \inf\{\sum_{j=1}^{m} v(z_j) : z_1, \ldots, z_m \in A, D_1 \rightarrow D_2 \subseteq \langle z_1, \ldots, z_m \rangle\} = v'(D_1) + v'(D_1 \rightarrow D_2),$$

that is, $v'(D_1 \to D_2) \geq v'(D_2) - v'(D_1)$.

If $a \in A$ and $x_1, \ldots, x_n \in A$ such that $<a> \subseteq <x_1, \ldots, x_n>$, then $(x_1, \ldots, x_n\ ;\ a) = 1$ hence $v(a) \leq \sum_{i=1}^{n} v(x_i)$, (by Remark 5.7.3 and c_{42}), so $v(a) \leq \inf\{\sum_{i=1}^{n} v(x_i) : x_1, \ldots, x_n \in A, <a> \subseteq <x_1, \ldots, x_n>\} = v'(<a>)$.

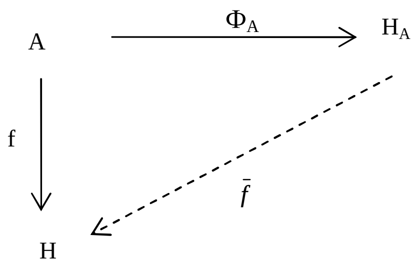
Since $<a> \subseteq <a>$ it follows that $v'(<a>) \leq v(a)$, hence $v'(<a>) = v(a)$, that is, $v' \circ j_A = v$. ■

We consider now the problem of extensions of pseudo-valuations in the case of Hertz algebras.

For a Hilbert algebra A, in §4 (see Theorem 4.4.11), we have proved the existence of a Hertz algebra H_A and a morphism of Hilbert algebras $\Phi_A : A \to H_A$ with the following properties:

(i) H_A is generated (as Hertz algebra) by $\Phi_A(A)$;

(ii) For every Hertz algebra H and every morphism of Hilbert algebras $f : A \to H$, there is a unique morphism of Hertz algebras $\bar{f} : H_A \to H$ such that the diagram



is commutative (i.e, $\bar{f} \circ \Phi_A = f$).

Proposition 5.7.13. For a Hilbert algebra A, the following are equivalent:

(i) A is Hertz algebra;

(ii) For every $x_1, x_2 \in A$ there is $a \in <x_1, x_2>$ such that $(x_1, x_2\ ;\ x) = a \to x$ for every $x \in A$;

(iii) Every finitely generated deductive system of A is principal;

(iv) For every $x_1, x_2 \in A$, $\Phi_A(x_1) \wedge \Phi_A(x_2) \in \Phi_A(A)$.

Proof .(i)⇒(iii). If $x_1, \ldots, x_n \in A$ and $a = x_1 \wedge \ldots \wedge x_n$, then from a_{14} and a_1 we deduce that $a \in <x_1, \ldots, x_n>$ and $<a> = <x_1, \ldots, x_n>$.

(iii)$\Rightarrow$(i). If $x_1, x_2 \in A$, then $<x_1, x_2> = <a>$, for some $a \in A$. Then $x_1, x_2 \in <x_1, x_2> = <a>$, hence $a \le x_1, x_2$ and $x_1 \to (x_2 \to a) = 1$(by Theorem 5.2.18).

If we have $a' \in A$ such that $a' \le x_1$, $a' \le x_2$, then $1 = a' \to 1 = a' \to [x_1 \to (x_2 \to a)] = (a' \to x_1) \to [(a' \to x_2) \to (a' \to a)] = 1 \to [1 \to (a' \to a)] = a' \to a$, hence $a' \le a$, that is, $a = x_1 \wedge x_2$, so A is a Hertz algebra. Hence (i)$\Leftrightarrow$(iii).

(i)$\Rightarrow$(ii). If A is a Hertz algebra and $x_1, x_2 \in A$, if we take $a = x_1 \wedge x_2 \in <x_1, x_2>$, then by c_{34} we have $(x_1, x_2 ; x) = a \to x$, for every $x \in A$.

(ii)$\Rightarrow$(iv). Let $x_1, x_2 \in A$ and $a \in <x_1, x_2>$ such that $(x_1, x_2 ; x) = a \to x$, for every $x \in A$.

Since $(x_1, x_2 ; x_1) = (x_1, x_2 ; x_2) = 1$ we deduce that $a \to x_1 = a \to x_2 = 1$, hence $\{a\} \to \{x_1, x_2\} = I$, where $I = \{1\}$ (see the proof of Theorem 5.4.13).

Since $(x_1, x_2 ; a) = a \to a = 1$, we deduce that $\{x_1, x_2\} \to \{a\} = I$, hence $\{x_1, x_2\} / \rho_A = \{a\} / \rho_A$ and $\Phi_A(x_1) \wedge \Phi_A(x_2) = \{x_1\} / \rho_A \wedge \{x_2\} / \rho_A = \{x_1, x_2\} / \rho_A = \{a\} / \rho_A = \Phi_A(a) \in \Phi_A(A)$.

(iv)$\Rightarrow$(i). Let $x_1, x_2 \in A$ and $a \in A$ such that $\Phi_A(x_1) \wedge \Phi_A(x_2) = \Phi_A(a)$. Then $\{x_1, x_2\} / \rho_A = \{a\} / \rho_A$, hence $x_1 \to (x_2 \to a) = 1$ and $a \to x_1 = a \to x_2 = 1$ (that is, $a \in <x_1, x_2>$ and $a \le x_1, x_2$). To prove that $a = x_1 \wedge x_2$ let $a' \in A$ such that $a' \le x_1, x_2$. As in the case of implication (iii)$\Rightarrow$(i), we deduce that $a' \le a$, hence $a = x_1 \wedge x_2$, that is, A is a Hertz algebra. ■

Proposition 5.7.14. Let $X = \{x_1, \ldots, x_n\}$, $Y = \{y_1, \ldots, y_m\} \in F(A)$ and $Z = X \to Y = \{y'_1, \ldots, y'_m\}$ (where $y'_j = (x_1, \ldots, x_n ; y_j)$, $1 \le j \le m$). Then

(i) $X / \rho_A \le Y / \rho_A \Leftrightarrow <Y> \subseteq <X>$;

(ii) $(\Phi_A(y_1), \ldots, \Phi_A(y_m) ; X / \rho_A) = 1 \Leftrightarrow <X> \subseteq <Y>$;

(iii) $<Z> = \cap \{D \in Ds(A) : <Y> \subseteq <X> \vee D\}$;

(iv) $H_A = <\Phi_A(A)>$.

Proof .(i). We have $X / \rho_A \le Y / \rho_A \Leftrightarrow X / \rho_A \to Y / \rho_A = \mathbf{1} \Leftrightarrow Z / \rho_A = I / \rho_A \Leftrightarrow y'_j = 1, 1 \le j \le m \Leftrightarrow y_j \in < X >, 1 \le j \le m$ (by c_{25}) $\Leftrightarrow <Y> \subseteq <X>$.

(ii). We have $(\Phi_A(y_1), \ldots, \Phi_A(y_m); X / \rho_A) = \mathbf{1} \Leftrightarrow \Phi_A(y_1) \wedge \ldots \wedge \Phi_A(y_m) \le X / \rho_A$ (by c_{34}) $\Leftrightarrow \{y_1, \ldots, y_m\} / \rho_A \le X / \rho_A \Leftrightarrow Y / \rho_A \le X / \rho_A \Leftrightarrow < X > \subseteq < Y >$.

(iii). If $y \in <Y>$, then $(y_1, \ldots, y_m ; y) = 1$ and by a_{14} we deduce that $(y'_1, \ldots, y'_m ; (x_1, \ldots, x_n ; y)) = ((x_1, \ldots, x_n ; y_1), \ldots, (x_1, \ldots, x_n ; y_m) ; (x_1, \ldots, x_n ; y)) = (x_1, \ldots, x_n , y_1, \ldots, y_m ; y) = (x_1, \ldots, x_n ; (y_1, \ldots, y_m ; y)) = (x_1, \ldots, x_n ; 1) = 1$, hence $(x_1, \ldots, x_n ; y) \in <Z>$. So $y \in <X> \vee <Z>$ (by Theorem 5.2.18), that is, $<Y> \subseteq <X> \vee <Z>$.

If we have $D \in \mathbf{Ds}(A)$ such that $<Y> \subseteq <X> \vee D$, then for $1 \le j \le m$, $y_j \in <X> \vee D$, hence $y'_j = (x_1, \ldots, x_n ; y_j) \in D$, so, $<Z> = <y'_1, \ldots, y'_m> \subseteq D$, hence $<Z> = \cap \{D \in \mathbf{Ds}(A) : <Y> \subseteq <X> \vee D\}$.

(iv). If $X = \{x_1, \ldots, x_n\} \in F(A)$, then $(\Phi_A(x_1), \ldots, \Phi_A(x_n) ; X / \rho_A) = (\Phi_A(x_1) \wedge \ldots \wedge \Phi_A(x_n)) \to X / \rho_A = \{x_1, \ldots, x_n\} / \rho_A \to X / \rho_A = X / \rho_A \to X / \rho_A = \mathbf{1}$, hence $H_A = <\Phi_A(A)>$ (see Theorem 5.2.18). ■

Theorem 5.7.15. Let A be a Hilbert algebra and $v : A \to \mathbb{R}$ a pseudo-valuation on A. Then there is a pseudo-valuation $v' : H_A \to \mathbb{R}$ such that $v' \circ \Phi_A = v$.

Proof. For $X \in F(A)$ we define

$$v'(X / r_A) = \inf_{Y=\{y_1,\ldots,y_m\}\in F(A)} \{\sum_{j=1}^{m} v(y_j) : (\Phi_A(y_1),\ldots,\Phi_A(y_m); X / r_A) = 1\}$$

$$= \inf_{Y=\{y_1,\ldots,y_m\}\in F(A)} \{\sum_{j=1}^{m} v(y_j) : Y / r_A \le X / r_A)\}$$

$$= \inf_{Y=\{y_1,\ldots,y_m\}\in F(A)} \{\sum_{j=1}^{m} v(y_j) :< X > \subseteq < Y >\} \text{ (by Proposition 5.7.14).}$$

If we have $X' \in F(A)$ such that $X / \rho_A = X' / \rho_A$, then $<X> = <X'>$ (by Proposition 5.7.14), hence v' is correctly defined.

Clearly $v'(\mathbf{1}) = v'(\{1\} / \rho_A) = v(1) = 0$, since $<1> \subseteq <1> = \{1\}$.

If $a \in A$, since $<a> \subseteq <a>$, we deduce that $v'(\{a\} / \rho_A) = v(a)$.

Let $Y = \{y_1, \ldots, y_m\} \in F(A)$ such that $<a> \subseteq <Y>$.

Then, in particular, $a \in <Y>$. By Remark 5.7.3 we obtain that $v(a) \le \sum_{j=1}^{m} v(y_j)$, hence $v(a) \le \inf_{Y=\{y_1,\ldots,y_m\}\in F(A)} \{\sum_{j=1}^{m} v(y_j) :< a > \subseteq < Y >\} = v'(\{a\} / \rho_A)$,

so $v'(\{a\} / \rho_A) = v(a)$, that is, $v' \circ \Phi_A = v$.

To prove that v' verifies a_{29} let $X, X' \in F(A)$ and $Y = \{y_1, \ldots, y_m\}$, $Y' = \{y'_1, \ldots, y'_p\} \in F(A)$ such that $(\Phi_A(y_1), \ldots, \Phi_A(y_m) ; X / \rho_A) =$

(Φ_A (y'_1), …, Φ_A (y'_p) ; X / ρ_A → X′ / ρ_A) = 1 (then <X> ⊆ <Y> and <X → X′> ⊆ <Y′>).

By c_{15} we deduce that ($\Phi_A(y'_1)$, …, $\Phi_A(y'_p)$; X / ρ_A) ≤ ($\Phi_A(y'_1)$, …, $\Phi_A(y'_p)$; X′ / ρ_A), hence ($\Phi_A(y_1)$, …, $\Phi_A(y_m)$; ($\Phi_A(y'_1)$, …, Φ_A (y'_p) ; X / ρ_A)) ≤ ($\Phi_A(y_1)$, …, $\Phi_A(y_m)$; ($\Phi_A(y'_1)$, …, $\Phi_A(y'_p)$; X′ / ρ_A)) ⇒ **1** = ($\Phi_A(y'_1)$, …, $\Phi_A(y'_p)$; ($\Phi_A(y_1)$, …, $\Phi_A(y_m)$; X / ρ_A)) ≤ ($\Phi_A(y_1)$, …, $\Phi_A(y_m)$; ($\Phi_A(y'_1)$, …, $\Phi_A(y'_p)$; X′ / ρ_A)) ⇒ ($\Phi_A(y_1)$, …, $\Phi_A(y_m)$, $\Phi_A(y'_1)$, …, $\Phi_A(y'_p)$; X′ / ρ_A) = **1** ⇒ $v'(X'/r_A) \le \sum_{j=1}^{m} v(y_j) + \sum_{k=1}^{p} v(y'_k)$.

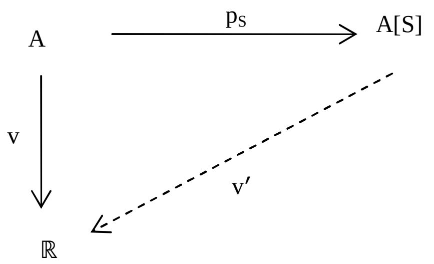
So, $v'(X'/r_A) \le \inf_{Y=\{y_1,...,y_m\}\in F(A)} \{\sum_{j=1}^{m} v(y_j) : <X> \subseteq <Y>\} +$

$+ \inf_{Y'=\{y'_1,...,y'_p\}\in F(A)} \{\sum_{k=1}^{p} v(y'_k) : <X \to X'> \subseteq <Y'>\} =$ v′(X / ρ_A) + v′(X / ρ_A → X′ / ρ_A), hence v′(X / ρ_A → X′ / ρ_A) ≥ v′(X′ / ρ_A) - v′(X / ρ_A), that is, v′ is a pseudo-valuation on H_A. ■

For an ⊻- closed system S of A we have defined Hilbert algebra of fractions of A relative to S, A[S] = A / θ_S (see Definition 5.6.5). We recall that by p_S : A → A[S] we have denoted the canonical morphism of Hilbert algebras defined by $p_S(a) = a / \theta_S$, for every a ∈ A.

Theorem 5.7.16. For a ⊻ - closed system S ⊆ A and a pseudo-valuation v : A → ℝ, the following are equivalent:

(i) There is a valuation v′ : A[S] → ℝ such that the diagram



is commutative (i.e, v′ ∘ p_S = v);

(ii) v(s*) = 0 for every s ∈ S.

Proof. (i)⇒(ii). Let $v' : A[S] \to \mathbb{R}$ be a valuation such that $v' \circ p_S = v$ and $s \in S$. Since $s^* \to s^* = 1 = s^* \to 1$ we deduce that $(s^*, 1) \in \theta_S$ hence $p_S(s^*) = p_S(1)$; then $v(s^*) = (v' \circ p_S)(s^*) = v'(p_S(s^*)) = v'(p_S(1)) = (v' \circ p_S)(1) = v(1) = 0$.

(ii) ⇒ (i). For $x \in A$ we define $v'(x / \theta_S) = v(x)$.

If $x, y \in A$ and $x / \theta_S = y / \theta_S$, then there is $s \in S$ such that $x \veebar s = y \veebar s$, hence $s^* \to x = s^* \to y$.

Since $s^* \to (x \to y) = (s^* \to x) \to (s^* \to y) = (s^* \to x) \to (s^* \to x) = 1$, we obtain that $s^* \leq x \to y$ and analogously $s^* \leq y \to x$. Then $v(x \to y) \leq v(s^*) = 0$ and $v(y \to x) \leq v(s^*) = 0$, hence $v(x \to y) = v(y \to x) = 0$.

Since v is a valuation we deduce that $x \to y = y \to x = 1$, hence $x = y$ and $v(x) = v(y)$, so v' is correctly defined; clearly $v' \circ p_S = v$.

We have $v'(1 / \theta_S) = v(1) = 0$ and $v'(x / \theta_S \to y / \theta_S) = v'((x \to y) / \theta_S) = v(x \to y) \geq v(y) - v(x) = v'(y / \theta_S) - v'(x / \theta_S)$, that is, v' is a pseudo-valuation on A. To prove v' is a valuation, let $x \in A$ such that $v'(x / \theta_S) = 0$. Then $v(x) = 0$ hence $x = 1$, that is, $x / \theta_S = \mathbf{1}$.■

c. Residuated lattices

5. 8. Definitions. Examples. Rules of calculus

Definition 5.8.1. An algebra $(A, \vee, \wedge, \odot, \to, 0, 1)$ of type (2, 2, 2, 2, 0, 0) will be called *residuated lattice* if :

Lr_1: $(A, \vee, \wedge, 0, 1)$ is a bounded lattice;

Lr_2: $(A, \odot, 1)$ is a commutative monoid;

Lr_3: For every $x, y \in L$, $x \leq y \to z \Leftrightarrow x \odot y \leq z$.

The axiom Lr_3 is called *axiom of residuation* (or *Galois correspondence*) and for every $x, y \in A$, $x \to y = \sup \{z \in A : x \odot z \leq y\}$.

Remark 5.8.2. The axiom of residuation is a particular case of *loin of residuation* ([8]). More precisely, let $(P, \leq)$ and $(Q, \leq)$ two posets and $f : P \to Q$ a function. We say that f is *residuated* if there is a function $g : Q \to P$ such that for every $p \in P$ and $q \in Q$, $f(p) \leq q \Leftrightarrow p \leq g(q)$. We say that (f, g) is a pair *of residuation.*

If we consider A a residuated lattice, P = Q = A, and for every a∈A, f_a, g_a : A → A, f_a (x) = x ⊙ a and g_a (x) = a → x, x∈A, then (f_a, g_a) form a pair of residuation.

Examples

1. Let p be a fixed natural number and A = [0, 1] the real unit interval. If for x, y ∈ A, we define x ⊙ y = 1 – min{1, $\sqrt[p]{(1-x)^p + (1-y)^p}$ } and x → y = sup {z ∈ [0, 1] : x ⊙ z ≤ y}, then (A, max, min, ⊙, →, 0, 1) is a residuated lattice.

2. If we preserve the notation from Example 1, and we define for x, y ∈ A, x ⊙ y = $\sqrt[p]{\max\{0, x^p + y^p - 1\}}$ and x → y = min {1, $\sqrt[p]{1 - x^p + y^p}$ }, then (A, max, min, ⊙, →, 0, 1) becomes a residuated lattice called *generalized Łukasiewicz structure* (for p = 1 we obtain the notion of *Łukasiewicz structure*).

3. If on A = [0, 1] for x, y ∈ A, we define x ⊙ y = min {x, y} and x → y = 1 if x ≤ y and y otherwise, then (A, max, min, ⊙, →, 0, 1) is a residuated lattice (called *Gődel structure*).

4. If consider on A = [0, 1], ⊙ to be the usual multiplication of real numbers and for x, y ∈ A, $x \to y = \begin{cases} 1, & if\ x \le y, \\ \frac{y}{x}, & if\ x > y \end{cases}$, then (A, max, min, ⊙, →, 0, 1) is another example of residuated lattice (called *Gaines structure*).

5. If (A, ∨, ∧,→ , 0) is a Heyting algebra, then (A, ∨, ∧, ⊙, →, 0, 1) becomes a residuated lattice, where ⊙ coincides with ∧ .

6. If (A, ∨, ∧, → , 0, 1) is a Boolean algebra, then if we define for x, y ∈ A, x ⊙ y = x ∧ y and x → y = x′ ∨ y, (A, ∨, ∧, ⊙, →, 0, 1) becomes a residuated lattice.

Examples 2 and 3 have some connections with the notion of t – norm.

We call *continuous t – norm* a continuous function ⊙ : [0, 1] × [0, 1] → [0, 1] such that ([0, 1], ⊙, 1) is an ordered commutative monoid.

So, there are three fundamental t-norms:

Łukasiewicz t-norm: x $\odot_L$ y = max {x + y – 1, 0};

Gődel t-norm: x $\odot_G$ y = min {x, y};

Product (or Gaines) t-norm: x $\odot_P$ y = x · y.

Since relative to natural ordering [0,1] becomes a complete lattice, every continuous t-norm induce a natural *residuum* (or *implication*) by

$$x \to y = \max \{z \in [0, 1] : x \odot z \le y\}.$$

So, the implications generated by the three norms mentioned before are

$x \to_L y = \min (y - x + 1, 1)$;

$x \to_G y = 1$ if $x \le y$ and y otherwise ;

$x \to_P y = 1$ if $x \le y$ and y / x otherwise.

The origin of residuated lattices is in Mathematical Logic without contraction .They have been investigated by Krull ([56]), Dilworth ([38]), Ward and Dilworth ([83]), Ward ([82]), Balbes and Dwinger ([2]) and Pavelka ([68]).

These lattices have been known under many names : *BCK-lattices* in [50], *full BCK-algebras* in [56],*FL_{ew}-algebras* in [67] and *integral, residuated, commutative l-monoids* in [6].

In [53] it is proved that the class of residuated lattices is equational.

Definition 5.8.3. A residuated lattice (A, ∨, ∧, ⊙, →, 0, 1) is called *BL-algebra* if the following two identities hold in A :

(BL_1) $x \odot (x \to y) = x \wedge y$;

(BL_2) $(x \to y) \vee (y \to x) = 1$.

Remark 5.8.4. 1.Łukasiewicz structure, Gödel structure and Product structure are BL-algebras. Not every residuated lattice, however, is a BL-algebra (see [81,p.16]).

Consider for example a residuated lattice defined on the unit interval I=[0,1], for all x, y, z ∈ I, such that $x \odot y = 0$ if $x+y \le \frac{1}{2}$ and $x \wedge y$ elsewhere, $x \to y = 1$ if $x \le y$ and $\max\{\frac{1}{2} - x, y\}$ elsewhere. Let $0 < y < x$, $x + y < \frac{1}{2}$. Then $y < \frac{1}{2} - x$ and $0 \ne y = x \wedge y$, but $x \odot (x \to y) = x \odot (\frac{1}{2} - x) = 0$. Therefore ($BL_1$) does not hold.

2. ([52]).We give an example of a (finite) residuated lattice which is not a BL-algebra, too.

Let A = {0, a, b, c, 1} with 0 < a, b < c < 1, but a and b are incomparable. A becomes a residuated lattice relative to the following operations :

→	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1

1	0	a	b	c	1

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

The condition $x \vee y = [(x \to y) \to y] \wedge [(y \to x) \to x]$, for all $x, y \in A$ is not verified, since $c = a \vee b \neq [(a \to b) \to b] \wedge [(b \to a) \to a] = (b \to b) \wedge (a \to a) = 1$, hence A is not a BL-algebra.

3. ([55]). We consider the residuated lattice A with the universe {0, f, e, d, c, b, a, 1}. Lattice ordering is such that $0<d<c<b<a<1$, $0<d<e<f<a<1$ and elements {b, c} and {e, f} are pairwise incomparable.

Multiplication is given in the table below, and lattice ordering is shown beside it.

$\odot$	1	a	b	c	d	e	f	0
1	1	a	b	c	d	e	f	0
a	a	c	c	c	0	d	d	0
b	b	c	c	c	0	0	d	0
c	c	c	c	c	0	0	0	0
d	d	0	0	0	0	0	0	0
e	e	d	0	0	0	d	d	0
f	f	d	d	0	0	d	d	0
0	0	0	0	0	0	0	0	0

Clearly, A contains {f, e, d, c, b, a} as sublattice, which shows that A is not distributive, and not even modular (see Theorems 2.3.4 and 2.3.8).

It is easy to see that $a \to 0 = d$, $b \to 0 = e$, $c \to 0 = f$, $d \to 0 = a$, $e \to 0 = b$ and $f \to 0 = c$.

In what follows by A we denote a residuated lattice .

Theorem 5.8.5. Let $x, x_1, x_2, y, y_1, y_2, z \in A$.
Then

r-c_1: $x = 1 \to x$;
r-c_2: $1 = x \to x$;
r-c_3: $x \odot y \leq x, y$;
r-c_4: $x \odot y \leq x \wedge y$;

r-c$_5$: $y \le x \to y$,

r-c$_6$: $x \odot y \le x \to y$;

r-c$_7$: $x \le y \Leftrightarrow x \to y = 1$;

r-c$_8$: $x \to y = y \to x = 1 \Leftrightarrow x = y$;
r-c$_9$: $x \to 1 = 1$;
r-c$_{10}$: $0 \to x = 1$;

r-c$_{11}$: $x \odot (x \to y) \le y \Leftrightarrow x \le (x \to y) \to y$;

r-c$_{12}$: $x \to y \le (x \odot z) \to (y \odot z)$;

r-c$_{13}$: If $x \le y$, then $x \odot z \le y \odot z$;
r-c$_{14}$: $x \to y \le (z \to x) \to (z \to y)$;
r-c$_{15}$: $x \to y \le (y \to z) \to (x \to z)$;

r-c$_{16}$: $x \le y \Rightarrow z \to x \le z \to y$;

r-c$_{17}$: $x \le y \Rightarrow y \to z \le x \to z$;

r-c$_{18}$: $x \to (y \to z) = (x \odot y) \to z$;
r-c$_{19}$: $x \to (y \to z) = y \to (x \to z)$;
r-c$_{20}$: $x_1 \to y_1 \le (y_2 \to x_2) \to [(y_1 \to y_2) \to (x_1 \to x_2)]$.

*Proof.***r-c$_1$.** Since $x \odot 1 = x \le x \Rightarrow x \le 1 \to x$.

If we have $z \in A$ such that $1 \odot z \le x$, then $z \le x$ and so $x = \sup\{z \in A : 1 \odot z \le x\} = 1 \to x$.

r-c$_2$. From $1 \odot x = x \le x \Rightarrow 1 \le x \to x$; since $x \to x \le 1 \Rightarrow x \to x = 1$.
r-c$_3$. Follows from r-c$_2$ and Lr$_2$.
r-c$_4$. Follows from r-c$_3$ and Lr$_2$.
r-c$_5$. Follows from r-c$_2$ and Lr$_2$.
r-c$_6$. Follows from r-c$_4$ and r-c$_5$.

r-c$_7$. We have $x \le y \Leftrightarrow x \odot 1 \le y \Leftrightarrow 1 \le x \to y \Leftrightarrow x \to y = 1$.
r-c$_8$, (r-c$_9$), (r-c$_{10}$). It follows from r-c$_7$.
r-c$_{11}$. It follows immediately from Lr$_3$.

r-c$_{12}$. By Lr$_3$ we have $x \to y \le (x \odot z) \to (y \odot z) \Leftrightarrow (x \to y) \odot x \odot z \le y \odot z \Leftrightarrow (x \to y) \odot x \le z \to (y \odot z)$. But by r-c$_{11}$ we have $(x \to y) \odot x \le y$ and $y \le z \to (y \odot z)$, hence $(x \to y) \odot x \le z \to (y \odot z)$.
r-c$_{13}$. It follows from r-c$_{12}$.

r-c$_{14}$. By (Lr$_3$) we have $x \to y \le (z \to x) \to (z \to y) \Leftrightarrow (x \to y) \odot (z \to x) \le z \to y \Leftrightarrow (x \to y) \odot (z \to x) \odot z \le y$.
Indeed, by r-c$_{11}$ and r-c$_{13}$ we have that $(x \to y) \odot (z \to x) \odot z \le (x \to y) \odot x \le y$.
r-c$_{15}$. As in the case of r-c$_{14}$.
r-c$_{16}$. It follows from r-c$_{14}$.

r-c$_{17}$. It follows from r-c$_{15}$.

r-c$_{18}$. We have $(x \to (y \to z)) \odot (x \odot y) \le (y \to z) \odot y \le z$, hence $x \to (y \to z) \le (x \odot y) \to z$. On the other hand, from $((x \odot y) \to z) \odot (x \odot y) \le z$, we deduce that $((x \odot y) \to z) \odot x \le z \to y$, therefore $(x \odot y) \to z \le x \to (z \to y)$, so we obtain the requested equality.

r-c$_{19}$. It follows from r-c$_{18}$.

r-c$_{20}$. We have to prove that $(x_1 \to y_1) \odot (y_2 \to x_2) \odot (y_1 \to y_2) \odot x_1 \le x_2$; this inequality is a consequence by applying several times r-c$_{11}$. ■

In a residuated lattice A, for $x \in A$ and a natural number n we define $x^* = x \to 0$, $(x^*)^* = x^{**}$, $x^0 = 1$ and for $n \ge 1$, $x^n = x \odot \ldots \odot x$ (n terms).

Theorem 5.8.6. If $x, y \in A$, then :

r-c$_{21}$: $x \odot x^* = 0$;
r-c$_{22}$: $x \le x^{**}$;
r-c$_{23}$: $1^* = 0$, $0^* = 1$;
r-c$_{24}$: $x \to y \le y^* \to x^*$;
r-c$_{25}$: $x^{***} = x^*$.

Proof. **r-c$_{21}$.** We have, $x^* \le x \to 0 \Leftrightarrow x \odot x^* \le 0$, hence $x \odot x^* = 0$.

r-c$_{22}$. We have $x \to x^{**} = x \to (x^* \to 0) = x^* \to (x \to 0) = x^* \to x^* = 1$.

r-c$_{23}$. Clearly.

r-c$_{24}$. It follows from r-c$_{15}$ for $z = 0$.

r-c$_{25}$. From r-c$_{22}$ we deduce that $x^* \le x^{***}$ and from $x \le x^{**}$ we deduce that $x^{**} \to 0 \le x \to 0 \Leftrightarrow x^{***} \le x^*$, therefore $x^{***} = x^*$. ■

By *bi-residuum* on a residuated lattice A we understand the derived operation $\leftrightarrow$ defined for $x, y \in A$ by $x \leftrightarrow y = (x \to y) \wedge (y \to x)$.

Theorem 5.8.7. If $x, y, x_1, y_1, x_2, y_2 \in A$, then

r-c$_{26}$: $x \leftrightarrow 1 = x$;
r-c$_{27}$: $x \leftrightarrow y = 1 \Leftrightarrow x = y$;
r-c$_{28}$: $x \leftrightarrow y = y \leftrightarrow x$;
r-c$_{29}$: $(x \leftrightarrow y) \odot (y \leftrightarrow z) \le x \leftrightarrow z$;
r-c$_{30}$: $(x_1 \leftrightarrow y_1) \wedge (x_2 \leftrightarrow y_2) \le (x_1 \wedge x_2) \leftrightarrow (y_1 \wedge y_2)$;
r-c$_{31}$: $(x_1 \leftrightarrow y_1) \wedge (x_2 \leftrightarrow y_2) \le (x_1 \vee x_2) \leftrightarrow (y_1 \vee y_2)$;
r-c$_{32}$: $(x_1 \leftrightarrow y_1) \odot (x_2 \leftrightarrow y_2) \le (x_1 \odot x_2) \leftrightarrow (y_1 \odot y_2)$;
r-c$_{33}$: $(x_1 \leftrightarrow y_1) \odot (x_2 \leftrightarrow y_2) \le (x_1 \leftrightarrow x_2) \leftrightarrow (y_1 \leftrightarrow y_2)$.

Proof .**r-c$_{26}$ - r-c$_{29}$,** are immediate consequences of Theorem 5.8.5 .

r-c$_{30}$. If we denote $a = x_1 \leftrightarrow y_1$ and $b = x_2 \leftrightarrow y_2$, using the above rules of calculus we deduce that $(a \wedge b) \odot (x_1 \wedge x_2) \leq [(x_1 \to y_1) \wedge (x_2 \to y_2)] \odot (x_1 \wedge x_2) \leq [(x_1 \to y_1) \odot x_1] \wedge [(x_2 \to y_2) \odot x_2 \leq y_1 \wedge y_2$, hence $a \wedge b \leq (x_1 \wedge x_2) \to (y_1 \wedge y_2)$.

Analogously we deduce $a \wedge b \leq (y_1 \wedge y_2) \to (x_1 \wedge x_2)$, hence $a \wedge b \leq (x_1 \wedge x_2) \leftrightarrow (y_1 \wedge y_2)$;

r-c$_{31}$. With the notations from r-c$_{30}$ we have

$(a \wedge b) \odot (x_1 \vee x_2) = [(a \wedge b) \odot x_1] \vee [(a \wedge b) \odot x_2] \leq$

$\leq [(x_1 \to y_1) \odot x_1] \vee [(x_2 \to y_2) \odot x_2] \leq y_1 \vee y_2$,

hence $a \wedge b \leq (x_1 \vee x_2) \to (y_1 \vee y_2)$.

From here the proof is similar with the proof of r-c$_{30}$.

r-c$_{32}$. We have that $(a \odot b) \odot (x_1 \odot x_2) \leq [(x_1 \to y_1) \odot x_1] \odot [(x_2 \to y_2) \odot x_2] \leq y_1 \odot y_2$, hence $(a \odot b) \leq (x_1 \odot x_2) \to (y_1 \odot y_2)$.

From here the proof is similar with the proof of r-c$_{30}$.

r-c$_{33}$. We have $(a \odot b) \odot (x_1 \to x_2) \leq (y_1 \to x_1) \odot (x_2 \to y_2) \odot (x_1 \to x_2) \leq (y_1 \to x_2) \odot (x_2 \to y_2) \leq y_1 \to y_2$, and from here the proof is similar with the proof of r-c$_{30}$. ■

Theorem 5.8.8. If A is a complete residuated lattice, x ∈ A and $(y_i)_{i \in I}$ a family of elements of A, then :

r-c$_{34}$: $x \odot (\bigvee_{i \in I} y_i) = \bigvee_{i \in I} (x \odot y_i)$;

r-c$_{35}$: $x \odot (\bigwedge_{i \in I} y_i) \leq \bigwedge_{i \in I} (x \odot y_i)$;

r-c$_{36}$: $x \to (\bigwedge_{i \in I} y_i) = \bigwedge_{i \in I} (x \to y_i)$;

r-c$_{37}$: $(\bigvee_{i \in I} y_i) \to x = \bigwedge_{i \in I} (y_i \to x)$;

r-c$_{38}$: $\bigvee_{i \in I} (y_i \to x) \leq (\bigwedge_{i \in I} y_i) \to x$;

r-c$_{39}$: $\bigvee_{i \in I} (x \to y_i) \leq x \to (\bigvee_{i \in I} y_i)$;

r-c$_{40}$: $(\bigvee_{i \in I} y_i)^* = \bigwedge_{i \in I} y_i^*$;

r-c$_{41}$: $(\bigwedge_{i \in I} y_i)^* \geq \bigvee_{i \in I} y_i^*$.

Proof. **r-c$_{34}$.** Clearly $\bigvee_{i \in I} (x \odot y_i) \leq x \odot (\bigvee_{i \in I} y_i)$.

Conversely, since for every $i \in I$, $x \odot y_i \leq \underset{i\in I}{\vee}(x \odot y_i) \Rightarrow y_i \leq x \rightarrow [\underset{i\in I}{\vee}(x \odot y_i)]$, then $\underset{i\in I}{\vee} y_i \leq x \rightarrow [\underset{i\in I}{\vee}(x \odot y_i)]$, therefore $x \odot (\underset{i\in I}{\vee} y_i) \leq \underset{i\in I}{\vee}(x \odot y_i)$, so we obtain the requested equality.

r-c$_{35}$. Clearly.

r-c$_{36}$. Let $y = \underset{i\in I}{\wedge} y_i$. Since for every $i \in I$, $y \leq y_i$, we deduce that $x \rightarrow y \leq x \rightarrow y_i$, hence $x \rightarrow y \leq \underset{i\in I}{\wedge}(x \rightarrow y_i)$.

On the other hand, the inequality $\underset{i\in I}{\wedge}(x \rightarrow y_i) \leq x \rightarrow y$ is equivalent with $x \odot [\underset{i\in I}{\wedge}(x \rightarrow y_i)] \leq y$.

This is true because by r-c$_{35}$ we have

$$x \odot [\underset{i\in I}{\wedge}(x \rightarrow y_i)] \leq \underset{i\in I}{\wedge}[x \odot (x \rightarrow y_i)] \leq \underset{i\in I}{\wedge} y_i = y.$$

r-c$_{37}$. Let $y = \underset{i\in I}{\vee} y_i$; since for every $i \in I$, $y_i \leq y \Rightarrow y \rightarrow x \leq y_i \rightarrow x \Rightarrow y \rightarrow x \leq \underset{i\in I}{\wedge}(y_i \rightarrow x)$.

Conversely, $\underset{i\in I}{\wedge}(y_i \rightarrow x) \leq y \rightarrow x \Leftrightarrow y \odot [\underset{i\in I}{\wedge}(y_i \rightarrow x)] \leq x$.

By r-c$_{35}$ we have $y \odot [\underset{i\in I}{\wedge}(y_i \rightarrow x)] \leq \underset{i\in I}{\wedge}[y \odot (y_i \rightarrow x)] \leq$ $\leq \underset{i\in I}{\wedge}[y_i \odot (y_i \rightarrow x)] \leq \underset{i\in I}{\wedge} x = x$, so we obtain the requested equality.

The others subpoints of the theorem are immediate. ■

Proposition 5.8.9. If $x, x', y, y', z \in A$, then :

r-c$_{42}$: $x\odot(y\rightarrow z) \leq y\rightarrow(x\odot z) \leq (x\odot y) \rightarrow (x\odot z)$;

r-c$_{43}$: $x\vee y = 1$ implies $x\odot y = x\wedge y$;

r-c$_{44}$: $x\rightarrow (y\rightarrow z) \geq (x\rightarrow y) \rightarrow (x\rightarrow z)$;

r-c$_{45}$: $x\vee(y\odot z)\geq(x\vee y)\odot(x\vee z)$, hence $x^m\vee y^n \geq (x\vee y)^{mn}$, for any $m, n\geq 1$;

r-c$_{46}$: $(x\rightarrow y) \odot (x'\rightarrow y') \leq (x\vee x') \rightarrow (y\vee y')$;

r-c$_{47}$: $(x\rightarrow y) \odot (x'\rightarrow y') \leq (x\wedge x') \rightarrow (y\wedge y')$.

Proof. **r-c$_{42}$.** The first inequality follows from $x\odot y\odot(y\rightarrow z) \leq x\odot z$ and the second from r-c$_{17}$.

r-c$_{43}$. Suppose $x\vee y = 1$.Clearly $x\odot y\leq x$ and $x\odot y\leq y$.

Let now $t\in A$ such that $t\leq x$ and $t\leq y$. By r-c$_{42}$ we deduce that $t\rightarrow(x\odot y)\geq x\odot(t\rightarrow y) = x\odot 1=x$ and $t\rightarrow(x\odot y)\geq y\odot(t\rightarrow x) = y\odot 1 = y$, so $t\rightarrow(x\odot y)\geq x\vee y=1$, hence $t\rightarrow(x\odot y) = 1\Leftrightarrow t\leq x\odot y$, that is, $x\odot y= x\wedge y$.

r-c$_{44}$. By r-c$_{18}$ we have $x\rightarrow (y\rightarrow z) = (x\odot y) \rightarrow z$ and $(x\rightarrow y) \rightarrow (x\rightarrow z) = [x\odot (x\rightarrow y)]\rightarrow z$. But $x\odot y\leq x\odot(x\rightarrow y)$, so we obtain $(x\odot y)\rightarrow z\geq[x\odot (x\rightarrow y)]\rightarrow z \Leftrightarrow x\rightarrow (y\rightarrow z)\geq(x\rightarrow y) \rightarrow (x\rightarrow z)$.

r-c$_{45}$. By r-c$_{34}$ we deduce $(x\vee y)\odot(x\vee z) = x^2\vee(x\odot y)\vee(x\odot z)\vee(y\odot z) \leq x\vee (x\odot y)\vee(x\odot z)\vee(y\odot z) = x\vee(y\odot z)$.

r-c$_{46}$. From the inequalities $x\odot(x\rightarrow y)\odot(x'\rightarrow y') \leq x\odot (x\rightarrow y) \leq x\wedge y \leq y\vee y'$ and $x'\odot(x\rightarrow y)\odot(x'\rightarrow y') \leq x'\odot (x'\rightarrow y')\leq x'\wedge y'\leq y\vee y'$ we deduce that $(x\rightarrow y)\odot(x'\rightarrow y') \leq x\rightarrow (y\vee y')$ and $(x\rightarrow y)\odot(x'\rightarrow y') \leq x'\rightarrow (y\vee y')$, so $(x\rightarrow y)\odot(x'\rightarrow y') \leq (x\rightarrow (y\vee y'))\wedge(x'\rightarrow (y\vee y')) = (x\vee x') \rightarrow (y\vee y')$.

r-c$_{47}$. As in the case of r-c$_{46}$. ■

If $B = \{a_1, a_2, \ldots, a_n\}$ is a finite subset of A we denote

$$\prod B = a_1\odot a_2\odot\ldots\odot a_n.$$

Proposition 5.8.10. ([5],[7]). Let $A_1, A_2, \ldots, A_n$ finite subsets of A .

r-c$_{48}$: If $a_1\vee a_2\vee \ldots\vee a_n = 1$, for all $a_i\in A_i$, $i\in\{1,2,\ldots,n\}$, then $(\prod A_1) \vee \ldots\vee (\prod A_n) = 1$.

Proof. For n=2 it is proved in [5] and for n=2, A_1 a singleton and A_2 a doubleton in [7]. The proof for an arbitrary n is a simple mathematical induction argument. ■

Corollary 5.8.11. Let $a_1, a_2, \ldots, a_n \in A$.

r-c$_{49}$: If $a_1\vee a_2\vee \ldots\vee a_n = 1$, then $a_1^k \vee a_2^k \vee \ldots\vee a_n^k = 1$ for every natural number k .

Proposition 5.8.12. Let $x, y_1, y_2, z_1, z_2 \in A$.
If $x \leq y_1 \leftrightarrow y_2$ and $x \leq z_1 \leftrightarrow z_2$, then $x^2 \leq (y_1 \rightarrow z_1) \leftrightarrow (y_2 \rightarrow z_2)$.

Proof. From $x \leq y_1 \leftrightarrow y_2 \Rightarrow x \leq y_2 \rightarrow y_1 \Rightarrow x \odot y_2 \leq y_1$ and analogously we deduce that $x \odot z_1 \leq z_2$.

Then $x \odot x \leq (y_1 \rightarrow z_1) \rightarrow (y_2 \rightarrow z_2) \Leftrightarrow x \odot x \odot (y_1 \rightarrow z_1) \leq (y_2 \rightarrow z_2) \Leftrightarrow x \odot x \odot (y_1 \rightarrow z_1) \odot y_2 \leq z_2$.

Indeed, $x \odot x \odot (y_1 \to z_1) \odot y_2 \leq x \odot (y_1 \to z_1) \odot y_1 \leq x \odot z_1 \leq z_2$ and analogously $x \odot x \leq (y_2 \to z_2) \to (y_1 \to z_1)$, therefore we obtain the requested inequality. ■

Proposition 5.8.13. Suppose that A is complete and $x, x_i, y_i \in A$ ($i \in I$). If for every $i \in I$, $x \leq x_i \leftrightarrow y_i$, then $x \leq (\bigwedge_{i\in I} x_i) \leftrightarrow (\bigwedge_{i\in I} y_i)$.

Proof. Since $x \leq x_i \to y_i$ for every $i \in I$, we deduce that $x \odot x_i \leq y_i$ and then $x \odot (\bigwedge_{i\in I} x_i) \leq \bigwedge_{i\in I} (x \odot x_i) \leq (\bigwedge_{i\in I} y_i)$, hence $x \leq (\bigwedge_{i\in I} x_i) \to (\bigwedge_{i\in I} y_i)$.

Analogously, $x \leq (\bigwedge_{i\in I} y_i) \to (\bigwedge_{i\in I} x_i)$, therefore we obtain the requested inequality. ■

Taking as a guide line the case of BL-algebras ([81]) , a residuated lattice A will be called *G-algebra* if $x^2 = x$ for every $x \in A$.

Remark 5.8.14. In a G-algebra A , $x \odot y = x \wedge y$ for any $x,y \in A$.

Proposition 5.8.15. In a residuated lattice A the following assertions are equivalent :
(i) A is a G-algebra
(ii) $x \odot (x \to y) = x \odot y = x \wedge y$ for any $x,y \in A$.

Proof. (i)⇒(ii). Let $x, y \in A$. By r-c_{42} we have $x \odot (x \to y) \leq$ $\leq (x \odot x) \to (x \odot y) \Leftrightarrow x \odot (x \to y) \leq x \to (x \odot y) \Leftrightarrow x \to y \leq x \to (x \to (x \odot y)) =$ $= x^2 \to (x \odot y) = x \to (x \odot y) \Rightarrow x \odot (x \to y) \leq x \odot y$.

Since $y \leq x \to y$, then $x \odot y \leq x \odot (x \to y)$, so $x \odot (x \to y) = x \odot y$.

Clearly $x \odot y \leq x, y$. To prove $x \odot y = x \wedge y$, let $t \in A$ such that $t \leq x$ and $t \leq y$. Then $t^2 \leq x \odot y$, that is, $x \odot y = x \wedge y$.

(ii)⇒(i).In particular for x=y we obtain $x \odot x = x \wedge x = x \Leftrightarrow x^2 = x$. ■

Proposition 5.8.16.For a residuated lattice $(A, \vee, \wedge, \odot, \to, 0, 1)$ the following assertions are equivalent :
(i) $(A, \to, 1)$ is a Hilbert algebra;
(ii) A is a G-algebra.

Proof. (i)⇒(ii). Suppose that (A, 1) is a Hilbert algebra, then for every x, y, z ∈ A we have $x\to(y\to z) = (x\to y)\to(x\to z)$.

But by r-c_{18}, $x\to(y\to z) = (x\odot y)\to z$ and $(x\to y)\to(x\to z) = (x\odot(x\to y))\to z$, hence $x\odot y = x\odot(x\to y)$, so we obtain $(x\odot y)\to z = (x\odot(x\to y))\to z$, so $x\odot y = (x\odot(x\to y))$; for x = y we obtain $x^2 = x$, that is, A is a G-algebra.

(ii)⇒(i).Follows from Proposition 5.8.13. ■

5. 9. Boolean center of a residuated lattice

If $(L,\wedge,\vee,0,1)$, is a bounded lattice, we recall (see Chapter 2) that an element a ∈ L is called *complemented* if there is an element b ∈ L such that $a\wedge b = 0$ and $a\vee b = 1$; if such element exists it is called a complement of a . We will denote $b = a'$ and the set of all complemented elements in A by **B**(A).Complements are, generally, not unique, unless the lattice is distributive (see Lemma 2.6.2).

In residuated lattices, altough the underlying lattices need not be distributive (see Remark 5.7.4.(3)), the complements are unique.

Lemma 5.9.1.([55]) Suppose that a ∈ A has a complement b∈ A.Then the following hold :

(i) If c is another complement of a in A , then c = b ;
(ii) a´ = b and b´ = a ;
(iii) $a^2 = a$.

Lemma 5.9.2. If e∈ B(A) , then e´ = e* and e = e .**

Proof. If e∈**B**(A), and we denote a = e´, then $e\vee a = 1$ and $e\wedge a = 0$. Since $e\odot a \le e\wedge a = 0$, then $e\odot a = 0$, hence $a\le e\to 0 = e^*$.

On the other hand, $e^* = 1 = 1\odot e^* = (e\vee a)\odot e^* = (e\odot e^*)\vee(a\odot e^*) = 0\vee(a\odot e^*) = a\odot e^*$, hence $e^*\le a$, that is, $e^*=a$.

The equality $e^{**} = e$ follows from Lemma 5.9.1,(ii). ■

Remark 5.9.3.([55]). If e, f ∈ **B**(A), then $e\wedge f$, $e\vee f$ ∈ **B(**A).

Morover, $(e\vee f)' = e'\wedge f'$ and $(e\wedge f)' = e'\vee f'$.

So, $e\to f = e'\vee f$ ∈ **B**(A) and

r-c$_{50}$: $e\odot x = e\wedge x$, for every x∈ A.

Corollary 5.9.4. ([55]).The set B(A) is the universe of a Boolean subalgebra of A.

Proposition 5.9.5. For e∈A the following are equivalent :

(i) e∈B(A);

(ii) e∨e* = 1 .

Proof. (i)⇒(ii). If e∈**B**(A), by Lemma 5.9.2, e∨e´ = e∨e* = 1.

(ii)⇒(i). Suppose that e∨e*=1. We have 0 = 1* = (e∨e*)* = e*∧e** ≥ ≥e∧ e*, hence e∧ e*=0, that is e∈**B**(A) ■

Proposition 5.9.6. For e∈A we consider the following assertions :

(1) e∈B(A);
(2) e^2 = e and e = e;**
(3) e^2 = e and e* → e = e;
(4) (e→x)→e = e for every x ∈ A;
(5) e∧ e* = 0 .
Then :

(i) (1)⇒(2), (3), (4) and (5);

(ii) (2)⇏(1), (3) ⇏ (1), (4) ⇏ (1), (5) ⇏ (1).

Proof. (i). (1)⇒(2).Follows from Lemma 5.9.1,(iii) and Lemma 5.9.2.

(1)⇒(3). If e∈**B**(A), then e∨e*=1. Since 1=e∨e*≤[(e→e*) →e*]∧[(e*→ e) →e] we deduce that (e→ e*) →e*=(e*→e)→e=1, hence e→ e*≤ e* and e*→ e ≤ e, that is, e→ e*= e* and e*→ e = e.

(1)⇒(4). If x∈A, then from 0≤x we deduce e*≤e→ x hence (e→ x) → e≤ e*→ e=e, by (1)⇒(3). Since e≤(e→ x) → x we obtain (e→x)→e = e.

(1)⇒(5). Follows from Lemma 5.9.2.

(ii). Consider the residuated lattice A = {0, a, b, c, 1} from Remark 5.7.4 (2).; it is easy to verify that **B**(A) = {0,1}.

(2)⇏(1). We have a^2=a,a*=b, b*=a,hence a**=b*=a, but a∉**B**(A).

(3)⇏(1). We have a^2=a and a*→a = b→a=a, but a∉**B**(A).

(4)⇏(1). It is easy to verify that (a→x) →a = a for every x∈A, but a∉**B**(A).

(5)$\nRightarrow$(1).We have $a\wedge a^* = a\wedge b = 0$, but $a\vee a^* = a\vee b = c\neq 1$, hence $a\notin \mathbf{B}(A)$. ■

Remark 5.9.7. ([81]). If A is a BL-algebra , then all assertions (1)-(5) from the above proposition are equivalent.

Proposition 5.9.8. If e, f $\in$ B(A) and x, y $\in$A, then :

r-c$_{51}$: $x\odot (x\to e) = e\wedge x$, $e\odot (e\to x) = e\wedge x$;

r-c$_{52}$: $e\vee (x\odot y) = (e\vee x) \odot (e\vee y)$;

r-c$_{53}$: $e\wedge (x\odot y) = (e\wedge x) \odot (e\wedge y)$;

r-c$_{54}$: $e\odot (x\to y) = e\odot [(e\odot x) \to (e\odot y)]$;

r-c$_{55}$: $x\odot (e\to f) = x\odot [(x\odot e) \to (x\odot f)]$

r-c$_{56}$: $e\to (x\to y) = (e\to x) \to (e\to y)$.

Proof. **r-c$_{51}$.** Since $e\leq x\to e$, then $x\odot e\leq x\odot(x\to e)$, hence $x\wedge e\leq x\odot(x\to e)$. From $x\odot(x\to e)\leq x$, e we deduce the other inequality $x\odot(x\to e) \leq x\wedge e$, so $x\odot(x\to e) = x\wedge e$. Analogously for the second equality.

r-c$_{52}$. We have $(e\vee x)\odot(e\vee y) = [(e\vee x)\odot e]\vee[(e\vee x)\odot y] = [(e\vee x)\odot e]\vee[(e\odot y)\vee(x\odot y)] = [(e\vee x)\wedge e]\vee[(e\odot y)\vee(x\odot y)] =$

$= e\vee(e\odot y)\vee (x\odot y) = e\vee(x\odot y)$.

r-c$_{53}$. As above, $(e\wedge x)\odot(e\wedge y) = (e\odot x)\odot(e\odot y) = (e\odot e)\odot(x\odot y) = e\odot(x\odot y) = e\wedge(x\odot y)$.

The rest of rules **r-c$_{54-56}$** are left for the reader.■

5. 10. Deductive systems of a residuated lattice

In this section we put in evidence the congruences of a residuated lattice and characterize the subdirectly irreducible residuated lattices.

Definition 5.10.1. Let A be a residuated lattice. A non-empty subset F$\subseteq$ A will be called *implicative filter* if

Lr$_4$: For every x, y $\in$ A with $x\leq y$, $x\in F \Rightarrow y\in F$;

Lr$_5$: If $x, y \in F \Rightarrow x \odot y \in F$.

We remark that an implicative filter of A is a filter for the underlying lattice $L(A) = (A, \vee, \wedge)$, but the converse is not true (see [81]).

Remark 5.10.2. ([81]). If A is a residuated lattice, then a non-empty subset $F \subseteq A$ is an implicative filter iff

Lr_6: $1 \in F$;

Lr_7: $x, x \to y \in F \Rightarrow y \in F$.

Following Remark 5.10.2 an implicative filter will be called *deductive system* (**ds** on short). So ,to avoid confusion we reserve,howewer,the name filter to lattice filters in this book.

For a residuated lattice A we denote by **Ds**(A) the set of all deductive systems (implicative filters) of A.

Clearly, {1}, A $\in$ **Ds**(A) and any intersection of deductive systems is also a deductive system.

In what follows we will take into consideration the connections between the congruence of a residuated lattice A and the deductive systems of A.

For D$\in$ **Ds**(A) we denote by θ_D the binary relation on A:

$$(x, y) \in \theta_D \Leftrightarrow x \to y, y \to x \in D.$$

For a congruence ρ on A (that is, $\rho \in$ **Con**(A) - see Chapter 3) we denote

$$D_\rho = \{x \in A: (x, 1) \in \rho\}.$$

As in the case of lattices we have the following result:

Theorem 5.10.3.Let A be a residuated lattice,D$\in$Ds(A) and $\rho\in$Con (A). Then :

(i) $\theta_D \in$ Con (A) and $D_\rho \in$Ds(A);

(ii) The assignments D ⇝ θ_D and ρ ⇝ D_ρ give a latticeal isomorphism between Ds(A) and Con(A).

For D$\in$ **Ds**(A) and a$\in$A let a/D the equivalence class of a modulo θ_D.If we denote by A/D the quotient set A/ θ_D , then A/D becomes a residuated lattice with the natural operations induced from those of A (see Chapter 3).Clearly, in A/D, **0** = 0/D and **1=** 1/D .

The following result is immediate:

Proposition 5.10.4. Let D$\in$ Ds(A),and a,b$\in$A,then :

(i) a/D = 1/D iff a $\in$D , hence a/D $\neq$ 1 iff a$\notin$ D ;

(ii) a/D = 0/D iff a* $\in$D;

(iii) If D is proper and a/D = 0 , then a$\notin$ D;

(iv) a/D$\leq$ b/D iff a$\to$b$\in$D.

It follows immediately from the above , that a residuated lattice A (see and Chapter 3) is subdirectly irreducible iff it has the second smallest **ds**,i.e,the smallest ds among all **ds** except {1}.The next theorem characterises internally subdirectly irreducible and simple residuated lattices.

Theorem 5.10.5. ([55]) A residuated lattice A is

(i) ***subdirectly irreducible*** **(*si* on short) iff there exists an element a $<$ 1 such that for any x$<$1 there exists a natural number n $\geq$ 1 such that $x^n \leq$ a ;**

(ii) ***simple*** **iff a can be taken to be 0 .**

Proposition 5.10.6. ([55]) In any *si* residuated lattice, if x$\vee$y = 1, then x =1 or y =1 holds.

Therefore, every **si** residuated lattice has at most one *co-atom* (see Chapter 2). The next result characterises these **si** residuated lattices which have co-atoms.

Theorem 5.10.7.([55]) A residuated lattice A has the unique co-atom iff there exists an element a$<$1 and a natural number n such that $x^n \leq$a holds for any x<1.

Directly indecomposable residuated lattices also have quite a handly description.It was obtained for a subvariety of residuated lattices, called *product algebras* .

For arbitrary residuated lattices we have :

Theorem 5.10.8. ([55]) A nontrivial residuated lattice A is directly indecomposable iff B(A) = {0, 1}.

5.11. The lattice of deductive systems of a residuated lattice

In this section we present new results relative to lattice of deductive systems of a residuated lattice.We also characterize the residuated lattices for which the lattice of deductive systems is a Boolean algebra.

For a non-empty subset X of a residuated lattice A we denote by <X> the *deductive system of A generated by X* (that is, < X > = $\cap$ {D$\in$ **Ds(**A**)** : X $\subseteq$ D}).

For D$\in$ **Ds(**A**)** and a $\in$ A we denote by D(a) = < D $\cup$ {a}> .

Proposition 5.11.1. If X $\subseteq$ A is a non-empty subset,then <X> = {x $\in$ A : x $\geq$ $x_1 \odot \ldots \odot x_n$, with $x_1, \ldots, x_n \in$ X}.

Proof. If we denote by $\overline{X}$ the set from the right part of the equality from the enounce, it is immediate that this is an implicative filter which contains the set

X, hence $<X> \subseteq \overline{X}$. Now let D∈**Ds**(A) such that X ⊆ D and $x \in \overline{X}$. Then there are $x_1, \ldots, x_n \in X$ such that $x \geq x_1 \odot \ldots \odot x_n$. Since $x_1, \ldots, x_n \in D \Rightarrow x_1 \odot \ldots \odot x_n \in D \Rightarrow x \in D$, hence $\overline{X} \subseteq D$; we deduce that $\overline{X} \subseteq \cap D = <X>$, that is, $<X> = \overline{X}$.
■

Corollary 5.11.2. Let a∈A,D,D_1,D_2∈ Ds(A).
Then:

(i) $<a> = <\{a\}> = \{x \in L : a^k \leq x$, for every natural number $k\}$;

(ii) $D(a) = \{x \in A : x \geq d \odot a^n$, with $d \in D$ and $n \geq 1\} = \{x \in A : a^n \rightarrow x \in D$, for some $n \geq 1\}$;

(iii) $<D_1 \cup D_2> = \{ x \in A : x \geq d_1 \odot d_2$ for some $d_1 \in D_1$ and $d_2 \in D_2\}$;

(iv) (Ds(A),⊆) is a complete lattice, where, for a family $(D_i)_{i \in I}$ of deductive systems, $\bigwedge_{i \in I} D_i = \bigcap_{i \in I} D_i$ and $\bigvee_{i \in I} D_i = < \bigcup_{i \in I} D_i >$.

Proposition 5.11.3. If a, b∈A, then
(i) $<a> = [a)$ iff $a^2 = a$;
(ii) $a \leq b$ implies $<a> \subseteq <b>$;
(iii) $<a> \cap <b> = <a \vee b>$;
(iv) $<a> \vee <b> = <a \wedge b> = <a \odot b>$;
(v) $<a> = \{1\}$ iff a = 1 .

Proof. (i), (ii). Straightforward.

(iii). Since $a \vee b \leq a, b$, by (ii) $<a \vee b> \subseteq <a>$, $<b>$, hence $<a \vee b> \subseteq <a> \cap <b>$. Let now $x \in <a> \cap <b>$; then $x \geq a^m$, $x \geq b^n$ for some natural numbers m, n ≥ 1, hence $x \geq a^m \vee b^n \geq (a \vee b)^{mn}$, by r-$c_{30}$, so $x \in <a \vee b>$. Hence $<a \vee b> = = <a> \cap <b>$.

(iv). Since $a \odot b \leq a \wedge b \leq a, b$, by (ii), we deduce that $<a>, <b> \subseteq <a \wedge b> \subseteq <a \odot b>$, hence $<a> \vee <b> \subseteq <a \wedge b> \subseteq <a \odot b>$.

For the converse inclusions, let $x \in <a \odot b>$. Then for some natural number $n \geq 1$, $x \geq (a \odot b)^n = a^n \odot b^n \in <a> \vee <b>$ (since $a^n \in <a>$ and $b^n \in <b>$), (by Proposition 5.11.1), hence $x \in <a> \vee <b>$, that is $<a \odot b> \subseteq <a> \vee <b>$, so $<a> \vee <b> = <a \wedge b> = <a \odot b>$.

(v). Obviously. ■

Corollary 5.11.4. If we denote by $Ds_p(A)$ the family of all principal ds of A, then $Ds_p(A)$ is a bounded sublattice of Ds(A).

Proof. Apply Proposition 5.11.3, (iii), (iv) and the fact that {1} = <1> ∈ $\mathbf{Ds_p(A)}$ and A = <0> ∈ $\mathbf{Ds_p(A)}$. ■

Propostion 5.11.5. The lattice (Ds(A), ⊆) is a complete Brouwerian lattice (hence distributive), the compact elements being exactly the principal ds of A .

Proof. Clearly, if $(D_i)_{i\in I}$ is a family of **ds** from A, then the infimum of this family is $\wedge_{i\in I} D_i = \bigcap_{i\in I} D_i$ and the supremum is $\vee_{i\in I} D_i =< \bigcup_{i\in I} D_i >$ = {x ∈ A : x ≥ x_{i_1} ⊙ …⊙ x_{i_m} , where $i_1, \dots, i_m \in I$, $x_{i_j} \in D_{i_j}$, 1 ≤ j ≤ m}, that is **Ds(**A**)** is complete.

We will prove that the compacts elements of **Ds(**A**)** are exactly the principal filters of A. Let D be a compact element of **Ds(**A**)**. Since D = $\vee_{a\in D} < a >$, there are m ≥ 1, and $a_1, \dots, a_m \in A$, such that D = $<a_1> \vee \dots \vee <a_m> = < a_1 \odot \dots \odot a_m >$, by Proposition 5.11.3, (iv). Hence D is a principal ds of A.

Conversely, let a ∈ A and $(D_i)_{i\in I}$ be a family of ds of A such that <a> ⊆ $\vee_{i\in I} D_i$. Then a ∈ $\vee_{i\in I} D_i =< \bigcup_{i\in I} D_i >$, so we deduce that are m ≥ 1, $i_1, \dots, i_m \in I$, $x_{i_j} \in D_{i_j}$ (1 ≤ j ≤ m) such that a ≥ $x_{i_1} \odot \dots \odot x_{i_m}$.

It follows that a ∈ $< D_{i_1} \cup \dots \cup D_{i_m} >$, so <a> ∈ $< D_{i_1} \cup \dots \cup D_{i_m} > = D_{i_1} \vee \dots \vee D_{i_m}$.

For any ds D we have D = $\vee_{a\in D} < a >$, so the lattice **Ds(**A**)** is algebraic.

In order to prove that **Ds(**A) is Brouwerian we must show that for every ds D and every family $(D_i)_{i\in I}$ of ds, $D \wedge (\vee_{i\in I} D_i) = \vee_{i\in I}(D \wedge D_i) \Leftrightarrow D \cap (\vee_{i\in I} D_i) =< \bigcup_{i\in I}(D \cap D_i) >$.

Clearly, $D \cap (\vee_{i\in I} D_i) \supseteq < \bigcup_{i\in I}(D \cap D_i) >$.

Let now x ∈ $D \cap (\vee_{i\in I} D_i)$. Then x ∈ D and there exist $i_1, \dots, i_m \in I$, $x_{i_j} \in D_{i_j}$ (1 ≤ j ≤ m) such that x ≥ $x_{i_1} \odot \dots \odot x_{i_m}$. Then x = x ∨ ($x_{i_1} \odot \dots \odot x_{i_m}$) ≥ (x∨ x_{i_1})⊙ …⊙ (x ∨ x_{i_m}), by lr-c_{30}. Since x ∨ $x_{i_j} \in D \cap D_{i_j}$ for every 1 ≤ j ≤ m, we deduce that x ∈ $\bigcup_{i\in I}(D \cap D_i)$, hence $D \cap (\vee_{i\in I} D_i) \subseteq < \bigcup_{i\in I}(D \cap D_i) >$, that is $D \cap (\vee_{i\in I} D_i) =< \bigcup_{i\in I}(D \cap D_i) >$. ■

For $D_1, D_2 \in$ **Ds**(A) we define $D_1 \to D_2 = \{x \in A : [x) \cap D_1 \subseteq D_2\}$.

Lemma 5.11.6. If A is a Hilbert algebra and $D, D_1, D_2 \in$ Ds(A), then
(i) $D_1 \to D_2 \in$ Ds (A) ;
(ii) $D_1 \cap D \subseteq D_2$ iff $D \subseteq D_1 \to D_2$.

Proof. (i).Since $<1> = \{1\}$ and $<1> \cap D_1 \subseteq D_2$, we deduce that $1 \in D_1 \to D_2$.

Let $x, y \in A$ such that $x \le y$ and $x \in D_1 \to D_2$, that is $<x> \cap D_1 \subseteq D_2$. Then $<y> \subseteq <x>$, so $<y> \cap D_1 \subseteq <x> \cap D_1 \subseteq D_2$, hence $<y> \cap D_1 \subseteq D_2$, that is $y \in D_1 \to D_2$.

To prove that Lr_5 is verified , let $x, y \in A$ such that $x, y \in D_1 \to D_2$, hence $<x> \cap D_1 \subseteq D_2$ and $<y> \cap D_1 \subseteq D_2$. We deduce $(<x> \cap D_1) \vee (<y> \cap D_1) \subseteq D_2$, hence by Proposition 5.11.5, $(<x> \vee <y>) \cap D_1 \subseteq D_2$.

By Proposition 5.11.3 we deduce that $<x \odot y> \cap D_1 \subseteq D_2$, hence $x \odot y \in D_1 \to D_2$, that is $D_1 \to D_2 \in$ **Ds**(A)**.**

(ii). Suppose $D \cap D_1 \subseteq D_2$ and let $x \in D$. Then $<x> \subseteq D$, hence $<x> \cap D_1 \subseteq D \cap D_1 \subseteq D_2$, so $x \in D_1 \to D_2$, that is $D \subseteq D_1 \to D_2$.

Suppose $D \subseteq D_1 \to D_2$ and let $x \in D \cap D_1$. Then $x \in D$, hence $x \in D_1 \to D_2$, that is $<x> \cap D_1 \subseteq D_2$. Since $x \in <x> \cap D_1 \subseteq D_2$ we obtain $x \in D_2$, that is $D \cap D_1 \subseteq D_2$. ■

For $D_1, D_2 \in$ **Ds**(A) we denote $D_1 * D_2 = \{x \in A : x \vee y \in D_2$ for all $y \in D_1\}$.

Proposition 5.11.7. For all $D_1, D_2 \in$ Ds(A), $D_1 * D_2 = D_1 \to D_2$.

Proof. Let $x \in D_1 * D_2$ and $z \in <x> \cap D_1$, that is, $z \in D_1$ and $z \ge x^n$ for some natural $n \ge 1$. Then $x \vee z \in D_2$. Since $z = x^n \vee z \ge (x \vee z)^n$, by r-$c_{30}$, we deduce that $z \in D_2$, hence $x \in D_1 \to D_2$, so $D_1 * D_2 \subseteq D_1 \to D_2$.

For converse inclusion, let $x \in D_1 \to D_2$. Thus $<x> \cap D_1 \subseteq D_2$, so if $y \in D_1$, then $x \vee y \in <x> \cap D_1$, hence $x \vee y \in D_2$.

We deduce that $x \in D_1 * D_2$, so $D_1 \to D_2 \subseteq D_1 * D_2$ we deduce that $D_1 * D_2 = D_1 \to D_2$. ■

Remark 5.11.8. From Lemma 5.10.6 we deduce that (**Ds**(A),$\vee$, $\wedge$, {1}, A) is a Heyting algebra, where for D $\in$ **Ds**(A), D* = D $\to$ 0 = D $\to$ {1} = {x $\in$ A : x$\vee$y = 1 for every y $\in$ D}, so, for a $\in$ A, <a>* = {x $\in$ A: x $\vee$ a = 1}.

Proposition 5.11.9. If x,y $\in$ A , then <x>*$\cap$<y>* = <x$\odot$y>*.

Proof. If a $\in$ <x$\odot$y>*, then a $\vee$ (x$\odot$y) = 1. Since x$\odot$y $\leq$ x, y then a $\vee$ x = 1 and a $\vee$ y = 1, hence a $\in$ <x>*$\cap$<y>*, that is <x$\odot$y>* $\subseteq$ <x>*$\cap$<y>*.

Let now a $\in$ <x>*$\cap$<y>*, that is a $\vee$ x = 1 and a $\vee$ y = 1.

By r-c_{30} we deduce that a $\vee$ (x $\odot$ y) $\geq$ (a $\vee$ x) $\odot$ (a $\vee$ y) = 1, hence a $\vee$ (x $\odot$ y) = 1, that is a $\in$ <x$\odot$y>*. It follows that <x>*$\cap$<y>* $\subseteq$ <x$\odot$y>*, hence <x>*$\cap$<y>* = <x$\odot$y>*. ■

Theorem 5.11.10. The following assertions are equivalent:

(i) (Ds(A),$\vee$, $\wedge$, *, {1}, A) is a Boolean algebra ;

(ii) Every ds of A is principal and for every a$\in$A there exists n$\geq$1 such that a$\vee$(a^n) * = 1 .

Proof. (i) $\Rightarrow$(ii). Let D $\in$ **Ds**(A); since **Ds**(A) is a Boolean algebra, then D $\vee$ D* = A. So, for 0 $\in$ A, there exist a $\in$ D and b $\in$ D* such that a $\odot$ b = 0.

Since b $\in$ D*, by Remark 5.11.8, it follows that a$\vee$b = 1.

By r-c_{28} we deduce that a $\wedge$ b = a $\odot$ b = 0, that is b is the complement of a in L(A). Hence a, b $\in$ **B**(A) = **B**(L(A)).

If x $\in$ D, since b $\in$ D*, we have b$\vee$x = 1. Since a = a $\wedge$ (b$\vee$x) = (a $\wedge$ b) $\vee$ (a$\wedge$x) = a$\wedge$x we deduce that a $\leq$ x, that is, D = <a>. Hence every ds of A is principal.

Let now x $\in$ A; since **Ds**(A) is a Boolean algebra, then <x> $\vee$ <x>* = A $\Leftrightarrow$ <x>* (x) = A $\Leftrightarrow$ {a $\in$ A : a $\geq$ c $\odot$ x^n, with c $\in$ <x>* and n $\geq$ 1} = A .

So, since 0 $\in$ A, there exist c $\in$ <x>* and n $\geq$ 1 such that c $\odot$ x^n = 0. Since c $\in$ <x>*, then x$\vee$c = 1. By r-c_{15} , from c $\odot$ x^n = 0 we deduce c $\leq$ (x^n)*. So 1= x$\vee$c $\leq$ x $\vee$ (x^n)*, hence x $\vee$ (x^n)* = 1 .

(ii) ⇒(i). By Remark 5.11.8, **Ds**(A) is a Heyting algebra. To prove that **Ds**(A) is a Boolean algebra, we must show that for D ∈ **Ds**(A), D* ={1} only for D=A.By hypothesis, every **ds** of A is principal, so we have a∈A such that D = <a>.

Also, by hypothesis, for a ∈ A, there is n ≥ 1 such that $a \vee (a^n)^* = 1$.

By Remark 5.11.8, $(a^n)^* \in <a>^* = \{1\}$, hence $(a^n)^* = 1$, that is $a^n = 0$. We deduce that 0 ∈ D, hence D = A. ■

5.12. The Spectrum of a residuated lattice

This section contains new characterization for meet-irreducible and completely meet-irreducible **ds** of a residuated lattice A (see Definition 2.3.12).

Lemma 5.12.1. Let D ∈ Ds(A) and a, b∈ A such that a∨b∈ A.
Then D(a)∩D(b) = D.

Proof. Clearly, D⊆D(a)∩D(b). To prove converse inclusion, let x∈ D(a)∩D(b). Then there are $d_1, d_2 \in D$ and m, n ≥ 1 such that $x \geq d_1 \odot a^m$ and $x \geq d_2 \odot b^n$. Then $x \geq (d_1 \odot a^m) \vee (d_2 \odot b^n) \geq (d_1 \vee d_2) \odot (d_1 \vee b^n) \odot (d_2 \vee a^m) \odot (a \vee b)^{mn}$, hence x ∈ D, that is, D(a)∩D(b) ⊆ D, so we obtain the desired equality.
■

Corollary 5.12.2. For D∈ Ds(A) the following are equivalent :
(i) If $D = D_1 \cap D_2$, with $D_1, D_2 \in$ Ds(A), then $D = D_1$ or $D = D_2$;
(ii) For a, b∈A, if a∨b∈ D, then a∈D or b∈D.

Proof. (i) ⇒(ii). If a, b ∈ A such that a ∨ b ∈ D, then by Proposition 5.12.1, D(a)∩D(b) **=** D, hence D = D(a) or D = D(b), hence a ∈ D or b ∈ D.

(ii) ⇒(i). Let $D_1, D_2 \in$ **Ds**(A) such that $D = D_1 \cap D_2$. If by contrary, $D \neq D_1$ and $D \neq D_2$ then there are $a \in D \setminus D_1$ and $b \in D \setminus D_2$. If denote c = a ∨ b, then $c \in D_1 \cap D_2 = D$, hence a ∈ D or b ∈ D, a contradiction. ■

Definition 5.12.3. We say that P∈ Ds(A) is *prime* if P≠A and P verifies one of the equivalent assertions from Corollary 5.12.2.

Remark 5.12.4. Following Corollary 5.12.2, $P\in$ **Ds**(A), $P\neq A$, is prime iff P is a proper meet-irreducible element in the lattice (**Ds**(A), $\subseteq$).

We denote by **Spec**(A) the set of all prime **ds** of A.**Spec**(A) will be called the *spectrum* of A .

Theorem 5.12.5. (*Prime ds theorem*). If $D\in$Ds(A) and I is an ideal of the lattice L(A) such that $D\cap I = \emptyset$, then there exists a prime ds P of A such $D\subseteq P$ and $P\cap I = \emptyset$.

Proof. Let $F_D = \{D' \in Ds(A) : D\subseteq D'$ and $D'\cap I = \emptyset\}$. A routine application of Zorn's lemma shows that F_D has a maximal element P. Suppose by contrary that P is not a prime **ds**, that is, there are a, $b\in A$ such that $a\vee b\in P$ but $a\notin P$ and $b\notin P$. By the maximality of P we deduce that P(a), $P(b)\notin F_D$, hence $P(a)\cap I\neq\emptyset$ and $P(b)\cap I\neq\emptyset$.

Then there are $p_1\in P(a)\cap I$ and $p_2\in P(b)\cap I$.

By Corollary 5.11.2, $p_1\geq f\odot a^m$ and $p_2\geq g\odot b^n$, with f, $g\in P$ and m, n natural numbers.Then $p_1\vee p_2 \geq (f\odot a^m)\vee(g\odot b^n)\geq(f\vee g)\odot(g\vee a^m)\odot(f\vee b^n)\odot(b^n\vee a^m) \geq (f\vee g)\odot(g\vee a^m)\odot(f\vee b^n)\odot(a\vee b)^{m.n}$.

Since $f\vee g$, $g\vee a^m$, $f\vee b^n$, $a\vee b\in P$ we deduce that $p_1\vee p_2 \in P$; but $p_1\vee p_2 \in I$, hence $P\cap I \neq \emptyset$, a contradiction. Hence P is a prime **ds.** ■

Corollary 5.12.6. (i) If A is a non-trivial, then every proper ds of A can be extended to a prime ds;

(ii) If $D\in$Ds(A) is proper and $a\in A\setminus D$,then there exists $P\in$Spec(A) such that $D\subseteq P$ and $a\notin P$;

(iii) If $a\in A$, $a \neq 0$, then there exists $P\in$Spec(A) such that $a\in P$;

(iv) Every proper ds D of A is the intersection of all prime ds which contain D;

(v) $\cap$Spec(A) = {1}.

Proof. (i).It is an immediate consequence of Theorem 5.12.5.

(ii).Consider $I = (a]$ .The condition $a\in A\setminus D$ is equivalent with $D\cap I = \emptyset$, so we can apply Theorem 5.12.5.

(iii).Consider $D = \langle a\rangle$, $I = \{0\}$ and apply Theorem 5.12.5.

(iv). Let $D' = \{P \in$ **Spec**(A): $D \subseteq P\}$; clearly $D \subseteq D'$.

To prove another inclusion we shall prove the inclusion of the complementaries.If a ∉ D, then by (iii) there is P∈ **Spec(**A) such that D⊆P and a ∉ P. There results that a∉{P ∈ **Spec**(A): D ⊆ P} = D′, so a ∉ D′, hence D′⊆ D, that is, D = D′.

(v). Straightforward. ■

Examples

1. Consider the example from Remark 5.8.4 (1) of residuated lattice A = [0, 1] which is not a BL-algebra. If x ∈ [0, 1] , x > $\frac{1}{4}$, then x + x > $\frac{1}{2}$, hence x ⊙ x = x ∧ x = x, so <x> = [x) = [x, 1]. If a, b ∈ A = [0, 1] and a ∨ b ∈<x> = [x, 1], then a ∨ b = max{a, b} ≥ x, hence a ≥ x or b ≥ x. So, a ∈ <x> or b ∈ <x>, that is, <x> ∈ **Spec**(A).

2. Consider the residuated lattice A = {0, a, b, c, 1} from Remark 5.8.4 (2). It is immediate that **Ds**(A) = {{1}, {1, c}, {1, a, c}, {1, b, c}, A} and **Spec**(A) = {{1},{1, a, c},{1, b, c}}, since {1,c}= {1, a, c} ∩ {1, b, c}, then {1, c} ∉ **Spec**(A). Since ⊙ = ∧, the **ds** of A coincide with the filters of the associated lattice L(A).

Proposition 5.12.7. For a proper ds P of A we consider the following assertions :

(1) P∈Spec(A);

(2) If a, b∈A, and a∨b =1, then a∈P or b∈P;

(3) For all a, b∈A, a→ b∈P or b→ a∈P;

(4) A/P is a chain;

Then :

(i) (1) ⇒(2) but (2) ⇏ (1);

(ii) (3) ⇒ (1) but (1) ⇏ (3);

(iii) (4) ⇒ (1) but (1) ⇏ (4).

Proof . (i). (1) ⇒(2) is clear by Corollary 5.12.2 (since 1∈D).

(2) ⇏ (1).Consider the residuated lattice A = {0, a, b, c, 1} from Remark 5.8.4 (example 2).Then D = {1, c} ∉ **Spec**(A).Clearly , if x, y∈A and x∨y = 1 , then x = 1 or y = 1, hence x∈D or y∈D , but D ∉ **Spec**(A).

(ii).To prove (3) ⇒ (1) let a,b ∈A such that a∨b∈P.

From r-c_{11} we deduce that $a \vee b \le [(a \to b) \to b] \wedge [(b \to a) \to a]$, hence $(a \to b) \to b$, $(b \to a) \to a \in P$. If $a \to b \in P$, then $b \in P$; if $b \to a \in P$, then $a \in P$, that is, $P \in$ **Spec**(A).

(1) $\not\Rightarrow$ (3). Consider also the residuated lattice A = {0, a, b, c, 1} from Remark 5.8.4 (Example 2). Then $P=\{1\} \in$ **Spec**(A) .We have $a \to b = b \neq 1$ and $b \to a = a \neq 1$, hence $a \to b$ and $b \to a \notin P$.

(iii).To prove (4) $\Rightarrow$ (1), let a, b$\in$A. Since A/P is supposed chain,$a/P \le b/P$ or $b/P \le a/P$ $\Leftrightarrow$(by Proposition 5.10.4.) $a \to b \in P$ or $b \to a \in P$ and we apply (ii).

(1) $\not\Rightarrow$ (4).Consider A as above ; then $P=\{1\} \in$ **Spec**(A) and $A/P \approx A$ is not chain. ■

Remark 5.12.8. If A is a BL-algebra, then all assertions (1)-(4) from Proposition 5.12.7 are equivalent (see [81]).

As in the case of Hilbert algebra (see Theorem 5.3.11) we have :

Theorem 5.12.9. For P$\in$ Ds(A), P$\neq$A, the following assertions are equivalent :

(i) P $\in$ Spec(A)

(ii) For any x, y $\notin$ P there is z $\notin$ P such that x $\le$ z and y $\le$ z.

Theorem 5.12.10. For P $\in$ Ds(A), P$\neq$A, the following are equivalent :

(i) P$\in$Spec(A) ;

(ii) For every H $\in$ Ds (A), H $\to$ P = P or H $\subseteq$ P ;

(iii) If x, y $\in$ A and $\langle x \rangle \cap \langle y \rangle \subseteq$ P, then x $\in$ P or y $\in$ P ;

(iv) For α, β $\in$ A / P, $\alpha \neq 1$, $\beta \neq 1$, there is $\gamma \in$ A / P such that $\gamma \neq 1$ and $\alpha, \beta \le \gamma$.

Proof. (i)$\Rightarrow$(ii). Suppose that P is a prime and let H $\in$ **Ds**(A); since **Ds**(A) is a Heyting algebra, by Theorem 5.1.9. we deduce that $P = (H \to P) \cap ((H \to P) \to P)$.Since P is meet-irreducible, then by Corollary 5.12.2 , $P = P \to H$ or $P = (H \to P) \to P$; in the second case, since $H \subseteq (H \to P) \to P$ we deduce that $H \subseteq P$.

(ii)$\Rightarrow$(i). Let $D_1, D_2 \in$ **Ds**(A) such that $P = D_1 \cap D_2$; then $D_1 \subseteq D_2 \to P$, so, if $D_2 \subseteq P$, then $D_2 = P$ and if $D_2 \to P = P$, then $D_1 = P$.Hence (i)$\Leftrightarrow$(ii).

(i)⇒(iii). Let $x, y \in A$ such that $\langle x\rangle \cap \langle y\rangle \subseteq P$ and suppose that $x \notin P$, $y \notin P$; by Theorem 5.12.9 there is $z \notin P$ such that $x \le z$ and $y \le z$. Then $z \in \langle x\rangle \cap \langle y\rangle \subseteq D$, hence $z \in D$, a contradiction !

(iii)⇒(ii). Let $H \in$ **Ds**(A) such that $H \nsubseteq D$ and we shall prove that $H \to P = P$. Let $x \in H \to P$; then $\langle x\rangle \cap H \subseteq P$ and if $y \in H \setminus P$, then $\langle y\rangle \subseteq H$, hence $\langle x\rangle \cap \langle y\rangle \subseteq \langle x\rangle \cap H \subseteq D$. Since $y \notin P$, we deduce that $x \in P$, hence $H \to P = P$.

(i)⇒(iv). Let $\alpha, \beta \in A / P$, $\alpha \ne 1$, $\beta \ne 1$; then $\alpha = x / P$, $\beta = y / P$ with $x, y \notin P$. By Theorem 5.12.9 there is $z \notin P$ such that $x \le z$ and $y \le z$. If we take $\gamma = z / P \in A / P$, $\gamma \ne 1$ and $\alpha, \beta \le \gamma$, since $x \to z = y \to z = 1 \in P$.

(iv) ⇒ (i). Let $x, y \notin P$; if we take $\alpha = x / P$, $\beta = y / P$, $\alpha, \beta \in A / P$, $\alpha \ne 1$, $\beta \ne 1$, hence there is $\gamma = z / P$, $\gamma \ne 1$, (hence $z \notin P$) such that $\alpha, \beta \le \gamma$.

Thus $x \to z$, $y \to z \in P$.If consider $t = (y \to z) \to ((x \to z) \to z)$, then by r-$c_{11}$,we deduce that $x,y \le t$. Since $z \notin P$, then $t \notin P$,hence $P \in$ **Spec**(A) (by Theorem 5.12.9). ■

Corollary 5.12.11. If D ∈ Spec(A), then in Heyting algebra Ds(A), D is dense or regular element.

Proof. If $H = D^* \in$ **Ds**(A), by Theorem 5.12.10, (ii) we have $D^* \subseteq D$ or $D^* \to D = D$; in the first case we obtain that $D^* \to D = 1$ or $D^{**} = 1$, hence $D^* = 0$, so D is a dense element in **Ds**(A); in the second case we deduce that $D^* \to D = D \Leftrightarrow D^{**} = D$, hence D is a regular element in **Ds**(A). ■

Theorem 5.12.12. If every D ∈ Ds(A) has a unique representation as an intersection of elements from Spec(A), then Ds(A) is a Boolean algebra.

Proof. To prove **Ds**(A) is a Boolean algebra, let $D \in$ **Ds**(A) and consider $D' = \{M \in$ **Spec**(A): $D \nsubseteq M\} \in$ **Ds**(A).

We have to prove that D' is the complement of D in Heyting algebra **Ds**(A).

Clearly $D \cap D' = \{1\}$; if $D \vee D' \ne A$, then by Corollary 5.12.6 there is $D'' \in$ **Spec**(A) such that $D \vee D' \subseteq D''$, hence D has two distinct representation as intersection of elements from **Spec**(A):

$D' = \cap \{M \in$ **Spec**$(A): D \nsubseteq M\}$ and

$D' = D'' \cap (\cap \{M \in$ **Spec**$(A): D \nsubseteq M\})$, a contradiction, hence $D \vee D' = A$, that is, **Ds**(A) is a Boolean algebra. ■

Remark 5.12.13. For the case of lattices with 0 and 1 we have an analogous result of Hashimoto (see [47]).

As an immediate consequence of Zorn's lemma we obtain :

Proposition 5.12.14. If $D \in Ds(A)$ and $a \notin D$, there is a deductive system M_a maximal with the property that $D \subseteq M_a$ and $a \notin M_a$(we say that M_a is *maximal relative to a*).

Theorem 5.12.15. Let $D \in Ds(A)$, $D \neq A$ and $a \in A \setminus D$. Then the following are equivalent :

(i) D is maximal relative to a ;

(ii) $a \notin D$ and ($x \notin D$ implies $x^n \to a \in D$ for some $n \geq 1$).

Proof. (i)⇒(ii). Clearly $a \notin D$. Let $x \in A \setminus D$. If $a \notin D(x)$, since $D \subset D(x)$ then by the maximality of D we deduce that $D(x) = A$, hence $a \in D(x)$, a contradiction !. We deduce that $a \in D(x)$, hence $a \geq d \odot x^n$, with $d \in D$ and $n \geq 1$.

Then $d \leq x^n \to a$, hence $x^n \to a \in D$.

(ii)⇒(i). Suppose by contrary that there is $D' \in$ **Ds**(A), $D' \neq A$ such that $a \notin D'$ and $D \subset D'$. Then there is $x_0 \in D'$ such that $x_0 \notin D$, hence by hypothesis there is $n \geq 1$ such that $x_0^n \to a \in D \subset D'$.

Thus from $x_0^n \to a \in D'$ and $x_0^n \in D'$ we deduce that $a \in D'$, a contradiction ! ■

Theorem 5.12.16. For $D \in Ds(A)$, $D \neq A$ the following assertions are equivalent :

(i) D is completely meet-irreducibile ;

(ii) There is $a \notin D$ such that D is *maximal relative to a* .

Proof. (i)⇒(ii). See [43, p.248] (since by Proposition 5.11.6, **Ds**(A) is an algebraic lattice).

(ii)⇒(i). Let $D\in$**Ds**(A) maximal relative to a and suppose $D = \bigcap_{i\in I} D_i$ with $D_i \in$ **Ds**(A) for every $i\in I$. Since $a\notin D$ there is $j\in I$ such that $a\notin D_j$. So, $a\notin D_j$ and $D\subseteq D_j$.

By the maximality of D we deduce that $D = D_j$, that is, D is completely meet-irreducible ■

Theorem 5.12.17. For $D \in Ds(A)$ the following are equivalent :

(i) D is meet-completely irreducible;

(ii) If $\bigcap_{x\in I\subseteq A} [x) \subseteq D$, then $I \cap D \neq \emptyset$;

(iii) In the set A/D \{1} there exists an element p with the property that for every $\alpha \in A/D\setminus\{1\}$ there is $n \geq 1$ such that $\alpha^n \leq p$.

Proof. (i)⇒(ii). Straightforward.

(ii)⇒(i). Let $D = \bigcap_{i\in I} D_i$ with $D_i \in Ds(A)$ for every $i\in I$, and suppose that for every $i \in I$ there exist $x_i \in D_i \setminus D$. Since $<x_i> \subseteq D_i$ for every $i \in I$, we deduce that $\bigcap_{i\in I} (<x_i>) \subseteq \bigcap_{i\in I} D_i = D$, so, by hypothesis there is $i\in I$ such that $x_i\in D$, a contradiction ! .

(i) ⇒ (iii). By Theorem 5.12.16, D is maximal relative to an element $a \notin D$; hence if denote $p = a / D \in A/D$, $p\neq 1$ (since $a\notin D$) and for every $\alpha = b / D \in A / D$ with $\alpha \neq 1$ (hence $b \notin D$) by Theorem 5.12.15 there is $n \geq 1$ such that $b^n \to a\in D$, that is, $\alpha^n \leq p$.

(iii) ⇒ (i). Let $p = a/D \in A/D \setminus\{1\}$ (that is, $a \notin D$) and $\alpha = b/D\in A/D \setminus\{1\}$ (that is, $b \notin D$).

By hypothesis there is $n \geq 1$ such that $\alpha^n \leq p \Leftrightarrow b^n \to a\in D$.

Then by Theorems 5.12.15 and 5.12.16 we deduce that D is completely meet-irreducible.■

d. Wajsberg algebras

5.13. Definition. Examples. Properties.Rules of calculus

Definition 5.13.1.([81]). An algebra (L, →, * ,1) of type (2, 1, 0) will be called *Wajsberg algebra* if for every x, y, z ∈ L the following axioms are verified:

w_1: $1 \to x = x$;
w_2: $(x \to y) \to [(y \to z) \to (x \to z)] = 1$;
w_3: $(x \to y) \to y = (y \to x) \to x$;
w_4: $(x^* \to y^*) \to (y \to x) = 1$.

A first example of Wajsberg algebra is offered by a Boolean algebra $(L, \vee, \wedge, ', 0, 1)$, where for $x, y \in L$, $x \to y = x' \vee y$.

For more information about Wajsberg algebras,I recommend to the reader the paper [39].

If L is a Wajsberg algebra, on L we define the relation $x \le y \Leftrightarrow x \to y = 1$; it is immediate that we obtain an order on L (called *natural ordering*). Relative to natural ordering ,1 is the greatest element of L.

Theorem 5.13.2. Let L be a Wajsberg algebra and x, y, z ∈L. Then

w-c_1: If $x \le y$, then $y \to z \le x \to z$;

w-c_2: $x \le y \to x$;

w-c_3: If $x \le y \to z$, then $y \le x \to z$;

w-c_4: $x \to y \le (z \to x) \to (z \to y)$;

w-c_5: $x \to (y \to z) = y \to (x \to z)$;

w-c_6: If $x \le y$, then $z \to x \le z \to y$;

w-c_7: $1^* \le x$;
w-c_8: $x^* = x \to 1^*$.

Proof. **w-c_1.** From w_2 we deduce that $x \to y \le (y \to z) \to (x \to z)$; since $x \to y = 1$, then $(y \to z) \to (x \to z) = 1$, hence $y \to z \le x \to z$.

w-c_2. From $y \le 1$ and w-c_1 we deduce that $1 \to x \le y \to x$, hence $x \le y \to x$.

w-c_3. If $x \le y \to z$, then $(y \to z) \to z \le x \to z$. By w_3 we deduce that $(z \to y) \to y \le x \to z$. Since $y \le (z \to y) \to y \Rightarrow y \le x \to z$.

w-c_4. By w_2 we have that $z \to x \le (x \to y) \to (z \to y)$, so by w-$c_3$ we deduce that $x \to y \le (z \to x) \to (z \to y)$.

w-c_5. We have $y \le (z \to y) \to y = (y \to z) \to z$.

By w-c_4 we deduce that $(y \to z) \to z \le (x \to (y \to z)) \to (x \to z)$, hence $y \le (x \to (y \to z)) \to (x \to z)$, therefore $x \to (y \to z) \le y \to (x \to z)$.

Analogously another inequality, from where it follows the required equality.

w-c_6. It follows immediately from w-c_4.

w-c$_7$. We have $x^* \to 1^* \leq 1 \to x = x \Rightarrow 1^* \leq x$.
w-c$_8$. We have $x^* \leq (1^*)^* \to x^* \leq x \to 1^*$ (by w-4).

On the other hand, $x^* \to 1^* \leq 1 \to x = x \Rightarrow x \to 1^* \leq (x^* \to 1^*) \to 1^* =$ $= (1^* \to x^*) \to x^* \Rightarrow 1^* \to x^* \leq (x \to 1^*) \to x^*$ (by w-c$_3$).

Since $1^* \leq x^*$ (by w-c$_6$) $\Rightarrow 1 = (x \to 1^*) \to x^*$, hence $x \to 1^* \leq x^*$, so $x \to 1^* = x^*$. ■

We deduce that 1^* is the lowest element of Wajsberg algebra L relative to natural ordering, that is, $1^* = 0$.

As in the case of residuated lattices, for $x \in L$ we denote $x^{**} = (x^*)^*$.

The following result is straightforward:

Proposition 5.13.3. If L is a Wajsberg algebra and x, y $\in$ L, then
w-c$_9$: $x^{**} = x$;
w-c$_{10}$: $x^* \to y^* = y \to x$, $x^* \to y = y^* \to x$;
w-c$_{11}$: $x \leq y \Leftrightarrow y^* \leq x^*$.

Proposition 5.13.4. Let L be a Wajsberg algebra. Relative to the natural ordering, L become lattice, where for x, y $\in$ L, $x \vee y = (x \to y) \to y$ and $x \wedge y = (x^* \vee y^*)^*$.

Proof. From w-c$_2$ we deduce that $x, y \leq (x \to y) \to y$. If $z \in L$ is such that $x, y \leq z$ then $x \to z = 1$ and by w$_1$ we deduce that $(x \to z) \to z = z$. Also, $z \to x \leq y \to x$ hence $(y \to x) \to x \leq (z \to x) \to x = (x \to z) \to z = z$ or $(x \to y) \to y \leq z$, therefore $x \vee y = (x \to y) \to y$.

To prove that $x \wedge y = (x^* \vee y^*)^*$, we observe that from $x^*, y^* \leq$ $x^* \vee y^* \Rightarrow (x^* \vee y^*)^* \leq x^{**} = x, y^{**} = y$.

Now let $z \in L$ such that $z \leq x, y$. Then $x^*, y^* \leq z^* \Rightarrow x^* \vee y^* \leq z^* \Rightarrow$ $z = z^{**} \leq (x^* \vee y^*)^*$, hence $x \wedge y = (x^* \vee y^*)^*$. ■

Corollary 5.13.5. If L is a Wajsberg algebra and x, y $\in$ L, then
w-c$_{12}$: $(x \wedge y)^* = x^* \vee y^*$;
w-c$_{13}$: $(x \vee y)^* = x^* \wedge y^*$.

In what follows we want to mark some connections between Wajsberg algebras and residuated lattices.

If L is a Wajsberg algebra, for $x, y \in L$ we define $x \odot y = (x \to y^*)^*$.

Theorem 5.13.6. If (L, $\to$, *, 1) is a Wajsberg algebra, then (L, $\vee$, $\wedge$, $\odot$, $\to$, 0, 1) is a residuated lattice.

Proof. To prove that the triple (L, $\odot$, 1) is a commutative monoid, let $x, y, z \in L$. We have $x \odot y = (x \to y^*)^* = (x^{**} \to y^*)^* = (y \to x^{***})^* = (y \to x^*)^* = y \odot x$, hence the operation $\odot$ is commutative.

For the associativity of $\odot$ we have: $x \odot (y \odot z) = x \odot (z \odot y) = x \odot (z \to y^*)^* = [x \to (z \to y^*)^{**}]^* = [x \to (z \to y^*)]^* = [z \to (x \to y^*)]^* = [z \to (x \to y^*)^{**}]^* = z \odot (x \to y^*)^* = z \odot (x \odot y) = (x \odot y) \odot z$.

Also, $x \odot 1 = (x \to 1^*)^* = (x \to 0)^* = x^{**} = x$.

We have to prove $x \odot y \le z \Leftrightarrow x \le y \to z$.

Indeed, $x \odot y \le z \Leftrightarrow (x \to y^*)^* \le z \Leftrightarrow z^* \le x \to y^* \Leftrightarrow x \le z^* \to y^* = y \to z \Leftrightarrow x \le y \to z$. ■

Corollary 5.13.7. If L is a Wajsberg algebra and $x, y, z \in L$, then

w-c$_{14}$: $(x \vee y) \to z = (x \to z) \wedge (y \to z)$;

w-c$_{15}$: $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$;

w-c$_{16}$: $(x \to y) \vee (y \to x) = 1$;

w-c$_{17}$: $(x \wedge y) \to z = (x \to y) \to (x \to z)$.

Proof. **w-c$_{14}$, w-c$_{15}$.** Follows from Theorem 2.13.6.

w-c$_{16}$. We have $(y \to x) \to (x \to y) = [(x \vee y) \to x] \to [(x \vee y) \to y] = [x^* \to (x \vee y)^*] \to [y^* \to (x \vee y)^*] = y^* \to \{[x^* \to (x \vee y)^*] \to (x \vee y)^*\} = y^* \to [x^* \vee (x \vee y)^*] = [x^* \vee (x \vee y)^*]^* \to y = [x \wedge (y \vee x)] \to y = x \to y$, hence $(x \to y) \vee (y \to x) = [(x \to y) \to (y \to x)] \to (y \to x) = (y \to x) \to (y \to x) = 1$.

w-c$_{17}$. We have $(x \wedge y) \to z = (x^* \vee y^*)^* \to (z^*)^* = z^* \to (x^* \vee y^*) = z^* \to [(y^* \to x^*) \to x^*] = z^* \to [(x \to y) \to x^*] = (x \to y) \to (z^* \to x^*) = (x \to y) \to (x \to z)$. ■

Theorem 5.13.8. Let (L, $\vee$, $\wedge$, $\odot$, $\to$, 0, 1) be a residuated lattice. Then (L, $\to$, *, 1) is a Wajsberg algebra iff $(x \to y) \to y = (y \to x) \to x$, for every $x, y \in L$, where $x^* = x \to 0$.

Proof. "$\Rightarrow$". Straightforward.

"⇐". From $(x \to 0) \to 0 = (0 \to x) \to x$ we deduce that $x^{**} = 1 \to x = x$, hence $x^{**} = x$, for $x \in L$. So, if we take into consideration the calculus rules r-c_1 – r-c_{20} from Theorem 5.8.5, we deduce that w_2 is true.

For w-c_5 : $x^* \to y^* = (x \to 0) \to (y \to 0) = y \to [(x \to 0) \to 0] = y \to x^{**} = y \to x$ and the proof is complete. ■

Remark 13.9. For an example of residuated lattice which is not an Wajsberg algebra see [81, p.39].

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