Contents

Introduction

Residuation is a fundamental concept of ordered structures .In this survey we consider the consequences of adding a residuated monoid operation to lattice. The residuated lattices have been studied in several brances of mathematics, including the areas of lattice -ordered groups, ideal lattices of rings, linear logic and multi-valued logic.

The origin of residuated lattices is in Mathematical Logic without contraction. They have been investigated by Krull ([**93**]), Dilworth ([**52**]), Ward and Dilworth ([**136**]), Ward ([**135**]), Balbes and Dwinger ([**2**]) and Pavelka ([**111**]).

In [**80**], Idziak prove that the class of residuated lattices is equational. These lattices have been known under many names: *BCK- latices* in [**79**], *full BCK- algebras* in [**93**], FL_{ew}- *algebras* in [**107**], and *integral, residuated, commutative l-monoids* in [**13**].

Apart from their logical interest, residuated lattices have interesting algebraic properties (see [**12**], [**52**], [**92**], [**108**], [**135**], [**136**]).

A *residuated lattice* is an algebra $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$of type (2,2,2,2,0,0) equipped with an order $\leq$ satisfying the following:

$(A, \wedge, \vee, 0, 1)$ is a bounded lattice;

$(A, \odot, 1)$ is a commutative ordered monoid;

$\odot$ and $\rightarrow$ form an adjoint pair, i.e. $c \leq a \rightarrow b$ iff $a \odot c \leq b$ for all $a, b, c \in A$.

Importante examples of residuated lattices structures are BL-algebras (corresponding to Hajek's basic fuzzy logics, see [**75**]) and MV-algebras (corresponding to Łukasiewicz many-valued logic, see [**45**]). All these examples (with the exception of residuated lattices are *hoops*, i.e. they satisfy the equation $a \odot (a \rightarrow b) = b \odot (b \rightarrow a)$.

$BL-$algebras are exactly the residuated lattices satisfying $a \wedge b = a \odot (a \rightarrow b)$ and $(a \rightarrow b) \vee (b \rightarrow a) = 1$, for all $a, b \in A$ and $MV-$ algebras, are exactly those residuated lattices where $a \vee b$ and $(a \rightarrow b) \rightarrow b$ coincide (which is a relativized version of the law of double negation $a^{**} = a$). Also, if in a $BL-$ algebra, $a^{**} = a$ for all $a \in A$, and for $a, b \in A$ we denote $a \oplus b = (a^* \odot b^*)^*$, (where $a^* = a \rightarrow 0$), we obtain an $MV-$ algebra $(A, \oplus, ^*, 0)$. So, $MV-$ algebras will turn to be particular case of $BL-$ algebras.

In this book we begin a sistematic algebraic investigation of some algebras of fuzyy logics (residuated lattices and particulares cases: MV and BL-algebras, pseudo MV and BL-algebras).

$MV-$algebras were originally introduced by Chang in [**42**] in order to give an algebraic proof of the completeness theorem for the infinite - valued Łukasiewicz calculus [**127**], but their theory was also developed from an algebraic point of view. Just take a quick view over this domain.

The most popular example of MV-algebra is the interval $[0, 1]$ of the abelian l-group $(R, \max, \min, +, -, 0)$ equiped with the continuous t-conorm $\oplus$ defined by

$x \oplus y = \min(1, x + y)$ (as a model of disjunction), with the continuous t-norm $\odot$ defined by $x \odot y = \max(0, x + y - 1)$ (as a model of conjunction), with the negation $^-$ defined by $x^- = 1 - x$ and with the real numbers 0 and 1 as first and last element of the lattice $([0, 1], \max, \min)$. The connection with fuzzy was made and today we use to speak about the *algebra of fuzzy logic*.

In 1958, Chang defined the MV-algebras and in 1959 he proved the completeness theorem which stated the real unit interval $[0, 1]$ as a standard model of this logic. The structures directly obtained from Łukasiewicz logic, in the sense that the basic operations coincide with the basic logical connectives (implication and negation), were defined by Font, Rodriguez and Torrens in [**62**] under the name of *Wajsberg algebras*. One great event in the theory of MV-algebras was Mundici's theorem from 1986: the category of MV-algebras is equivalent to the category of abelian lattice-ordered groups with strong unit [**105**]. Through its consequences, this theorem can be identified at the origins of a considerable number of results on MV-algebras.

In the last 15 years the number of papers devoted to Chang's MV -algebras has been increasing so rapidly that, since the year 2000 the AMS Classification Index contains the special item 06D35 for MV -algebras. To quote just a handful of books, the monograph [**44**] is entierly devoted to MV -algebras, Hajek's monograph [**75**] and Gottwald's book [**72**] devote ample space to these algebras.

As shown in the book [**59**] and in the pioneering textbook [**119**], $MV-$ algebras also provide an important specimen of ,,quantum structures". The second volume of the Handbook of Measure Theory [**110**] includes several chapters on $MV-$ algebraic measure theory. As the algebras of Łukasiewicz infinite - valued logic, $MV-$ algebras are also considered in various surveys, e.g. [**109**] and [**102**].

Equivalents of MV -algebras are found in the literature under various names, including *bounded commutative BCK-algebras*, [**134**], [**88**], [**128**], *Bosbach's bricks* [**16**], *Buff's s-algebras* [**17**], *Komori's CN-algebras* [**90**], *Wajsberg algebras* [**62**].

Also, in the last years, one can distinguish fruitful research directions, coexisting and communicating with deeper and deeper researches on MV-algebras.

One direction is concerned with structures obtained by adding operations to the MV-algebra structure, or even combining MV-algebras with other structures in order to obtain more expressive models and powerful logical systems.

Another direction is centered on the non-commutative extensions of $MV-$ algebras,starting from arbitrary l-groups instead of abelian l-groups. In 1999, *pseudo* $MV-$*algebras* ($psMV$-algebras, shortly) where introduced to extend the concept of MV - algebra to the non-commutative case, see [**66**], [**68**]; they can be taken as algebraic semantics for a non-commutative generalization of a multiple valued reasoning.

Immediately, A. Dvurecenskij proved that the category of pseudo MV-algebras is equivalent to the category of l-groups with strong unit, this result extending the fundamental theorem of Mundici.

The third direction we want to emphasize began with Hájek's book, where BL-logic and BL-algebras were defined [**74**], [**75**].

A natural question was then to obtain a general fuzzy logical system arising from the structure of $[0, 1]$ introduced by a continuous t-norm and its associated residum. In 1998, Hajek [**75**] introduced a very general many-valued logic, called *Basic Logic* (or BL), with the idea to formalize the many-valued semantics introduced by a continuous t-norm on the real unit interval $[0, 1]$. This Basic Logic turns to be a

fragment common to three important many-valued logics: $\aleph_0-$ valued Łukasiewicz logic, Gödel logic and Product logic.

The Lindenbaum-Tarski algebras for Basic Logic are called $BL-$*algebras.* Apart from their logical interest, $BL-$ algebras have important algebraic properties and they have been intensively studied from an algebraic point of view. $BL-$ algebras form an equational class of residuated lattices.

Juste notice that Łukasiewicz logics is an axiomatic extension of BL-logic and, consequently, MV-algebras are a particular class of BL-algebras; MV-algebras are categorically equivalent to BL-algebras with the property $x^{**} = x$.

The next step in the research was then immediately made by establishing the connection between $BL-$ algebras and pseudo MV- algebras. In 2000, G. Georgescu and A. Iorgulescu defined the non-commutative extension of BL- algebras, called *pseudo BL-algebras* (introduced in [**53**], [**54**]); the class of pseudo BL- algebras contains the pseudo MV- algebras.

A remarkable construction in ring theory is the *localization ring* $A_{\mathcal{F}}$ associated with a Gabriel topology $\mathcal{F}$ on a ring A; for certain issues connected to the therm *localization* we have in view Chapter IV : *Localization* in N. Popescu' s book [**112**].

For some informal explanations of the notion of *localization* see [**106**], [**113**], [**114**].

In Lambek' s book [**96**] it is introduced the notion of *complete ring of quotients* of a commutative ring, as a particular case of localization ring (relative to the *dense ideals*).

Starting from the example of the ring, J. Schmid introduces in [**121**], [**122**] the notion of *maximal lattice of quotients* for a distributive lattice. The central role in this construction is played by the concept of *multipliers,* defined by W. H. Cornish in [**47**].

Using the model of localization ring, in [**64**] is defined for a bounded distributive lattice L the *localization lattice* $L_{\mathcal{F}}$ *of* L *with respect to a topology* $\mathcal{F}$ *on* L and is proved that the maximal lattice of quotients for a distributive lattice is a lattice of localization (relative to the topology of *regular* ideals).

The same theory is also valid for the *lattice of fractions* of a distributive lattice with 0 and 1 relative to an $\wedge$-closed system.

The book is organized in two parts.

In the first part we review the basic definitions and results of this algebras with more details and examples; we make connections between theses algebras; we study the homomorphisms, the filters (ideals, prime and maximal).

The main aim of the last part is to develop a theory of localization for BL-algebras and MV-algebras, to extend this theory to the non-commutative case (pseudo MV-algebras) and to translate the theory of localization in categories of abelian and nonabelian l-groups with strong unit (a subject which has never been approached in the mathematical literature).

For the basic notions relative to these categories of algebras we followed the monographies: [**45**], [**75**], [**129**] as well as the paper : [**68**].

I shall now give a chronological survey of this book.

Chapter 1 is dedicated to basic notions of residuated lattices, which turn out to be fundamental in manny applications.

We recall the basic definition of residuated lattices with more details and examples and we put in evidence many rules of calculus. For a residuated lattice A

we denote by $Ds(A)$ the lattice of all deductive systems (implicative filters) of A; we put in evidence characterisations for the maximal and prime elements on $Ds(A)$ and some properties of the lattice $(Ds(A), \subseteq)$. Also, we characterize the residuated lattices for which the lattice of deductive systems is a Boolean lattice.

Archimedean and hyperarchimedean residuated lattices are introduced and caracterized; we prove some theorems of Nachbin type for residuated lattices.

For more details we recommemd [**113**] and [**129**].

Chapter 2 contains all the necessary algebraic results we need to be able to prove in details a the category of $MV-$ algebras; also we study Wajsberg algebas and show their mutual equivalence. $MV-$ algebras are particular residuated lattices, however, from application point of view they posses the best properties as we will see. The result we study are due to J.M.Font, A.J. Rodriguez, A. Torrens, R. Cignolli, D. Mundici, I.M.L. D'Ottaviano For further reading on $MV-$ algebras we recommend [**45**].

We recall some basic definitions and results.

For an MV-algebra, we denote by $Id(A)$ the set of ideals of A and we present some known basic definitions and results relative to the lattice of ideals of A. For $I_1, I_2 \in Id(A)$ we define $I_1 \wedge I_2 = I_1 \cap I_2$, $I_1 \vee I_2 =$ the ideal generated by $I_1 \cup I_2$ and for $I \in Id(A)$, $I^* = \{a \in A : a \wedge x = 0, \text{ for every } x \in I\}$. Theorem 2.17 characterizes the MV-algebras for which the lattice of ideals $(Id(A), \wedge, \vee, ^*, \{0\}, A)$ is a Boolean algebra.

We study the prime spectrum $Spec(A)$ and the maximal spectrum $Max(A)$ of an MV-algebra.

For any class of structures, the representation theorems have a special significace.

The Chang's Subdirect Representation Theorem is a fundamental result.

The idea of associating a totally ordered abelian group to any MV- algebra A is due to Chang, who in [**42**] and [**43**] gave first purely algebraic proof of the completeness of the Łukasiewicz axioms for the infinite-valued calculus. In [**45**] is proved the Chang completeness theorem starting that if an equation holds in the unit real interval $[0, 1]$, then the the equation holds in every MV- algebra. This proof is elementary, and use the good sequences; good sequences and Γ functor were first introduced in [**105**].

An applications is the equivalence between MV- algebras and lattice ordered abelian groups with strong unit.

We also prove that there is one-to-one correspondence between MV- algebras and Wajsberg algebras; each MV- algebra can be seen as Wajsberg algebra and conversely. MV- algebras will turn out to be particular residuated lattices.

Chapter 3 contains results on $BL-$ algebras.

For more details we recommemd [**113**] and [**129**].

We recall the basic definitions, examples and rules of calculus in $BL-$ algebras; we also prove some results about injective objects in the category $\mathcal{BL}$ of $BL-$ algebras; the principal role is played by the *MV-center* of a BL-algebra, defined by Turunen and Sessa in [**132**]; this is a very useful construction, which associates an MV-algebra with every BL-algebra. In this way, many properties can be transferred from MV-algebras to BL-algebras and backwards.

So, we prove that :

The category $\mathcal{MV}$ of MV-algebras is a reflective subcategory of the category $\mathcal{BL}$ of BL-algebras and the reflector $\mathcal{R} : \mathcal{BL} \to \mathcal{MV}$ preserves monomorphisms (Theorem 3.12).

As consequence, we obtain that *if A is a complete and divisible MV-algebra, then A is an injective object in the category $\mathcal{BL}$* (Theorem 3.14).

For a BL-algebra A we denote by $Ds(A)$ the lattice of all deductive systems of A. We put in evidence characterizations for the meet-irreducible elements of $Ds(A)$. For the lattice $Ds(A)$ (which is distributive) we denote by $Spec(A)$ the set of all (finitely) meet-irreducible (hence meet-prime) elements ($Spec(A)$ is called the *spectrum* of A) and by $Irc(A)$ the set of all (completely) meet-irreducible elements of the lattice $Ds(A)$ and we put in evidence characterizations for elements of $Spec(A)$ and $Irc(A)$.

Relative to the uniqueness of deductive systems as intersection of primes we prove that this is possible only in the case of Boolean algebras.

The notions of *archimedean* and *hyperarchimedean* $BL-$ algebras are introduced and characterized. A *Nachbin* type theorem is obtained: *for a BL-algebra A, A is hyperarchimedean iff any prime deductive system is minimal prime* (Theorem 3.56).

Chapters 4 and **5** (Pseudo MV - algebras, respectively, Pseudo BL - algebras) presents the general theory of Pseudo MV - algebras and Pseudo BL - algebras, algebras which are generalization of $MV(BL)$ -algebras.

In 1999, Georgescu and Iorgulescu (see [**66**], [**68**]) defined pseudo $MV-$ algebras as a non-commutative extensions of $MV-$ algebras. Dvurečenskij extended Mundici's equivalence results. In [**58**], he proved that every pseudo $MV-$ algebra is isomorphic with an interval in an l-group and he established the categorical equivalence between pseudo $MV-$ algebras and l-groups with strong unit.

For a detailed study of pseudo $MV-$ algebras one can see [**68**], [**58**].

For an exhaustive theory of l-groups we refer to [**10**].

In [**67**], [**53**], [**54**], A. Di Nola, G. Georgescu and A. Iorgulescu defined the pseudo BL-algebras as a non-commutative extension of BL algebras (the class of pseudo BL - algebras contains the pseudo MV-algebras, see [**66**], [**68**]).

We begin the investigation of filters and congruences. We define the filters (ideals) of a pseudo $BL(MV)$-algebra; for a pseudo BL-algebra A we denote by $F(A)(F_{\rm n}(A))$ the lattice of all filters (normal filters) of A and we put in evidence some results about the lattice $F(A)(F_{\rm n}(A))$. By using the two distance functions we define two binary relations on , $\equiv_{L(F)}$ and $\equiv_{R(F)}$, related to a filter F of A; these two relations are equivalence relations, but they are not congruences. The quotient set $A/L(F)$ and $A/R(F)$ are bounded distributive lattices. We characterize the prime and maximal filters of A, we prove the prime filter theorem and we give characterizations for the maximal and prime elements on $F(A)(F_{\rm n}(A))$. We characterize the pseudo BL-algebras for which the lattice of filters (normal filters) is a Boolean lattice. Archimedean and hyperarchimedean pseudo $BL(MV)$-algebras are characterized. In end we prove a theorem of Nachbin type for pseudo BL-algebras.

In **Chapter 6** we develop the theory of localization for $BL(MV)$ -algebras. We denote by A a BL -algebra and by $B(A)$ the set of all boolean elements of $L(A)$.

In **Section 1**, for an $\wedge-$closed system $S \subseteq A$ ($1 \in S$ and $x, y \in S$ implies $x \wedge y \in S$) we consider the congruence θ_S on A defined by:

$$(x, y) \in \theta_S \text{ iff there exists } e \in S \cap B(A) \text{ such that } x \wedge e = y \wedge e.$$

Then $A[S] = A/\theta_S$ verifies the following property of universality: *If A' is a BL-algebra and $f : A \to A'$ is a morphism of BL-algebras such that $f(S \cap B(A)) = \{\mathbf{1}\}$, then there exists an unique morphism of BL-algebras $f' : A[S] \to A'$ such that the diagram*

$$\begin{array}{ccc} A & \xrightarrow{p_S} & A[S] \\ & \searrow^{f} \quad \swarrow_{f'} & \\ & A' & \end{array}$$

is commutative (i.e. $f' \circ p_S = f$), where $p_S : A \to A[S]$ is the canonical onto morphism of $BL-$ algebras.

This result suggests us to call $A[S]$ the *BL-algebra of fractions relative to the $\wedge-$closed system S*. If $BL-$ algebra A is in particular an MV -algebra, then $A[S]$ is an $MV-$ algebra.

In **Section 2** we define the notion of *strong multiplier* on a $BL-$ algebra A. We denote by $\mathcal{I}(A)$ the set of all *order ideals* of A:

$$\mathcal{I}(A) = \{I \subseteq A : \text{if } x, y \in A, x \leq y \text{ and } y \in I, \text{then } x \in I\}.$$

By a *partial strong multiplier* on A we mean a map $f : I \to A$, where $I \in \mathcal{I}(A)$, which verifies the following conditions:

$(sm - BL_1)$ $f(e \odot x) = e \odot f(x)$, for every $e \in B(A)$ and $x \in I$,
$(sm - BL_2)$ $f(x) \leq x$, for every $x \in I$,
$(sm - BL_3)$ If $e \in I \cap B(A)$, then $f(e) \in B(A)$,
$(sm - BL_4)$ $x \wedge f(e) = e \wedge f(x)$, for every $e \in I \cap B(A)$ and $x \in I$ (note that $e \odot x \in I$ since $e \odot x \leq e \wedge x \leq x$).

For $I \in \mathcal{I}(A)$, we denote $M(I, A) = \{f : I \to A \mid f \text{ is a strong multiplier on } A\}$ and $M(A) = \underset{I \in \mathcal{I}(A)}{\cup} M(I, A)$.

If $I_1, I_2 \in \mathcal{I}(A)$ and $f_i \in M(I_i, A), i = 1, 2$, we define $f_1 \wedge f_2$, $f_1 \vee f_2$, $f_1 \boxdot f_2$, $f_1 \to f_2 : I_1 \cap I_2 \to A$ by $(f_1 \wedge f_2)(x) = f_1(x) \wedge f_2(x)$, $(f_1 \vee f_2)(x) = f_1(x) \vee f_2(x)$, $(f_1 \boxdot f_2)(x) = f_1(x) \odot [x \to f_2(x)] = f_2(x) \odot [x \to f_1(x)]$, $(f_1 \to f_2)(x) = x \odot [f_1(x) \to f_2(x)]$, for every $x \in I_1 \cap I_2$ and we obtain a BL-algebra $(M(A), \wedge, \vee, \boxdot, \to, \mathbf{0}, \mathbf{1})$.

If $BL-$ algebra $(A, \wedge, \vee, \odot, \to, 0, 1)$ is an $MV-$ algebra $(A, \oplus, ^*, 0)$ (i.e. $x^{**} = x$, for all $x \in A$), then $BL-$ algebra $(M(A), \wedge, \vee, \boxdot, \to, \mathbf{0}, \mathbf{1})$ is an $MV-$ algebra $(M(A), \boxplus, ^*, \mathbf{0})$. If $I_1, I_2 \in \mathcal{I}(A)$ and $f_i \in M(I_i, A), i = 1, 2$, we have $f_1 \boxplus f_2 : I_1 \cap I_2 \to A$, $(f_1 \boxplus f_2)(x) = (f_1(x) \oplus f_2(x)) \wedge x$, for every $x \in I_1 \cap I_2$ and for $I \in \mathcal{I}(A)$ and $f \in M(I, A)$ we have $f^* : I \to A$, $f^*(x) = (f \to \mathbf{0})(x) = x \odot (f(x) \to \mathbf{0}(x)) = x \odot (f(x) \to 0) = x \odot (f(x))^*$, for every $x \in I$.

We prove that the algebra of multipliers $M_{\mathcal{BL}}(A)$ for $BL-$ algebras (defined in [**33**]) is in fact a generalization of the algebra of multipliers $M_{\mathcal{MV}}(A)$ for $MV-$ algebras (defined in [**26**]) (although they are defined different because of the different choice of the term language).

So, if $BL-$ algebra A is an $MV-$ algebra, then $M_{\mathcal{BL}}(A) = M_{\mathcal{MV}}(A)$.

If we denote by $\mathcal{R}(A) = \{I \subseteq A : I \text{ is a regular subset of } A\}$, then $M_r(A) = \{f \in M(A) : dom(f) \in \mathcal{I}(A) \cap \mathcal{R}(A)\}$ is a BL-subalgebra of $M(A)$. Moreover, $M_r(A)$ *is a Boolean subalgebra of* $M(A)$.

On the Boolean algebra $M_r(A)$ we consider the congruence ρ_A defined by $(f_1, f_2) \in \rho_A$ iff f_1 and f_2 coincide on the intersection of their domains.

For $f \in M_r(A)$ with $I = dom(f) \in \mathcal{I}(A) \cap \mathcal{R}(A)$, we denote by $[f, I]$ the congruence class of f modulo ρ_A and by $Q(A)$ the BL-algebra $M_r(A)/\rho_A$ which is a Boolean algebra.

Let A be a $BL(MV)$-algebra. A $BL(MV)$-algebra F is called *$BL(MV)$-algebra of fractions of A* if:

$(BLfr_1)$ $B(A)$ is a $BL(MV)$-subalgebra of F (that is $B(A) \leq F$),

$(BLfr_2)$ For every $a', b', c' \in F, a' \neq b'$, there exists $e \in B(A)$ such that $e \wedge a' \neq e \wedge b'$ and $e \wedge c' \in B(A)$.

As a notational convenience, we write $A \preceq F$ to indicate that F is a $BL(MV)$-algebra of fractions of A.

A_M is said to be a *maximal $BL(MV)$-algebra of quotients of A* if $A \preceq A_M$ and for every $BL(MV)$-algebra F with $A \preceq F$ there exists an injective morphism of $BL(MV)$-algebras $i : F \to A_M$. If $A \preceq F$, then F is a Boolean algebra, hence A_M is a Boolean algebra.

If $BL(MV)$- algebra A is a Boolean algebra, then $B(A) = A$ and the axioms $sm - BL_1, sm - BL_2, sm - BL_3$ and $sm - BL_4$ are equivalent with $sm - BL_1$, hence A_M is in this case just the classical *Dedekind-MacNeille completion of A* (see [**122**], p.687).

The main result of **Section 3** asserts that *$Q(A) = M_r(A)/\rho_A$ is a maximal $BL(MV)$-algebra of quotients of A.*

An interesting remark is that we can replace the Boolean algebra $B(A)$ with a Boolean subalgebra $B \subseteq B(A)$ and finally we obtain that $Q(A)$ is just $Q(B) =$ *the MacNeille completion* of B. In particular for $B = B(A)$ we obtain the results of this chapter.

In **Sections 4** and **5** we study the *$BL(MV)-$ algebra of localization of A with respect to a topology $\mathcal{F}$ on A* (denoted by $A_{\mathcal{F}}$).

The notion of *topology* for $BL(MV)$ - algebras is introduced in a similar way as for rings, monoids or bounded distributive lattices. We define the notion of *$\mathcal{F}-$ multiplier,* where $\mathcal{F}$ is a topology on a $BL(MV)-$ algebra A. The $\mathcal{F}$ -multipliers will be used to construct the *localization $BL(MV)-$ algebra $A_{\mathcal{F}}$* with respect to a topology $\mathcal{F}$. We define the congruence $\theta_{\mathcal{F}}$ on A by

$$(x, y) \in \theta_{\mathcal{F}} \Leftrightarrow \text{there exists } I \in \mathcal{F} \text{ such that } e \wedge x = e \wedge y \text{ for any } e \in I \cap B(A).$$

An $\mathcal{F}-$ multiplier is a mapping $f : I \to A/\theta_{\mathcal{F}}$, where $I \in \mathcal{F}$ and for every $x \in I$ and $e \in B(A)$ the following axioms are fulfilled:

$(m - BL_1)$ $f(e \odot x) = e/\theta_{\mathcal{F}} \wedge f(x) = e/\theta_{\mathcal{F}} \odot f(x)$,

$(m - BL_2)$ $f(x) \leq x/\theta_{\mathcal{F}}$.

In order to obtain the maximal $BL(MV)$ -algebra of quotients $Q(A)$ (defined in Section 2 of this chapter) as a $BL(MV)$ -algebra of localization relative to a topology $\mathcal{F}$, we develope another theory of multipliers (meaning we add the two new axioms for $\mathcal{F}$-multipliers and will be so called strong $\mathcal{F}$-multipliers). These two new axioms are:

$(m - BL_3)$ If $e \in I \cap B(A)$, then $f(e) \in B(A/\theta_{\mathcal{F}})$,

$(m - BL_4)$ $(x/\theta_{\mathcal{F}}) \wedge f(e) = (e/\theta_{\mathcal{F}}) \wedge f(x)$, for every $e \in I \cap B(A)$ and $x \in I$.

Analogous as in the case of $\mathcal{F}-$ multipliers if we work with strong-$\mathcal{F}-$ multipliers we obtain a $BL(MV)-$ subalgebra of $A_{\mathcal{F}}$ denoted by $s - A_{\mathcal{F}}$ which will be called the *strong-localization $BL(MV)-$ algebra of A with respect to the topology $\mathcal{F}$.*

In **Section 6** we describe the localization $BL(MV)$-algebra $A_{\mathcal{F}}$ in some special instances. Contrary with the case of maximal $BL(MV)$ -algebra of quotients, in general $A_{\mathcal{F}}$ is not a Boolean algebra.

For example, if we consider $BL-$ algebra $A = I = [0, 1]$ and $\mathcal{F}$ is the topology $\mathcal{F}(I) = \{I' \in \mathcal{I}(A) : I \subseteq I'\}$ then $A_{\mathcal{F}}$ is not a Boolean algebra.

For $\mathcal{F} = \mathcal{I}(A) \cap \mathcal{R}(A)$, $s - A_{\mathcal{F}}$ is exactly a maximal $BL(MV)$-algebra $Q(A)$ of quotients of A, which is a Boolean algebra.

If $\mathcal{F}_S$ is the topology associated with an $\wedge-$closed system $S \subseteq A$, then the $BL(MV)$-algebra $s - A_{\mathcal{F}_S}$ is isomorphic with $B(A[S])$.

$MV-$ algebras can be studied within the context of abelian lattice-ordered groups with strong units (abelian lu-groups) and this point of view plays a crucial role in **Section 7**.

This point of view is possible by the fundamental result of Mundici (Theorem 2.60) [**105**] that the category of MV-algebras is equivalent with the category of lu-groups ([**3**], [**45**], [**105**]).

In this section we translate the theory of localization MV -algebras defined in Section 5 for $BL-$ algebras and in particular for $MV-$ algebras into the language of localization of abelian lu-groups.

In **Chapter 7** and **8**, we develop - taking as a guide-line the case of $BL(MV)$ -algebras - the theory of localization for pseudo $BL(MV)$ - algebras (which are non-commutative generalization of these). The main topic of this chapter is to generalize to pseudo $BL(MV)-$ algebras the notions of $BL(MV)-$ algebras of multipliers, $BL(MV)-$ algebra of fractions and maximal $BL(MV)-$ algebra of quotients. The structure, methods and techniques in this chapter are analogous to the structure, methods and techniques for $MV(BL)-$ algebras exposed in Chapter 6.

Following the categorical equivalence between the category of l-groups with a strong unit (lu-groups) and the category of pseudo MV -algebras ([**58**]) we translate the theory of localization of pseudo MV -algebras into the language of localization of lu-groups .

This was a short presentation of this book.

We hope that we convinced the reader that algebra of many-valued logic is a mathematically interesting theory, with connections with other branches of mathematics.

I think that this book is a base for future developments in the theory of localization for other algebras of fuzzy logic.

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CHAPTER 1

Residuated lattices

The origin of residuated lattices is in Mathematical Logic without contraction. They have been investigated by Krull ([**93**]), Dilworth ([**52**]), Ward and Dilworth ([**136**]), Ward ([**135**]), Balbes and Dwinger ([**2**]) and Pavelka ([**111**]).

In [**80**], Idziak prove that the class of residuated lattices is equational. These lattices have been known under many names: *BCK- latices* in [**79**], *full BCK- algebras* in [**93**], FL_{ew}- *algebras* in [**107**], and *integral, residuated, commutative l-monoids* in [**13**].

Apart from their logical interest, residuated lattices have interesting algebraic properties (see [**12**], [**52**], [**92**], [**108**], [**135**], [**136**]).

In this chapter we recall the basic definition of residuated lattices with more details and examples and we put in evidence many rules of calculus. For a residuated lattice A we denote by Ds(A) the lattice of all deductive systems (implicative filters) of A; we put in evidence characterisations for the maximal and prime elements on Ds(A) and some properties of the lattice $(Ds(A), \subseteq)$. Also, we characterize the residuated lattices for which the lattice of deductive systems is a Boolean lattice.

Archimedean and hyperarchimedean residuated lattices are introduced and characterized; we prove some theorems of Nachbin type for residuated lattices.

For the preliminaries in general lattice theory we strongly recommend for reader the very beautiful monograph Lattice theory of George Grätzer ([**73**]).

For further reading on residuated lattices we recommend [**75**] and [**129**].

1. Definitions and preliminaries

We review the basic definitions of residuated lattices, with more details and examples. Also we put in evidence some rules of calculus and the connection between residuated lattices and Hilbert algebras.

Definition 1.1. A *residuated lattice* is an algebra

$$(A, \wedge, \vee, \odot, \to, 0, 1)$$

of type (2,2,2,2,0,0) equipped with an order $\leq$ satisfying the following:

- (LR_1) $(A, \wedge, \vee, 0, 1)$ is a bounded lattice;
- (LR_2) $(A, \odot, 1)$ is a commutative ordered monoid*;*
- (LR_3) $\odot$ and $\to$ form an adjoint pair, i.e., $c \leq a \to b$ iff $a \odot c \leq b$, for all $a, b, c \in A$.

The relations between the pair of operations $\odot$ and $\to$ expressed by LR_3, is a particular case of the *law of residuation,* or *Galois correspondence* (see [**12**]) and for every $x, y \in A, x \to y = \sup\{z \in A : x \odot z \leq y\}$. Namely, let A and B two posets, and $f : A \to B$ a map. Then f is called *residuated* if there is a map $g : B \to A$, such that for any $a \in A$ and $b \in B$, we have $f(a) \leq b$ iff $b \leq g(a)$ (this is, also expressed by saying that the pair (f, g) is a *residuated pair*).

Now setting A a residuated lattice, $B = A$, and defining, for any $a \in A$, two maps $f_a, g_a : A \to A$, $f_a(x) = x \odot a$ and $g_a(x) = a \to x$, for any $x \in A$, we see that $x \odot a = f_a(x) \leq y$ iff $x \leq g_a(y) = a \to y$ for every $x, y \in A$, that is, for every $a \in A$, (f_a, g_a) is a *pair of residuation* .

The symbols $\Rightarrow$ and $\Leftrightarrow$ are used for logical implication and logical equivalence.

In [**80**] it is proved that the class $\mathcal{RL}$ of residuated lattices is equational; one of the equational axiomatizations of $\mathcal{RL}$ can be:

($\mathcal{L}$) Equations axiomatizing the variety of bounded lattices;
($\mathcal{M}$) Equations axiomatizing the variety of commutative monoids;
($\mathcal{R}_1$) $(x \odot y) \to z = x \to (y \to z)$;
($\mathcal{R}_2$) $[(x \to y) \odot x] \wedge y = (x \to y) \odot x$ (i.e., $(x \to y) \odot x \leq y$);
($\mathcal{R}_3$) $(x \wedge y) \to y = 1$.

EXAMPLE 1.1. *Let p be a fixed natural number and $I = [0, 1]$ the real unit interval. If for $x, y \in I$, we define $x \odot y = 1 - \min\{1, [(1 - x)^p + (1 - y)^p]^{1/p}\}$ and $x \to y = \sup\{z \in [0, 1] : x \odot z \leq y\}$, then $(I, \max, \min, \odot, \to, 0, 1)$ is a residuated lattice.*

EXAMPLE 1.2. *If we preserve the notation from Example 1, and we define for $x, y \in I$, $x \odot y = (\max\{0, x^p + y^p - 1\})^{1/p}$ and $x \to y = \min\{1, (1 - x^p + y^p)^{1/p}\}$, then $(I, \max, \min, \odot, \to, 0, 1)$ become a residuated lattice called* generalized Łukasiewicz structure. *For $p = 1$ we obtain the notion of* Łukasiewicz structure *($x \odot y = \max\{0, x + y - 1\}, x \to y = \min\{1, 1 - x + y\}$).*

EXAMPLE 1.3. *If on $I = [0, 1]$, for $x, y \in I$ we define $x \odot y = \min\{x, y\}$ and $x \to y = 1$ if $x \leq y$ and y otherwise, then $(I, \max, \min, \odot, \to, 0, 1)$ is a residuated lattice (called* Gődel structure*).*

EXAMPLE 1.4. *If consider on $I = [0, 1]$, $\odot$ to be the usual multiplication of real numbers and for $x, y \in I, x \to y = 1$ if $x \leq y$ and y/x otherwise, then $(I, \max, \min, \odot, \to, 0, 1)$ is a residuated lattice (called* Products structure *or* Gaines structure*).*

EXAMPLE 1.5. *If $(A, \vee, \wedge, ', 0, 1)$ is a Boolean algebra, then if we define for every $x, y \in A, x \odot y = x \wedge y$ and $x \to y = x' \vee y$, then $(A, \vee, \wedge, \odot, \to, 0, 1)$ become a residuated lattice.*

Examples 1.2, 1.3 and 1.4 have some connections with the notion of *t-norm*.

We call *continuous t-norm* a continuous function $\odot : [0, 1] \times [0, 1] \to [0, 1]$ such that $([0, 1], \odot, 1)$ is an ordered commutative monoid.

So, there are three fundamental t-norms:

Łukasiewicz t-norm: $x \odot_L y = \max\{0, x + y - 1\}$;

Gődel t-norm: $x \odot_G y = \min\{x, y\}$;

*Product (*or *Gaines) t-norm:* $x \odot_P y = x \odot y$.

Since relative to natural ordering on $[0, 1]$, $[0, 1]$ become a complete lattice, every continuous t-norm introduce a natural *residum* (or *implication*) by

$$x \to y = \max\{z \in [0, 1] : x \odot z \leq y\}.$$

So, the implications generated by the three norms mentioned before are

$x \to_L y = \min\{1, y - x + 1\}$;

$x \to_G y = 1$ if $x \leq y$ and y otherwise;

$x \to_P y = 1$ if $x \leq y$ and y/x otherwise.

DEFINITION 1.2. ([**129**]) A residuated lattice $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is called *BL-algebra,* if the following two identities hold in A :

(BL_4) $x \odot (x \rightarrow y) = x \wedge y$;
(BL_5) $(x \rightarrow y) \vee (y \rightarrow x) = 1$.

For more details about BL-algebras, see Chapter 3.

REMARK 1.1.
1. *Łukasiewicz structure, Gődel structure and Product structure are $BL-$ algebras;*
2. *Any boolean algebra can be regarded as a residuated lattice where the operations $\odot$ and $\wedge$ coincide and $x \rightarrow y = x^{'} \vee y$.*

REMARK 1.2. *If in a $BL-$ algebra A, $x^{**} = x$ for all $x \in A$,(where $x^* = x \rightarrow 0$), and for $x, y \in A$ we denote $x \oplus y = (x^* \odot y^*)^*$, then we obtain an algebra $(A, \oplus, ^*, 0)$ of type $(2, 1, 0)$ satisfying the following:*

$$x \oplus (y \oplus z) = (x \oplus y) \oplus z,$$

$$x \oplus y = y \oplus x,$$

$$x \oplus 0 = x,$$

$$x \oplus 0^* = 0^*,$$

$$(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x, \text{ for all } x, y \in A.$$

Then for all $x, y \in A, (y \rightarrow x) \rightarrow x = x \vee y = (x \rightarrow y) \rightarrow y$. $BL-$ algebras of this kind will turn out to be so called $MV-$ algebras (see [**129**] *and Chapter 2). Conversely, if $(A, \oplus, ^*, 0)$ is an MV-algebra, then $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a BL-algebra, where for $x, y \in A$:*

$$x \odot y = (x^* \oplus y^*)^*,$$

$$x \rightarrow y = x^* \oplus y, 1 = 0^*,$$

$$x \vee y = (x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x \text{ and } x \wedge y = (x^* \vee y^*)^*.$$

REMARK 1.3. ([**129**]) *A residuated lattice $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is an MV-algebra iff it satisfies the additional condition: $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$, for any $x, y \in A$ (see Theorem 2.70).*

EXAMPLE 1.6. ([**84**]) *We give an another example of a finite residuated lattice, which is not a BL-algebra. Let $A = \{0, a, b, c, 1\}$ with $0 < a, b < c < 1$, but a, b are incomparable. A become a residuated lattice relative to the following operations:*

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

,

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

.

The condition $x \vee y = [(x \rightarrow y) \rightarrow y] \wedge [(y \rightarrow x) \rightarrow x]$, for all $x, y \in A$ is not verified, since $c = a \vee b \neq [(a \rightarrow b) \rightarrow b] \wedge [(b \rightarrow a) \rightarrow a] = (b \rightarrow b) \wedge (a \rightarrow a) = 1$, hence A is not a BL-algebra.

EXAMPLE 1.7. ([**92**]) *We consider the residuate lattice A with the universe $\{0,a,b,c,d,e,f,1\}$. Lattice ordering is such that $0<d<c<b<a<1$, $0<d<e<f<a<1$ and elements $\{b,f\}$ and $\{c,e\}$ are pairwise incomparable. The operations of implication and multiplication are given by the tables below :*

$\rightarrow$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	d	1	a	a	f	f	f	1
b	e	1	1	a	f	f	f	1
c	f	1	1	1	f	f	f	1
d	a	1	1	1	1	1	1	1
e	b	1	a	a	a	1	1	1
f	c	1	a	a	a	a	1	1
1	1	a	b	c	d	e	f	1

,

$\odot$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	c	c	c	0	d	d	a
b	0	c	c	c	0	0	d	b
c	0	c	c	c	0	0	0	c
d	0	0	0	0	0	0	0	d
e	0	d	0	0	0	d	d	e
f	0	d	d	0	0	d	d	f
1	0	a	b	c	d	e	f	1

Clearly, A contains $\{a,b,c,d,e,f\}$ as a sublattice, and that is a copy of the so-called benzene ring, which shows that A is not distributive, and even not modular (see [**22**]*). But it is easy to see that $a^*=d, b^*=e, c^*=f, d^*=a$, $e^*=b$ and $f^*=c$.*

EXAMPLE 1.8. ([**92**]) *Let A be the residuate lattice with the universe $\{0,a,b,c,d,1\}$ such that $0<b<a<1$, $0<d<c<a<1$ and c and d are incomparable with b. The operations of implication and multiplication are given by the tables below :*

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	0	1	b	c	c	1
b	c	a	1	c	c	1
c	b	a	b	1	a	1
d	b	a	b	a	1	1
1	0	a	b	c	d	1

,

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	b	d	d	a
b	c	b	b	0	0	b
c	b	d	0	d	d	c
d	b	d	0	d	d	d
1	0	a	b	c	d	1

Then A is obtained from the nonmodular lattice N_5 (see [**22**]*), called the pentagon, by adding the new greatest element* 1*. Then A is another example of nondistributive residuated lattice.*

EXAMPLE 1.9. ([**84**]) *We give an example of a finite residuate lattice which is an non-linearly MV-algebra. Let $A=\{0,a,b,c,d,1\}$, with $0<a,b<c<1, 0<b<d<1$, but a,b and, respective c,d are incomparable. We define*

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	c	c	1	1	1	1
c	b	c	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

,

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

and so A become a $BL-$algebra.We have in A the following operations:

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	a	c	c	1	1
b	b	c	d	1	d	1
c	c	c	1	1	1	1
d	d	1	d	1	d	1
1	1	1	1	1	1	1

,

$*$	0	a	b	c	d	1
	1	d	c	b	a	0

It is easy to see that $0^* = 1, a^* = d, b^* = c, c^* = b, d^* = a, 1^* = 0$ *and* $x^{**} = x$, *for all* $x \in A$, *hence* A *is an* $MV-$ *algebra which is not chain.*

EXAMPLE 1.10. ([**84**]) *We give an another example of a finite residuate lattice* $A = \{0, a, b, c, d, e, f, g, 1\}$, *which is non-linearly* $MV-$ *algebra, with* $0 < a < b < e < 1, 0 < c < f < g < 1, a < d < g, c < d < e$, *but* $\{a, c\}, \{b, d\}, \{d, f\}, \{b, f\}$ *and, respective* $\{e, g\}$ *are incomparable. We define*

$\rightarrow$	0	a	b	c	d	e	f	g	1
0	1	1	1	1	1	1	1	1	1
a	g	1	1	g	1	1	g	1	1
b	f	g	1	f	g	1	f	g	1
c	e	e	e	1	1	1	1	1	1
d	d	e	e	g	1	1	g	1	1
e	c	d	e	f	g	1	f	g	1
f	b	b	b	e	e	e	1	1	1
g	a	b	b	d	e	e	g	1	1
1	0	a	b	c	d	e	f	g	1

,

$\odot$	0	a	b	c	d	e	f	g	1
0	0	0	0	0	0	0	0	0	0
a	0	0	a	0	0	a	0	0	a
b	0	a	b	0	a	b	0	a	b
c	0	0	0	0	0	0	c	c	c
d	0	0	a	0	0	a	c	c	d
e	0	a	b	0	a	b	c	d	e
f	0	0	0	c	c	c	f	f	f
g	0	0	a	c	c	d	f	f	g
1	0	a	b	c	d	e	f	g	1

and so A *become a residuated lattice. We have* $0^* = 1, a^* = g, b^* = f, c^* = e, d^* = d, e^* = c, f^* = b, g^* = a$.

EXAMPLE 1.11. ([**84**]) *We give an example of a finite residuate lattice which is an* MV*-algebra. Let* $A = \{0, a, b, c, d, 1\}$, *with* $0 < a < b < 1, 0 < c < d < 1$, *but* a, c *and, respective* b, d *are incomparable. We define*

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	d	1	1
b	c	d	1	c	d	1
c	b	b	b	1	1	1
d	a	b	b	d	1	1
1	0	a	b	c	d	1

,

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	a	0	0	a
b	0	a	b	0	a	b
c	0	0	0	c	c	c
d	0	0	a	c	c	d
1	0	a	b	c	d	1

It is easy to see that $0^* = 1, a^* = d, b^* = c, c^* = b, d^* = a$.

In what follows by A we denote a residuated lattice; for $x \in A$ and a natural number n, we define $x^* = x \rightarrow 0, (x^*)^* = x^{**}$, $x^0 = 1$ and $x^n = x^{n-1} \odot x$ for $n \geq 1$.

DEFINITION 1.3. An element a in A is called *idempotent* iff $a^2 = a$, and it is called *nilpotent* iff there exists a natural number n such that $a^n = 0$. The minimum n such that $a^n = 0$ is called *nilpotence order* of a and will be denoted by $ord(a)$; if there is no such n, then $ord(a) = \infty$. A residuated lattice A is called *locally finite*

if every $a \in A, a \neq 1$, has finite order. An element a in A is called *dense* iff $a^* = 0$, and it is called a *unity* iff for all natural numbers n, $(a^n)^*$ is nilpotent. The set of dense elements of A will be denoted by $D(A)$.

THEOREM 1.1. *Let $x, x_1, x_2, y, y_1, y_2, z \in A$. Then we have the following rules of calculus:*

$(lr - c_1)$ $1 \to x = x, x \to x = 1$;
$(lr - c_2)$ $x \odot y \leq x, y$, *hence* $x \odot y \leq x \wedge y$, $y \leq x \to y$ *and* $x \odot 0 = 0$;
$(lr - c_3)$ $x \odot y \leq x \to y$;
$(lr - c_4)$ $x \leq y$ *iff* $x \to y = 1$;
$(lr - c_5)$ $x \to y = y \to x = 1 \Leftrightarrow x = y; x \to 1 = 1, 0 \to x = 1$;
$(lr - c_6)$ $x \odot (x \to y) \leq y, x \leq (x \to y) \to y, ((x \to y) \to y) \to y = x \to y$;
$(lr - c_7)$ $x \to y \leq (x \odot z) \to (y \odot z)$;
$(lr - c_8)$ $x \leq y$ *implies* $x \odot z \leq y \odot z$;
$(lr - c_9)$ $x \to y \leq (z \to x) \to (z \to y)$;
$(lr - c_{10})$ $x \to y \leq (y \to z) \to (x \to z)$;
$(lr - c_{11})$ $x \leq y$ *implies* $z \to x \leq z \to y, y \to z \leq x \to z$ *and* $y^* \leq x^*$;
$(lr - c_{12})$ $x \odot (y \to z) \leq y \to (x \odot z) \leq (x \odot y) \to (x \odot z)$;
$(lr - c_{13})$ $x \to (y \to z) = (x \odot y) \to z = y \to (x \to z)$;
$(lr - c_{14})$ $x_1 \to y_1 \leq (y_2 \to x_2) \to [(y_1 \to y_2) \to (x_1 \to x_2)]$.

Proof. $(lr - c_1)$. Since $x \odot 1 = x \leq x \Rightarrow x \leq 1 \to x$. If we have $z \in A$ such that $1 \odot z = x$, then $z \leq x$ and so $x = sup\{z \in A : 1 \odot z \leq x\} = 1 \to x$;

From $1 \odot x = x \leq x \Rightarrow 1 \leq x \to x$; since $x \to x \leq 1 \Rightarrow x \to x = 1$.

$(lr - c_2)$. Follows from $lr - c_1$ and LR_2. As $x \odot y \leq y \Rightarrow y \leq x \to y$.

$(lr - c_3)$. Follows from $lr - c_1$ and $lr - c_2$: $x \odot y \leq y$ and $y \leq x \to y$ so $x \odot y \leq x \to y$.

$(lr - c_4)$. We have $x \leq y \Leftrightarrow x \odot 1 \leq y \Leftrightarrow 1 \leq x \to y \Leftrightarrow x \to y = 1$.

$(lr - c_5)$. Follows from $lr - c_4$.

$(lr - c_6)$. Follows immediately from LR_3.

$(lr - c_7)$. By LR_3 we have $x \to y \leq (x \odot z) \to (y \odot z) \Leftrightarrow (x \to y) \odot x \odot z \leq y \odot z \Leftrightarrow (x \to y) \odot x \leq z \to (y \odot z)$. But by $lr - c_6$, we have $(x \to y) \odot x \leq y$ and $y \leq z \to (y \odot z)$, hence $(x \to y) \odot x \leq z \to (y \odot z)$.

$(lr - c_8)$. Follows from $lr - c_7$.

$(lr - c_9)$. By LR_3 we have $x \to y \leq (z \to x) \to (z \to y) \Leftrightarrow (x \to y) \odot (z \to x) \leq z \to y \Leftrightarrow (x \to y) \odot (z \to x) \odot z \leq y$.

Indeed, by $lr - c_6$ we have that $(x \to y) \odot (z \to x) \odot z \leq (x \to y) \odot x \leq y$.

$(lr - c_{10})$. As in the case of $lr - c_9$.

$(lr - c_{11})$. It follows from $lr - c_9$ and $lr - c_{10}$.

$(lr - c_{12})$. The first equality follows from $x \odot y \odot (y \to z) \leq x \odot z$ and the second from $lr - c_{11}$.

$(lr - c_{13})$. We have $(x \to (y \to z)) \odot (x \odot y) \leq (y \to z) \odot y \leq z$, hence $x \to (y \to z) \leq (x \odot y) \to z$. On the another hand, from $((x \odot y) \to z) \odot (x \odot y) \leq z$, we deduce that $((x \odot y) \to z) \odot x \leq y \to z$, therefore $(x \odot y) \to z \leq x \to (y \to z)$, so we obtain the requested equality.

$(lr - c_{14})$. We have to prove that $(x_1 \to y_1) \odot (y_2 \to x_2) \odot (y_1 \to y_2) \odot x_1 \leq x_2$; this inequality is a consequence of applying several times $lr - c_6$. ■

REMARK 1.4. *From $lr - c_1$ and $lr - c_4$ we deduce that* 1 *is the greatest element of A.*

THEOREM 1.2. *If $x, y \in A$, then :*

$(lr - c_{15})$ $x \odot x^* = 0$ *and* $x \odot y = 0$ *iff* $x \leq y^*$;
$(lr - c_{16})$ $x \leq x^{**}, x^{**} \leq x^* \to x$;
$(lr - c_{17})$ $1^* = 0$, $0^* = 1$;
$(lr - c_{18})$ $x \to y \leq y^* \to x^*$;
$(lr - c_{19})$ $x^{***} = x^*, (x \odot y)^* = x \to y^* = y \to x^* = x^{**} \to y^*$.

Proof. $(lr - c_{15})$. We have, $x^* \leq x \to 0 \Leftrightarrow x \odot x^* \leq 0$, so $x \odot x^* = 0$.

$(lr - c_{16})$. We have $x \to x^{**} = x \to (x^* \to 0) = x^* \to (x \to 0) = x^* \to x^* = 1$ and $x^{**} \to (x^* \to x) = (x^{**} \odot x^*) \to x \overset{lr-c_{15}}{=} 0 \to x = 1$.

$(lr - c_{17})$. $1^* \leq 0 \Leftrightarrow 0 = 1 \to 0 \Leftrightarrow 0 \odot 1 \leq 0$, analogously, $0^* = 1$;

$(lr - c_{18})$. It follows from $lr - c_{10}$ for $z = 0 : 1 = (x \to y) \to (y^* \to x^*)$ hence $x \to y \leq y^* \to x^*$.

$(lr - c_{19})$. From $lr - c_{16}$ we deduce that $x^* \leq x^{***}$ and from $x \leq x^{**}$ we deduce that $x^{**} \to 0 \leq x \to 0 \Leftrightarrow x^{***} \leq x^*$, therefore $x^{***} = x^*$. ■

THEOREM 1.3. *If A is a complete residuated lattice, $x \in A$ and $(y_i)_{i \in I}$ a family of elements of A, then :*

$(lr - c_{20})$ $x \odot (\bigvee_{i \in I} y_i) = \bigvee_{i \in I} (x \odot y_i)$;
$(lr - c_{21})$ $x \odot (\bigwedge_{i \in I} y_i) \leq \bigwedge_{i \in I} (x \odot y_i)$;
$(lr - c_{22})$ $x \to (\bigwedge_{i \in I} y_i) = \bigwedge_{i \in I} (x \to y_i)$;
$(lr - c_{23})$ $(\bigvee_{i \in I} y_i) \to x = \bigwedge_{i \in I} (y_i \to x)$;
$(lr - c_{24})$ $\bigvee_{i \in I} (y_i \to x) \leq (\bigwedge_{i \in I} y_i) \to x$;
$(lr - c_{25})$ $\bigvee_{i \in I} (x \to y_i) \leq x \to (\bigvee_{i \in I} y_i)$;
$(lr - c_{26})$ $(\bigvee_{i \in I} y_i)^* = \bigwedge_{i \in I} y_i^*$;
$(lr - c_{27})$ $(\bigwedge_{i \in I} y_i)^* \geq \bigvee_{i \in I} y_i^*$.

Proof. $(lr - c_{20})$. Clearly, $x \odot y_i \leq x \odot (\bigvee_{i \in I} y_i)$, for each $i \in I$, therefore $\bigvee_{i \in I} (x \odot y_i) \leq x \odot (\bigvee_{i \in I} y_i)$.

Conversely, since for every $i \in I$, $x \odot y_i \leq \bigvee_{i \in I} (x \odot y_i) \Rightarrow y_i \leq x \to [\bigvee_{i \in I} (x \odot y_i)]$, then $\bigvee_{i \in I} y_i \leq x \to [\bigvee_{i \in I} (x \odot y_i)]$, therefore $x \odot (\bigvee_{i \in I} y_i) \leq \bigvee_{i \in I} (x \odot y_i)$, so we obtain the requested equality.

$(lr - c_{21})$. Clearly .

$(lr - c_{22})$. Let $y = \bigwedge_{i \in I} y_i$. Since for every $i \in I$, $y \leq y_i$, we deduce that $x \to y \leq x \to y_i$, hence $x \to y \leq \bigwedge_{i \in I} (x \to y_i)$; On the another hand, the inequality $\bigwedge_{i \in I} (x \to y_i) \leq x \to y$ is equivalent with $x \odot [\bigwedge_{i \in I} (x \to y_i)] \leq y$. This is true because by $lr - c_{21}$ we have $x \odot [\bigwedge_{i \in I} (x \to y_i)] \leq \bigwedge_{i \in I} [x \odot (x \to y_i)] \leq \bigwedge_{i \in I} y_i = y$.

$(lr - c_{23})$. Let $y = \bigvee_{i \in I} y_i$; since for every $i \in I$, $y_i \leq y \Rightarrow y \to x \leq y_i \to x \Rightarrow y \to x \leq \bigwedge_{i \in I} (y_i \to x)$; Conversely, $\bigwedge_{i \in I} (y_i \to x) \leq y \to x \Leftrightarrow y \odot [\bigwedge_{i \in I} (y_i \to x)] \leq x$.

By $lr-c_{21}$ we have $y\odot[\bigwedge_{i\in I}(y_i\to x)]\leq\bigwedge_{i\in I}[y\odot(y_i\to x)]\overset{lr-c_{20}}{=}\bigwedge_{i\in I}[\bigvee_{i\in I}(y_i\odot(y_i\to x))]\leq\bigwedge_{i\in I}x=x$, so we obtain the requested equality.

$(lr-c_{24})$. By $lr-c_{11}$, for every $i\in I$, $y_i\to x\leq(\bigwedge_{i\in I}y_i)\to x$ thus $\bigvee_{i\in I}(y_i\to x)\leq(\bigwedge_{i\in I}y_i)\to x$.

$(lr-c_{25})$. Similary with $lr-c_{24}$.

$(lr-c_{26})$. In particular by taking $x=0$ in $lr-c_{23}$ we obtain $(\bigvee_{i\in I}y_i)^*=\bigwedge_{i\in I}y_i^*$.

$(lr-c_{27})$. In particular by taking $x=0$ in $lr-c_{24}$ we obtain $(\bigwedge_{i\in I}y_i)^*\geq\bigvee_{i\in I}y_i^*$. ■

COROLLARY 1.4. *If $x,x',y,y',z\in A$ then:*

$(lr-c_{28})$ $x\vee y=1$ *implies* $x\odot y=x\wedge y$;
$(lr-c_{29})$ $x\to(y\to z)\geq(x\to y)\to(x\to z)$;
$(lr-c_{30})$ $x\vee(y\odot z)\geq(x\vee y)\odot(x\vee z)$, *hence* $x\vee y^n\geq(x\vee y)^n$ *and* $x^m\vee y^n\geq(x\vee y)^{mn}$, *for any* m,n *natural numbers;*
$(lr-c_{31})$ $(x\to y)\odot(x'\to y')\leq(x\vee x')\to(y\vee y')$;
$(lr-c_{32})$ $(x\to y)\odot(x'\to y')\leq(x\wedge x')\to(y\wedge y')$.

Proof. $(lr-c_{28})$. Suppose $x\vee y=1$. Clearly $x\odot y\leq x$ and $x\odot y\leq y$. Let now $t\in A$ such that $t\leq x$ and $t\leq y$. By $lr-c_{12}$ we have $t\to(x\odot y)\geq x\odot(t\to y)=x\odot 1=x$ and $t\to(x\odot y)\geq y\odot(t\to x)=y\odot 1=y$, so $t\to(x\odot y)\geq x\vee y=1$, hence $t\to(x\odot y)=1\Leftrightarrow t\leq x\odot y$, that is, $x\odot y=x\wedge y$.

$(lr-c_{29})$. We have by $lr-c_{13}$: $x\to(y\to z)=(x\odot y)\to z$ and $(x\to y)\to(x\to z)=[x\odot(x\to y)]\to z$. But $x\odot y\leq x\odot(x\to y)$, so we obtain $(x\odot y)\to z\geq[x\odot(x\to y)]\to z\Leftrightarrow x\to(y\to z)\geq(x\to y)\to(x\to z)$.

$(lr-c_{30})$. By $lr-c_{20}$ we deduce $(x\vee y)\odot(x\vee z)=x^2\vee(x\odot y)\vee(x\odot z)\vee(y\odot z)\leq$ $\leq x\vee(x\odot y)\vee(x\odot z)\vee(y\odot z)=x\vee(y\odot z)$.

$(lr-c_{31})$. From the inequalities:
$x\odot(x\to y)\odot(x'\to y')\leq x\odot(x\to y)\leq x\wedge y\leq y\vee y'$ and
$x'\odot(x\to y)\odot(x'\to y')\leq x'\odot(x'\to y')\leq x'\wedge y'\leq y\vee y'$ we deduce that
$(x\to y)\odot(x'\to y')\leq x\to(y\vee y')$ and $(x\to y)\odot(x'\to y')\leq x'\to(y\vee y')$.
So, $(x\to y)\odot(x'\to y')\leq[x\to(y\vee y')]\wedge[x'\to(y\vee y')]\overset{lr-c_{23}}{=}(x\vee x')\to(y\vee y')$.

$(lr-c_{32})$. From the inequalities:
$(x\wedge x')\odot(x\to y)\odot(x'\to y')\leq x\odot(x\to y)\overset{lr-c_6}{\leq}y$ and
$(x\wedge x')\odot(x\to y)\odot(x'\to y')\leq x'\odot(x'\to y')\overset{lr-c_6}{\leq}y'$ we deduce that
$(x\to y)\odot(x'\to y')\leq(x\wedge x')\to y$ and $(x\to y)\odot(x'\to y')\leq(x\wedge x')\to y'$.
So, $(x\to y)\odot(x'\to y')\leq[(x\wedge x')\to y]\wedge[(x\wedge x')\to y']\overset{lr-c_{22}}{=}(x\wedge x')\to(y\wedge y')$.■

If $B=\{a_1,a_2,...,a_n\}$ is a finite subset of A we denote $\Pi B=a_1\odot...\odot a_n$.

PROPOSITION 1.5. *Let $A_1,...,A_n$ finite subsets of A.*

$(lr-c_{33})$ *If $a_1\vee...\vee a_n=1$, for all $a_i\in A_i, i\in\{1,...,n\}$, then*

$$(\Pi A_1)\vee...\vee(\Pi A_n)=1.$$

Proof. For $n=2$ it is proved in [**14**] and for $n=2$, A_1 a singleton and A_2 a doubleton in [**11**] (Lemma 6.4). The proof for arbitrary n is a simple mathematical induction argument. ■

COROLLARY 1.6. *Let* $a_1, ..., a_n \in A$.

$(lr - c_{34})$ *If* $a_1 \vee ... \vee a_n = 1$, *then* $a_1^k \vee ... \vee a_n^k = 1$, *for every natural number* k.

PROPOSITION 1.7. *Suppose* A *is a locally finite residuated lattice. Then for all* $a, b \in A, a \vee b = 1$ *iff* $a = 1$ *or* $b = 1$.

Proof. Assume $a \vee b = 1$. Then, since $a \vee b \leq [(a \to b) \to b] \wedge [(b \to a) \to a]$ we deduce that $(a \to b) \to b = (b \to a) \to a = 1$, hence $a \to b = b$ and $b \to a = a$. Let now $a \neq 1$. Since the residuated lattice A is locally finite (under consideration) there is a natural number m such that $a^m = 0$. Now $b = a \to b = a \to (a \to b) = a^2 \to b = ... = a^m \to b = 0 \to b = 1$. ■

PROPOSITION 1.8. *In any locally finite residuated lattice* A, *for all* $x \in A$

(i) $0 < x < 1$ *iff* $0 < x^* < 1$;
(ii) $x^* = 0$ *iff* $x = 1$;
(iii) $x^* = 1$ *iff* $x = 0$.

Proof. (i). Assume $0 < x < 1$, $ord(x) = m \geq 2$. Then, $x^{m-1} \odot x = 0, x^{m-2} \odot x \neq 0$, so by the definition of $x^*, 0 < x^{m-1} \leq x^* < x^{m-2} \leq 1$. Conversely, let $0 < x^* < 1$, $ord(x^*) = n \geq 2$. Then by similar argument, $0 < (x^*)^{n-1} \leq x^{**} < (x^*)^{n-2} \leq 1$.

If now $x = 0$, then $x^* = 1$, a contradiction. Therefore $0 < x \leq x^{**} < 1$.

(ii). If $x^* = 0$ but $x \neq 1$, then $0 < x < 1$, which leads to a contradiction $x^* \neq 0$. Thus $x = 1$.

(iii). Analogously as (ii). ■

By *bi-residuum* on a residuated lattice A we understand the derived operation $\longleftrightarrow$ defined for $x, y \in A$ by $x \longleftrightarrow y = (x \to y) \wedge (y \to x)$. Bi-residumm will offer us an elegant way to interpret fuzzy logic equivalence.

THEOREM 1.9. *If* A *is a residuated lattice and* $x, y, x_1, y_1, x_2, y_2 \in A$, *then*

$(birez_1)$ $x \longleftrightarrow 1 = x$;
$(birez_2)$ $x \longleftrightarrow y = 1 \Leftrightarrow x = y$;
$(birez_3)$ $x \longleftrightarrow y = y \longleftrightarrow x$;
$(birez_4)$ $(x \longleftrightarrow y) \odot (y \longleftrightarrow z) \leq x \longleftrightarrow z$;
$(birez_5)$ $(x_1 \longleftrightarrow y_1) \wedge (x_2 \longleftrightarrow y_2) \leq (x_1 \wedge x_2) \longleftrightarrow (y_1 \wedge y_2)$;
$(birez_6)$ $(x_1 \longleftrightarrow y_1) \wedge (x_2 \longleftrightarrow y_2) \leq (x_1 \vee x_2) \longleftrightarrow (y_1 \vee y_2)$;
$(birez_7)$ $(x_1 \longleftrightarrow y_1) \odot (x_2 \longleftrightarrow y_2) \leq (x_1 \odot x_2) \longleftrightarrow (y_1 \odot y_2)$;
$(birez_8)$ $(x_1 \longleftrightarrow y_1) \odot (x_2 \longleftrightarrow y_2) \leq (x_1 \longleftrightarrow x_2) \longleftrightarrow (y_1 \longleftrightarrow y_2)$.

Proof. $(birez_1) - (birez_3)$. Are immediate consequences of Theorem 1.1.

$(birez_4)$. By $lr - c_{10}$, $(x \to y) \odot (y \to z) \leq x \to z$, therefore $(x \longleftrightarrow y) \odot (y \longleftrightarrow z) \leq (x \to y) \odot (y \to z) \leq x \to z$. Similarly, $(x \longleftrightarrow y) \odot (y \longleftrightarrow z) \leq z \to x$. We conclude that $(x \longleftrightarrow y) \odot (y \longleftrightarrow z) \leq x \longleftrightarrow z$.

$(birez_5)$. If we denote $a = x_1 \longleftrightarrow y_1$ and $b = x_2 \longleftrightarrow y_2$, using the above rules of calculus we deduce $(a \wedge b) \odot (x_1 \wedge x_2) \leq [(x_1 \to y_1) \wedge (x_2 \to y_2)] \odot (x_1 \wedge x_2) \leq [(x_1 \to y_1) \odot x_1] \wedge [(x_2 \to y_2) \odot x_2] \leq y_1 \wedge y_2$, hence $a \wedge b \leq (x_1 \wedge x_2) \to (y_1 \wedge y_2)$.

Analogously we deduce $a \wedge b \leq (y_1 \wedge y_2) \to (x_1 \wedge x_2)$, hence $a \wedge b \leq (x_1 \wedge x_2) \longleftrightarrow (y_1 \wedge y_2)$.

$(birez_6)$. With the notations from $birez_5$ we have

$(a \wedge b) \odot (x_1 \vee x_2) = [(a \wedge b) \odot x_1] \vee [(a \wedge b) \odot x_2] \leq [(x_1 \to y_1) \odot x_1] \vee [(x_2 \to y_2) \odot x_2] \leq y_1 \wedge y_2$, hence $a \wedge b \leq (x_1 \vee x_2) \to (y_1 \vee y_2)$.

Analogously we deduce $a\wedge b\leq (y_1\vee y_2)\to (x_1\vee x_2)$, hence $a\wedge b\leq (x_1\vee x_2)\longleftrightarrow (y_1\vee y_2)$.

$(birez_7)$. We have $(a\odot b)\odot(x_1\odot x_2)\leq [(x_1\to y_1)\odot x_1]\odot[(x_2\to y_2)\odot x_2]\leq y_1\odot y_2$, hence $a\odot b\leq (x_1\odot x_2)\to (y_1\odot y_2)$.

Analogously we deduce that $a\odot b\leq (y_1\odot y_2)\to (x_1\odot x_2)$, so $a\odot b\leq (x_1\odot x_2)\longleftrightarrow (y_1\odot y_2)$.

$(birez_8)$. We have $(a\odot b)\odot (x_1\to x_2)\leq (y_1\to x_1)\odot (x_2\to y_2)\odot (x_1\to x_2)\leq (y_1\to x_2)\odot (x_2\to y_2)\leq y_1\wedge y_2$, and from here the proof is similary with the proof of $birez_5$. ■

PROPOSITION 1.10. *Let A be a residuated lattice and $x,y_1,y_2,z_1,z_2\in A$. If $x\leq y_1\longleftrightarrow y_2$ and $x\leq z_1\longleftrightarrow z_2$, then $x^2\leq (y_1\longleftrightarrow z_1)\longleftrightarrow (y_2\longleftrightarrow z_2)$.*

Proof. From $x\leq y_1\longleftrightarrow y_2\Rightarrow x\leq y_2\to y_1\Rightarrow x\odot y_2\leq y_1$ and analogously we deduce that $x\odot z_1\leq z_2$.

Then $x\odot x\leq (y_1\to z_1)\to (y_2\to z_2)\Leftrightarrow x\odot x\odot (y_1\to z_1)\leq (y_2\to z_2)\Leftrightarrow x\odot x\odot (y_1\to z_1)\odot y_2\leq z_2$.

Indeed, $x\odot x\odot (y_1\to z_1)\odot y_2\leq x\odot (y_1\to z_1)\odot y_1\leq x\odot z_1\leq z_2$ and analogously $x\odot x\leq (y_2\to z_2)\to (y_1\to z_1)$, therefore we obtain the inequality requested . ■

PROPOSITION 1.11. *Suppose A is complete and $x,x_i,y_i\in L$ $(i\in I)$. If $x\leq x_i\longleftrightarrow y_i$ for every $i\in I$, then $x\leq (\bigwedge\limits_{i\in I}x_i)\longleftrightarrow (\bigwedge\limits_{i\in I}y_i)$.*

Proof. Since $x\leq x_i\longleftrightarrow y_i$ for every $i\in I$, we deduce that $x\odot x_i\leq y_i$ and then $x\odot (\bigwedge\limits_{i\in I}x_i)\leq \bigwedge\limits_{i\in I}(x\odot x_i)\leq \bigwedge\limits_{i\in I}y_i$, hence $x\leq (\bigwedge\limits_{i\in I}x_i)\to (\bigwedge\limits_{i\in I}y_i)$.

Analogously, $x\leq (\bigwedge\limits_{i\in I}y_i)\to (\bigwedge\limits_{i\in I}x_i)$, therefore we obtain the requested inequality.

■

2. Boolean center of a residuated lattice

Let $(L,\vee,\wedge,0,1)$ be a bounded lattice. Recall (see [**73**]) that an element $a\in L$ is called *complemented* if there is an element $b\in L$ such that $a\vee b=1$ and $a\wedge b=0$; if such element b exists it is called a *complement* of a. We will denote $b=a'$ and the set of all complemented elements in L by $B(L)$. Complements are generally not unique, unless the lattice is distributive.

In residuated lattices however, although the underlying lattices need not be distributive, the complements are unique.

LEMMA 1.12. *Suppose that $a\in A$ have a complement $b\in A$. Then, the following hold:*

(i) If c is another complement of a in A, then $c=b$;
(ii) $a'=b$ and $b'=a$;
(iii) $a^2=a$.

Proof. See [**92**], Lema 1.3, p.14. ■

Let $B(A)$ the set of all complemented elements of the lattice $L(A)=(A,\wedge,\vee,0,1)$.

LEMMA 1.13. *If $e\in B(A)$, then $e'=e^*$ and $e^{**}=e$.*

Proof. If $e\in B(A)$, and $a=e'$, then $e\vee a=1$ and $e\wedge a=0$. Since $e\odot a\leq e\wedge a=0$, then $e\odot a=0$, hence $a\leq e\to 0=e^*$. On the another hand,

$e^* = 1 \odot e^* = (e \vee a) \odot e^* \overset{lr-c_{20}}{=} (e \odot e^*) \vee (a \odot e^*) = 0 \vee (a \odot e^*) = a \odot e^*$, hence $e^* \leq a$, that is $e^* = a$. The equality $e^{**} = e$ follows from Lemma 1.12, (ii).■

REMARK 1.5. *If $e, f \in B(A)$, then $e \wedge f, e \vee f \in B(A)$. Moreover, $(e \vee f)' = e' \wedge f'$ and $(e \wedge f)' = e' \vee f'$. So, $e \to f = e' \vee f \in B(A)$.*

Proof. See [**92**], Lema 1.7, p.15. ■

LEMMA 1.14. *If $e \in B(A)$, then*

$(lr - c_{35})$ *$e \odot x = e \wedge x$, for every $x \in A$.*

Proof. See [**92**], Lema 1.6, p.15. ■

COROLLARY 1.15. *The set $B(A)$ is the universe of a Boolean subalgebra of A (called the* Boolean center *of A).*

Proof. We prove that for any $x, y, z \in B(A)$, the distributive law holds. By $lr - c_{35}$ and properties of residuated lattices, we ave the following series of identities: $x \wedge (y \vee z) = x \odot (y \vee z) = (x \odot y) \vee (x \odot z) = (x \wedge y) \vee (x \wedge z)$. ■

PROPOSITION 1.16. *For $e \in A$ the following are equivalent:*

(i) $e \in B(A)$;
(ii) $e \vee e^* = 1$.

Proof. $(i) \Rightarrow (ii)$. If $e \in B(A)$, by Lemma 1.13, $e \vee e' = e \vee e^* = 1$.

$(ii) \Rightarrow (i)$. Suppose that $e \vee e^* = 1$. We have: $0 = 1^* = (e \vee e^*)^* \overset{lr-c_{26}}{=} e^* \wedge e^{**} \geq e^* \wedge e$, (by $lr - c_{16}$), hence $e^* \wedge e = 0$, that is, $e \in B(A)$.■

DEFINITION 1.4. A totally ordered (linearly ordered) residuated lattice will be called *chain* .

REMARK 1.6. *If A is a chain, then $B(A) = \{0, 1\}$.*

PROPOSITION 1.17. *For $e \in A$ we consider the following assertions:*

(1) $e \in B(A)$;
(2) $e^2 = e$ *and* $e = e^{**}$;
(3) $e^2 = e$ *and* $e^* \to e = e$;
(4) $(e \to x) \to e = e$, *for every* $x \in A$;
(5) $e \wedge e^* = 0$.

Then:

(i) $(1) \Rightarrow (2), (3), (4)$ *and* (5),
(ii) $(2) \nRightarrow (1), (3) \nRightarrow (1), (4) \nRightarrow (1), (5) \nRightarrow (1)$,
(iii) *If A is a $BL-$algebra then the conditios* $(1) - (5)$ *are equivalent.*

Proof. (i). $(1) \Rightarrow (2)$. Follows from Lemma 1.12 (iii), and Lemma 1.13.

$(1) \Rightarrow (3)$. If $e \in B(A)$, then $e \vee e^* = 1$. Since $1 = e \vee e^* \leq [(e \to e^*) \to e^*] \wedge [(e^* \to e) \to e]$, by $lr - c_6$ and $lr - c_1$.

We deduce that $(e \to e^*) \to e^* = (e^* \to e) \to e = 1$, hence $e \to e^* \leq e^*$ and $e^* \to e \leq e$ (by $lr - c_4$), that is, $e \to e^* = e^*$ and $e^* \to e = e$ (by $lr - c_2$).

$(1) \Rightarrow (4)$. If $x \in A$, then from $0 \leq x$ we deduce $e^* \leq e \to x$ hence $(e \to x) \to e \leq e^* \to e = e$, by $(1) \Rightarrow (3)$. Since $e \leq (e \to x) \to e$ we obtain $(e \to x) \to e = e$.

$(1) \Rightarrow (5)$. Follows from Proposition 1.16 (since by Lemma 1.13, $e' = e^*$).

(ii). Consider the residuated lattice $A = \{0, a, b, c, 1\}$ from the Example 1.6; it is easy to verify that $B(A) = \{0, 1\}$.

(2) $\not\Rightarrow$ (1). We have $a^2 = a, a^* = b, b^* = a$, hence $a^{**} = b^* = a$, but $a \notin B(A)$.

(3) $\not\Rightarrow$ (1). We have $a^2 = a$ and $a^* \to a = b \to a = a$, but $a \notin B(A)$.

(4) $\not\Rightarrow$ (1). It is easy to verify that $(a \to x) \to a = a$ for every $x \in A$, but $a \notin B(A)$.

(5) $\not\Rightarrow$ (1). We have $a \wedge a^* = a \wedge b = 0$, but $a \vee a^* = a \vee b = c \neq 1$, hence $a \notin B(A)$. ■

REMARK 1.7.
1. *If $A = \{0, a, b, c, 1\}$, is the residuated lattice from Example 1.6, then $B(A) = \{0, 1\}$;*
2. *If $A = \{0, a, b, c, d, e, f, 1\}$, is the residuated lattice from Example 1.7, then $B(A) = \{0, 1\}$; also $B(A) = \{0, 1\}$, where A is the residuated lattice from Example 1.8;*
3. *If $A = \{0, a, b, c, d, 1\}$, is the residuated lattice from Example 1.9, then $B(A) = \{0, a, d, 1\}$;*
4. *If $A = \{0, a, b, c, d, e, f, g, 1\}$, is the residuated lattice from Example 1.10, then $B(A) = \{0, b, f, 1\}$;*
5. *If $A = \{0, a, b, c, d, 1\}$, is the residuated lattice from Example 1.11, then $B(A) = \{0, b, c, 1\}$.*

LEMMA 1.18. *If $e, f \in B(A)$ and $x, y \in A$, then:*

$(lr - c_{36})$ $x \odot (x \to e) = e \wedge x, e \odot (e \to x) = e \wedge x$;
$(lr - c_{37})$ $e \vee (x \odot y) = (e \vee x) \odot (e \vee y)$;
$(lr - c_{38})$ $e \wedge (x \odot y) = (e \wedge x) \odot (e \wedge y)$;
$(lr - c_{39})$ $e \odot (x \to y) = e \odot [(e \odot x) \to (e \odot y)]$;
$(lr - c_{40})$ $x \odot (e \to f) = x \odot [(x \odot e) \to (x \odot f)]$;
$(lr - c_{41})$ $e \to (x \to y) = (e \to x) \to (e \to y)$.

Proof. $(lr - c_{36})$. Since $e \leq x \to e$, then $x \odot e \leq x \odot (x \to e)$, hence $x \wedge e \leq x \odot (x \to e)$. From $x \odot (x \to e) \leq x, e$ we deduce the another inequality $x \odot (x \to e) \leq x \wedge e$, so $x \odot (x \to e) = e \wedge x$.

Analogous for the sequend equality.

$(lr - c_{37})$. We have

$$(e \vee x) \odot (e \vee y) \overset{lr-c_{20}}{=} [(e \vee x) \odot e] \vee [(e \vee x) \odot y] = [(e \vee x) \odot e] \vee [(e \odot y) \vee (x \odot y)]$$
$$= [(e \vee x) \wedge e] \vee [(e \odot y) \vee (x \odot y)] = e \vee (e \odot y) \vee (x \odot y) = e \vee (x \odot y).$$

$(lr - c_{38})$. As above,

$$(e \wedge x) \odot (e \wedge y) = (e \odot x) \odot (e \odot y) = (e \odot e) \odot (x \odot y) = e \odot (x \odot y) = e \wedge (x \odot y).$$

$(lr - c_{39})$. By $lr - c_7$ we have $x \to y \leq (e \odot x) \to (e \odot y)$, hence $e \odot (x \to y) \leq e \odot [(e \odot x) \to (e \odot y)]$.

Conversely, $(e \odot x) \odot [(e \odot x) \to (e \odot y)] \leq e \odot y \leq y$ so $e \odot [(e \odot x) \to (e \odot y)] \leq x \to y$. Hence $e \odot [(e \odot x) \to (e \odot y)] \leq e \odot (x \to y)$.

$(lr - c_{40})$. We have $x \odot [(x \odot e) \to (x \odot f)] = x \odot [(x \odot e) \to (x \wedge f)] \overset{lr-c_{22}}{=}$
$x \odot [((x \odot e) \to x) \wedge ((x \odot e) \to f)] = x \odot [1 \wedge ((x \odot e) \to f)] = x \odot ((x \odot e) \to f) \overset{lr-c_{13}}{=}$
$\overset{lr-c_{13}}{=} x \odot [x \to (e \to f)] \overset{lr-c_{36}}{=} x \wedge (e \to f) = x \odot (e \to f)$, since $e \to f \in B(A)$, see Remark 1.5.

$(lr - c_{41})$. Follows from $lr - c_{13}$ and $lr - c_{36}$ since $e \wedge x = e \odot x$.■

COROLLARY 1.19. *If $e \in B(A)$ and $x, y \in A$, then:*
$(lr - c_{42})$ $e \wedge (x \vee y) = (e \wedge x) \vee (e \wedge y)$.

DEFINITION 1.5. Let A and B be residuated lattices. $f : A \to B$ is a *morphism of residuated lattices* if f is morphism of bounded lattices and for every $x, y \in A$: $f(x \odot y) = f(x) \odot f(y)$ and $f(x \to y) = f(x) \to f(y)$.

Following current usage, if f is one-one we shall equivalently say that f is an *injective homomorphism* , or an *embedding* . If the homomorphism $f : A \to B$ is onto, we say that f is *surjective* . A bijective morphism of residuated lattices will be called *isomorphism* of residuated lattices (we write $A \approx B$). The *kernel* of homomorphism $f : A \to B$ is the set $Ker(f) = f^{-1}(0) = \{x \in A : f(x) = 0\}$.

DEFINITION 1.6. A *Heyting algebra* is a lattice $(L, \vee, \wedge)$ with 0 such that for every $a, b \in L$, there exists an element $a \to b \in L$ (called the *pseudocomplement of a with respect to b*) such that for every $x \in L$, $a \wedge x \leq b$ iff $x \leq a \to b$ (that is, $a \to b = \sup\{x \in L : a \wedge x \leq b\}$).

DEFINITION 1.7. Following Diego ([**51**]), by *Hilbert algebra* we mean an algebra $(A, \to, 1)$ of type $(2, 0)$ satisfying the following identities:
(H_1) $x \to (y \to x) = 1$;
(H_2) $(x \to (y \to z)) \to ((x \to y) \to (x \to z)) = 1$;
(H_3) If $x \to y = y \to x = 1$, then $x = y$.

REMARK 1.8. *([**51**]) If $(L, \vee, \wedge, \to, 0)$ is a Heyting algebra, then $(L, \to, 1)$ is a Hilbert algebra, where $1 = a \to a$ for an element $a \in L$.*

Taking as a guide -line the case of $BL-$ algebras (see Example 3.9), a residuated lattice A will be called *G- algebra* if $x^2 = x$, for every $x \in A$.

REMARK 1.9. *In a G-algebra A, $x \odot y = x \wedge y$ for every $x, y \in A$.*

PROPOSITION 1.20. *In a residuated lattice A the following assertions are equivalent :*
(i) *$x^2 = x$ for every $x \in A$;*
(ii) *$x \odot (x \to y) = x \odot y = x \wedge y$ for every $x, y \in A$.*

Proof. $(i) \Rightarrow (ii)$. Let $x, y \in A$. By $lr - c_{12}$ we have

$$x \odot (x \to y) \leq (x \odot x) \to (x \odot y) \Leftrightarrow x \odot (x \to y) \leq x \to (x \odot y) \Leftrightarrow$$

$$x \to y \leq x \to (x \to (x \odot y)) = x^2 \to (x \odot y) = x \to (x \odot y) \Rightarrow$$

$$x \odot (x \to y) \leq x \odot y.$$

Since $y \leq x \to y$, then $x \odot y \leq x \odot (x \to y)$, so $x \odot (x \to y) \leq x \odot y$.
Clearly, $x \odot y \leq x, y$. To prove $x \odot y = x \wedge y$, let $t \in A$ such that $t \leq x$ and $t \leq y$. Then $t = t^2 \leq x \odot y$, that is, $x \odot y = x \wedge y$.
$(ii) \Rightarrow (i)$. In particular for $x = y$ we obtain $x \odot x = x \wedge x = x \Leftrightarrow x^2 = x$. ■

PROPOSITION 1.21. *For a residuated lattice $(A, \wedge, \vee, \odot, \to, 0, 1)$ the following are equivalent:*
(i) *$(A, \to, 1)$ is a Hilbert algebra;*
(ii) *$(A, \wedge, \vee, \odot, \to, 0, 1)$ is a G-algebra.*

Proof. $(i) \Rightarrow (ii)$. Suppose that $(A, \to, 1)$ is a Hilbert algebra, then for every $x, y, z \in A$ we have

$$x \to (y \to z) = (x \to y) \to (x \to z).$$

From $lr - c_{13}$ we have

$$x \to (y \to z) = (x \odot y) \to z \text{ and } (x \to y) \to (x \to z) = (x \odot (x \to y)) \to z,$$

so we obtain

$$(x \odot y) \to z = (x \odot (x \to y)) \to z$$

hence $x \odot y = x \odot (x \to y)$; for $x = y$ we obtain $x^2 = x$, that is, A is a G- algebra.

$(ii) \Rightarrow (i)$. Follows from Proposition 1.20. ■

3. The lattice of deductive systems of a residuated lattice

In this section we put in evidence the congruences of a residuated lattice and present some results relative to lattice of deductive systems of a residuated lattice. We characterize the subdirectly irreducible residuated lattices and the residuated lattices for which the lattice of deductive systems is a Boolean algebra.

DEFINITION 1.8. A non empty subset $D \subseteq A$ is called a *deductive system* of A, **ds** for short, if the following conditions are satisfied*:*

(Ds_1) $1 \in D$;
(Ds_2) If $x, x \to y \in D$, then $y \in D$.

Clearly $\{1\}$ and A are **ds** ; a **ds** D of A is called *proper* if $D \neq A$.

REMARK 1.10.
1. *A **ds** D is proper iff $0 \notin D$ iff no element $x \in A$ holds $x, x^* \in D$;*
2. *$x \in D$ iff $x^n \in D$ for every $n \geq 1$.*

REMARK 1.11. *A nonempty subset $D \subseteq A$ is a **ds** of A iff for all $x, y \in A$:*

(Ds_1') *If x, $y \in D$, then $x \odot y \in D$;*
(Ds_2') *If $x \in D, y \in A, x \leq y$, then $y \in D$.*

Indeed, assume that $D \subseteq A, D \neq \emptyset$ is a subset of A satisfy Ds_1' and Ds_2'. In such case there is an element $x \in D \subseteq A$ and as $x \leq 1$ we have $1 \in D$. Assume $x, x \to y \in D$. Then $x \odot (x \to y) \leq y \in D$ and so D is a **ds**. Let conversely, D be a **ds**. Assume $x, y \in D$. Since $x \to [y \to (x \odot y)] = 1 \in D$, we have $y \to (x \odot y) \in D$, therefore $x \odot y \in D$. Thus Ds_1 holds. To verify Ds_2 let $x \in D, x \leq y$. Then $x \to y = 1 \in D$, hence $y \in D$.

REMARK 1.12. *Deductive systems are called also* implicative (or congruence) filters *in literature. To avoid confusion we reserve, however, the name filter to lattice filters in this paper. From $lr - c_2$ and Remark 1.11 we deduce that every **ds** of A is a filter for $L(A)$, but filters of $L(A)$ are not, in general, deductive systems for A (see* [**129**]*).*

We denote by $Ds(A)$ the set of all deductive systems of A.

In wath follows we will take in consideration the connections between the congruences of a residuated lattice A and the implicative filtres (deductive systems) of A.

Whith any deductive systems D of A we can associate a congruence θ_D on A by defining : $(a, b) \in \theta_D$ iff $a \to b, b \to a \in D$ iff $(a \to b) \odot (b \to a) \in D$. Conversely,

for $\theta \in Con(A)$, the subset D_θ of A defined by $a \in D_\theta$ iff $(a, 1) \in \theta$ is a deductive system of A. Moreover the natural maps associated whith the above are mutually inverse and establish an isomorphism between the lattices $Ds(A)$ and $Con(A)$.

So, as in the case of lattices we have the following result:

THEOREM 1.22. *Let A be a residuated lattice, $D \in Ds(A)$ and $\theta \in Con(A)$. Then*

(*i*) *$\theta_D \in Con(A)$ and $D_\theta \in Ds(A)$;*
(*ii*) *The assignments $D \rightsquigarrow \theta_D$ and $\theta \rightsquigarrow D_\theta$ give a laticeal isomorphisms between $Ds(A)$ and $Con(A)$.*

For $a \in A$, let a/D be the equivalence class of a modulo θ_D. If we denote by A/D the quotient set A/θ_D, then A/D becomes a residuated lattice with the natural operations induced from those of A. Clearly, in A/D, $\mathbf{0} = 0/D$ and $\mathbf{1} = 1/D$.

PROPOSITION 1.23. *Let $D \in Ds(A)$, and $a, b \in A$, then*

(*i*) *$a/D = 1/D$ iff $a \in D$, hence $a/D \neq \mathbf{1}$ iff $a \notin D$;*
(*ii*) *$a/D = 0/D$ iff $a^* \in D$;*
(*iii*) *If D is proper and $a/D = 0/D$, then $a \notin D$;*
(*iv*) *$a/D \leq b/D$ iff $a \to b \in D$.*

Proof. (*i*). We have $a/D = 1/D$ iff $(a \to 1) \odot (1 \to a) \in D$ iff $1 \odot a = a \in D$.
(*ii*). We have $a/D = 0/D$ iff $(a \to 0) \odot (0 \to a) \in D$ iff $a^* \odot 1 = a^* \in D$.
(*iii*). Follow from Remark 1.10.
(*iv*). By $lr - c_4$ we have $a/D \leq b/D$ iff $a/D \to b/D = \mathbf{1}$ iff $(a \to b)/D = 1/D$ iff $a \to b \in D$ (by (*i*)). ■

We recall (see [**22**]) some fundamental concepts of Universal Algebra.

Let A and $(A_i)_{i \in I}$ be algebras of the same type. A *subdirect representation* of A with factors A_i is an embedding $f : A \to \prod_{i \in I} A_i$ such that each f_i defined by $f_i = \pi_i \circ f$ is onto A_i, for each $i \in I$. Here, π_i denotes the $i-$ th projection. Such an A is also called subdirect product of A_i.

An algebra A is *subdirectly irreducible* (**si** for short) iff it is non-trivial and for any subdirect representation $f : A \to \prod_{i \in I} A_i$, there exists a j such that f_j is an isomorphism of A onto A_j. A *fundamental subdirect representation theorem* of Birkhoff's says that every algebra has a subdirect representation with **si** factors.

Two other important types of algebras (see [**22**], Chapter 3) are: *directly indecomposable algebras*, i.e., these that cannot be nontrivially represented as direct products and *simple algebras,* i. e., these that have two-element congruence lattices (see [**22**], p.89).

Clearly, *simple* implies **si** implies *directly indecomposable*; neither of the converse implications holds in general.

By Proposition 1.23 it follows immediately that a residuated lattice A is subdirectly irreducible iff it has the second smallest **ds**, i.e. the smallest **ds** among all **ds** except $\{1\}$ (see and [**18**]).

The next theorem characterises internally subdirectly irreducible and simple residuated lattices.

THEOREM 1.24. ([**92**]) *A residuated lattice A is:*

(*i*) *subdirectly irreducible iff there exists an element $a < 1$ such that for any $x < 1$ there exists a natural number $n \geq 1$ such that $x^n \leq a$;*

(*ii*) *simple iff a can be taken to be* 0.

COROLLARY 1.25. ([**18**], [**92**]) *If A is subdirectly irreducible, then* $B(A) = \{0, 1\}$.

PROPOSITION 1.26. ([**92**]) *In any* ***si*** *residuated lattice, if* $x \vee y = 1$, *then either* $x = 1$ *or* $y = 1$ *holds.*

Therefore, every **si** residuated lattice has at most one *coatom* (recall that are element a of a lattice L with the greatest element 1 is a coatom if it is maximal among elements in $L\backslash\{1\}$).

The next result characterises these **si** residuated lattices which have the coatom:

THEOREM 1.27. ([**91**]) *A residuated lattice A has the unique coatom iff there exists an element* $a < 1$ *and a natural number n such that* $x^n \leq a$ *holds for any* $x < 1$.

Directly indecomposable residuated lattices also have quite a handly description. It was obtained for a subvariety of residuated latticers, called product algebras, by Cignoli and Torrens in [**46**].

For arbitrary residuated lattices we have:

THEOREM 1.28. ([**92**]) *A nontrivial residuated lattice A is directly indecomposable iff* $B(A) = \{0, 1\}$.

REMARK 1.13. *The lattices from Examples 1.6, 1.7 and 1.8 are directly indecomposable.*

For a nonempty subset $S \subseteq A$, the smallest **ds** of A which contains S, i.e. $\cap\{D \in Ds(A) : S \subseteq D\}$, is said to be *the* ***ds*** *of A generated by S* and will be denoted by $[S)$.

If $S = \{a\}$, with $a \in A$, we denote by $[a)$ the **ds** *generated by* $\{a\}$ ($[a)$ is called *principal*).

For $D \in Ds(A)$ and $a \in A$, we denote by $D(a) = [D \cup \{a\})$ (clearly, if $a \in D$, then $D(a) = D$).

PROPOSITION 1.29. *Let* $S \subseteq A$ *a nonempty subset of A,* $a \in A$, $D, D_1, D_2 \in Ds(A)$. *Then*

(*i*) *If S is a deductive system, then* $[S) = S$;
(*ii*) $[S) = \{x \in A : s_1 \odot ... \odot s_n \leq x$, *for some* $n \geq 1$ *and* $s_1, ..., s_n \in S\}$. *In particular,* $[a) = \{x \in A : x \geq a^n$, *for some* $n \geq 1\}$;
(*iii*) $D(a) = \{x \in A : x \geq d \odot a^n$, *whith* $d \in D$ *and* $n \geq 1\}$;
(*iv*) $[D_1 \cup D_2) = \{x \in A : x \geq d_1 \odot d_2$ *for some* $d_1 \in D_1$ *and* $d_2 \in D_2\}$;

Proof. (*i*). Obvioulsly.

(*ii*). If we denote by S' the set from the right part of equality from enounce, it is immediate that this is an deductive system which contain the set S, hence $[S) \subseteq S'$. Let now $D \in Ds(A)$ such that $S \subseteq D$ and $x \in S'$. Then there are $s_1, ..., s_n \in S$ such that $s_1 \odot ... \odot s_n \leq x$. Since $s_1, ..., s_n \in D \Rightarrow s_1 \odot ... \odot s_n \in D \Rightarrow x \in D$, hence $S' \subseteq D$; we deduce that $S' \subseteq \cap D = [S)$, that is, $[S) = S'$.

(*iii*), (*iv*). Following by (*ii*). ■

LEMMA 1.30. *Let* $D \in Ds(A)$ *and* $a \in A$. *Then* $D(a) = \{x \in A : a^n \to x \in D$, *for some* $n \geq 1\}$.

Proof. If $x \in D(a)$, then $x \geq d \odot a^n$, for some $n \geq 1$ and $d \in D$. Thus, $d \leq a^n \to x$, so $a^n \to x \in D$.

Conversely, assume that $d = a^n \to x \in D$ for some $n \geq 1$. We also have $(a^n \odot d) \to x = d \to (a^n \to x) = d \to d = 1$, hence $a^n \odot d \leq x$. Therefore, $x \in D(a)$. ■

PROPOSITION 1.31. *For any element x of a residuated lattice A, there is a proper ds D of A such that $x \in D$ iff $ord(x) = \infty$.*

Proof. Let D be a proper **ds** and $x \in D$. Then $x^n \in D$, whence $x^n \neq 0$ for any natural number n. Therefore $ord(x) = \infty$. Conversely, if $ord(x) = \infty$, then $D = [x) = \{y \in A : x^n \leq y \text{ for some natural number } n\}$ is a proper **ds** of A and $x \in D$. ■

For D_1, $D_2 \in Ds(A)$ we put

$$D_1 \wedge D_2 = D_1 \cap D_2 \text{ and } D_1 \vee D_2 = [D_1 \cup D_2).$$

PROPOSITION 1.32. *If $a, b \in A$, then*

(*i*) $[a) = \{x \in A : a \leq x\}$ *iff* $a \odot a = a$;
(*ii*) $a \leq b$ *implies* $[b) \subseteq [a)$;
(*iii*) $[a) \cap [b) = [a \vee b)$;
(*iv*) $[a) \vee [b) = [a \wedge b) = [a \odot b)$;
(*v*) $[a) = 1$ *iff* $a = 1$.

Proof. (*i*), (*ii*). Obviously.

(*iii*). Since $a, b \leq a \vee b$, by (*ii*), $[a \vee b) \subseteq [a), [b)$, hence $[a \vee b) \subseteq [a) \cap [b)$. Let now $x \in [a) \cap [b)$; then $x \geq a^m, x \geq b^n$ for some natural numbers $m, n \geq 1$, hence $x \geq a^m \vee b^n \geq (a \vee b)^{mn}$, (by $lr - c_{30}$), so $x \in [a \vee b)$, that is, $[a) \cap [b) \subseteq [a \vee b)$. Hence $[a) \cap [b) = [a \vee b)$.

(*iv*). Since $a \odot b \leq a \wedge b \leq a, b$, by (*ii*), we deduce that $[a), [b) \subseteq [a \wedge b) \subseteq [a \odot b)$, hence $[a) \vee [b) \subseteq [a \wedge b) \subseteq [a \odot b)$.

For the converse inclusions, let $x \in [a \odot b)$. Then for some natural number $n \geq 1$, $x \geq (a \odot b)^n = a^n \odot b^n \in [a) \vee [b)$ (since $a^n \in [a), b^m \in [b)$), (by Proposition 1.29, (*ii*)), hence $x \in [a) \vee [b)$, that is, $[a \odot b) \subseteq [a) \vee [b)$, so $[a) \vee [b) = [a \wedge b) = [a \odot b)$.

(*v*). Obviously. ■

DEFINITION 1.9. We recall ([**73**], p.93) that a lattice $(L, \vee, \wedge)$ is called *Brouwerian* if it satisfies the identity $a \wedge (\bigvee_i b_i) = \bigvee_i (a \wedge b_i)$) (whenever the arbitrary unions exists). Let L be a complete lattice and let a be an element of L. Then a is called *compact* if $a \leq \vee X$ for some $X \subseteq L$ implies that $a \leq \vee X_1$ for some finite $X_1 \subseteq X$. A complete lattice is called *algebraic* if every element is the join of compact elements (in the literature, algebraic lattices are also called *compactly generated lattices).*

PROPOSITION 1.33. *The lattice $(Ds(A), \subseteq)$ is a complete Brouwerian lattice (hence distributive), the compacts elements being exactly the principal **ds** of A.*

Proof. Clearly, if $(D_i)_{i \in I}$ is a family of **ds** from A, then the infimum of this family is $\underset{i \in I}{\wedge} D_i = \underset{i \in I}{\cap} D_i$ and the supremum is $\underset{i \in I}{\vee} D_i = [\underset{i \in I}{\cup} D_i) = \{x \in A : x \geq x_{i_1} \odot ... \odot x_{i_m}$, where $i_1, ..., i_m \in I, x_{i_j} \in D_{i_j}$, $1 \leq j \leq m\}$, that is, $Ds(A)$ is complete.

We will to prove that the compacts elements of $Ds(A)$ are exactly the principal **ds** of A. Let D be a compact element of $Ds(A)$. Since $D = \bigvee\limits_{a\in D}[a)$, there are $m \geq 1$ and $a_1, ..., a_m \in A$ such that $D = [a_1) \vee ... \vee [a_m) = [a_1 \odot .... \odot a_m)$, (by Proposition 1.32, (iv)). Hence D is a principal **ds** of A.

Conversely, let $a \in A$ and $(D_i)_{i\in I}$ be a family of **ds** of A such that $[a) \subseteq \bigvee\limits_{i\in I} D_i$. Then $a \in \bigvee\limits_{i\in I} D_i = [\bigcup\limits_{i\in I} D_i)$, so we deduce that there are $m \geq 1$, $i_1, ..., i_m \in I, x_{i_j} \in D_{i_j}$ $(1 \leq j \leq m)$ such that $a \geq x_{i_1} \odot ... \odot x_{i_m}$.

It follows that $a \in [D_{i_1} \cup ... \cup D_{i_m})$, so $[a) \subseteq [D_{i_1} \cup ... \cup D_{i_m}) = D_{i_1} \vee ... \vee D_{i_m}$.

For any **ds** D we have $D = \bigvee\limits_{a\in D}[a)$, so the lattice $Ds(A)$ is algebraic.

In order to prove that $Ds(A)$ is Brouwerian we must show that for every **ds** D and every family $(D_i)_{i\in I}$ of **ds**, $D \wedge (\bigvee\limits_{i\in I} D_i) = \bigvee\limits_{i\in I}(D \wedge D_i) \Leftrightarrow D \cap (\bigvee\limits_{i\in I} D_i) = [\bigcup\limits_{i\in I}(D \cap D_i))$. Clearly, $[\bigcup\limits_{i\in I}(D \cap D_i)) \subseteq D \cap (\bigvee\limits_{i\in I} D_i)$.

Let now $x \in D \cap (\bigvee\limits_{i\in I} D_i)$. Then $x \in D$ and there exist $i_1, ..., i_m \in I, x_{i_j} \in D_{i_j}$ $(1 \leq j \leq m)$ such that $x \geq x_{i_1} \odot ... \odot x_{i_m}$. Then $x = x \vee (x_{i_1} \odot ... \odot x_{i_m}) \geq (x \vee x_{i_1}) \odot .... \odot (x \vee x_{i_m})$ (by $lr - c_{30}$). Since $x \vee x_{i_j} \in D \cap D_{i_j}$, for every $1 \leq j \leq m$ we deduce that $x \in \bigvee\limits_{i\in I}(D \cap D_i)$, hence $D \cap (\bigvee\limits_{i\in I} D_i) \subseteq \bigvee\limits_{i\in I}(D \cap D_i)$, that is, $D \cap (\bigvee\limits_{i\in I} D_i) = \bigvee\limits_{i\in I}(D \cap D_i)$.■

COROLLARY 1.34. *If we denote by $Ds_p(A)$ the family of all principal **ds** of A, then $Ds_p(A)$ is a bounded sublattice of $Ds(A)$.*

Proof. Apply Proposition 1.32, (iii), (iv) and the fact that $\{1\} = [1) \in Ds_p(A)$ and $A = [0) \in Ds_p(A)$. ■

For D_1, $D_2 \in Ds(A)$ we put

$$D_1 \to D_2 = \{a \in A : D_1 \cap [a) \subseteq D_2\}.$$

LEMMA 1.35. *If D_1, $D_2 \in Ds(A)$ then*

(i) $D_1 \to D_2 \in Ds(A)$;

(ii) If $D \in Ds(A)$, then $D_1 \cap D \subseteq D_2$ iff $D \subseteq D_1 \to D_2$, that is,

$$D_1 \to D_2 = \sup\{D \in Ds(A) : D_1 \cap D \subseteq D_2\}.$$

Proof. (i). Since $[1) = \{1\}$ and $[1) \cap D_1 = \{1\} \subseteq D_2$ we deduce that $1 \in D_1 \to D_2$.

Let x, $y \in A$ such that $x \leq y$ and $x \in D_1 \to D_2$, that is, $[x) \cap D_1 \subseteq D_2$. Then $[y) \subseteq [x)$, so $[y) \cap D_1 \subseteq [x) \cap D_1 \subseteq D_2$, hence $[y) \cap D_1 \subseteq D_2$, that is, $y \in D_1 \to D_2$.

To proof that (Ds_1') is verified, let x, $y \in A$ such that $x, y \in D_1 \to D_2$, hence

$$[x) \cap D_1 \subseteq D_2 \text{ and } [y) \cap D_1 \subseteq D_2.$$

We deduce $([x) \cap D_1) \vee ([y) \cap D_1) \subseteq D_2$, hence by Proposition 1.33, $([x) \vee [y)) \cap D_1 \subseteq D_2$. By Proposition 1.32 we deduce that $[x \odot y) \cap D_1 \subseteq D_2$, hence, $x \odot y \in D_1 \to D_2$, that is, $D_1 \to D_2 \in Ds(A)$.

(ii). Suppose $D_1 \cap D \subseteq D_2$ and let $x \in D$. Then $[x) \subseteq D$, hence $[x) \cap D_1 \subseteq D \cap D_1 \subseteq D_2$, so $x \in D_1 \to D_2$, that is, $D \subseteq D_1 \to D_2$.

Suppose $D \subseteq D_1 \to D_2$ and let $x \in D_1 \cap D$. Then $x \in D$, hence $x \in D_1 \to D_2$, that is, $[x) \cap D_1 \subseteq D_2$. Since $x \in [x) \cap D_1 \subseteq D_2$ we obtain $x \in D_2$, that is, $D_1 \cap D \subseteq D_2$. ■

For D_1, $D_2 \in Ds(A)$, we denote

$$D_1 * D_2 = \{x \in A : x \vee y \in D_2, \text{ for all } y \in D_1\}.$$

PROPOSITION 1.36. *For all* D_1, $D_2 \in Ds(A), D_1 * D_2 = D_1 \to D_2$.

Proof. Let $x \in D_1 * D_2$ and $z \in [x) \cap D_1$, that is, $z \in D_1$ and $z \geq x^n$ for some $n \geq 1$. Then $x \vee z \in D_2$. Since $z = z \vee x^n \geq (z \vee x)^n$ (by $lr - c_{30}$) we deduce that $z \in D_2$, hence $x \in D_1 \to D_2$, so $D_1 * D_2 \subseteq D_1 \to D_2$.

For converse inclusion, let $x \in D_1 \to D_2$. Thus $[x) \cap D_1 \subseteq D_2$, so, if $y \in D_1$ then $x \vee y \in [x) \cap D_1$, hence $x \vee y \in D_2$. We deduce that $x \in D_1 * D_2$, so $D_1 \to D_2 \subseteq D_1 * D_2$. Since $D_1 * D_2 \subseteq D_1 \to D_2$ we deduce that $D_1 * D_2 = D_1 \to D_2$. ■

COROLLARY 1.37. $(Ds(A), \vee, \wedge, \to, \{1\})$ *is a Heyting algebra, where for* $D \in Ds(A)$,

$$D^* = D \to \mathbf{0} = D \to \{1\} = \{x \in A : x \vee y = 1, \text{ for every } y \in D\},$$

hence for every $x \in D$ *and* $y \in D^*, x \vee y = 1$. *In particular, for every* $a \in A$,

$$[a)^* = \{x \in A : x \vee a = 1\}.$$

PROPOSITION 1.38. *If* $x, y \in A$, *then* $[x \odot y)^* = [x)^* \cap [y)^*$.

Proof. If $a \in [x \odot y)^*$, then $a \vee (x \odot y) = 1$. Since $x \odot y \leq x, y$ then $a \vee x = a \vee y = 1$, hence $a \in [x)^* \cap [y)^*$, that is, $[x \odot y)^* \subseteq [x)^* \cap [y)^*$.

Let now $a \in [x)^* \cap [y)^*$, that is, $a \vee x = a \vee y = 1$.

By $lr - c_{30}$ we deduce $a \vee (x \odot y) \geq (a \vee x) \odot (a \vee y) = 1$, hence $a \vee (x \odot y) = 1$, that is, $a \in [x \odot y)^*$.

It follows that $[x)^* \cap [y)^* \subseteq [x \odot y)^*$, hence $[x \odot y)^* = [x)^* \cap [y)^*$. ■

THEOREM 1.39. *If* A *is a residuated lattice, then the following assertions are equivalent:*

(*i*) $(Ds(A), \vee, \wedge, ^*, \{1\}, A)$ *is a Boolean algebra;*
(*ii*) *Every* ***ds*** *of* A *is principal and for every* $a \in A$ *there exists* $n \geq 1$ *such that* $a \vee (a^n)^* = 1$.

Proof. $(i) \Rightarrow (ii)$. Let $D \in Ds(A)$; since $Ds(A)$ is supposed Boolean algebra, then $D \vee D^* = A$. So, since $0 \in A$, there exist $a \in D$, $b \in D^*$ such that $a \odot b = 0$.

Since $b \in D^*$, by Corollary 1.37, it follow that $a \vee b = 1$. By $lr - c_{28}$ we deduce that $a \wedge b = a \odot b = 0$, that is, b is the complement of a in $L(A)$. Hence $a, b \in B(A) = B(L(A))$.

If $x \in D$, since $b \in D^*$, we have $b \vee x = 1$. Since $a = a \wedge (b \vee x) \overset{lr-c_{42}}{=} (a \wedge b) \vee (a \wedge x) = a \wedge x$ we deduce that $a \leq x$, that is, $D = [a)$. Hence every **ds** of A is principal.

Let now $x \in A$; since $Ds(A)$ is a Boolean algebra, then $[x) \vee [x)^* = A \Leftrightarrow [x)^*(x) = A \Leftrightarrow \{a \in A : a \geq c \odot x^n$, with $c \in [x)^*$ and $n \geq 1\} = A$ (see Proposition 1.29, (ii)).

So, since $0 \in A$, there exist $c \in [x)^*$ and $n \in \omega$ such that $c \odot x^n = 0$. Since $c \in [x)^*$, then $x \vee c = 1$. By $lr - c_{15}$, from $c \odot x^n = 0$ we deduce $c \leq (x^n)^*$. So, $1 = x \vee c \leq x \vee (x^n)^*$, hence $x \vee (x^n)^* = 1$.

$(ii) \Rightarrow (i)$. By Corollary 1.37, $Ds(A)$ is a Heyting algebra. To prove $Ds(A)$ is a Boolean algebra, we must show that for $D \in Ds(A)$, $D^* = \{1\}$ only for $D = A$ ([**2**], p. 175). By hypothesis every **ds** of A is principal, so we have $a \in A$ such that $D = [a)$.

Also, by hypothesis, for $a \in A$, there is $n \in \omega$ such that $a \vee (a^n)^* = 1$. By Corollary 1.37, $(a^n)^* \in [a)^* = \{1\}$, hence $(a^n)^* = 1$, that is, $a^n = 0$. By Remark 1.10, we deduce that $0 \in D$, hence $D = A$. ■

4. The spectrum of a residuated lattice

This section contains some characterizations for meet-irreducible and completely meet-irreducible **ds** of a residuated lattice A.

DEFINITION 1.10. Let L be a lattice with the least element 0 and the greatest element 1. We recall that (see Definition **??**) an element $p < 1$ is *finitely meet-irreducible* if $p = x \wedge y$ implies $p = x$ or $p = y$; an element $p < 1$ is *meet-prime* if $x \wedge y \leq p$ implies $x \leq p$ or $y \leq p$. Dually is defined the notions of *join-irreducible* and *join-prime*.

REMARK 1.14. *If L is distributive, meet-irreducible and meet-prime elements are the same.*

These definitions can be extended to arbitrary meets and we obtain the concepts of *completely meet (join)-irreducible* and *completely meet (join)-prime* elements, which are no longer equivalent.

We denote by $Ir(L)$ ($Irc(L)$) the set of all meet-irreducible (completely meet-irreducible) elements of L.

PROPOSITION 1.40. *Let $D \in Ds(A)$ and $a, b \in A$ such that $a \vee b \in D$. Then $D(a) \cap D(b) = D$.*

Proof. Clearly, $D \subseteq D(a) \cap D(b)$. To prove converse inclusion, let $x \in D(a) \cap D(b)$. Then there are $d_1, d_2 \in D$ and $m, n \geq 1$ such that $x \geq d_1 \odot a^m$ and $x \geq d_2 \odot b^n$. Then $x \geq (d_1 \odot a^m) \vee (d_2 \odot b^n) \geq (d_1 \vee d_2) \odot (d_1 \vee b^n) \odot (d_2 \vee a^m) \odot (a \vee b)^{mn}$, hence $x \in D$, that is, $D(a) \cap D(b) \subseteq D$, so we obtain the desired equality.■

COROLLARY 1.41. *For $D \in Ds(A)$ the following are equivalent:*

(i) If $D = D_1 \cap D_2$ whith $D_1, D_2 \in Ds(A)$, then $D = D_1$ or $D = D_2$;

(ii) For $a, b \in A$, if $a \vee b \in D$, then $a \in D$ or $b \in D$.

Proof. $(i) \Rightarrow (ii)$. If $a, b \in A$ such that $a \vee b \in D$, then, by Proposition 1.40, $D(a) \cap D(b) = D$, hence $D = D(a)$ or $D = D(b)$, so $a \in D$ or $b \in D$.

$(ii) \Rightarrow (i)$. Let $D_1, D_2 \in Ds(A)$ such that $D = D_1 \cap D_2$. If by contrary $D \neq D_1$ and $D \neq D_2$ then there are $a \in D_1 \backslash D$ and $b \in D_2 \backslash D$.

If denote $c = a \vee b$, then $c \in D_1 \cap D_2 = D$, so $a \in D$ or $b \in D$, a contradiction. ■

DEFINITION 1.11. We say that $P \in Ds(A)$ is *prime* if $P \neq A$ and P verify one of the equivalent assertions from Corollary 1.41.

REMARK 1.15. *Following Corollary 1.41, $P \in Ds(A)$, $P \neq A$ is prime iff P is a meet-irreducible **ds** in the lattice $(Ds(A), \subseteq)$.*

We denote $Spec(A) = Ir(Ds(A))\backslash\{A\}$ and by $Irc(A) = Irc(Ds(A))\backslash\{A\}$.

EXAMPLE 1.12. *Consider the example from Remark 1.1 (2) of residuated lattice $I = [0,1]$ which is not a BL− algebra. If $x \in [0,1], x > \frac{1}{4}$, then $x + x > \frac{1}{2}$, hence $x \odot x = x \wedge x = x$, so $[x) = [x,1]$. If $a, b \in I$ and $a \vee b \in [x) = [x,1]$, then $a \vee b = \max\{a,b\} \geq x$, hence $a \geq x$ or $b \geq x$. So, $a \in [x)$ or $b \in [x)$, that is, $[x) \in Spec(I)$.*

EXAMPLE 1.13. *Consider the residuated lattice $A = \{0, a, b, c, 1\}$ from Example 1.6. It is imediate to prove that*

$$Ds(A) = \{\{1\}, \{1,c\}, \{1,a,c\}, \{1,b,c\}, A\}$$

and

$$Spec(A) = \{\{1\}, \{1,a,c\}, \{1,b,c\}\}.$$

*Since $\{1,c\} = \{1,a,c\} \cap \{1,b,c\}$, then $\{1,c\} \notin Spec(A)$. Since $\odot$ coincide with $\wedge$, the **ds** of A coincide with the filters of the associated lattice $L(A)$.*

PROPOSITION 1.42. *For a proper ds P of A consider the following assertions:*

(1) *$P \in Spec(A)$;*
(2) *If $a, b \in A$, and $a \vee b = 1$, then $a \in P$ or $b \in P$;*
(3) *For all $a, b \in A$, $a \to b \in P$ or $b \to a \in P$;*
(4) *A/P is a chain.*

Then

(*i*) (1) $\Rightarrow$ (2) *but* (2) $\not\Rightarrow$ (1);
(*ii*) (3) $\Rightarrow$ (1) *but* (1) $\not\Rightarrow$ (3);
(*iii*) (4) $\Rightarrow$ (1) *but* (1) $\not\Rightarrow$ (4).

Proof. (*i*).(1) $\Rightarrow$ (2) is clearly by Corollary 1.41, (since $1 \in P$).

(2) $\not\Rightarrow$ (1) Consider A from Example 1.6. Then $D = \{1,c\} \notin Spec(A)$. Clearly, if $x, y \in A$ and $x \vee y = 1$, then $x = 1$ or $y = 1$, hence $x \in D$ or $y \in D$, but $D \notin Spec(A)$.

(*ii*). To prove (3) $\Rightarrow$ (1), let $a, b \in A$ such that $a \vee b \in P$. By $lr - c_6$ we obtain $a \vee b \leq [(a \to b) \to b] \wedge [(b \to a) \to a]$, hence $(a \to b) \to b, (b \to a) \to a \in P$. If $a \to b \in P$ then $b \in P$; if $b \to a \in P$, then $a \in P$, that is, $P \in Spec(A)$.

(1) $\not\Rightarrow$ (3) Consider A from Example 1.6. Then $P = \{1\} \in Spec(A)$. We have $a \to b = b \neq 1$ and $b \to a = a \neq 1$, hence $a \to b$ and $b \to a \notin P$.

(*iii*). To prove (4) $\Rightarrow$ (1) let $a, b \in A$. Since A/P is supposed chain, $a/P \leq b/P$ or $b/P \leq a/P \Leftrightarrow$ (by Proposition 1.23) $a \to b \in P$ or $b \to a \in P$ and we apply (*ii*).

(1) $\not\Rightarrow$ (4) Consider A from Example 1.6 and $P = \{1\} \in Spec(A)$. Then $A/P \approx A$ is not a chain. ■

REMARK 1.16. *If A is a BL− algebra, then all assertions from the above proposition are equivalent, see Theorem 3.23 and Proposition 3.30.*

REMARK 1.17. *If in Example 1.6 we consider $P = \{1,a,c\}$ or $P = \{1,b,c\}$, then $P \in Spec(A)$, and $A/P \approx L_2 = \{0,1\}$.*

REMARK 1.18. 1. *In general, in a residuated lattice A, if P is a prime **ds** and Q is a proper **ds** such that $P \subseteq Q$, then Q is not a prime **ds**. For example, if consider $A = \{0,a,b,c,1\}$ from Example 1.6. We have $P = \{1\}, Q = \{1,c\} \in Ds(A)$, $P \subseteq Q, P = \{1\} \in Spec(A)$ but Q is not a prime **ds** (see Example 1.13);*

2. *If the residuated lattice A is a $BL-$ algebra and P is a prime **ds**, Q is a proper **ds** such that $P \subseteq Q$, then Q is a prime **ds**, (see Theorem 3.25).*

REMARK 1.19. *If P is a prime **ds** of A, then $A\backslash P$ is an ideal in the lattice $L(A) = (A, \wedge, \vee, 0, 1)$.*

Proof. Since P is proper, $0 \notin P$, hence we have $0 \in A\backslash P$. If $a \leq b$ and $b \in A\backslash P$, then $a \in A\backslash P$, since P is a **ds** of A. If $a, b \in A\backslash P$ (that is, $a \notin P$ and $b \notin P$), then $a \vee b \in A\backslash P$, since P is a prime **ds**. ■

THEOREM 1.43. *(Prime **ds** theorem) If $D \in Ds(A)$ and I is an ideal of the lattice $L(A)$ such that $D \cap I = \varnothing$, then there is a prime **ds** P of A such that $D \subseteq P$ and $P \cap I = \varnothing$.*

Proof. Let $F_D = \{D' \in Ds(A) : D \subseteq D' \text{ and } D' \cap I = \emptyset\}$. A routine application of Zorn's lemma shows that F_D has a maximal element P. Suppose by contrary that P is not a prime deductive system, that is, there are $a, b \in A$ such that $a \vee b \in P$, but $a \notin P, b \notin P$ (see Corollary 1.41).

By the maximality of P we deduce that $P(a), P(b) \notin F_D$, hence $P(a) \cap I \neq \emptyset$ and $P(b) \cap I \neq \emptyset$, that is, there are $p_1 \in P(a) \cap I$ and $p_2 \in P(b) \cap I$. By Proposition 1.29, $p_1 \geq f \odot a^m$ and $p_2 \geq g \odot b^n$, whith $f, g \in P$ and m, n natural numbers.

Then $p_1 \vee p_2 \geq (f \odot a^m) \vee (g \odot b^n) \overset{lr-c_{30}}{\geq} (f \vee g) \odot (g \vee a^m) \odot (f \vee b^n) \odot (b^n \vee a^m) \overset{lr-c_{30}}{\geq} \overset{lr-c_{30}}{\geq} (f \vee g) \odot (g \vee a^m) \odot (f \vee b^n) \odot (a \vee b)^{m+n}$. Since $f \vee g, g \vee a^m, f \vee b^n \in P$ we deduce that $p_1 \vee p_2 \in P$; but $p_1 \vee p_2 \in I$, hence $P \cap I \neq \varnothing$, a contradiction.

Hence P is a prime **ds**. ■

REMARK 1.20. *If A is a nontrivial residuated lattice, then any proper **ds** of A can be extended to a prime **ds**.*

REMARK 1.21. *In general, if A is a residuated lattice, the set of proper ds including a prime **ds** P of A is not a chain, but if the residuated lattice is a $BL-$ algebra, then the set of proper **ds** including a prime **ds** P of A is a chain, (see Theorem 3.26). If we consider the residuated lattice from Example 1.6 and the prime **ds** $P = \{1\}$. The set of proper **ds** including a prime **ds** $P = \{1\}$ of A is $\{\{1, c\}, \{1, a, c\}, \{1, b, c\}\}$, but $\{1, a, c\} \nsubseteq \{1, b, c\}$ and $\{1, b, c\} \nsubseteq \{1, b, c\}$, so $\{\{1, c\}, \{1, a, c\}, \{1, b, c\}\}$ is not a chain.*

COROLLARY 1.44. *Let $D \in Ds(A)$ and $a \in A\backslash D$. Then:*

(i) There is $P \in Spec(A)$ such that $D \subseteq P$ and $a \notin P$;
*(ii) D is the intersection of those prime **ds** which contain D;*
(iii) $\cap Spec(A) = \{1\}$.

PROPOSITION 1.45. *For a proper **ds** $P \in Ds(A)$ the following are equivalent:*

(i) $P \in Spec(A)$;
(ii) For every $x, y \in A\backslash P$ there is $z \in A\backslash P$ such that $x \leq z$ and $y \leq z$.

Proof. $(i) \Rightarrow (ii)$. Let $P \in Spec(A)$ and $x, y \in A\backslash P$. If by contrary, for every $a \in A$ with $x \leq a$ and $y \leq a$ then $a \in P$, since $x, y \leq x \vee y$ we deduce that $x \vee y \in P$, hence, $x \in P$ or $y \in P$, a contradiction.

$(ii) \Rightarrow (i)$. Suppose by contrary that there exist $D_1, D_2 \in Ds(A)$ such that $D_1 \cap D_2 = P$ and $P \neq D_1$, $P \neq D_2$. So, we have $x \in D_1\backslash P$ and $y \in D_2\backslash P$. By hypothesis there is $z \in A\backslash P$ such that $x \leq z$ and $y \leq z$.

We deduce $z \in D_1 \cap D_2 = P$, a contradiction. ■

COROLLARY 1.46. *For a proper* ***ds*** $P \in Ds(A)$ *the following are equivalent:*

(*i*) $P \in Spec(A)$;
(*ii*) *If* $x, y \in A$ *and* $[x) \cap [y) \subseteq P$, *then* $x \in P$ *or* $y \in P$.

Proof. $(i) \Rightarrow (ii)$. Let $x, y \in A$ such that $[x) \cap [y) \subseteq P$ and suppose by contrary that $x,\ y \notin P$. Then by Proposition 1.45 there is $z \in A \backslash P$ such that $x \leq z$ and $y \leq z$. Hence $z \in [x) \cap [y) \subseteq P$, so $z \in P$, a contradiction.

$(ii) \Rightarrow (i)$. Let $x, y \in A$ such that $x \vee y \in P$. Then $[x \vee y\ ) \subseteq P$.

Since $[x \vee y) = [x) \cap [y)$ (by Proposition 1.32, (iii)) we deduce that $[x) \cap [y) \subseteq P$, hence, by hypothesis, $x \in P$ or $y \in P$, that is, $P \in Spec(A)$. ■

COROLLARY 1.47. *For a proper* ***ds*** $P \in Ds(A)$ *the following are equivalent:*

(*i*) $P \in Spec(A)$;
(*ii*) *For every* $\alpha, \beta \in A/P, \alpha \neq \mathbf{1}, \beta \neq \mathbf{1}$ *there is* $\gamma \in A/P, \gamma \neq \mathbf{1}$ *such that* $\alpha \leq \gamma,\ \beta \leq \gamma$.

Proof. $(i) \Rightarrow (ii)$. Clearly, by Proposition 1.45 and Proposition 1.23, since if $\alpha = a/P$, with $a \in A$, then the condition $\alpha \neq \mathbf{1}$ is equivalent with $a \notin P$.

$(ii) \Rightarrow (i)$. Let $\alpha, \beta \in A/P$. Then in A/P, $\alpha = a/P \neq \mathbf{1}$ and $\beta = b/P \neq \mathbf{1}$. By hypothesis there is $\gamma = c/P \neq \mathbf{1}$ (that is, $c \notin P$) such that $\alpha, \beta \leq \gamma$ equivalent with $a \to c, b \to c \in P$. If consider $d = (b \to c) \to ((a \to c) \to c)$, then by $lr - c_6$, we deduce that $a, b \leq d$. Since $c \notin P$ we deduce that $d \notin P$, hence by Proposition 1.45, we deduce that $P \in Spec(A)$. ■

THEOREM 1.48. *For a proper* ***ds*** $P \in Ds(A)$ *the following are equivalent:*

(*i*) $P \in Spec(A)$;
(*ii*) *For every* $D \in Ds(A)$, $D \to P = P$ *or* $D \subseteq P$.

Proof. $(i) \Rightarrow (ii)$. Let $P \in Spec(A)$. Since $Ds(A)$ is a Heyting algebra (by Corollary 1.37) for $D \in Ds(A)$ we have

$$P = (D \to P) \cap ((D \to P) \to P)$$

and so $P = D \to P$ or $P = (D \to P) \to P$. If $P = (D \to P) \to P$ then $D \subseteq P$.

$(ii) \Rightarrow (i)$. Let $D_1, D_2 \in Ds(A)$ such that $D_1 \cap D_2 = P$. Then $D_1 \subseteq D_2 \to P$ (see Lemma 1.35, (ii)) and so, if $D_2 \subseteq P$, then $P = D_2$ and if $D_2 \to P = P$, then $P = D_1$, hence $P \in Spec(A)$. ■

DEFINITION 1.12. *(*[**73**], p.58) Let $(L, \vee, \wedge)$ a lattice with 0 and $x \in L$. An element $x^* \in L$ is a *pseudocomplement* of x if $x \wedge x^* = 0$, and $x \wedge y = 0$ implies that $y \leq x^*$ (that is, $x^* = \sup\{y \in L : x \wedge y = 0\}$). The lattice L is called *pseudocomplemented* if every element $x \in L$ has a pseudocomplement $x^* \in L$.

REMARK 1.22. *If* $(L, \vee, \wedge, \to, 0)$ *is a Heyting algebra, then* $(L, \vee, \wedge, ^*, 0)$ *is a pseudocomplemented lattice, where for every* $x \in L$, $x^* = x \to 0$.

We recall that if $(L, \vee, \wedge, ^*, 0, 1)$ is a *pseudocomplemented distributive lattice*, then two subsets associated with L ([**2**], p.153) are

$$Rg(L) = \{x \in L : x^{**} = x\} \text{ and}$$

$$D(L) = \{x \in L : x^* = 0\}.$$

The elements of $Rg(L)$ are called *regular* and those of $D(L)$ *dense*. Note that $\{0,1\} \subseteq Rg(L), 1 \in D(L)$ and $D(L)$ is a filter in L and $Rg(L)$ is a Boolean algebra under the operations induced by the ordering on L ([**2**], p.157).

COROLLARY 1.49. *For a residuated lattice A, $Spec(A) \subseteq D(Ds(A)) \cup Rg(Ds(A))$.*

Proof. Let $P \in Spec(A)$ and $D = P^* \in Ds(A)$; then by Theorem 1.48, $D \subseteq P$ or $D \to P = P$ equivalent with $P^* \subseteq P$ or $P^* \to P = P$. Since $Ds(A)$ is a Heyting algebra then $P^* \to P = P^{**}$, so $P^{**} = A$ or $P^{**} = P$ equivalent with $P^* = \{1\}$ or $P^{**} = P$, that is $P \in D(Ds(A)) \cup Rg(Ds(A))$. ■

Relative to the uniqueness of deductive systems as intersection of primes we have:

THEOREM 1.50. *If every $D \in Ds(A)$ has a unique representation as an intersection of elements of $Spec(A)$, then $(Ds(A), \vee, \wedge, ^*, \{1\}, A)$ is a Boolean algebra.*

Proof. Let $D \in Ds(A)$ and $D' = \cap\{M \in Spec(A) : D \nsubseteq M\} \in Ds(A)$. By Corollary 1.44, (ii), $D \cap D' = \cap\{M \in Spec(A)\} = \{1\}$; if $D \vee D' \neq A$, then by Corollary 1.44, (i), there exists $D'' \in Spec(A)$ such that $D \vee D' \subseteq D''$ and $D'' \neq A$. Consequently, D' has two representations $D' = \cap\{M \in Spec(A) : D \nsubseteq M\} = D'' \cap (\cap\{M \in Spec(A) : D \nsubseteq M\})$, which is contradictory. Therefore $D \vee D' = A$ and so $Ds(A)$ is a Boolean algebra. ■

REMARK 1.23. *For the case of distributive lattice see* [**73**]*, p.77.*

As an immediate consequence of Zorn's lemma we obtain:

LEMMA 1.51. *If $D \in Ds(A)$, $D \neq A$ and $a \notin D$, then there exists $D_a \in Ds(A)$ maximal with the property that $D \subseteq D_a$ and $a \notin D_a$.*

Proof. Let $\mathcal{F}_{D,a} = \{D' \in Ds(A) : D \subseteq D' \text{ and } a \notin D'\}$; clearly $\mathcal{F}_{D,a} \neq \emptyset$, because $D \in \mathcal{F}_{D,a}$.

If $\mathbf{C}$ is a chain in $\mathcal{F}_{D,a}$ then $\cup\mathbf{C} \in \mathcal{F}_{D,a}$. By Zorn's lemma there exists a ds D_a which is maximal subject to containing D and $a \notin D_a$. ■

DEFINITION 1.13. $D \in Ds(A)$, $D \neq A$ is called *maximal relative to a* if $a \notin D$ and if $D' \in Ds(A)$ is proper such that $a \notin D'$, and $D \subseteq D'$, then $D = D'$.

If in Lemma 1.51 we consider $D = \{1\}$ we obtain:

COROLLARY 1.52. *For any $a \in A, a \neq 1$, there is a **ds** D_a maximal relative to a.*

THEOREM 1.53. *For $D \in Ds(A), D \neq A$ the following are equivalent:*

(i) $D \in Irc(A)$;

(ii) There is $a \in A$ such that D is maximal relative to a.

Proof. $(i) \Rightarrow (ii)$. See ([**69**], p.248), since by Proposition 1.33, $Ds(A)$ is an algebraic lattice.

$(ii) \Rightarrow (i)$. Let $D \in Ds(A)$ maximal relative to a and suppose $D = \bigcap_{i \in I} D_i$ with $D_i \in Ds(A)$ for every $i \in I$. Since $a \notin D$ there is $j \in I$ such that $a \notin D_j$. So, $a \notin D_j$ and $D \subseteq D_j$. By the maximality of D we deduce that $D = D_j$, that is, $D \in Irc(A)$. ■

THEOREM 1.54. *Let $D \in Ds(A)$ be a **ds**, $D \neq A$ and $a \in A \backslash D$. Then the following are equivalent:*

(*i*) *D is maximal relative to a;*
(*ii*) *For every $x \in A\backslash D$ there is $n \geq 1$ such that $x^n \to a \in D$.*

Proof. $(i) \Rightarrow (ii)$. Let $x \in A\backslash D$. If $a \notin D(x) = D \vee [x)$, since $D \subset D(x)$ then $D(x) = A$ (by the maximality of D) hence $a \in D(x)$, a contradiction. We deduce that $a \in D(x)$, hence $a \geq d \odot x^n$, with $d \in D$ and $n \geq 1$. Then $d \leq x^n \to a$, hence $x^n \to a \in D$.

$(ii) \Rightarrow (i)$. We suppose by contrary that there is $D' \in Ds(A), D' \neq A$ such that $a \notin D'$ and $D \subset D'$. Then there is $x_0 \in D'$ such that $x_0 \notin D$, hence by hypothesis there is $n \geq 1$ such that $x_0^n \to a \in D \subset D'$. Thus from $x_0^n \to a \in D'$ and $x_0^n \in D'$, we deduce that $a \in D'$, a contradiction. ■

Corollary 1.55. *For a ds $D \in Ds(A), D \neq A$ the following are equivalent:*

(*i*) *$D \in Irc(A)$;*
(*ii*) *In the set $A/D\backslash\{\mathbf{1}\}$ we have an element $p \neq \mathbf{1}$ with the property that for every $\alpha \in A/D\backslash\{\mathbf{1}\}$ there is $n \geq 1$ such that $\alpha^n \leq p$.*

Proof.$(i) \Rightarrow (ii)$. By Theorem 1.53, D is maximal relative to an element $a \notin D$; then, if denote $p = a/D \in A/D$, $p \neq \mathbf{1}$ (since $a \notin D$) and for every $\alpha = b/D$, $\alpha \neq \mathbf{1}$ (that is $b \notin D$) by Theorem 1.54 there is $n \geq 1$ such that $b^n \to a \in D$, that is, $\alpha^n \leq p$.

$(ii) \Rightarrow (i)$. Let $p = a/D \in A/D\backslash\{\mathbf{1}\}$, (that is, $a \notin D$) and $\alpha = b/D \in A/D\backslash\{\mathbf{1}\}$, (that is, $b \notin D$). By hypothesis there is $n \geq 1$ such that $\alpha^n \leq p$ equivalent with $b^n \to a \in D$. Then, by Theorem 1.54, we deduce that $D \in Irc(A)$. ■

Definition 1.14. A **ds** P of A is a *minimal prime* if $P \in Spec(A)$ and, whenever $Q \in Spec(A)$ and $Q \subseteq P$, we have $P = Q$.

Proposition 1.56. *If P is a minimal prime **ds**, then for any $a \in P$ there is $b \in A\backslash P$ such that $a \vee b = 1$.*

Proof. Let P be a minimal prime **ds** and $a \in P$.
We define the set

$$S_a = \{x \in A : \text{ there is } b \in A\backslash P \text{ such that } a \vee b \geq x\}.$$

If $b \in A\backslash P$ then $a \vee b \geq b$, so $b \in S_a$, that is, $A\backslash P \subseteq S_a$. Moreover, $a \in S_a$ because $a \vee 0 = a \geq a$ and $0 \in A\backslash P$.

We shall prove that S_a is an ideal of the lattice $L(A)$.

Let $x, y \in A$ such that $y \in S_a$ and $x \leq y$. Thus, there is $b \in A\backslash P$ such that $a \vee b \geq y \geq x$, hence $a \vee b \geq x$, so $x \in S_a$.

If $x, y \in S_a$ then there are $b, c \in A\backslash P$ such that $a \vee b \geq x$ and $a \vee c \geq y$. If we suppose that $b \vee c \in P$ we get $b \in P$ or $c \in P$ because P is a prime **ds**. Thus, $b \vee c \in A\backslash P$ and $a \vee (b \vee c) \geq x \vee y$, so $x \vee y \in S_a$, hence S_a is an ideal.

Now, we suppose that $1 \notin S_a$. It follows that $\{1\} \cap S_a = \varnothing$ so, by Theorem 1.43, there is a prime **ds** Q such that $S_a \cap Q = \varnothing$. Since $A\backslash P \subseteq S_a$, we get $Q \subseteq P$. But Q is prime and P is minimal prime, so $P = Q$. On the other hand, $a \in S_a$, so $a \notin Q$. We get $a \in P\backslash Q$, which contradicts the fact that $P = Q$. Thus, our assumption that $1 \notin S_a$ is false. We conclude that $1 \in S_a$ and the proof is finished. ■

5. Maximal deductive systems; archimedean and hyperarchimedean residuated lattices

In this section we introduce the notions of archimedean and hyperarchimedean residuated lattice and prove two theorems of Nachbin type for residuated lattices.

DEFINITION 1.15. A **ds** of A is *maximal* if it is proper and it is not contained in any other proper **ds**.

The following result is an immediate consequence of Zorn's lemma:

PROPOSITION 1.57. *In a nontrivial residuated lattice A, every proper **ds** can be extended to a maximal **ds**.*

We shall denote by $Max(A)$ the set of all maximal **ds** of A.

PROPOSITION 1.58. $Max(A) \subseteq Spec(A)$.

Proof. Let $M \in Max(A)$ and $D_1, D_2 \in Ds(A)$ such that $M = D_1 \cap D_2$. By the maximality of M we deduce that $M = D_1$ or $M = D_2$, hence $M \in Spec(A)$ (see Corollary 1.41). ■

We have:

THEOREM 1.59. *If D is a proper **ds** of A, then the following are equivalent:*

*(i) D is a maximal **ds**;*

(ii) For any $x \notin D$ there exist $d \in D, n \geq 1$ such that $d \odot x^n = 0$.

Proof. $(i) \Rightarrow (ii)$. If $x \notin D$, then $[D \cup \{x\}) = A$, hence $0 \in [D \cup \{x\})$. By Proposition 1.29, (iii), there exist $n \geq 1$ and $d \in D$ such that $d \odot x^n \leq 0$. Thus $d \odot x^n = 0$.

$(ii) \Rightarrow (i)$. Assume there is a proper **ds** D' such that $D \subset D'$. Then there exists $x \in D'$ such that $x \notin D$. By hypothesis there exist $d \in D, n \geq 1$ such that $d \odot x^n = 0$. But $x, d \in D'$ hence we obtain $0 \in D'$, a contradiction. ■

COROLLARY 1.60. *If M is a proper **ds** of A, then the following are equivalent:*

*(i) M is a maximal **ds**;*

(ii) For any $x \in A, x \notin M$ iff $(x^n)^* \in M$, for some $n \geq 1$.

THEOREM 1.61. *If M is a proper ds of A, then the following are equivalent:*

*(i) M is a maximal **ds**,*

(ii) A/M is locally finite.

Proof. $(i) \Leftrightarrow (ii)$. It follows by observing that the condition (ii) can be reformulated in the following way: for any $x \in A, x/M \neq 1/M$ (that is, $x \notin M$), $(x/M)^n = 0/M$, for some $n \geq 1 \Leftrightarrow x^n/M = 0/M \Leftrightarrow (x^n)^* \in M$. ■

DEFINITION 1.16. The intersection of the maximal **ds** of A is called the *radical* of A and will be denoted by $Rad(A)$. It is obvious that $Rad(A) \in Ds(A)$.

EXAMPLE 1.14. *Let A be the $5-$element residuated lattice from Example 1.6. It is easy to see that A has two maximal **ds**: $\{1, a, c\}$ and $\{1, b, c\}$, hence $Rad(A) = \{1, c\}$.*

For any $n \geq 1$ and $a \in A$ we denote $\widetilde{n}a = [(a^*)^n]^*$.

THEOREM 1.62. *([**63**], [**92**])*

(*i*) $Rad(A) = \{x \in A :$ *for any* $n \geq 1$ *there exists* $m \geq 1$ *such that* $\widetilde{m}(x^n) = 1\} = \{x \in A :$ *for any* $n \geq 1$ *there is* $k_n \geq 1$ *such that* $[(x^n)^*]^{k_n} = 0\}$ *(that is,* $Rad(A)$ *is the set of unityes of* A*);*
(*ii*) $D(A) \in Ds(A)$ *and* $D(A) \subseteq Rad(A)$.

For a residuated lattice A we make the following notation:

$$Rad_{BL}(A) = \{a \in A : (a^n)^* \leq a, \text{ for every } n \geq 1\}.$$

PROPOSITION 1.63. *For a residuated lattice* A, $Rad_{BL}(A) \subseteq Rad(A)$.

Proof. Let $a \notin Rad(A)$, hence there is a maximal **ds** M with $a \notin M$. Then there is n such that $(a^n)^* \in M$, (by Corollary 1.60). If suppose $a \in Rad_{BL}(A)$ then in particular for this n we have $(a^n)^* \leq a$, hence $a \in M$, by (Ds_2'), a contradiction. Hence $(a^n)^* \not\leq a$, i.e. $a \notin Rad_{BL}(A)$, that is, $Rad_{BL}(A) \subseteq Rad(A)$. ■

REMARK 1.24. *If* A *is a* $BL-$ *algebra, then* $Rad(A) = Rad_{BL}(A)$.

PROPOSITION 1.64. *If* A *is a residuated lattice, then* $B(A) \cap Rad(A) = \{1\}$.

Proof. Obviously, $1 \in B(A) \cap Rad(A)$. Let $e \in B(A), e \neq 1$. By Theorem 1.43, there is a prime **ds** P of A such that $e \notin P$. By Proposition 1.16, (ii), we have $e \vee e^* = 1 \in P$, so $e^* \in P$ (since P is prime and $e \notin P$). By Proposition 1.57, there is a maximal **ds** M such that $P \subseteq M$. It follows that $e^* \in M$, so $e \notin M$. Thus, $e \notin Rad(A)$. ■

DEFINITION 1.17. An element a of a residuated lattice A is called *infinitesimal* if $a \neq 1$ and $a^n \geq a^*$ for any $n \geq 1$.

We denote by $Inf(A)$ the set of all infinitesimals of A.

EXAMPLE 1.15. *If* $A = \{0, a, b, c, 1\}$ *is the* $5-$*element residuated lattice from Example 1.6, then* a *is not infinitesimal (since* $a^* = b$ *and* $a \not\geq b$*); analogously, we deduce that* b *is not infinitesimal Since* $c^* = 0$*, then* $c^n = c \geq 0 = c^*$*, for every natural number* n*, hence* c *is an infinitesimal element of* A*. So,* $Inf(A) = \{c\}$.

LEMMA 1.65. *For every* $a, b \in A$*, we have:*
$(lr - c_{43})$ $a^{**} \odot b^{**} \leq (a \odot b)^{**}$.

Proof. By $lr - c_{19}$, $(a \odot b)^* = a \to b^*$, so $(a \odot b)^* \odot a \leq b^*$. By $lr - c_{11}$ we deduce that $b^{**} \leq [(a \odot b)^* \odot a]^* = (a \odot b)^* \to a^*$, so $b^{**} \odot (a \odot b)^* \leq a^*$. Then $a^{**} \leq [b^{**} \odot (a \odot b)^*]^* = b^{**} \to (a \odot b)^{**}$, that is, $a^{**} \odot b^{**} \leq (a \odot b)^{**}$. ■

COROLLARY 1.66. *For every* $a \in A$ *and* $n \geq 1$ *we have:*
$(lr - c_{44})$ $(a^{**})^n \leq (a^n)^{**}$.

PROPOSITION 1.67. *For every nonunit element* a *of* A $(a \neq 1)$, a *is infinitesimal implies* $a \in Rad(A)$.

Proof. Let $a \neq 1$ be an infinitesimal and suppose $a \notin Rad(A)$. Thus, there is a maximal **ds** M of A such that $a \notin M$. By Corollary 1.60, there is $n \geq 1$ such that $(a^n)^* \in M$. By hypothesis $a^n \geq a^*$ hence $(a^n)^* \leq a^{**}$, so $a^{**} \in M$. By $lr - c_{44}$ we deduce that $(a^{**})^n \leq (a^n)^{**}$, hence $(a^n)^{**} \in M$. If denote $b = (a^n)^*$ we conclude that $b, b^* \in M$, hence $0 = b^* \odot b \in M$, that is, $M = A$, a contradiction.■

COROLLARY 1.68. $Inf(A) \subseteq Rad(A)$.

REMARK 1.25. 1. *If A is the residuated lattice from Example 1.6, then $Inf(A) \subset Rad(A)$, since $Inf(A) = \{c\}$ and $Rad(A) = \{1, c\}$ (see Examples 1.14 and 1.15).*

2. *In general, $Rad(A)\backslash\{1\} \nsubseteq Inf(A)$. Indeed, let A be the residuated lattice from Example 1.7. Then the **ds** of A are $\{1\}, \{1, a, b, c\}$ and A. It is easy to see that A has two prime **ds**: $\{1\}, \{1, a, b, c\}$ and a unique maximal **ds** $\{1, a, b, c\}$; hence $Rad(A) = \{1, a, b, c\}$. Obviously, a is an infinitesimal element of A ($a^n = c$, for every $n \geq 1$, $a^* = d$ and $c \geq d$). But ($b^2 = c, b^* = e$ and c, e are incomparable), ($c^2 = c, c^* = f$ and c, f are incomparable), ($d^2 = 0, d^* = a$ and $a > 0$), ($e^2 = d, e^* = b$ and $d < b$), ($f^2 = d, f^* = c$ and $d < c$), ($0^2 = 0, 0^* = 1$ and $0 < 1$), so we conclude that $b, c, d, e, f, 0 \notin Inf(A)$. It follows that $Inf(A) = \{a\}$. Thus $Inf(A) \subseteq Rad(A)$ and $Rad(A)\backslash\{1\} \nsubseteq Inf(A)$.*

REMARK 1.26. *If A is a BL algebra, then $Rad(A)\backslash\{1\} = Inf(A)$, see Proposition 3.52.*

PROPOSITION 1.69. *For $a \in A$ and $n \geq 1$, the following assertions are equivalent:*

(i): $a^n \in B(A)$;
(ii): $a \vee (a^n)^* = 1$.

Proof. $(i) \Rightarrow (ii)$. Since $a^n \in B(A)$, by Proposition 1.16 we deduce that $a^n \vee (a^n)^* = 1$. But $a^n \leq a$, so $1 = a^n \vee (a^n)^* \leq a \vee (a^n)^*$. We obtain that $a \vee (a^n)^* = 1$.

$(ii) \Rightarrow (i)$. Since $a \vee (a^n)^* = 1 \overset{lr-c_{34}}{\Rightarrow} a^n \vee [(a^n)^*]^n = 1$. Since $[(a^n)^*]^n \leq (a^n)^*$, we obtain $1 = a^n \vee [(a^n)^*]^n \leq a^n \vee (a^n)^*$, so $a^n \vee (a^n)^* = 1$. By Proposition 1.16 we deduce that $a^n \in B(A)$. ■

LEMMA 1.70. *If $a \in A$ and $n \geq 1$ then the following hold: $a^n \in B(A)$ and $a^n \geq a^*$, implies $a = 1$.*

Proof. By Proposition 1.69, $a^n \in B(A) \Leftrightarrow a \vee (a^n)^* = 1$. By hypothesis, $a^n \geq a^*$. By $lr - c_{12}$ we obtain $(a^n)^* \leq a^{**}$, so $1 = a \vee (a^n)^* \leq a \vee a^{**} = a^{**}$, hence $a^{**} = 1$, that is, $a^* = 0$.

Then $(a \odot a) \to 0 = a \to (a \to 0) = a \to 0 = a^* = 0$, so we deduce that $(a^2)^* = 0$. Recursively we obtain that $(a^n)^* = 0$. Then $a \vee (a^n)^* = a \vee 0 = 1$, hence $a = 1$.■

LEMMA 1.71. *In any residuated lattice A the following are equivalent:*

(i) For every $a \in A, a^n \geq a^*$ for any $n \geq 1$ implies $a = 1$;
(ii) For every $a, b \in A, a^n \geq b^*$ for any $n \geq 1$ implies $a \vee b = 1$;
(iii) For every $a, b \in A, a^n \geq b^*$ for any $n \geq 1$ implies $a \to b = b$ and $b \to a = a$.

Proof. $(i) \Rightarrow (ii)$. Let $a, b \in A$ such that $a^n \geq b^*$ for any $n \geq 1$. We get $(a \vee b)^* = a^* \wedge b^* \leq b^* \leq a^n \leq (a \vee b)^n$, hence $(a \vee b)^n \geq (a \vee b)^*$, for any $n \geq 1$. By hypothesis, $a \vee b = 1$.

$(ii) \Rightarrow (iii)$. Since $1 = a \vee b \leq [(b \to a) \to a] \wedge [(a \to b) \to b]$ we deduce that $(b \to a) \to a = (a \to b) \to b = 1$, that is, $a \to b = b$ and $b \to a = a$.

$(iii) \Rightarrow (i)$. Let $a \in A$ such that $a^n \geq a^*$ for any $n \geq 1$. If consider $b = a$ we obtain $a \vee b = 1 \Leftrightarrow a \vee a = 1 \Leftrightarrow a = 1$.■

DEFINITION 1.18. A residuated lattice A is called *archimedean* if the equivalent conditions from Lemma 1.71 are satisfied.

One can easily remark that a residuated lattice is archimedean iff it has no infinitesimals.

EXAMPLE 1.16.
1. *Consider $A = \{0, a, b, c, 1\}$ the residuated lattice from Example 1.6. Since $c^n = c$ for every natural number n, and $c^* = 0$ we deduce that $c^n \geq c^*$ for every $n \geq 1$ but $c \neq 1$, hence A is not archimedean;*
2. *Consider $A = \{0, a, b, c, d, e, f, 1\}$ the residuated lattice from Example 1.7. We have $a^* = d, b^* = e, c^* = f, d^* = a, e^* = b$ and $f^* = c$. Since $a \geq d = a^*$ and $a^n = c$ for every $n \geq 2$ and $c \geq d = a^*$ we deduce that $a^n \geq a^*$, for every $n \geq 1$, hence A is also not archimedean;*
3. *Consider $A = \{0, a, b, c, d, 1\}$ the residuated lattice from Example 1.9. We have $a^n = a$ for every $n \geq 1$ and $a^* = d$ hence $a^n \not\geq a^*$ for every $n \geq 1; b^n = 0$ for every $n \geq 1$ and $b^* = c$ hence $b^n \not\geq b^*$ for every $n \geq 1$; $c^2 = a \not\geq c^* = b$, $d^n = d$ for every $n \geq 1$ and $d^* = a$, hence $d^n \not\geq d^* = a$, for every $n \geq 1$. Hence if $x \in A$ and $x^n \geq x^*$, for every $n \geq 1$, then $x = 1$, that is, A is archimedean.*

DEFINITION 1.19. Let A be a residuated lattice. An element $a \in A$ is called *archimedean* if it satisfy the condition*:*

$$\textit{there is } n \geq 1 \textit{ such that } a^n \in B(A),$$

(equivalent by Proposition 1.69 with $a \vee (a^n)^* = 1$). A residuated lattice A is called *hyperarchimedean* if all its elements are archimedean.

REMARK 1.27. *If the residuated lattice A is a $BL(MV)-$ algebra we obtain the Definition 3.12(respectively, 2.7 and 2.8).*

EXAMPLE 1.17.
1. *Consider $A = \{0, a, b, c, d, 1\}$ the residuated lattice from Example 1.9; By Example 1.16 we deduce that A is archimedean. By Remark 1.7 (3) we have $B(A) = \{0, a, d, 1\}$. Since $a^2 = a \in B(A)$, $b^2 = 0 \in B(A), c^2 = a \in B(A)$ and $d^2 = d \in B(A)$ we deduce that A is even hyperarchimedean.*
2. *Consider $A = \{0, a, b, c, d, e, f, g, 1\}$ the residuated lattice from Example 1.10; we have $B(A) = \{0, b, f, 1\}$ (see Remark 1.7). Since $a^2 = 0 \in B(A), b^2 = b \in B(A), c^2 = 0 \in B(A)$, $d^2 = 0 \in B(A), e^2 = b \in B(A), f^2 = f \in B(A)$ and $g^2 = f \in B(A)$ we deduce that A is hyperarchimedean.*
3. *If consider $A = \{0, a, b, c, d, 1\}$ the residuated lattice from Example 1.8 we deduce that $B(A) = \{0, 1\}$. Since $a^n = a \notin B(A)$, for every $n \geq 1$, we deduce that A is not hyperarchimedean; since $a^* = 0$, then $a^n = a \geq 0 = a^*$, for every $n \geq 1$, but $a \neq 1$, so A is not even archimedean.*

From Lemma 1.70 we deduce:

COROLLARY 1.72. *Every hyperarchimedean residuated lattice is archimedean.*

THEOREM 1.73. *For a residuated lattice A, if A is hyperarchimedean, then for any **ds** D, the quotient residuated lattice A/D is archimedean.*

Proof. To prove A/D is archimedean, let $x = a/D \in A/D$ such that $x^n \geq x^*$ for any $n \geq 1$. By hypothesis, there is $m \geq 1$ such that $a \vee (a^m)^* = 1$, i.e. $a^m \in B(A)$.

It follows that $x \vee (x^m)^* = 1$ (in A/D), i.e. $x^m \in B(A/D)$. In particular we have $x^m \geq x^*$, so by Lemma 1.70 we deduce that $x = 1$, that is, A/D is archimedean. ■

We recall a theorem of *Nachbin type* for lattices (see [**2**], p.73):

THEOREM 1.74. *A distributive lattice is relatively complemented iff every prime ideal is maximal.*

Now, we present an analogously theorem of Theorem 1.74 for residuated lattices:

THEOREM 1.75. *For a residuated lattice A the following assertions are equivalent:*

(*i*) *A is hyperarchimedean;*
(*ii*) *$Spec(A) = Max(A)$;*
(*iii*) *Any prime **ds** is minimal prime.*

Proof. $(i) \Rightarrow (ii)$. Since $Max(A) \subseteq Spec(A)$, we only have to prove that any prime **ds** of A is maximal. Let $P \in Spec(A)$. To prove $P \in Max(A)$, let $x \notin P$. Since A is hyperarchimedean there is $n \geq 1$ such that $x^n \in B(A)$, hence $x \vee (x^n)^* = 1$, (by Proposition 1.69). Since $1 \in P$ we deduce that $x \vee (x^n)^* \in P$. Since $x \notin P$, by Corollary 1.41 we deduce that $(x^n)^* \in P$, that is, $P \in Max(A)$ (see Corollary 1.60).

$(ii) \Rightarrow (iii)$. Let P, Q prime **ds** such that $P \subseteq Q$. By hypothesis, P is maximal, so $P = Q$. Thus Q is minimal prime.

$(iii) \Rightarrow (i)$. Let a be a nonunit element from A. We shall prove that a is an archimedean element. If we denote

$$D = [a)^* = \{x \in A : a \vee x = 1\} \text{ (by Corollary 1.37),}$$

then $D \in Ds(A)$. Since $a \neq 1$, then $a \notin D$ and we consider

$$D' = D(a) = \{x \in A : x \geq d \odot a^n \text{ for some } d \in D \text{ and } n \geq 1\}.$$

If we suppose that D' is a proper **ds** of A, then by Corollary 1.43, there is a prime ds P such that $D' \subseteq P$. By hypothesis, P is a minimal prime. Since $a \in P$, using Proposition 1.56 , we infer that there is $x \in A \backslash P$ such that $a \vee x = 1$. It follows that $x \in D \subseteq D' \subseteq P$, hence $x \in P$, so we get a contradiction.

Thus D' is not proper, so $0 \in D'$, hence there is $n \geq 1$ and $d \in D$ such that $d \odot a^n = 0$. Thus $d \leq (a^n)^*$. We get $a \vee d \leq a \vee (a^n)^*$. But $a \vee d = 1$ (since $d \in D$), so we obtain that $a \vee (a^n)^* = 1$, that is a is an archimedean element. ■

In the end of this section we recall another theorem of Nachbin for lattices (see [**73**], p. 76):

THEOREM 1.76. *Let L be a distributive lattice with 0 and 1. Then L is a Boolean lattice iff $P(L)$ is unordered (where $P(L)$ is there set of all prime ideals of L).*

Now, we present an analogously theorem of Theorem 1.76 for residuated lattices:

THEOREM 1.77. *For a residuated lattice A the following assertions are equivalent:*

(*i*) *A is hyperarchimedean,*
(*ii*) *$(Spec(A), \subseteq)$ is unordered.*

Proof. $(i) \Rightarrow (ii)$. Let A be hyperarchimedean, and suppose by contrary that there are $P, Q \in Spec(A), P \subset Q$. Chose $a \in Q \backslash P$. Then $a^n \in Q$ for every $n \geq 1$, hence $(a^n)^* \notin Q$ and $(a^n)^* \notin P$ for every $n \geq 1$. Since A is hyperarchimedean, there exists $n \geq 1$ such that $a \vee (a^n)^* = 1$ (see Proposition 1.69). Then $a \vee (a^n)^* = 1 \in P$, hence $(a^n)^* \in P$ (since $a \notin P$, see Corollary 1.41), a contradiction.

$(ii) \Rightarrow (i)$. Now let $(Spec(A), \subseteq)$ be unordered and $a \in A$, and let us assume that a is not archimedean element, that is, $a^n \notin B(A)$ for every $n \geq 1$.

The set $I_a = \{x \in A : x \odot a^n = 0 \text{ for some } n \geq 1\}$ is an ideal for the lattice $L(A) = (A, \wedge, \vee, 0, 1)$. Indeed, if $x, y \in A, y \in I_a$ and $x \leq y$, then $y \odot a^n = 0$ for some $n \geq 1$. Since $x \odot a^n \leq y \odot a^n = 0$ we deduce that $x \odot a^n = 0$, hence $x \in I_a$. Also, if $x, y \in I_a$ then there are $m, n \geq 1$ such that $x \odot a^m = y \odot a^n = 0$. If denote $p = m+n$, then $x \odot a^p = y \odot a^p = 0$. By $lr - c_{20}$, $(x \vee y) \odot a^p = (x \odot a^p) \vee (y \odot a^p) = 0 \vee 0 = 0$, hence $x \vee y \in I_a$, so I_a is an ideal of $L(A)$. Consider $I =$ the ideal of $L(A)$ generated by $I_a \cup \{a\} = (I_a \cup \{a\}] = (a] \vee I_a$ (where $(a] = \{x \in A : x \leq a\}$).

Clearly $I = \{x \in A : x \leq y \vee a \text{ with } y \in I_a\}$. The ideal I does not contain 1, since if suppose $1 \in I$, then there exist $y \in I_a$ and $n \geq 1$ such that $y \vee a = 1$ and $y \odot a^n = 0$. By $lr - c_{34}$, $y^n \vee a^n = 1$. Since $y^n \leq y$ we deduce that $y^n \odot a^n = 0$, so by $lr - c_{28}$, $y^n \wedge a^n = 0$, hence $a^n \in B(A)$, a contradiction.

Following Theorem 1.43 (with $D = \{1\}$), there is a prime **ds** P of A (i.e. $P \in Spec(A)$) such that $P \cap I = \emptyset$.

Consider $P_a =$ the **ds** generated by $P \cup \{a\}$ (i. e. $P_a = P(a) = [a) \vee P = \{x \in A : x \geq y \odot a^n \text{ with } y \in P \text{ and } n \geq 1\}$); here, we recall that $[a)$ is the ds generated by $\{a\}-$ see Proposition 1.29, (ii).

Note that $0 \notin P_a$, otherwise $y \odot a^n = 0$ for some $y \in P$ and $n \geq 1$. Then $y \in P \cap I = \emptyset$.

Following Theorem 1.43 (with $D = P_a$), there exists a prime **ds** Q of A (i.e. $Q \in Spec(A)$) such that $0 \notin Q$ and $P_a \subseteq Q$.

Then $P \subset Q$, a contradiction since $(Spec(A), \subseteq)$ is supposed unordered. ■

6. Residuated lattice of fractions relative to a $\wedge-$ closed system

In this section, taking as a guide-line the case of rings we introduce for a residuated lattice A the notion of residuated lattice of fractions relative to a $\wedge$-closed system S. In particular if A is an MV-algebra (pseudo MV-algebra), BL-algebra, (pseudo BL-algebra) we obtain the results from Chapters 6, 7 and 8 (see Remarks 1.31 and **??**).

Definition 1.20. A nonempty subset $S \subseteq A$ is called $\wedge-$*closed system in* A if $1 \in S$ and $x, y \in S$ implies $x \wedge y \in S$.

If $\mathcal{P}$ is a prime ideal of the underlying lattice $L(A) = (A, \wedge, \vee)$ (that is $\mathcal{P} \neq A$ and if $x, y \in A$ such that $x \wedge y \in \mathcal{P}$, then $x \in \mathcal{P}$ or $y \in \mathcal{P}$), then $S = A \backslash \mathcal{P}$ is a $\wedge-$closed system.

We denote by $S(A)$ the set of all $\wedge-$closed system of A (clearly $\{1\}, A \in S(A)$).

For $S \in S(A)$, on A we consider the relation θ_S defined by $(x, y) \in \theta_S$ iff there is $e \in S \cap B(A)$ such that $x \wedge e = y \wedge e$.

Lemma 1.78. *The relation θ_S is a congruence on A.*

Proof. The reflexivity (since $1 \in S \cap B(A)$) and the symmetry of θ_S are immediately. To prove the transitivity of θ_S, let $(x, y), (y, z) \in \theta_S$. Thus there are $e, f \in S \cap B(A)$ such that $x \wedge e = y \wedge e$ and $y \wedge f = z \wedge f$. If denote $g = e \wedge f \in S \cap B(A)$, then $g \wedge x = (e \wedge f) \wedge x = (e \wedge x) \wedge f = (y \wedge e) \wedge f = (y \wedge f) \wedge e = (z \wedge f) \wedge e = z \wedge (f \wedge e) = z \wedge g$, hence $(x, z) \in \theta_S$.

To prove the compatibility of θ_S with the operations $\wedge, \vee, \odot$ and $\rightarrow$, let $x, y, z, t \in A$ such that $(x, y) \in \theta_S$ and $(z, t) \in \theta_S$. Thus there are $e, f \in S \cap B(A)$ such that $x \wedge e = y \wedge e$ and $z \wedge f = t \wedge f$; we denote $g = e \wedge f \in S \cap B(A)$, see Remark 1.5.

We obtain:

$$(x \wedge z) \wedge g = (x \wedge z) \wedge (e \wedge f) = (x \wedge e) \wedge (z \wedge f) = (y \wedge e) \wedge (t \wedge f) = (y \wedge t) \wedge g,$$

hence $(x \wedge z, y \wedge t) \in \theta_S$ and

$$(x \vee z) \wedge g \overset{lr-c_{35}}{=} (x \vee z) \odot g \overset{lr-c_{20}}{=} (x \odot g) \vee (z \odot g) \overset{lr-c_{35}}{=} [(e \wedge f) \wedge x] \vee [(e \wedge f) \wedge z] =$$

$$= [(e \wedge x) \wedge f] \vee [e \wedge (f \wedge z)] = [(e \wedge y) \wedge f] \vee [e \wedge (f \wedge t)] =$$

$$= [(e \wedge f) \wedge y] \vee [(e \wedge f) \wedge t] \overset{lr-c_{35}}{=} (y \odot g) \vee (t \odot g) \overset{lr-c_{20}}{=} (y \vee t) \odot g \overset{lr-c_{30}}{=} (y \vee t) \wedge g.$$

hence $(x \vee z, y \vee t) \in \theta_S$.

By $lr - c_{35}$ we obtain:

$$(x \odot z) \wedge g = (x \odot z) \odot g = (x \odot e) \odot (z \odot f) = (x \wedge e) \odot (z \wedge f) = (y \wedge e) \odot (t \wedge f) =$$

$$= (y \odot e) \odot (t \odot f) = (y \odot t) \odot g = (y \odot t) \wedge g,$$

hence $(x \odot z, y \odot t) \in \theta_S$ and by $lr - c_{39}$:

$$(x \rightarrow z) \wedge g = (x \rightarrow z) \odot g = g \odot [(g \odot x) \rightarrow (g \odot z)] = g \odot [(g \wedge x) \rightarrow (g \wedge z)] =$$

$$= g \odot [(g \wedge y) \rightarrow (g \wedge t)] = g \odot [(g \odot y) \rightarrow (g \odot t)] = (y \rightarrow t) \odot g = (y \rightarrow t) \wedge g,$$

hence $(x \rightarrow z, y \rightarrow t) \in \theta_S$.■

For $x \in A$ we denote by x/S the equivalence class of x relative to θ_S and by $A[S] = A/\theta_S$. By $p_S : A \rightarrow A[S]$ we denote the canonical map defined by $p_S(x) = x/S$, for every $x \in A$. Clearly, $A[S]$ become a residuated lattice, where $\mathbf{0} = 0/S$, $\mathbf{1} = 1/S$ and for every $x, y \in A, x/S \wedge y/S = (x \wedge y)/S, x/S \vee y/S = (x \vee y)/S, x/S \odot y/S = (x \odot y)/S, x/S \rightarrow y/S = (x \rightarrow y)/S$. So, p_S is an onto morphism of residuated lattices.

REMARK 1.28. *Since for every $s \in S \cap B(A)$, $s \wedge s = s \wedge 1$ we deduce that $s/S = 1/S = \mathbf{1}$, hence $p_S(S \cap B(A)) = \{\mathbf{1}\}$.*

REMARK 1.29. *If $S = \{1\}$ or S is such that $1 \in S$ and $S \cap (B(A) \backslash \{1\}) = \varnothing$, then for $x, y \in A, (x, y) \in \theta_S \Longleftrightarrow x \wedge 1 = y \wedge 1 \Longleftrightarrow x = y$, hence in this case $A[S] = A$.*

REMARK 1.30. *If S is an $\wedge-$closed system such that $0 \in S$ (for example $S = A$ or $S = B(A)$), then for every $x, y \in A$, $(x, y) \in \theta_S$ (since $x \wedge 0 = y \wedge 0$ and $0 \in S \cap B(A)$), hence in this case $A[S] = \mathbf{0}$.*

PROPOSITION 1.79. *If $a \in A$, then $a/S \in B(A[S])$ iff there is $e \in S \cap B(A)$ such that $a \vee a^* \geq e$. So, if $e \in B(A)$, then $e/S \in B(A[S])$.*

Proof. For $a \in A$, we have by Proposition 1.16, $a/S \in B(A[S]) \Leftrightarrow a/S \vee (a/S)^* = 1 \Leftrightarrow (a \vee a^*)/S = 1/S$ iff there is $e \in S \cap B(A)$ such that $(a \vee a^*) \wedge e = 1 \wedge e = e \Leftrightarrow a \vee a^* \geq e$.

If $e \in B(A)$, since $1 \in S \cap B(A)$ and $1 = e \vee e^* \geq 1$, we deduce that $e/S \in B(A[S])$. ■

THEOREM 1.80. *If A' is a residuated lattice and $f : A \to A'$ is an morphism of residuated lattices such that $f(S \cap B(A)) = \{\mathbf{1}\}$, then there is an unique morphism of residuated lattices $f' : A[S] \to A'$ such that the diagram*

$$\begin{array}{ccc} A & \xrightarrow{p_S} & A[S] \\ & \searrow^{f} \quad \swarrow_{f'} & \\ & A' & \end{array}$$

is commutative (i.e. $f' \circ p_S = f$).

Proof. If $x, y \in A$ and $p_S(x) = p_S(y)$, then $(x, y) \in \theta_S$, hence there is $e \in S \cap B(A)$ such that $x \wedge e = y \wedge e$. Since f is morphism of residuated lattices, we obtain that $f(x \wedge e) = f(y \wedge e) \Leftrightarrow f(x) \wedge f(e) = f(y) \wedge f(e) \Leftrightarrow f(x) \wedge \mathbf{1} = f(y) \wedge \mathbf{1} \Leftrightarrow f(x) = f(y)$.

From this remark, we deduce that the map $f' : A[S] \to A'$ defined for $x \in A$ by $f'(x/S) = f(x)$ is correct defined. Clearly, f' is an morphism of residuated lattices. The unicity of f' follows from the fact that p_S is an onto map.■

DEFINITION 1.21. Theorem 1.80 allows us to call $A[S]$ the *residuated lattice of fractions relative to the $\wedge-$closed system S.*

REMARK 1.31. *If the residuated lattice A is a $BL-$ algebra, then $x/S \wedge y/S = (x \wedge y)/S = (x \odot (x \to y))/S = x/S \odot (x/S \to y/S)$ and $(x/S \to y/S) \vee (y/S \to x/S) = ((x \to y) \vee (y \to x))/S = 1/S = \mathbf{1}$, hence $A[S]$ is a $BL-$ algebra. In this case, $A[S]$ is the BL-algebra of fractions relative to the $\wedge-$closed system S, and we obtain the Theorem 6.3 Analogous if A is a pseudo $BL-$ algebra, so we obtain the Theorem 8.3.*

Suppose now that $\mathcal{P}$ is a prime ideal of the underlying lattice $L(A)$. Then $\mathcal{P} \neq A$ and $S = A \backslash \mathcal{P}$ is a $\wedge-$closed system in A;we denote $A[S]$ by $A_{\mathcal{P}}$ and $I_{\mathcal{P}} = \{x/S : x \in \mathcal{P}\}$.

LEMMA 1.81. *If $x \in A$ such that $x/S \in I_{\mathcal{P}}$, then $x \in \mathcal{P}$.*

Proof. If $x/S \in I_{\mathcal{P}}$, then $x/S = y/S$ with $y \in \mathcal{P} \Rightarrow$ there is $e \in S \cap B(A)$ such that $x \wedge e = y \wedge e \leq y \Rightarrow x \wedge e \in \mathcal{P} \Rightarrow x \in \mathcal{P}$ (since $\mathcal{P}$ is prime and $e \in S = A \backslash \mathcal{P}$, hence $e \notin \mathcal{P}$). ■

PROPOSITION 1.82. *The set $I_{\mathcal{P}}$ is a proper prime ideal of the underlying lattice $L(A_{\mathcal{P}})$.*

Proof. If $x, y \in \mathcal{P}$, then $x/S \vee y/S = (x \vee y)/S \in A_{\mathcal{P}}$ (since $x \vee y \in \mathcal{P}$). Consider now $x \in \mathcal{P}$ and $y \in A$ such that $y/S \leq x/S$. Then $y/S \to x/S = 1/S \Leftrightarrow (y \to x)/S = 1/S \Leftrightarrow$ there is $e \in S \cap B(A)$ such that $e \wedge (y \to x) = e \wedge 1 = e$, hence $e \leq y \to x \Leftrightarrow e \odot y \leq x \Leftrightarrow e \wedge y \leq x$. Then $e \wedge y \in \mathcal{P}$, hence $y \in \mathcal{P}$, so $y/S \in I_{\mathcal{P}}$, that is, $I_{\mathcal{P}}$ is an ideal of $A_{\mathcal{P}}$.

If by contrary, $I_{\mathcal{P}} = A_{\mathcal{P}}$, then $1/S \in I_{\mathcal{P}}$, hence $1 \in \mathcal{P}$ (by Lemma 1.81) $\Leftrightarrow \mathcal{P} = A$, a contradiction.

To prove that $I_{\mathcal{P}}$ is prime, let $x, y \in A$ such that $x/S \wedge y/S \in I_{\mathcal{P}}$. Then $(x \wedge y)/S \in I_{\mathcal{P}} \Rightarrow x \wedge y \in \mathcal{P}$, by Lemma 1.81 $\Rightarrow x \in \mathcal{P}$ or $y \in \mathcal{P} \Rightarrow x/S \in I_{\mathcal{P}}$ or $y/S \in I_{\mathcal{P}}$, hence $I_{\mathcal{P}}$ is a proper prime ideal in lattice $L(A_{\mathcal{P}})$. ■

REMARK 1.32. *Following the model of commutative rings, the process of passing from A to $A_{\mathcal{P}}$ is called localization at $\mathcal{P}$ (taking as a guide-line the case of rings, see* [**81**]*).*

CHAPTER 2

MV-algebras

MV-algebras are particular residuated lattices.

MV-algebras were originally introduced by Chang in [**42**] in order to give an algebraic counterpart of the Łukasiewicz many valued logic (MV = many valued). Just take a quick view over this domain. In 1958, Chang defined the MV-algebras and in 1959 he also proved the completeness theorem which stated the real unit interval $[0, 1]$ as a standard model of this logic. The structures directly obtained from Łukasiewicz logic, in the sense that the basic operations coincide with the basic logical connectives (implication and negation), were defined by Font, Rodriguez and Torrens in [**62**] under the name of Wajsberg algebras. Wajsberg algebras and MV - algebras are categorically isomorphic. One great event in the theory of MV-algebras was Mundici's theorem from 1986: the category of MV-algebras is equivalent to the category of abelian lattice-ordered groups with strong unit [**105**]. Through its consequences, this theorem can be identified at the origins of a considerable number of results on MV-algebras.

In the last years, one can distinguish at least three fruitful research directions, coexisting and communicating with deeper and deeper researches on MV-algebras.

One direction is concerned with structures obtained by adding operations to the MV-algebra structure, or even combining MV-algebras with other structures in order to obtain more expressive models and powerful logical systems.

Another direction is centered on the non-commutative extensions of MV-algebras, called pseudo MV-algebras (psMV-algebras, for short), introduced by Georgescu and Iorgulescu in 1999 [**66**], [**68**].

Finally, the third direction I want to emphasize began with Hájek's book, where BL-logic and BL-algebras were defined [**74**], [**75**]. Juste notice that Łukasiewicz logic in an axiomatic extension of BL-logic and, consequently, MV-algebras are a particular class of BL-algebras (see Remark 3.4). The non-commutative corresponding structures, called pseudo BL-algebras, were introduced by Di Nola, Georgescu and Iorgulescu [**67**], [**53**], [**54**].

The standard reference for the domain of MV-algebras is the monograph [**45**].

In this chapter, we recall some basic definitions and results abut MV-algebras.

For an MV-algebra A, we denote by Id(A) the set of ideals of A. We present some known basic definitions and results relative to the lattice of ideals of A.

We study the prime spectrum Spec(A) and the maximal spectrum Max(A) of an MV-algebra.

For any class of structures, the representation theorems have a special significace.

The Chang's Subdirect Representation Theorem is a fundamental result.

The idea of associating a totally ordered abelian group to any MV- algebra A is due to Chang, who in [**42**] and [**43**] gave first purely algebraic proof of the completeness of the Łukasiewicz axioms for the infinite-valued calculus. In [**45**] is proved the Chang completeness theorem starting that if an equation holds in the unit real interval $[0, 1]$, then the the

equation holds in every MV- algebra. This proof is elementary, and use the good sequences; good sequences and Γ functor were first introduced in [**105**].

An applications is the equivalence between MV- algebras and lattice ordered abelian groups with strong unit.

We also prove the one-to-one correspondence between MV- algebras and Wajsberg algebras; each MV- algebra can be seen as Wajsberg algebra and conversely. MV- algebras will turn out to be particular residuated lattices.

For further reading on MV- algebras we recommend [**45**].

1. Definitions and first properties. Some examples. Rules of calculus

We introduce MV- algebras by means of a small number of simple equations, in an attempt to capture certain properties of the real unit interval $[0, 1]$ equipped with addition $x \oplus y = \min\{1, x + y\}$ and negation $1 - x$ see Remark 1.2. We show that every MV-algebra contains a natural lattice-order. An main result is Chang's Subdirect Representation Theorem, stating that if an equation holds in all totally ordered MV- algebras, then the equation holds in all MV- algebras.

DEFINITION 2.1. An *MV-algebra* is an algebra $\mathcal{A} = (A, \oplus, ^*, 0)$ of type $(2, 1, 0)$ satisfying the following equations:

(MV_1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
(MV_2) $x \oplus y = y \oplus x$;
(MV_3) $x \oplus 0 = x$;
(MV_4) $x^{**} = x$;
(MV_5) $x \oplus 0^* = 0^*$;
(MV_6) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$, for all $x, y, z \in A$.

Note that axioms MV_1-MV_3 state that $(A, \oplus, 0)$ is an abelian monoid.

REMARK 2.1. *If in MV_6 we put $y = 0$ we obtain $x^{**} = 0^{**} \oplus x$, so, if $0^{**} = 0$ then $x^{**} = x$ for every $x \in A$. Hence, the axiom MV_4 is equivalent with* (MV_4') $0^{**} = 0$.

In order to simplify the notation, an MV-algebra $\mathcal{A} = (A, \oplus, ^*, 0)$ will be referred by its support set, A. An MV-algebra is *trivial* if its support is a singleton. On an MV-algebra A we define the constant 1 and the auxiliary operations $\odot, \ominus$ and $\rightarrow$ as follows :

$$1 = 0^*,$$
$$x \odot y = (x^* \oplus y^*)^*,$$
$$x \ominus y = x \odot y^* = (x^* \oplus y)^*,$$
$$x \rightarrow y = x^* \oplus y^*,$$

for any $x, y \in A$.

We consider the operation * more binding that any other operation, and $\odot$ more binding that $\oplus$ and $\ominus$.

REMARK 2.2. *([**82**]) In MV-algebra $\mathcal{A} = (A, \oplus, ^*, 0)$ we have:*

(MV_1') $x \odot (y \odot z) = (x \odot y) \odot z$;
(MV_2') $x \odot y = y \odot x$;
(MV_3') $x \odot 1 = x$;
(MV_4') $x^{**} = x$;

(MV_5') $x \odot 1^* = 1^*$;
(MV_6') $(x^* \odot y)^* \odot y = (y^* \odot x)^* \odot x$, for all $x, y \in A$, that is $(A, \odot, ^*, 1)$ is an $MV-$ algebra.

A *subalgebra* of an MV- algebra A is a subset A' of A containing the zero element of A, closed under the operations of A and equipped with the restriction to A' of these operations.

In the sequel, we provide some basic examples of MV-algebras.

EXAMPLE 2.1. *A singleton* $\{0\}$ *is a trivial example of an* MV*-algebra; an* MV*-algebra is said nontrivial provided its universe has more that one element.*

EXAMPLE 2.2. *Any Boolean algebra is an* MV*-algebra in which the operations* $\oplus$ *and* $\vee$ *coincide and* * *is the Boolean negation.*

We recall that a *lattice-ordered group (l-group)* (see [**10**]) is a structure $(G, +, 0, \leq)$ such that $(G, +, 0)$ is a group, $(G, \leq)$ is a lattice and the following property is satisfied:

$$\text{for any } x, y, a, b \in G, x \leq y \Rightarrow a + x + b \leq a + y + b.$$

For any l-group, $(G, +, 0, \leq)$ and for any $g \geq 0$ in G we denote

$$[0, g] = \{x \in G : 0 \leq x \leq g\}.$$

If G is an l-group, then a *strong unit* is an element $u > 0$ such that for any $x \in G$ there is a natural number n such that $-nu \leq x \leq nu$, see Definition 2.11.

In the sequel, an lu-group will be a pair (G, u) where G is an l-group and u is a strong unit of G. If (G, u) and (H, v) are lu-groups then an *lu-group homomorphism* is an l-group homomorphism $h : G \to H$ such that $h(u) = v$.

EXAMPLE 2.3. *Let* $(G, +, 0, \leq)$ *be an abelian l-group and* $u \in G$, $u > 0$. *If we define*

$$x \oplus y = u \wedge (x + y)$$

and

$$x^* = u - x,$$

for $x, y \in [0, u]$, *then* $[0, u]_G = ([0, u], \oplus, ^*, 0)$ *is an* MV*-algebra. For any* $x, y \in [0, u]$, *we get*

$$x \odot y = (x - u + y) \vee 0,$$

$$x \to y = (u - x + y) \wedge u,$$

$$x \ominus y = (x - y) \vee 0,$$

and the lattice operations coincide to those of G. *In particular, if we consider the real unit interval* $[0, 1]$ *and for all* $x, y \in [0, 1]$ *we define*

$$x \oplus y = \min\{1, x + y\}$$

and

$$x^* = 1 - x,$$

then $([0, 1], \oplus, ^*, 0)$ *is an* MV*-algebra.*

EXAMPLE 2.4. *The rational numbers in* $[0, 1]$, *and, for each integer* $n \geq 2$, *the* n*-element set* $L_n = \left\{0, \frac{1}{n-1}, ..., \frac{n-2}{n-1}, 1\right\}$ *yield examples of* $MV-$*subalgebras of* $[0, 1]$.

EXAMPLE 2.5. *Given an MV-algebra $(A, \oplus, ^*, 0)$ and a nonempty set X, the set A^X of all functions $f : X \longrightarrow A$ becomes an MV-algebra with the pointwise operations, i.e., if $f, g \in A^X$ then $(f \oplus g)(x) = f(x) \oplus g(x), f^*(x) = [f(x)]^*$ for any $x \in X$ and 0 is the constant function associated with $0 \in A$. The continuous functions from $[0,1]$ into $[0,1]$ form a subalgebra of the MV-algebra $[0,1]^{[0,1]}$.*

EXAMPLE 2.6. *(Chang's MV-algebra C - see [**42**]) Let $\{c, 0, 1, +, -\}$ be a set of formal symbols. For any $n \in N$ we define the following abbreviations:*

$$nc := \begin{cases} 0, \text{ if } n = 0, \\ c, \text{ if } n = 1, \\ c + (n-1)c, \text{ if } n > 1, \end{cases} \qquad 1 - nc := \begin{cases} 1, \text{ if } n = 0, \\ 1 - c, \text{ if } n = 1, \\ 1 - (n-1)c - c, \text{ if } n > 1. \end{cases}$$

We consider $C = \{nc : n \in N\} \cup \{1 - nc : n \in N\}$ and define the MV-algebra operations as follows:

($\oplus$1) *if $x = nc$ and $y = mc$ then $x \oplus y := (m+n)c$;*
($\oplus$2) *if $x = 1 - nc$ and $y = 1 - mc$ then $x \oplus y := 1$;*
($\oplus$3) *if $x = nc$ and $y = 1 - mc$ and $m \leq n$ then $x \oplus y := 1$;*
($\oplus$4) *if $x = nc$ and $y = 1 - mc$ and $n < m$ then $x \oplus y := 1 - (m-n)c$;*
($\oplus$5) *if $x = 1 - mc$ and $y = nc$ and $m \leq n$ then $x \oplus y := 1$;*
($\oplus$6) *if $x = 1 - mc$ and $y = nc$ and $n < m$ then $x \oplus y := 1 - (m-n)c$;*
(*1) *if $x = nc$ then $x^* := 1 - nc$;*
(*2) *if $x = 1 - nc$ then $x^* := nc$.*

Then, the structure $(C, \oplus, ^*, 0)$ is an MV-algebra, which is called the Chang's algebra.

THEOREM 2.1. *If $x, y, z \in A$ then the following hold :*

($mv - c_1$) $1^* = 0$;
($mv - c_2$) $x \oplus y = (x^* \odot y^*)^*$;
($mv - c_3$) $x \oplus 1 = 1, x \odot 1 = x, x \odot 0 = 0$;
($mv - c_4$) $(x \ominus y) \oplus y = (y \ominus x) \oplus x$;
($mv - c_5$) $x \oplus x^* = 1, x \odot x^* = 0, (x \odot y)^* = x^* \oplus y^*, (x \oplus y)^* = x^* \odot y^*, x \odot (x^* \oplus y) = y \odot (y^* \oplus x), x \odot (y \odot z) = (x \odot y) \odot z$;
($mv - c_6$) $x \ominus 0 = x$, $0 \ominus x = 0$, $x \ominus x = 0$, $1 \ominus x = x^*$, $x \ominus 1 = 0$;
($mv - c_7$) $x \oplus x = x$ *iff* $x \odot x = x$.

Proof. ($mv - c_1$). Obviously, $1^* = 0^{**} = 0$.

($mv - c_2$). We have $x^* \odot y^* = (x^{**} \oplus y^{**})^* = (x \oplus y)^*$, so $x \oplus y = (x \oplus y)^{**} = (x^* \odot y^*)^*$.

($mv - c_3$). We have $x \oplus 1 = x \oplus 0^* = 0^* = 1, x \odot 1 = (x^* \oplus 1^*)^* = x^{**} = x$ and $x \odot 0 = (x^* \oplus 0^*)^* = 1^{**} = 0$.

($mv - c_5$). By MV_4 we have $1 = (x^* \oplus 1)^* \oplus 1 = (1^* \oplus x)^* \oplus x = x^* \oplus x = x \oplus x^*$ and $x \odot x^* = (x^* \oplus x^{**})^* = 1^* = 0$.

Also $(x \odot y)^* = (x^* \oplus y^*)^{**} = x^* \oplus y^*$, $(x \oplus y)^* = (x^* \odot y^*)^{**} = x^* \odot y^*$, $x \odot (x^* \oplus y) = [x^* \oplus (x^* \oplus y)^*]^* = [x^* \oplus (x^* \oplus y^{**})^*]^* = [(x^{**} \oplus y^*)^* \oplus y^*]^* = (x \oplus y^*)^{**} \odot y^{**} = (x \oplus y^*) \odot y$

and $(x \odot y) \odot z = [(x \odot y)^* \oplus z^*]^* = (x^* \oplus y^* \oplus z^*)^* = [x^* \oplus (y^* \oplus z^*)]^* = x \odot (y^* \oplus z^*)^* = x \odot (y \odot z)$.

The other relations follows similarly. ■

LEMMA 2.2. *For $x, y \in A$, the following conditions are equivalent:*

(i) $x^* \oplus y = 1$;
(ii) $x \odot y^* = 0$;
(iii) $y = x \oplus (y \ominus x)$;
(iv) *There is an element* $z \in A$ *such that* $x \oplus z = y$.

Proof. $(i) \Rightarrow (ii)$. Follows from MV_4 and $mv - c_1 : x \odot y^* = (x^* \oplus y^{**})^* = (x^* \oplus y)^* = 1^* = 0$.

$(ii) \Rightarrow (iii)$. Follows from MV_3 and $mv - c_4 : x \oplus (y \ominus x) = x \oplus (y \odot x^*) = y \oplus (x \odot y^*) = y \oplus 0 = y$

$(iii) \Rightarrow (iv)$. Take in (iii), $z = y \ominus x$.

$(iv) \Rightarrow (i)$. We have $x^* \oplus y = x^* \oplus (x \oplus z) = (x^* \oplus x) \oplus z \overset{mv-c_5}{=} 1 \oplus z \overset{mv-c_3}{=} 1$. ■

For any two elements $x, y \in A$ let us agree to write $x \leq y$ iff x and y satisfy the equivalent conditions $(i) - (iv)$ in the above lemma. So, $\leq$ is an order relation on A (called the *natural order* on A).

Indeed, reflexivity is equivalent to $mv - c_5, (x \leq x$ iff $x^* \oplus x = 1)$ antisimetry follows from conditions (ii) and (iii) (if $x \leq y$ and $y \leq x$ then $y = x \oplus (x^* \odot y)$ and $x = y \oplus (y^* \odot x)$ but by MV_6, $x \oplus (x^* \odot y) = y \oplus (y^* \odot x)$, so $x = y$) and transitivity follows from condition (iv) (if $x \leq y$ and $y \leq z$ then there exist $u, v \in A$ such that $y = x \oplus u$ and $z = y \oplus v$, so $z = x \oplus u \oplus v$, that is, $x \leq z$).

We will say that an MV-algebra A is an MV-*chain* if it is linearly ordered relative to natural order.

The order relation in Chang's algebra C is defined by: $x \leq y$ iff $[x = nc$ and $y = 1 - mc\ ]$ or $[x = nc$ and $y = mc$ and $n \leq m\ ]$ or $[x = 1 - nc$ and $y = 1 - mc$ and $m \leq n\ ]$. In conclusion, C is a linearly ordered MV-algebra:

$$C = \{0, c, ..., nc, ..., 1 - nc, ..., 1 - c, 1\}.$$

THEOREM 2.3. *If* $x, y, z \in A$ *then the following hold :*

$(mv - c_8)$ $x \leq y$ *iff* $y^* \leq x^*$;
$(mv - c_9)$ *If* $x \leq y$, *then* $x \oplus z \leq y \oplus z$ *and* $x \odot z \leq y \odot z$;
$(mv - c_{10})$ *If* $x \leq y$, *then* $x \ominus z \leq y \ominus z$ *and* $z \ominus y \leq z \ominus x$;
$(mv - c_{11})$ $x \ominus y \leq x, x \ominus y \leq y^*$;
$(mv - c_{12})$ $(x \oplus y) \ominus x \leq y$;
$(mv - c_{13})$ $x \odot z \leq y$ *iff* $x \leq z^* \oplus y$;
$(mv - c_{14})$ $x \oplus y \oplus x \odot y = x \oplus y$.

Proof. $(mv - c_8)$. Follows from Lema 2.2, (i), since $x^* \oplus y = (y^*)^* \oplus x^*$.

$(mv - c_9)$. We get $y \oplus z \oplus (x \oplus z)^* = y \oplus (z \oplus x^* \odot z^*) = y \oplus (x^* \oplus x \odot z) = (y \oplus x^*) \oplus x \odot z = 1 \oplus x \odot z = 1$, so $x \oplus z \leq y \oplus z$. The other inequality follows similarly.

$(mv - c_{13})$. We have $x \odot z \leq y \Leftrightarrow (x \odot z)^* \oplus y = 1 \Leftrightarrow x^* \oplus z^* \oplus y = 1 \Leftrightarrow x \leq z^* \oplus y$.

The other relations follows similarly. ■

LEMMA 2.4. *On* A, *the natural order determines a bounded distributive lattice structure. Specifically, the join* $x \vee y$ *and the meet* $x \wedge y$ *of the elements* x *and* y *are given by:*

$$x \vee y = (x \ominus y) \oplus y = (y \ominus x) \oplus x = x \odot y^* \oplus y = y \odot x^* \oplus x,$$

$$x \wedge y = (x^* \vee y^*)^* = x \odot (x^* \oplus y) = y \odot (y^* \oplus x).$$

Clearly, $x \odot y \leq x \wedge y \leq x, y \leq x \vee y \leq x \oplus y$.

Proof. Obviously, $y \leq (x \ominus y) \oplus y$ and $x \leq (x \ominus y) \oplus y \overset{mv-c_4}{=} (y \ominus x) \oplus x$.

Suppose $x \leq z$ and $y \leq z$. By Lema 2.2, (i) and (iii), $x^* \oplus z = 1$ and $z = (z \ominus y) \oplus y$.

Then, $((x \ominus y) \oplus y)^* \oplus z = ((x \ominus y)^* \ominus y) \oplus y \oplus (z \ominus y) = (y \ominus (x \ominus y)^*) \oplus (x \ominus y)^* \oplus (z \ominus y) = (y \ominus (x \ominus y)^*) \oplus x^* \oplus y \oplus (z \ominus y) = (y \ominus (x \ominus y)^*) \oplus x^* \oplus z = 1$. It follows that $(x \ominus y) \oplus y \leq z$, so $x \vee y = (x \ominus y) \oplus y$.

We now imediately obtain the second equality as consequance of first equality together with $mv - c_8$. ■

We shall denote this distributive lattice with 0 and 1 by $L(A)$ (see [**42**], [**45**]).

We recall that:

Definition 2.2. An MV-algebra is called *Kleene algebra* iff it satisfies the additional condition:

$$x \wedge x^* \leq y \vee y^*.$$

Proposition 2.5. ([**45**]) *In any MV-algebra A, the following properties hold:*

(i) $(A, \odot, 1)$ *is an abelian monoid;*
(ii) $(A, \vee, \wedge, 0, 1)$ *is a bounded distributive lattice;*
(iii) $(A, \vee, \wedge, ^*, 0, 1)$ *is a Kleene algebra;*
(iv) $(A, \vee, \wedge, \odot, \rightarrow, 0, 1)$ *is a residuated lattice.*

For each $x \in A$, let $0x = 0, x^0 = 1$ and for each integer $n \geq 0, (n+1)x = nx \oplus x$, and $x^{n+1} = x^n \odot x$, respectively.

We say that the element $x \in A$ has *order* n, and we write $ord(x) = n$, if n is the last natural number such that $nx = 1$. We say that the element x has a *finite order* , and we write $ord(x) < \infty$, if x has order n for some $n \in N$. If no such n exists, we say that x has *infinite order* and we write $ord(x) = \infty$. An MV -algebra A is *locally finite* if every non-zero element of A has finite order.

Theorem 2.6. *If $x, x_1, ...x_n, y, z, (x_i)_{i \in I}$ are elements of A, then the following hold:*

$(mv - c_{15})$ $(x \vee y)^* = x^* \wedge y^*, (x \wedge y)^* = x^* \vee y^*$;

$(mv - c_{16})$ $x \oplus y = (x \vee y) \oplus (x \wedge y), x \odot y = (x \vee y) \odot (x \wedge y)$;

$(mv - c_{17})$ $x \oplus \left(\bigwedge_{i \in I} x_i \right) = \bigwedge_{i \in I} (x \oplus x_i)$;

$(mv - c_{18})$ $x \odot \left(\bigvee_{i \in I} x_i \right) = \bigvee_{i \in I} (x \odot x_i)$;

$(mv - c_{19})$ $x \wedge \left(\bigvee_{i \in I} x_i \right) = \bigvee_{i \in I} (x \wedge x_i)$;

$(mv - c_{20})$ $x \vee \left(\bigwedge_{i \in I} x_i \right) = \bigwedge_{i \in I} (x \vee x_i)$;

$(mv - c_{21})$ $x \oplus \left(\bigvee_{i \in I} x_i \right) = \bigvee_{i \in I} (x \oplus x_i)$;

$(mv - c_{22})$ $x \odot \left(\bigwedge_{i \in I} x_i \right) = \bigwedge_{i \in I} (x \odot x_i)$, *(if all suprema and infima exist).*

If $I = \{1, 2, ..., n\}$ then

$(mv - c_{23})$ $x \vee (x_1 \odot ... \odot x_n) \geq (x \vee x_1) \odot ... \odot (x \vee x_n)$; *in particular $x^m \vee y^n \geq (x \vee y)^{mn}$ for every $m, n \geq 0$;*

$(mv - c_{24})$ $x \wedge (x_1 \oplus ... \oplus x_n) \leq (x \wedge x_1) \oplus ... \oplus (x \wedge x_n)$; *in particular $(mx) \wedge (ny) \leq mn(x \wedge y)$ for every $m, n \geq 0$;*

$(mv - c_{25})$ *If* $x \vee y = 1$ *then* $x^m \vee y^n = 1$ *for every* $m, n \geq 0$;
$(mv - c_{26})$ *If* $x \wedge y = 0$, *then for every integers* $m, n \geq 0, (mx) \wedge (ny) = 0$.

Proof. $(mv - c_{15})$. $(x \vee y)^* = (x \oplus x^* \odot y)^* = x^* \odot (x \oplus y^*) = x^* \odot (x^{**} \oplus y^*) = x^* \wedge y^*$ and $(x \wedge y)^* = [x \odot (x^* \oplus y)]^* = x^* \oplus (x \odot y^*) = x^* \oplus (x^{**} \odot y^*) = x^* \vee y^*$.

$(mv - c_{16})$. We get $(x \vee y) \oplus (x \wedge y) = x \oplus (x^* \odot y) \oplus [y \odot (x \oplus y^*)] = x \oplus (x^* \odot y) \oplus [y \odot (x^* \odot y)^*] = x \oplus y \oplus (y^* \odot x^* \odot y) = x \oplus y \oplus 0 = x \oplus y$ and $x \odot y = (x^* \oplus y^*)^* = [(x^* \vee y^*) \oplus (x^* \wedge y^*)]^* = (x^* \vee y^*)^* \odot (x^* \wedge y^*)^* = (x^{**} \wedge y^{**}) \odot (x^{**} \vee y^{**}) = (x \vee y) \odot (x \wedge y)$.

$(mv - c_{17})$. It is obvious that $x \oplus \left(\bigwedge_{i \in I} x_i \right) \leq x \oplus x_i$, for any $i \in I$. Let $z \in A$ such that $z \leq x \oplus x_i$, for any $i \in I$. Then $z \leq x^{**} \oplus x_i$, for any $i \in I$, so $z \odot x^* \leq x_i$, for any $i \in I$.

Thus we have that $z \odot x^* \leq \bigwedge_{i \in I} x_i$, so $z \leq x \oplus \left(\bigwedge_{i \in I} x_i \right)$. Hence $x \oplus \left(\bigwedge_{i \in I} x_i \right) = \bigwedge_{i \in I} (x \oplus x_i)$.

$(mv - c_{18})$. It is obvious that $x \odot x_i \leq x \odot \left(\bigvee_{i \in I} x_i \right)$, for any $i \in I$. Let $z \in A$ such that $x \odot x_i \leq z$, for any $i \in I$. Then $x_i \leq x^* \oplus z$, for any $i \in I$, so $\bigvee_{i \in I} x_i \leq x^* \oplus z$.

Thus we have that $(\bigvee_{i \in I} x_i) \odot x \leq z$. We deduce that $x \odot \left(\bigvee_{i \in I} x_i \right) = \bigvee_{i \in I} (x \odot x_i)$.

$(mv - c_{19})$. By definition, $x \wedge \left(\bigvee_{i \in I} x_i \right) = \left(\bigvee_{i \in I} x_i \right) \odot \left[\left(\bigvee_{i \in I} x_i \right)^* \oplus x \right] \overset{mv - c_{18}}{=} \bigvee_{i \in I} \left[x_i \odot \left(\left(\bigvee_{i \in I} x_i \right)^* \oplus x \right) \right]$.

For any $i \in I$ we have $x_i \leq \bigvee_{j \in I} x_j$, so $\left(\bigvee_{j \in I} x_j \right)^* \leq x_i^* \Rightarrow \left(\bigvee_{j \in I} x_j \right)^* \oplus x \leq x_i^* \oplus x \Rightarrow x_i \odot \left[\left(\bigvee_{j \in I} x_j \right)^* \oplus x \right] \leq x_i \odot (x_i^* \oplus x) = x \wedge x_i$.

We deduce that $\bigvee_{i \in I} \left(x_i \odot \left[\left(\bigvee_{j \in I} x_j \right)^* \oplus x \right] \right) \leq \bigvee_{i \in I} (x \wedge x_i)$.

Another inequality is obviously.

$(mv - c_{20})$. It follows smilarly with $mv - c_{19}$.

$(mv - c_{21})$. We remark that $x \oplus x_i \leq x \oplus \left(\bigvee_{i \in I} x_i \right)$, for any $i \in I$. Let z such that $x \oplus x_i \leq z$ for any $i \in I$. We get $x^* \wedge x_i = x^* \odot (x \oplus x_i) \leq x^* \odot z$, for any $i \in I$. Using $mv - c_{19}$, it folows that $x^* \wedge \left(\bigvee_{i \in I} x_i \right) = \bigvee_{i \in I} (x^* \wedge x_i) \leq x^* \odot z$. Thus, $x \oplus \left[x^* \wedge \left(\bigvee_{i \in I} x_i \right) \right] \leq x \oplus (x^* \odot z) = x \vee z = z$.

Using $mv - c_{17}$, it folows that $x \oplus \left[x^* \wedge \left(\bigvee_{i \in I} x_i \right) \right] = (x \oplus x^*) \wedge \left[x \oplus \left(\bigvee_{i \in I} x_i \right) \right] = 1 \wedge \left[x \oplus \left(\bigvee_{i \in I} x_i \right) \right] = x \oplus \left(\bigvee_{i \in I} x_i \right)$.

Thus, $x \oplus \left(\bigvee_{i\in I} x_i\right) \leq z$.

$(mv - c_{22})$. It is obvious that $x \odot \left(\bigwedge_{i\in I} x_i\right) \leq x \odot x_i$, for any $i \in I$. Let $z \in A$ such that $z \leq x \odot x_i$, for any $i \in I$. Then $x^* \oplus z \leq x^* \oplus (x \odot x_i) = x^* \vee x_i$, for any $i \in I$, so $x^* \oplus z \leq \bigwedge_{i\in I}(x^* \vee x_i) = x^* \vee \left(\bigwedge_{i\in I} x_i\right)$.

Thus we have that $x\odot(x^* \oplus z) \leq x\odot\left[x^* \vee \left(\bigwedge_{i\in I} x_i\right)\right] = (x\odot x^*)\vee\left[x \odot \left(\bigwedge_{i\in I} x_i\right)\right] = 0 \vee \left[x \odot \left(\bigwedge_{i\in I} x_i\right)\right] = x \odot \left(\bigwedge_{i\in I} x_i\right)$. We deduce that

$z = x \wedge z \leq x \odot \left(\bigwedge_{i\in I} x_i\right)$.

The other relations follows similarly. ■

Remark 2.3. $(x \odot y^*) \wedge (y \odot x^*) = 0$, *for any* $x, y \in A$.

Proof. Indeed, $0 = (x \wedge y) \odot (x \wedge y)^* = (x \wedge y) \odot (x^* \vee y^*) = [x \odot (x^* \vee y^*)] \wedge [y \odot (x^* \vee y^*)] = [(x \odot x^*) \vee (x \odot y^*)] \wedge [(y \odot x^*) \vee (y \odot y^*)] = [0 \vee (x \odot y^*)] \wedge [(y \odot x^*) \vee 0] = (x \odot y^*) \wedge (y \odot x^*)$. ■

Lemma 2.7. *If* a, b, x *are elements of* A, *then:*

$(mv - c_{27})$ $[(a \wedge x) \oplus (b \wedge x)] \wedge x = (a \oplus b) \wedge x$;

$(mv - c_{28})$ $a^* \wedge x \geq x \odot (a \wedge x)^*$.

Proof. $(mv - c_{27})$. By $mv - c_{17}$ we have

$$[(a \wedge x) \oplus (b \wedge x)] \wedge x = ((a \wedge x) \oplus b) \wedge ((a \wedge x) \oplus x) \wedge x =$$

$$= ((a \wedge x) \oplus b) \wedge x = (a \oplus b) \wedge (x \oplus b) \wedge x = (a \oplus b) \wedge x.$$

$(mv - c_{28})$. We have

$$x \odot (a \wedge x)^* = x \odot (a^* \vee x^*) \overset{mv-c_{18}}{=} (x \odot a^*) \vee (x \odot x^*) \overset{mv-c_5}{=}$$

$$\overset{mv-c_5}{=} (x \odot a^*) \vee 0 = x \odot a^* \leq a^* \wedge x.\blacksquare$$

For any MV-algebra A we shall denote by $B(A)$ the set of all complemented elements of $L(A)$; the elements of $B(A)$ are called the *boolean elements* of A.

Theorem 2.8. *For every element* e *in an* MV*-algebra* A, *the following conditions are equivalent:*

(i) $e \in B(A)$;

(ii) $e \vee e^* = 1$;

(iii) $e \wedge e^* = 0$;

(iv) $e \oplus e = e$;

(v) $e \odot e = e$.

Proof. First we prove the folowing implications: $(iv) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (v) \Rightarrow (iv)$.

$(iv) \Rightarrow (iii)$. $e \wedge e^* = e^* \odot (e^{**} \oplus e) = e^* \odot (e \oplus e) = e^* \odot e = 0$.

$(iii) \Rightarrow (ii)$. $1 = 0^* = (e \wedge e^*)^* = e^* \vee e$.

$(ii) \Rightarrow (v)$. $e = e \odot 1 = e \odot (e \vee e^*) = (e \odot e) \vee (e \odot e^*) = (e \odot e) \vee 0 = e \odot e$.

$(v) \Rightarrow (iv)$. By hypothesis, $e^* \oplus e^* = e^*$. It follows that $e = e \oplus 0 = e \oplus (e \odot e^*) = e \oplus e \odot (e^* \oplus e^*) = e \oplus (e \wedge e^*) = (e \oplus e) \wedge (e \oplus e^*) = (e \oplus e) \wedge 1 = e \oplus e$.

For the equivalence $(i) \Leftrightarrow (iii)$ see the proof of Theorem 4.11 (the equivalence $(i) \Leftrightarrow (iv)$) .■

THEOREM 2.9. *If $e \in B(A)$ and $x \in A$ then $e \oplus x = e \vee x$ and $e \odot x = e \wedge x$.*

Proof. Obviously, $e \oplus x \geq e \vee x$ and $e \odot x \leq e \wedge x$. We shall prove that $e \oplus x \leq e \vee x$ and $e \wedge x \leq e \odot x$. Thus, $(e \oplus x) \odot (e \vee x)^* = (e \oplus x) \odot (e^* \wedge x^*) = [(e \oplus x) \odot e^*] \wedge [(e \oplus x) \odot x^*] = x \wedge e^* \wedge e \wedge x^* = 0$ and $(e \wedge x) \odot (e \odot x)^* = (e \wedge x) \odot (e^* \oplus x^*) = [e \odot (e^* \oplus x^*)] \wedge [x \odot (e^* \oplus x^*)] = e \wedge x^* \wedge x \wedge e^* = 0$. ■

COROLLARY 2.10. ([**45**])

(*i*) *$B(A)$ is subalgebra of the MV-algebra A. A subalgebra B of A is a boolean algebra iff $B \subseteq B(A)$,*

(*ii*) *An MV-algebra A is a boolean algebra iff the operation $\oplus$ is idempotent, i.e., the equation $x \oplus x = x$ is satisfied by A.*

EXAMPLE 2.7.

1. *If A is an MV-chain, then $B(A) = \{0, 1\} = L_2$.*
2. *If A is an MV-algebra and X a nonempty set, then $B(A^X) = (B(A))^X$ (see Example 2.5). In particular, if $A = [0, 1]$ then $B(A) = L_2$, hence $B([0, 1]^X) = (L_2)^X$ for every nonempty set X.*

REMARK 2.4. *For $e \in B(A)$ we denote $A(e) = \{x \in A : x \leq e\} = (e]$ (see Proposition 2.12); for $x \in A$ we introduce $x^{\sim} = x^* \wedge e \in A(e)$. Then $(A(e), \oplus, ^{\sim}, 0, e)$ is an MV-algebra.*

COROLLARY 2.11. *If $a \in B(A)$ and $x, y \in A$, then:*

$(mv - c_{29})$ $a^* \wedge x = x \odot (a \wedge x)^*$;
$(mv - c_{30})$ $a \wedge (x \oplus y) = (a \wedge x) \oplus (a \wedge y)$;
$(mv - c_{31})$ $a \vee (x \oplus y) = (a \vee x) \oplus (a \vee y)$.

Proof. $(mv - c_{29})$. See the proof of $mv - c_{28}$.

$(mv - c_{30})$. We have:

$$(a \wedge x) \oplus (a \wedge y) \overset{mv-c_{17}}{=} [(a \wedge x) \oplus a] \wedge [(a \wedge x) \oplus y] =$$

$$= [(a \wedge x) \vee a] \wedge [(a \oplus y) \wedge (x \oplus y)] = a \wedge (a \oplus y) \wedge (x \oplus y) = a \wedge (x \oplus y).$$

$(mv - c_{31})$. We have $(a \vee x) \oplus (a \vee y) = (a \oplus x) \oplus (a \oplus y) = (a \oplus a) \oplus (x \oplus y) = a \oplus (x \oplus y) = a \vee (x \oplus y)$. ■

DEFINITION 2.3. Let A and B be $MV-$algebras. A function $f : A \to B$ is a *morphism of $MV-$algebras* iff it satisfies the following conditions, for every $x, y \in A$:

(MV_7) $f(0) = 0$;
(MV_8) $f(x \oplus y) = f(x) \oplus f(y)$;
(MV_9) $f(x^*) = (f(x))^*$.

REMARK 2.5. *One can immediately prove that:*

$$f(1) = 1,$$
$$f(x \odot y) = f(x) \odot f(y),$$
$$f(x \vee y) = f(x) \vee f(y),$$
$$f(x \wedge y) = f(x) \wedge f(y),$$

for every $x, y \in A$.

Recall that, following current usage, if f is one-one we shall equivalently say that f is an *injective homomorphism*, or an *embedding*. If the homomorphism $f : A \to B$ is onto B we say that f is surjective. The kernel of a homomorphism $f : A \to B$ is the set $Ker(f) = f^{-1}(0) = \{x \in A : f(x) = 0\}$.

We denote by $\mathcal{MV}$ the category whose objects are $MV-$algebras and whose morphisms are $MV-$algebras homomorphisms. Since $\mathcal{MV}$ is an equational category, then the monomorphisms in $\mathcal{MV}$ are exactly the injective morphisms ([**2**]). If A and B are $MV-$ algebras we write $A \approx B$ iff there is an *isomorphism* of $MV-$ algebras from A onto B (that is a bijective morphism of $MV-$ algebras).

2. The lattice of ideals of an MV-algebra

For an MV-algebra, we denote by $Id(A)$ the set of ideals of A. We present some known basic definitions and results relative to the lattice of ideals of A. For $I_1, I_2 \in Id(A)$ we define $I_1 \wedge I_2 = I_1 \cap I_2$, $I_1 \vee I_2$= the ideal generated by $I_1 \cup I_2$ and for $I \in Id(A)$, $I^* = \{a \in A : a \wedge x = 0, \text{ for every } x \in I\}$. Theorem 2.17 characterizes the MV-algebras for which the lattice of ideals $(Id(A), \wedge, \vee, ^*, \{0\}, A)$ is a Boolean algebra.

Definition 2.4. An *ideal* of an MV-algebra A is a nonempty subset I of A satisfying the following conditions:

(I_1) If $x \in I$, $y \in A$ and $y \leq x$, then $y \in I$;
(I_2) If $x, y \in I$, then $x \oplus y \in I$.

Remark 2.6. *If I is an ideal then $0 \in I; x, y \in I \Rightarrow x \vee y \in I; x \oplus y \in I \Leftrightarrow x \vee y \in I$.*

We denote by $Id(A)$ the set of ideals of an MV-algebra A.

The intersection of any family of ideals of A is an ideal of A. For a nonempty set $M \subseteq A$, we denote by $(M]$ the *ideal of A generated by M* , i.e., the intersection of all ideals $I \supseteq M$. If $M = \{a\}$ with $a \in A$, we denote by $(a]$ the ideal generated by $\{a\}$ ($(a]$ is called *principal*).

Note that $(0] = \{0\}$ and $(1] = A$.

An ideal I of an $MV-$ algebra A is proper iff $I \neq A$.

Example 2.8. *(The ideals of $[0,1]$) Let $A = [0,1]$ be the MV-algebra from Example 2.3 and $I \subseteq [0,1]$ an ideal. Suppose that there is $x \in [0,1]$ such that $x \neq 0$. It follows that there is $n \in N$ such that $nx = \underbrace{x \oplus ... \oplus x}_{n \text{ ori}} = (\underbrace{x \oplus ... \oplus x}_{n \text{ ori}}) \wedge 1 = 1$. Since I is an ideal, it follows that $nx = 1 \in I$ and $I = [0,1]$. We deduce that for $x \neq 0, (x] = A$, thus $Id(A) = \{\{0\}, [0,1]\}$.*

Example 2.9. *If consider the MV-algebra $A = L_3^2$ from Example 2.4 , then $Id(A) = \{I_1 = \{(0,0)\}; I_2 = ((0,1)] = \{(0,0),(0,1)\}; I_3 = ((1,0)] = \{(0,0),(1,0)\}; I_4 = ((0,\frac{1}{2})] = \{(0,0),(0,\frac{1}{2}),(0,1)\}; I_5 = ((\frac{1}{2},0)] = \{(0,0),(\frac{1}{2},0),(1,0)\}; I_6 = ((\frac{1}{2},\frac{1}{2})] = A\}$.*

Remark 2.7. *If $f : A \to B$ be an MV-algebras homomorphism then $Ker(f)$ is a proper ideal of A. Indeed, since $f(0) = 0$ we deduce that $0 \in Ker(f)$. If $a, b \in A$ such that $a \leq b$ and $b \in Ker(f)$, then $f(a) \leq f(b)$ and $f(b) = 0$. We get $f(a) = 0$, so $b \in Ker(f)$. If $a, b \in Ker(f)$ then $f(a \oplus b) = f(a) \oplus f(b) = 0 \oplus 0 = 0$, so $a \oplus b \in Ker(f)$. Hence $Ker(f)$ is an ideal. Since $f(1) = 1$, we get $1 \notin Ker(f)$, so $Ker(f)$ is a proper ideal.*

PROPOSITION 2.12. *(i) If $M \subseteq A$ is a nonempty set, then*

$$(M] = \{x \in A : x \leq x_1 \oplus ... \oplus x_n \text{ for some } x_1, ..., x_n \in M\}.$$

In particular, for $a \in A$, $(a] = \{x \in A : x \leq na$ for some integer $n \geq 0\}$; if $e \in B(A)$, then $(e] = \{x \in A : x \leq e\}$;

(ii) If $I_1, I_2 \in Id(A)$, then

$$I_1 \vee I_2 \stackrel{def}{=} (I_1 \cup I_2] = \{a \in A : a \leq x_1 \oplus x_2 \text{ for some } x_1 \in I_1 \text{and } x_2 \in I_2\};$$

(iii) If $x, y \in A$, then $(x] \cap (y] = (x \wedge y]$.

Proof. (i). We denote $I = \{x \in A : x \leq x_1 \oplus ... \oplus x_n$ for some $x_1, ..., x_n \in M\}$ and we prove that I is the smallest ideal containing M. We remark that $M \subseteq I$, so I is non empty. Let $a \leq b$ and $b \in I$, so there are $n \geq 1$ and $x_1, ..., x_n \in M$ such that $a \leq b \leq x_1 \oplus ... \oplus x_n$. It follows that $a \in I$.

Let now, $a, b \in I$. Then $a \leq x_1 \oplus ... \oplus x_n$ and $b \leq y_1 \oplus ... \oplus y_m$ for some $x_1, ..., x_n, y_1, ..., y_m \in M$. We get $a \oplus b \leq x_1 \oplus ... \oplus x_n \oplus y_1 \oplus ... \oplus y_m$ so $a \oplus b \in I$. Thus I is an ideal containing M. Let I' another ideal of A that contains M and let a an arbitrary element from I. Hence $a \leq x_1 \oplus ... \oplus x_n$ and $x_1, ..., x_n \in M \subseteq I'$. Because I' is an ideal, it follows that $x_1 \oplus ... \oplus x_n \in I'$, so $a \in I'$ and $I \subseteq I'$. We proved that I is the smallest ideal containing M, so $(M] = I$.

(ii). Follows by (i).

(iii). Obviously, $x \in (x]$ and $y \in (y]$. Since $x \wedge y \leq x, y$ we get $x \wedge y \in (x]$ and $x \wedge y \in (y]$; then $x \wedge y \in (x] \cap (y]$, which is an ideal. Then $(x \wedge y] \subseteq (x] \cap (y]$.

Conversely, suppose that $z \in (x] \cap (y]$; then $z \leq nx$ and $z \leq my$ for some $m, n \geq 1$. It follows that $z \leq nx \wedge my \leq n(x \wedge my) \leq nm(x \wedge y)$, by $mv - c_{24}$; thus $z \in (x \wedge y]$. ■

For $I \in Id(A)$ and $a \in A \setminus I$ we denote by $I(a) = (a] \vee I = (I \cup \{a\}]$.

REMARK 2.8. *For $I(a)$ we have the next characterization:*

$$I(a) = \{x \in A : x \leq y \oplus na, \text{ for some } y \in I \text{ and integer } n \geq 0\}.$$

COROLLARY 2.13. *Let $I \in Id(A)$ and $a, b \in A \backslash I$; then $I(a) \cap I(b) = I(a \wedge b)$.*

Proof. Since $a \wedge b \leq a, b$ we deduce $a \wedge b \in I(a) \cap I(b)$, hence $I(a) \cap I(b) \supseteq I(a \wedge b)$. Let now $x \in I(a) \cap I(b)$. Then $x \leq x_1 \oplus ma$ and $x \leq x_2 \oplus nb$ for some $x_1, x_2 \in I$ and positive integers m, n. If $y = x_1 \oplus x_2 \in I$, and $p = m + n$, then $x \leq (x_1 \oplus ma) \wedge (x_2 \oplus nb) \leq (y \oplus pa) \wedge (y \oplus pb) \stackrel{mv-c_{17}}{=} y \oplus (pa \wedge pb) \leq y \oplus p^2(a \wedge b)$ (by $mv - c_{24}$), hence $x \in I(a \wedge b)$, that is $I(a) \cap I(b) \subseteq I(a \wedge b)$. We deduce $I(a) \cap I(b) = I(a \wedge b)$.■

COROLLARY 2.14. *If $x, y \in A$ then $(x] \vee (y] = (x \oplus y]$.*

Proof. It is suffices to show the inclusion $(x \oplus y] \subseteq (x] \vee (y]$. If $z \in (x \oplus y]$ then $z \leq n(x \oplus y)$ for some integer $n \geq 0$. But $n(x \oplus y) = (nx) \oplus (ny)$ and so $z \leq (nx) \oplus (ny)$. Since $nx \in (x]$ and $ny \in (y]$ we deduce that $z \in (x] \vee (y]$ that is $(x \oplus y] \subseteq (x] \vee (y]$. ■

For $I_1, I_2 \in Id(A)$, we put

$$I_1 \wedge I_2 = I_1 \cap I_2,$$

$$I_1 \vee I_2 = (I_1 \cup I_2],$$

$$I_1 \to I_2 = \{a \in A : (a] \cap I_1 \subseteq I_2\}.$$

Then $(Id(A), \vee, \wedge, \{0\}, A)$ is a *complete Brouwerian lattice* (see Definition 1.9).

LEMMA 2.15. *If $I_1, I_2 \in Id(A)$, then*

(i) $I_1 \to I_2 \in Id(A)$,

(ii) If $I \in Id(A)$, then $I_1 \cap I \subseteq I_2$ iff $I \subseteq I_1 \to I_2$ (that is,

$$I_1 \to I_2 = \sup\{I \in Id(A) : I_1 \cap I \subseteq I_2\}).$$

Proof. *(i)* Since $(0] \cap I_1 \subseteq I_2$ we deduce that $0 \in I_1 \to I_2$. If $x, y \in A$, $x \leq y$ and $y \in I_1 \to I_2$, then $(y] \cap I_1 \subseteq I_2$. Since $(x] \subseteq (y]$ we deduce that $(x] \cap I_1 \subseteq (y] \cap I_1 \subseteq I_2$, hence $x \in I_1 \to I_2$. Let now $x, y \in I_1 \to I_2$; then $(x] \cap I_1 \subseteq I_2$ and $(y] \cap I_1 \subseteq I_2$. We deduce $((x] \cap I_1) \vee ((y] \cap I_1) \subseteq I_2$ hence $((x] \vee (y]) \cap I_1 \subseteq I_2$, so $(x \oplus y] \cap I_1 \subseteq I_2$ (by Corollary 2.14), that is $x \oplus y \in I_1 \to I_2$.

(ii) $(\Rightarrow)$ Let $I \in Id(A)$ such that $I_1 \cap I \subseteq I_2$. If $x \in I$ then $(x] \cap I_1 \subseteq I \cap I_1 \subseteq I_2$ hence $x \in I_1 \to I_2$, that is $I \subseteq I_1 \to I_2$.

$(\Leftarrow)$ We suppose $I \subseteq I_1 \to I_2$ and let $x \in I_1 \cap I$; then $x \in I$, hence $x \in I_1 \to I_2$ that is $(x] \cap I_1 \subseteq I_2$. Since $x \in (x] \cap I_1$ then $x \in I_2$ that is $I_1 \cap I \subseteq I_2$.■

REMARK 2.9. *From Lemma 2.15 we deduce that $(Id(A), \vee, \wedge, \to, \{0\})$ is a Heyting algebra; for $I \in Id(A)$,*

$$I^* = I \to \{0\} = \{x \in A : (x] \cap I = \{0\}\}.$$

COROLLARY 2.16. *(i) For every $I \in Id(A)$, $I^* = \{x \in A : x \wedge y = 0$ for every $y \in I\}$ (see [**68**], p.114);*

(ii) For any $x \in A$, $(x]^* = \{y \in A : (y] \cap (x] = \{0\}\} = \{y \in A : x \wedge y = 0\}$(by Proposition 2.12, (iii)).

THEOREM 2.17. *If A is an MV-algebra, then the following conditions are equivalent:*

(i) $(Id(A), \vee, \wedge, ^*, \{0\}, A)$ is a Boolean algebra;

(ii) Every ideal of A is principal and for every $x \in A$, there is $n \in N$ such that $x \wedge (nx)^* = 0 \Leftrightarrow x^* \vee nx = 1$.

Proof. $(i) \Rightarrow (ii)$. If $I \in Id(A)$, because $Id(A)$ is supposed Boolean lattice then $I \vee I^* = A$, hence $1 \in I \vee I^*$. By Proposition 2.12 (ii), $1 = a \oplus b$ with $a \in I$ and $b \in I^*$. By Corollary 2.16 (i), $x \wedge b = 0$ for every $x \in I$.

So $(x^* \vee b^*)^* = 0 \Leftrightarrow x^* \vee b^* = 1 \Leftrightarrow (x \oplus b^*)^* \oplus b^* = 1 \Leftrightarrow x \oplus b^* \leq b^* \Leftrightarrow x \oplus b^* = b^*$ for every $x \in I$.

Since $a \oplus b = 1$ we obtain $b^* \leq a$ hence $x \oplus b^* = b^* \leq a$ for every $x \in I$. Finally, we obtain $x \leq x \oplus b^* \leq a$, hence $x \leq a$ for every $x \in I$, that is $I = (a]$.

Let $x \in A$; since $Id(A)$ is a Boolean algebra then $(x] \vee (x]^* = A$. By Corollary 2.16 (ii), we have

$$(x] \vee (x]^* = (x]^*(x) = A \Leftrightarrow$$

$$\Leftrightarrow \{a \in A : a \leq y \oplus nx, \text{for some } y \in (x]^* \text{and } n \in N\} = A.$$

(see Remark 2.8).

So, since $1 \in A$, there exists $y \in (x]^*$ and $n \in N$ such that $y \oplus nx = 1$. Since $y \in (x]^*$, then $y \wedge x = 0$.

By Lemma 2.2, from $y \oplus nx = 1$ we deduce $(nx)^* \leq y$. So,

$$(nx)^* \wedge x \leq y \wedge x = 0,$$

hence $(nx)^* \wedge x = 0 \Leftrightarrow x^* \vee nx = 1$.

$(ii) \Rightarrow (i)$. By Remark 2.9, $Id(A)$ is a Heyting algebra. To prove $Id(A)$ is a Boolean lattice we must show $I^* = \{0\}$ only for $I = A$ ([**2**], p.175).

By hypothesis, every ideal is principal, then $I = (a]$ for some $a \in A$. Also, for $a \in A$, there is $n \in N$ such that $a \wedge (na)^* = 0$. By Corollary 2.16 (ii), $(na)^* \in (a]^* = \{0\}$, hence $(na)^* = 0$, that is $na = 1$. By Proposition 2.12 (i), we deduce that $1 \in I$, hence $I = A$. ■

The *distance function* $d : A \times A \to A$ is defined by

$$d(x, y) = (x \odot y^*) \oplus (y \odot x^*) = (x \ominus y) \oplus (y \ominus x).$$

THEOREM 2.18. *In every MV-algebra we have:*

(i) $d(x, y) = 0$ *iff* $x = y$;
(ii) $d(x, 0) = 0, d(x, 1) = x^*$;
(iii) $d(x^*, y^*) = d(x, y)$;
(iv) $d(x, y) = d(y, x)$;
(v) $d(x, z) \leq d(x, y) \oplus d(y, z)$;
(vi) $d(x \oplus u, y \oplus v) \leq d(x, y) \oplus d(u, v)$;
(vii) $d(x \odot u, y \odot v) \leq d(x, y) \oplus d(u, v)$.

Proof. (i). If $x = y$ then it is obvius that $d(x, y) = 0$. Conversely, if $d(x, y) = 0$ then $x \odot y^* = y \odot x^* = 0$. We get that $x \leq y$ and $y \leq x$, so $x = y$.

(ii), (iii). Follows by easy computations.

(iv). We get $d(x, y) = (x \ominus y) \oplus (y \ominus x) = (y \ominus x) \oplus (x \ominus y) = d(y, x)$.

(v). We firstly prove that $x \odot z^* \leq (x \odot y^*) \oplus (y \odot z^*)$.

Indeed, $(x \odot z^*)^* \oplus (x \odot y^*) \oplus (y \odot z^*) = x^* \oplus z \oplus (x \odot y^*) \oplus (y \odot z^*) = [x^* \oplus (x \odot y^*)] \oplus [z \oplus (y \odot z^*)] = (x^* \vee y^*) \oplus (z \vee y) \geq y^* \oplus y = 1$.

Now, $d(x, z) = (x \odot z^*) \oplus (z \odot x^*) \leq (x \odot y^*) \oplus (y \odot z^*) \oplus (z \odot y^*) \oplus (y \odot x^*) = d(x, y) \oplus d(y, z)$.

(vi). We firstly prove that $(*) : (x \oplus u)^* \odot (y \oplus v) \leq (x^* \odot y) \oplus (u^* \odot v)$.

We have $[(x \oplus u)^* \odot (y \oplus v)]^* \oplus (x^* \odot y) \oplus (u^* \odot v) = x \oplus u \oplus (y^* \odot v^*) \oplus (x^* \odot y) \oplus (u^* \odot v) =$

$[x \oplus (x^* \odot y)] \oplus u \oplus (y^* \odot v^*) \oplus (u^* \odot v) = [y \oplus (y^* \odot x)] \oplus u \oplus (y^* \odot v^*) \oplus (u^* \odot v) = [y \oplus (y^* \odot v^*)] \oplus [u \oplus (u^* \odot v)] \oplus (y^* \odot x) = (y \vee v^*) \oplus (u \vee v) \oplus (y^* \odot x)$

$\geq (y \vee v^*) \oplus (u \vee v) \geq v^* \oplus v = 1$.

Now we prove (vi) using the inequality $(*) : d(x \oplus u, y \oplus v) = (x \oplus u)^* \odot (y \oplus v) \oplus (y \oplus v)^* \odot (x \oplus u) \leq [(x^* \odot y) \oplus (u^* \odot v)] \oplus [(y^* \odot x) \oplus (v^* \odot u)] = d(x, y) \oplus d(u, v)$.

(vii). Follows by (iii) and (vi) : $d(x \odot u, y \odot v) = d((x^* \oplus u^*)^*, (y^* \oplus v^*)^*) = d(x^* \oplus u^*, y^* \oplus v^*) \leq d(x^*, y^*) \oplus d(u^*, v^*) = d(x, y) \oplus d(u, v)$. ■

As an immediate consequence we have:

PROPOSITION 2.19. *If $f : A \to B$ is an $MV-$algebras homomorphism then the following assertions are equivalent:*

(i) *f is injective;*
(ii) $Ker(f) = \{0\}$.

Proof. $(i) \Rightarrow (ii)$. We suppose that f is injective and let $a \in Ker(f)$. Then $f(a) = f(0) = 0$, so $a = 0$.

$(ii) \Rightarrow (i)$. Convesely, let $Ker(f) = \{0\}$ and $a, b \in A$ such that $f(a) = f(b)$. It follows that $f(d(a, b)) = d(f(a), f(b)) = 0$. Since $Ker(f) = \{0\}$ we get $d(a, b) = 0$ so $a = b$. Thus f is an injective homomorphism. ■

EXAMPLE 2.10. *(i) I want to determine all the homomorphisms of MV-algebras $f:[0,1]\to[0,1]$. Let $A=[0,1]$ be the MV-algebra from Example 2.3 and $f:[0,1]\to[0,1]$ be an MV-algebras homomorphism. By Remark 2.7, $Ker(f)$ is a proper ideal of $[0,1]$, so by Example 2.8, $Ker(f)=\{0\}$. Thus, by Proposition 2.19, f is injective. We remark that $f(\frac{1}{2})=1-f(\frac{1}{2})$, so $f(\frac{1}{2})=\frac{1}{2}$. Since f is increasing, $f([0,\frac{1}{2}])\subseteq[0,\frac{1}{2}]$ and so to determine f it is suffice to determine $f_{|[0,\frac{1}{2}]}$. We have $\frac{1}{4}\oplus\frac{1}{4}=\frac{1}{2}\Rightarrow f(\frac{1}{4})\oplus f(\frac{1}{4})=f(\frac{1}{2})=\frac{1}{2}\Rightarrow f(\frac{1}{4})=\frac{1}{4}$. By induction we prove that $f(\frac{1}{2^n})=\frac{1}{2^n}$ for every $n\geq 1$. For $x=\frac{k}{2^n}\leq\frac{1}{2}$ we get $f(x)=f(\underbrace{\frac{1}{2^n}\oplus...\oplus\frac{1}{2^n}}_{k\ ori})=\underbrace{f(\frac{1}{2^n})\oplus...\oplus f(\frac{1}{2^n})}_{k\ ori}=\underbrace{\frac{1}{2^n}\oplus...\oplus\frac{1}{2^n}}_{k\ ori}=\underbrace{\frac{1}{2^n}+...+\frac{1}{2^n}}_{k\ ori}=\frac{k}{2^n}$. We deduce that $f(\frac{k}{2^n})=\frac{k}{2^n}$ for $\frac{k}{2^n}\leq\frac{1}{2}$. Let $x\in[0,\frac{1}{2}]$. We know that there are two sequences $(a_n)_{n\geq1},(b_n)_{n\geq1}\in[0,\frac{1}{2}]$ by the form $\frac{k}{2^n}$ such that $a_n\leq x\leq b_n$ with $a_n<a_{n+1}\leq x\leq b_{n+1}<b_n$ for every n . Since f is increasing we get $a_n<a_{n+1}\leq f(x)\leq b_{n+1}<b_n$. Thus, $f(x)=\lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}b_n=x$. We proved that $f(x)=x$ for any $x\in[0,1]$. In conclusion, the only MV-algebras homomorphism $f:[0,1]\to[0,1]$ is the identity.*

(ii) A similar conclusion can be obtained if we consider the MV-algebra $Q\cap[0,1]$ or the MV-algebras L_n with $n\geq2$. Indeed, if $f:L_n\to L_n$ be an MV-algebras homomorphism, then $\underbrace{\frac{1}{n-1}\oplus...\oplus\frac{1}{n-1}}_{n-1\ ori}=1\Rightarrow\underbrace{f(\frac{1}{n-1})\oplus...\oplus f(\frac{1}{n-1})}_{n-1\ ori}=1\Rightarrow f(\frac{1}{n-1})\neq0$. Suppose that $f(\frac{1}{n-1})=\frac{k}{n-1},k>1$. Then $f(\frac{n-2}{n-1})=f(\underbrace{\frac{1}{n-1}\oplus...\oplus\frac{1}{n-1}}_{n-2\ ori})=\underbrace{f(\frac{1}{n-1})\oplus...\oplus f(\frac{1}{n-1})}_{n-2\ ori}$. We get $f(\frac{1}{n-1})\geq\frac{2}{n-1}\Rightarrow f(\frac{n-2}{n-1})\geq\underbrace{\frac{2}{n-1}\oplus...\oplus\frac{2}{n-1}}_{n-2\ ori}=1\Rightarrow f(\frac{n-2}{n-1})=1$. But $\frac{n-2}{n-1}=\left(\frac{1}{n-1}\right)^*\Rightarrow f\left(\frac{n-2}{n-1}\right)=1-f\left(\frac{1}{n-1}\right)<1$, a contradiction. In conclusion, the only MV-algebras homomorphism $f:L_n\to L_n$ is the identity.

DEFINITION 2.5. An equivalence relation $\sim$ on a MV-algebra A is a *congruence* if the following properties are satisfied:

$(Con-mv_1)$ $x\sim y\Rightarrow x^*\sim y^*$;
$(Con-mv_2)$ $x\sim y, x'\sim y'\Rightarrow x\oplus x'\sim y\oplus y'$, *for every* $x,x',y,y'\in A$.

PROPOSITION 2.20. *Let I be an ideal of an MV-algebra A. Then the binary relation $\sim_I$ on A defined by $x\sim_I y$ iff $d(x,y)\in I$ (equivalent with $x\odot y^*\in I$ and $y\odot x^*\in I$) is a congruence relation on A. Moreover, $I=\{x\in A:x\sim_I 0\}$.*

Proof. Firstly we prove that $\sim_I$ is an equivalence on A. The reflexivity, $x\sim_I x$ follows by the fact that $d(x,x)=0\in I$, for any $x\in I$; The reflexivity, $x\sim_I y\Rightarrow y\sim_I x$, follows by the fact that $d(x,y)=d(y,x)$; in order to prove the transitivity,

we suppose that $x \sim_I y$ and $y \sim_I z$, that is $d(x,y), d(y,z) \in I$. By Theorem 2.18, $(v), d(x,z) \leq d(x,y) \oplus d(y,z) \in I$, so $d(x,z) \in I$ and $x \sim_I z$.

Now we have to prove the congruence properties. If $x \sim_I y$ then $d(x,y) = d(x^*,y^*) \in I$, so $x^* \sim_I y^*$.

Suppose $x \sim_I y, x' \sim_I y'$. Then $d(x,y), d(x',y') \in I$. By Theorem 2.18, $(vi), d(x \oplus x', y \oplus y') \leq d(x,y) \oplus d(x',y') \in I$, so $d(x \oplus x', y \oplus y') \in I$. Hence $x \oplus x' \sim_I y \oplus y'$. ■

PROPOSITION 2.21. *Conversely, if θ is a congruence relation on A, then $I_\theta = \{x \in A : (x,0) \in \theta\} \in Id(A)$ and $(x,y) \in \theta$ iff $(d(x,y),0) \in \theta$.*

Proof. Because θ is reflexive we get $0 \in I_\theta$, so I_θ is non empty. If $x \leq y$ and $y \in I_\theta$ then $x = x \wedge y$ and $(x = x \wedge y, x \wedge 0 = 0) \in \theta$, so $(x,0) \in \theta$ and $x \in I_\theta$. If $x, y \in I_\theta$, then $(x,0) \in \theta, (y,0) \in \theta$ so, by $Con - mv_2$, $(x \oplus y, 0) \in \theta$ so $x \oplus y \in I_\theta$. Hence I_θ is an ideal. ■

PROPOSITION 2.22. *The assignement $I \rightsquigarrow \sim_I$ is a bijection from the set $Id(A)$ of ideals of A onto the set of congruences on A; more precisely, the function $\alpha : Id(A) \to Con(A)$ defined by $\alpha(I) = \sim_I$ is an isomorphism of partially orderd sets.*

Proof. Let I and J be too ideals such that $\sim_I = \sim_J$. If $a \in A$ we get $a = d(a,0) \in I \Leftrightarrow a \sim_I 0 \Leftrightarrow a \sim_J 0 \Leftrightarrow d(a,0) \in J$ so $I = J$. Thus, α is injective. The map α is also surjective since for any $\sim \in Con(A)$ we have $\alpha(I_\sim) = \sim$.

The proof is complete showing that $I \subseteq J \Leftrightarrow \sim_I \subseteq \sim_J$.■

If I is an ideal of A and $x \in A$, the congruence class of x with respect to $\sim_I$ will be denoted by x/I, i.e. $x/I = \{y \in A : x \sim_I y\}$; one can eassy to see that $x \in I$ iff $x/I = 0/I$. We shall denote the quotient set $A/\sim_I$ by A/I. Since $\sim_I$ is a congruence on A, the MV-algebra operations on A/I given by

$$x/I \oplus y/I \stackrel{def}{=} (x \oplus y)/I$$

and

$$(x/I)^* \stackrel{def}{=} x^*/I,$$

are well defined. Hence, the system $(A/I, \oplus, ^*, 0/I)$ becomes an $MV-$ algebra, called *the quotient algebra* of A by the ideal I. The assignement $x \to x/I$ defines a homomorphism p_I from A onto the quotient algebra A/I, which is called the *natural homomorphism* from A onto A/I; we remark that $Ker(p_I) = I$.

Clearly, if $x, y \in A$ then $x/I \leq y/I$ iff $(x^* \oplus y)/I = 1/I$ iff $(x^* \oplus y)^* \in I$ iff $x \ominus y \in I$ iff $x \odot y^* \in I$.

3. The spectrum and the maximal ideals

In this Subsection we study the prime spectrum Spec(A) and the maximal spectrum Max(A) of an MV-algebra. If every ideal $I \in Id(A)$ has a unique representation as intersection of prime ideals then Id(A) is a Boolean algebra (see Theorem 2.39). We give a new characterizations for prime ideals of an MV-algebra (see Theorem 2.40, Theorem 2.41, Corollary 2.42 and Theorem 2.43).

REMARK 2.10. *An ideal proper P is finitely meet-irreducible in $Id(A)$ iff $I \cap J \subseteq P \Rightarrow I \subseteq P$ or $J \subseteq P$, for all $I, J \in Id(A)$. Indeed, let $I, J \in Id(A)$ such that $I \cap J = P$. We deduce that $I \cap J \subseteq P$ so, $I \subseteq P$ or $J \subseteq P$. But since $I \cap J = P$ we have that $P \subseteq I, J$. Finally, we obtain $I = P$ or $J = P$, so P is finitely meet-irreducible in $Id(A)$.*

DEFINITION 2.6. A proper ideal P of A is *prime* if it satisfies the following condition:

$$\textit{for each } x \textit{ and } y \textit{ in } A, \textit{either } x \ominus y = x \odot y^* \in P \textit{ or } y \ominus x = y \odot x^* \in P.$$

Following tradition, we denote by $Spec(A)$ the set of all prime ideals of A. $Spec(A)$ is called the *spectrum* of A.

An ideal I of an MV-algebra A is called *maximal* iff it is proper and no proper ideal of A strictly contains I, i.e., for each ideal $J \neq I$, if $I \subseteq J$, then $J = A$. We denote by $Max(A)$ the set of all maximal ideals of A.

The next lemma summarize, some easy relations between ideals and kernels of homomorphisms.

LEMMA 2.23. ([**45**]) *Let A, B be $MV-$ algebras and $f : A \to B$ a homomorphism. Then the following properties hold:*

- *(i) For each ideal $J \in Id(B)$, the set $f^{-1}(J) = \{x \in A : f(x) \in J\}$ is an ideal of A. Thus, in particular, $Ker(f) \in Id(A)$;*
- *(ii) $f(x) \leq f(y)$ iff $x \ominus y \in Ker(f)$;*
- *(iii) f is injective iff $Ker(f) = \{0\}$;*
- *(iv) $Ker(f) \neq A$ iff B is nontrivial;*
- *(v) $Ker(f) \in Spec(A)$ iff B is nontrivial and the image $f(A)$, as a subalgebra of B, is an $MV-$ chain.*

The well-known isomorphism theorems have corresponding versions for $MV-$ algebras. We only enounce the first and the second isomorphism theorem, since their proof follows directly from the classical ones, as an immediately consequence of Lema 2.23.

THEOREM 2.24. *(The first isomorphism theorem) If A and B are two $MV-$ algebras and $f : A \to B$ is a homomorphism, then $A/Ker(f)$ and $\mathrm{Im}(f)$ are isomorphic $MV-$ algebras.*

THEOREM 2.25. *(The second isomorphism theorem) If A is an $MV-$ algebra and I, J are two ideals such that $I \subseteq J$, then $(A/I)/p_I(J)$ and A/J are isomorphic $MV-$ algebras.*

THEOREM 2.26. *For a proper ideal $P \in Id(A)$ the following are equivalent:*

- *(i) P is finitely meet-irreducible in $Id(A)$, (equivalently by Remark 2.10 with $I \cap J \subseteq P \Rightarrow I \subseteq P$ or $J \subseteq P$, for all $I, J \in Id(A)$);*
- *(ii) $P \in Spec(A)$;*
- *(iii) A/P is chain;*
- *(iv) If $x \wedge y \in P$, then $x \in P$ or $y \in P$;*
- *(v) If $x \wedge y = 0$, then $x \in P$ or $y \in P$.*

Proof. We prove the equivalences $(i) \Leftrightarrow (iv), (ii) \Leftrightarrow (iv)$.

$(i) \Rightarrow (iv)$. Let $x, y \in A$ such that $x \wedge y \in P$. Then $(x] \cap (y] = (x \wedge y] \subseteq P$, so $x \in P$ or $y \in P$.

$(iv) \Rightarrow (i)$. Let $I, J \in Id(A)$ such that $I \cap J \subseteq P$. If we suppose that $I \nsubseteq P$ and $J \nsubseteq P$ then there are $x \in I \backslash P$ and $y \in J \backslash P$. We get $x \wedge y \in I \cap J \subseteq P$, and by hypothesis, $x \in P$ or $y \in P$, a contradiction. Thus, $I \subseteq P$ or $J \subseteq P$.

$(ii) \Rightarrow (iv)$. Suppose that $x \wedge y \in P$ and $x \odot y^* \in P$. It follows that $(x \odot y^*) \oplus (x \wedge y) \in P$. But $(x \odot y^*) \oplus (x \wedge y) = [(x \odot y^*) \oplus x] \wedge [(x \odot y^*) \oplus y] = [(x \odot y^*) \oplus x] \wedge (x \vee y) \geq x$. We get $x \in P$. Similarly, if $x^* \odot y \in P$ we infere that $y \in P$.

$(iv) \Rightarrow (ii)$. Obviously, since $(x \odot y^*) \wedge (y \odot x^*) = 0 \in P$.

We prove the equivalence $(iii) \Leftrightarrow (iv) \Leftrightarrow (v)$.

$(iii) \Rightarrow (iv)$. $x \wedge y \in P \Rightarrow x/P \wedge y/P = 0/P \Rightarrow x/P = 0/P$ or $y/P = 0/P \Rightarrow x \in P$ or $y \in P$.

$(iv) \Rightarrow (v)$. Obviously, $x \wedge y = 0 \in P$, so by (iv) we deduce that $x \in P$ or $y \in P$.

$(v) \Rightarrow (iii)$. Let $x/P, y/P \in A/P$; since $(x \odot y^*) \wedge (y \odot x^*) = 0 \in P$ we deduce by (v) that $x/P \leq y/P$ or $y/P \leq x/P$, so A/P is totally ordered. ■

REMARK 2.11. *We have a directly proof for the implication $(ii) \Rightarrow (iii)$: Let $x/P, y/P \in A/P$ and suppose that $x \odot y^* \in P$. Then $(x/P) \odot (y/P)^* = (x \odot y^*)/P = 0/P$, so $x/P \leq y/P$.*

THEOREM 2.27. *If A is an $MV-$ algebra then the following properties hold:*

(i) Every proper ideal of A that contains a prime ideal is prime;

(ii) For each prime ideal I of A, the set $\mathcal{I} = \{J \in Id(A) : I \subseteq J$ and $J \neq A\}$ is totally ordered by inclusion.

Proof. (i). Let I and P proper ideals of A such that $I \subseteq P$ and P is prime. Let $x, y \in A$. Since P is prime it follows that $x \odot y^* \in P$ or $y \odot x^* \in P$. Because $P \subseteq I$ we deduce that $x \odot y^* \in I$ or $y \odot x^* \in I$, so I is a prime ideal of A.

(ii). Let $J, K \in \mathcal{I}$ and suppose that $J \nsubseteq K$ and $K \nsubseteq J$. Thus, there are two elements $x, y \in A$ such that $x \in J \backslash K$ and $y \in K \backslash J$. Since I is prime, we get $x \odot y^* \in I \subseteq K$ or $y \odot x^* \in I \subseteq J$. It follows that $x \vee y = y \oplus (x \odot y^*) \in K$ or $x \vee y = x \oplus (y \odot x^*) \in J$, so $x \in K$ or $y \in J$, which is a contradiction. Thus, $J \subseteq K$ and $K \subseteq J$ and $\mathcal{I}$ is linearly ordered. ■

COROLLARY 2.28. *Every prime ideal of an $MV-$ algebra A is contained in a unique maximal ideal of A.*

Proof. ([**45**]) Let $I \in Spec(A)$. The set $\mathcal{I} = \{J \in Id(A) : J \neq A$ and $I \subseteq J\}$ is totally orderd by inclusion. Therefore, $M = \bigcup_{J \in \mathcal{I}} J$ is an ideal. Further, M is a proper ideal, because $1 \notin M$; we conclude that M is the only maximal ideal containing I. ■

The next result will play an important role:

THEOREM 2.29. *(Prime ideal theorem) Let A be an $MV-$ algebra, $I \in Id(A)$ and $a \in A \backslash I$. Then there is $P \in Spec(A)$ such that $I \subseteq P$ and $a \notin P$. In particular for every element $a \in A, a \neq 0$ there is $P \in Spec(A)$ such that $a \notin P$.*

Proof. ([**45**]) A routine application of Zorn's Lemma shows that there is an ideal $P \in Id(A)$ which is maximal with respect to the property that $I \subseteq P$ and $a \notin P$. We shall show that P is a prime ideal. Let x, y be elements of A and suppose that both $x \ominus y \notin P$ and $y \ominus x \notin P$. Then the ideal $(P \cup \{x \ominus y\})$ must contain the element a. By Remark 2.8, $a \leq p \oplus n(x \ominus y)$, for some $p \in P$ and some integer $n \geq 1$. Similarly, there is an element $q \in P$ and an integer $m \geq 1$ such that $a \leq q \oplus m(y \ominus x)$. Let $u = p \oplus q$ and $s = \max\{n, m\}$. Then $u \in P$, $a \leq u \oplus s(x \ominus y)$ and $a \leq u \oplus s(y \ominus x)$. Hence by $mv - c_{17}$ and $mv - c_{18}$ we have $a \leq [u \oplus s(x \ominus y)] \wedge [u \oplus s(y \ominus x)] = u \oplus [s(x \ominus y) \wedge s(y \ominus x)] = u$, hence $a \in P$, a contradiction. ■

COROLLARY 2.30. *Any proper ideal I of A can be extended to a prime ideal.*

Proof. Apply Theorem 2.29.

THEOREM 2.31. *For any $MV-$ algebra A, the following are equivalent:*

(i) A is an $MV-$ chain;
(ii) Any proper ideal of A is prime;
(iii) $\{0\}$ is a prime ideal;
(iv) $Spec(A)$ is linearly ordered.

Proof. $(i) \Rightarrow (ii)$. Let $I \in Id(A)$, proper ideal. Since A is an $MV-$ chain and $p_I : A \to A/I$ is a surjective homomorphism we deduce that A/I is also an $MV-$ chain and by Lemma 2.23, (v), I is a prime ideal.

$(ii) \Rightarrow (iii)$. Is obvious.

$(iii) \Rightarrow (iv)$. By Theorem 2.27, (i) and the fact that $\{0\}$ is a prime ideal, we deduce that $Spec(A) = \{I \in Id(A) : I$ is proper and $\{0\} \subseteq I\}$. Hence by Theorem 2.27, (ii) $Spec(A)$ is linearly ordered.

$(iv) \Rightarrow (i)$. Let $x, y \in A$ and suppose that $x \not\leq y$ and $y \not\leq x$, so $x \odot y^* \neq 0$ or $y \odot x^* \neq 0$. By Theorem 2.29, there are P, Q prime ideals such that $x \odot y^* \notin P$ and $y \odot x^* \notin Q$. Hence $y \odot x^* \in P$ and $x \odot y^* \in Q$. By hypothesis, $Spec(A)$ is linearly ordered, so $P \subseteq Q$ or $Q \subseteq P$. Thus, $y \odot x^* \in Q$ or $x \odot y^* \in P$, which is a contradiction. We have $x \leq y$ or $y \leq x$, so A is an $MV-$ chain. ■

REMARK 2.12. *Relative to Theorem 2.31, $(i) \Rightarrow (iv)$, we have a more general result: If A is an $MV-$ chain, then the set $Id(A)$ is totally ordered by inclusion. Indeed, let $I, J \in Id(A)$ such that $I \not\subseteq J$ and $J \not\subseteq I$. Then there exists two elements $x, y \in A$ such that $x \in I \backslash J$ and $y \in J \backslash I$. Whence $x \not\leq y$ and $y \not\leq x$, a contradiction.*

COROLLARY 2.32. *If A is an $MV-$ algebra then:*

(i) For every $I \in Id(A), I = \cap\{P \in Spec(A) : I \subseteq P\}$;
(ii) $\cap\{P \in Spec(A)\} = \{0\}$.

Proof. Apply Theorem 2.29. If $a \neq 0$ there is a prime ideal $P \in Spec(A)$ such that $a \notin P$, so $a \notin \cap\{P \in Spec(A)\}$.■

The next proposition generalizes a well known property of maximal ideals in boolean algebras:

PROPOSITION 2.33. *If M is a proper ideal of A then the following are equivalent:*

(i) M is maximal;
(ii) for any $a \in A$, $a \notin M$ iff $(na)^* \in M$ for some integer $n \geq 1$.

Proof. $(i) \Rightarrow (ii)$. Suppose that M is a maximal ideal of A. Since $a \notin M$, then $(M \cup \{a\}] = A$, so there exist $x \in M$ and $n \geq 1$ such that $na \oplus x = 1$. We deduce that $(na)^* \leq x \in M$, so $(na)^* \in M$.

Conversely, if $a \in M$, then $na \in M$, for each integer $n \geq 1$; since M is proper we deduce that $(na)^* \notin M$.

$(ii) \Rightarrow (i)$. Let $I \neq M$ be an ideal of A such that $M \subset I$. Then for every $a \in I \backslash M$ we must have $(na)^* \in M$ for some integer $n \geq 1$. Hence $1 = na \oplus (na)^* \in I$, so $I = A$ and M is maximal. ■

PROPOSITION 2.34. *If M is a proper ideal of A then the following are equivalent:*

(i) M is maximal;

(*ii*) *for any* $a \in A$, *if* $a \notin M$ *then there is* $n \in N$ *such that* $(a^*)^n \in M$;
(*iii*) A/M *is locally finite (i.e., every nonzero element from* A/M *has a finite order).*

Proof. By Proposition 2.33, $(i) \Leftrightarrow (ii)$, since $(na)^* = (a^*)^n$, by $mv - c_5$.

$(i) \Rightarrow (iii)$. We proved that every nonzero element from A/M has a finite order. Let $a/M \neq 0/M$. Then $a \notin M$ so there is $n \in N$ such that $(a^*)^n \in M$. We deduce that $(a^*)^n/M = 0/M \Rightarrow n(a/M) = 1/M$. We obtain that A/M is locally finite.

$(iii) \Rightarrow (i)$. Let $I \neq M$ be an ideal of A such that $M \subset I$ and consider $a \in I \backslash M$. Because $a/M \neq 0/M$ we must have $n(a/M) = 1/M$, for some integer $n \geq 1$, so $(na)/M = 1/M$, thus $(na)^* \in M \subseteq I$. We have, $na, (na)^* \in I$ so $I = A$ and M is maximal. ■

Remark 2.13. *If* A *is locally finite, then* A *is a chain. Indeed, suppose that* $x, y \in A$ *such that* $x \not\leq y$ *and* $y \not\leq x$. *Then* $x \odot y^* \neq 0, y \odot x^* \neq 0$, *so there is* n *such that* $n(x \odot y^*) = 1, n(y \odot x^*) = 1$. *Since* $(x \odot y^*) \wedge (y \odot x^*) = 0$, *by* $mv - c_{26}$ *we deduce that* $[n(x \odot y^*)] \wedge [n(y \odot x^*)] = 0$, *but* $[n(x \odot y^*)] \wedge [n(y \odot x^*)] = 1$, *a contradiction. Conversely assertion, is not true. Indeed, the Chang* $MV-$ *algebra* C *is chain but* $nc < 1$ *for every* n.

The intersection of the maximal ideals of A is called the *radical* of A; it will be denote by $Rad(A)$.

From the Theorem 2.29 and Corollary 2.28 we deduce that:

Corollary 2.35. *Every nontrivial* $MV-$ *algebra has a maximal ideal.*

Lemma 2.36. *Any maximal ideal of an* $MV-$ *algebra is a prime ideal and any proper ideal of* A *can be extended to a maximal ideal.*

Proof. M maximal $\overset{\text{Proposition 2.34}}{\Rightarrow}$ A/M is locally finite $\overset{\text{Remark 2.13}}{\Rightarrow}$ A/M is a chain $\overset{\text{Theorem 2.26}}{\Rightarrow}$ M is prime. ■

Remark 2.14. M *prime ideal* $\nRightarrow$ M *maximal ideal. Indeed, in Chang* $MV-$ *algebra* C, $\{0\}$ *is a prime ideal but* $\{0, c, ..., nc, ...\}$ *is the only maximal ideal.*

Remark 2.15. *In* $[0, 1]$ *and* L_n, $\{0\}$ *is a maximal ideal.*

Definition 2.7. Let A be an $MV-$ algebra. An element $x \in A$ is called *archimedean* if there is $n \geq 1$ such that $nx \in B(A)$.

Lemma 2.37. *The following condition are equivalent:*

(*i*) x *is an archimedean element;*
(*ii*) *there is* $n \geq 1$ *such that* $x^* \vee nx = 1$;
(*iii*) *there is* $n \geq 1$ *such that* $nx = (n + 1)x$.

Proof. $(i) \Rightarrow (ii)$. Using Theorem 2.9, we have $x^* \vee nx = x^* \oplus nx = x^* \oplus x \oplus (n - 1)x = 1$.

$(ii) \Rightarrow (iii)$. We have $1 = x^* \vee nx = (nx) \oplus (nx)^* \odot x^* = (nx) \oplus (nx \oplus x)^* = (nx) \oplus [(n + 1)x]^* \Rightarrow (n + 1)x \leq nx$. Obviously, $nx \leq (n + 1)x$, so $nx = (n + 1)x$.

$(iii) \Rightarrow (i)$. $nx = (n+1)x = (n+2)x = ... = (2n)x$ implies $(nx) \oplus (nx) = nx$, so $nx \in B(A)$. ■

Definition 2.8. An $MV-$ algebra A is called *hyperarchimedean* if all its elements are archimedean.

REMARK 2.16. *Any finite $MV-$ algebra is hyperarchimedean.*

REMARK 2.17. *A is hyperarchimedean iff $Max(A) = Spec(A)$. For the proof see Theorem 3.56 for the case of $BL-$ algebras.*

PROPOSITION 2.38. *Let A and B be $MV-$ algebras, $f : A \to B$ a homomorphism of $MV-$ algebras and M be a maximal ideal of B. Then the inverse image $f^{-1}(M)$ is a maximal ideal of A.*

Proof. By Lemma 2.23, (i), $f^{-1}(M)$ is an ideal of A; $f^{-1}(M)$ is proper since $f(1) = 1 \notin M$.

Let $x \notin f^{-1}(M)$, so $f(x) \notin M$. By Proposition 2.33 there is an integer $n \geq 1$ such that $(nf(x))^* \in M$. It follows that $(nx)^* \in f^{-1}(M)$, whence by Proposition 2.33, $f^{-1}(M)$ is a maximal ideal of A. ■

Relative to the uniqueness of ideals as intersection of primes we have:

THEOREM 2.39. *If A is an $MV-$ algebra and every $I \in Id(A)$ has a unique representation as an intersection of elements of $Spec(A)$, then $(Id(A), \vee, \wedge, ^*, \{0\}, A)$ is a Boolean algebra.*

Proof. Let $I \in Id(A)$ and $I' = \cap\{P \in Spec(A) : I \nsubseteq P\} \in Id(A)$. By Corollary 2.32 (ii), $I \cap I' = \cap\{P \in Spec(A)\} = \{0\}$; if $I \vee I' \neq A$, then by Theorem 2.29 there exists $I'' \in Spec(A)$ such that $I \vee I' \subseteq I''$ and $I'' \neq A$. Consequently, I' has two representations $I' = \cap\{P \in Spec(A) : I \nsubseteq P\} = I'' \cap (\cap\{P \in Spec(A) : I \nsubseteq P\})$, which is contradictory. Therefore $I \vee I' = A$ and so $Id(A)$ is a Boolean algebra. ■

THEOREM 2.40. *For a proper ideal $P \in Id(A)$ the following assertions are equivalent:*

(i) $P \in Spec(A)$;
(ii) For every $x, y \in A\backslash P$ there exists $z \in A\backslash P$ such that $z \leq x$ and $z \leq y$.

Proof. $(i) \Rightarrow (ii)$. Let $P \in Spec(A)$ and $x, y \in A\backslash P$. If by contrary, for every $a \in A$ with $a \leq x$ and $a \leq y$ then $a \in P$, since $x \wedge y \leq x, y$ we deduce $x \wedge y \in P$. Hence, by Theorem 2.26 (iv), $x \in P$ or $y \in P$, a contradiction.

$(ii) \Rightarrow (i)$. I suppose by contrary that there exist $I_1, I_2 \in Id(A)$ such that $I_1 \cap I_2 = P$, and $P \neq I_1$, $P \neq I_2$. So, we have $x \in I_1\backslash P$ and $y \in I_2\backslash P$. By hypothesis there is $z \in A\backslash P$ such that $z \leq x$ and $z \leq y$.

We deduce $z \in I_1 \cap I_2 = P$ - a contradiction. ■

THEOREM 2.41. *Let A be an $MV-$ algebra and I a proper ideal of A. The next assertions are equivalent:*

(i) $I \in Spec(A)$;
(ii) If $x, y \in A$ and $(x] \cap (y] \subseteq I$, then $x \in I$ or $y \in I$.

Proof. $(i) \Rightarrow (ii)$. Let $x, y \in A$ such that $(x] \cap (y] \subseteq I$ and suppose by contrary that x, $y \notin I$. Then by Theorem 2.40, there is $z \in A\backslash I$ such that $z \leq x$ and $z \leq y$. Hence $z \in (x] \cap (y] \subseteq I$, so $z \in I$, a contradiction.

$(ii) \Rightarrow (i)$. Let $x, y \in A$ such that $x \wedge y \in I$. Then $(x \wedge y] \subseteq I$.

Since $(x] \cap (y] = (x \wedge y]$ (by Proposition 2.12, (iii)) we deduce that $(x] \cap (y] \subseteq I$, hence $x \in I$ or $y \in I$, that is $I \in Spec(A)$ (by Theorem 2.26 (iv)). ■

COROLLARY 2.42. *Let A be an $MV-$ algebra. For $I \in Id(A)$ the next assertions are equivalent:*

(*i*) $I \in Spec(A)$;
(*ii*) *For every* $x, y \in A/I, x \neq 0, y \neq 0$, *there exists* $z \in A/I, z \neq 0$ *such that* $z \leq x$ *and* $z \leq y$.

Proof. Clearly, by Theorem 2.40, since if $x = a/I$, with $a \in A$, then the condition $x \neq 0$ is equivalent with $a \notin I$. ■

As in the case of residuated lattices (see Theorem 1.48) we have:

THEOREM 2.43. *Let* A *be an* $MV-$ *algebra. For a proper ideal* $I \in Id(A)$ *the next assertions are equivalent:*

(*i*) $I \in Spec(A)$,
(*ii*) *For every* $J \in Id(A), J \to I = I$ *or* $J \subseteq I$.

THEOREM 2.44. ([**68**]) *Let* A *be an* $MV-$ *algebra and* $P \in Id(A)$. *Then* P *is meet-irreducible element in the lattice* $Id(A)$ *iff there is an element* $a \in A \backslash P$ *such that* P *is an maximal element in the set* $\{I \in Id(A) : a \notin I\}$.

4. Subdirect representation theorem

For any class of structures, the representation theorems have a special significance.

We denote by I an nonempty set. The *direct product* of family $\{A_i\}_{i \in I}$ of $MV-$ algebras, denoted by $\prod\limits_{i \in I} A_i$ is the $MV-$ algebra obtained by endowing the cartesian product of the family with the $MV-$ operations defined pointwise. In other words, $\prod\limits_{i \in I} A_i$ is the set of all functions $f : I \to \bigcup\limits_{i \in I} A_i$ such that $f(i) \in A_i$, for all $i \in I$, with the operations * and $\oplus$ defined by $f^*(i) = (f(i))^*$ and $(f \oplus g)(i) = f(i) \oplus g)(i)$. The zero element of $\prod\limits_{i \in I} A_i$ is the function $0 : I \to \bigcup\limits_{i \in I} A_i$ such that $0(i) = 0_{A_i} \in A_i$. For every $j \in I$ the map $\pi_j : \prod\limits_{i \in I} A_i \to A_j$ is defined by $\pi_j(f) = f(j)$; each π_j is a homomorphism onto A_j, called the j-th *projection function* . In particular for each $MV-$ algebra A and nonempty set X, the $MV-$ algebra A^X is the direct product of the family $\{A_x\}_{x \in X}$, where $A_x = A$ for all $x \in X$.

PROPOSITION 2.45. *Let a natural number* n *and* $e_1, ..., e_n \in B(A)$ *such that* $e_i \wedge e_j = 0$ *for any* $i \neq j$ *and* $\bigvee\limits_{i=1}^{n} e_i = 1$, *then* A *s isomorphic with the direct product of the family* $\{A(e_i)\}_{i=1,n}$ *and the isomorphism is given by* $f : A \to \prod\limits_{i=1}^{n} A(e_i), f(x) = (x \wedge e_1, ..., x \wedge e_n)$.

Proof. By Theorem 2.9 the function f is an morphism of $MV-$ algebras.

If $x, y \in A$ such that $f(x) = f(y)$ then $x \wedge e_i = y \wedge e_i$ for any $i = 1, ..., n$. We get that $x = x \wedge 1 = x \wedge (e_1 \vee ... \vee e_n) = (x \wedge e_1) \vee ... \vee (x \wedge e_n) = (y \wedge e_1) \vee ... \vee (y \wedge e_n) = y \wedge (e_1 \vee ... \vee e_n) = y \wedge 1 = y$, so f is injective. In order to prove the surjectivity, we consider $(x_1, ..., x_n) \in \prod\limits_{i=1}^{n} A(e_i)$. If we denote $x = \bigvee\limits_{i=1}^{n} x_i$, then $f(x) = (x_1, ..., x_n)$. We have proved that f is an $MV-$ algebra isomorphism. ■

PROPOSITION 2.46. *Let* $A = \prod\limits_{i=1}^{n} A_i$. *Then there exist* $e_1, ..., e_n \in B(A)$ *such that* $e_i \wedge e_j = 0$ *for any* $i \neq j$ *and* $\bigvee\limits_{i=1}^{n} e_i = 1$ *and* $A_i \approx A(e_i)$, *for all* $i = 1, ..., n$.

Proof. Let $e_1 = (1, 0, ..., 0), ..., e_n = (0, ..., 0, 1)$. ■

DEFINITION 2.9. An $MV-$ algebra A is *indecomposable* if $A \approx A_1 \times A_2$ implies A_1 or A_2 is trivial, where A_1 and A_2 are too $MV-$ algebras and $A_1 \times A_2$ is their direct product.

By Propositions 2.45 and 2.46 we obtain:

PROPOSITION 2.47. *An $MV-$ algebra A is indecomposable iff $B(A) = \{0, 1\}$.*

COROLLARY 2.48. *If A is an $MV-$ chain then A is indecomposable.*

EXAMPLE 2.11. *The $MV-$ algebra $[0, 1]$ and Chang's $MV-$ algebra C are indecomposable.*

DEFINITION 2.10. An $MV-$ algebra A is a *subdirect product* of a family $\{A_i\}_{i\in I}$ of $MV-$ algebras iff there exists a one-one homomorphism $h : A \to \prod_{i\in I} A_i$ such that for each $j \in I$ the composite map $\pi_j \circ h$ is a homomorphism onto A_j.

If A is a subdirect product of a family $\{A_i\}_{i\in I}$, then A is isomorphic to the subalgebra $h(A)$ of $\prod_{i\in I} A_i$; also, the restriction to $h(A)$ of each projection is a surjective map.

In [**45**] it is prove the following result, as particular case of a theorem of universal algebra, due to Birkhoff:

THEOREM 2.49. *An $MV-$ algebra A is a subdirect product of a family $\{A_i\}_{i\in I}$ of $MV-$ algebras iff there is a family $\{J_i\}_{i\in I}$ of ideals of A such that:*

(i) $A_i \cong A/J_i$ *for each* $i \in I$ *and*

(ii) $\bigcap_{i\in I} J_i = \{0\}$.

The following result is fundamental:

THEOREM 2.50. *(Chang's Subdirect Representation Theorem) Every nontrivial $MV-$ algebra is a subdirect product of $MV-$ chains.*

Proof. By Theorem 2.49 and Lemma 2.23 (v), an $MV-$ algebra A is a subdirect product of a family $\{A_i\}_{i\in I}$ of $MV-$ chains iff there is a family $\{P_i\}_{i\in I}$ of prime ideals of A such that $\bigcap_{i\in I} P_i = \{0\}$ (the monomorphism is $\Phi : A \to \prod_{P\in Spec(A)} A/P, \Phi(a) = (a/P)_{P\in Spec(A)}$. To prove apply Corollary 2.32, (i) to the ideal $\{0\}$. ■

5. MV-algebras and lu-groups; Chang completeness theorem

The idea of associating a totally ordered abelian group to any MV- algebra A is due to Chang, who in [**42**] and [**43**] gave first purely algebraic proof of the completeness of the Łukasiewicz axioms for the infinite-valued calculus. In [**45**] is proved the Chang completeness theorem starting that if an equation holds in the unit real interval $[0, 1]$, then the the equation holds in every MV- algebra. This proof is elementary, and use the good sequences; good sequences and Γ functor were first introduced in [**105**].

An applications is the categorical equivalence between MV- algebras and lattice ordered abelian groups with strong unit.

We recall the definition of an *lu*-group:

DEFINITION 2.11. An *lu-group* is an algebra $(G, +, -, 0, \vee, \wedge, u)$, where

$(lu - G_1)$ $(G, +, -, 0)$ is a group;
$(lu - G_2)$ $(G, \vee, \wedge)$ is a lattice;
$(lu - G_3)$ For any $x, y, a, b \in G, x \leq y$ implies $a + x + b \leq a + y + b$;
$(lu - G_4)$ $u > 0$ is a strong unit for G (that is, for all $x \in G$ there is some natural number $n \geq 1$ such that $-nu \leq x \leq nu$).

If G is abelian, then $(G, +, -, 0, \vee, \wedge, u)$ will be called *abelian lu-group*.

REMARK 2.18. ([**10**], *Propositions 1.2.2, 1.2.14*) *If G is an ordered group (l-group, see* [**10**]*) then*

(i) $(G, \vee, \wedge)$ *is a distributive lattice and for every* $x, y, z \in G$;
(ii) *If* $x \leq y$ *then* $z + x \leq z + y$ *and* $x + z \leq y + z$;
(iii) $x \leq y$ *iff* $-y \leq -x$;
(iv) $(x \vee y) + z = (x + z) \vee (y + z); z + (x \vee y) = (z + x) \vee (z + y)$;
(v) $(x \wedge y) + z = (x + z) \wedge (y + z); z + (x \wedge y) = (z + x) \wedge (z + y)$.

REMARK 2.19. *For each element x of an l-group G, the positive part x^+, the negative part x^- and the absolute value of x are defined as folows: $x^+ = 0 \vee x, x^- = 0 \vee (-x), |x| = x^+ + x^- = x^+ \vee x^-$; a strong unit u of G is an archimedean element of G, i.e. an element $u \in G$ such that for each $x \in G$ there is an integer $n \geq 0$ with $|x| \leq nu$.*

Following common usage, we let R, Q, Z denote the additive abelian groups of reals, rationales, integers with the natural order.

EXAMPLE 2.12. $(R, +)$ *with the natural order is an abelian lu-group, where for example* $u = 1$.

EXAMPLE 2.13. $(Q, +)$ *and* $(Z, +)$ *are abelian lu-groups with the natural order and* $u = 1$.

EXAMPLE 2.14. *Let (X, τ) be a topological space and $C(X)$ the aditive group of real - valued continuous functions defined on X. We make $C(X)$ an l-group by providing it with its usual pointwise order: $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in X$. If denote by $C_b(X)$ the subgroup of bounded elements of $C(X)$, then $C_b(X)$ is an abelian lu-group where order units are the elements $u \in C_b(X)$ with the property that there exists $c > 0$ such that $u(x) \geq c$ for every $x \in X$.*

PROPOSITION 2.51. ([**45**]) *If (G, u) is an abelian lu-group then for any $x \geq 0$ in G there are $x_{1,...,}x_n \in [0, u]$ such that $x = x_1 + ... + x_n$. Hence, any abelian lu-group is generated by its unit interval $[0, u]$.*

The best reference to general lattice ordered groups is [**10**] and [**50**].

We shall often write (G, u) to indicate that G is an abelian *lu*-group with *strong unit* u. If (G, u) is an abelian *lu*-group then the unit interval of G is

$$[0, u]_G = \{g \in G : 0 \leq g \leq u\}.$$

It has a canonical $MV-$ algebra structure given by Example 2.3. Mundici's result says that for any $MV-$ algebra A there is an abelian *lu*-group (G_A, u) such that A and $[0, u]_{G_A}$ are isomorphic. The categorical equivalence means that the entire theory of abelian *lu*-groups applies to $MV-$ algebras. The main work involved has the flavor of translation.

DEFINITION 2.12. Let G and G' be l-groups. A function $h : G \to G'$ is said to be *l-group homomorphism* iff h is both a group homomorphism and a lattice homomorphism i.e., for each $x, y \in G$, $h(x-y) = h(x) - h(y), h(x \vee y) = h(x) \vee h(y)$ and $h(x \wedge y) = h(x) \wedge h(y)$. If $0 < u \in G, 0 < u' \in G'$ and let $h : G \to G'$ is said to be l-group homomorphism such that $h(u) = u'$. Then h is said to be a *unital l-homomorphism* .

We recall that we denote by $\mathcal{MV}$ the category of $MV-$ algebras and by $\mathcal{UG}$ we denote the category whose objects are abelian *lu*-groups and whose morphisms are abelian *lu*-group homomorphisms. The definition of Mundici's functor

$$\Gamma : \mathcal{UG} \to \mathcal{MV}$$

is strainghtforward (see [**3**], [**45**]):

$$\Gamma(G, u) := [0, u]_G,$$

$$\Gamma(h) := h_{|[0,u]},$$

where (G, u) is an abelian *lu*-group and $h : (G, u) \to (H, v)$ is an abelian *lu*-group homomorphism.

EXAMPLE 2.15. *If $G = R$ and $u = 1$, then $\Gamma(R, 1) = [0, 1]$ (see Example 2.3).*

EXAMPLE 2.16. *If $G = Q$ and $u = 1$, then $\Gamma(Q, 1) = Q \cap [0, 1]$ (see Example 2.4).*

EXAMPLE 2.17. *If $G = Z$ and $u = 1$, then $\Gamma(Z, 1) = \{0, 1\} = L_2$.*

EXAMPLE 2.18. *If $G = Z$ and $u = n \geq 2$, then $\Gamma(Z, n)$ is isomorphic with MV -algebra $Ł_n$ (see Example 2.4). Also, $\Gamma(\frac{1}{n-1}Z, 1) = Ł_n$, where $\frac{1}{n-1}Z = \{\frac{z}{n-1} : z \in Z\}$.*

EXAMPLE 2.19. *Let $G = Z \times_{lex} Z$ be the lexicographical product, i.e. the group operations are defined on components but the order relation is lexicographic:*

$$(n_1, n_2) \leq (m_1, m_2) \text{ iff } n_1 < m_1 \text{ or } n_1 = m_1 \text{ and } n_2 \leq m_2.$$

We remark that G is a totally ordered abelian l-group and $u = (1, 0)$ is a strong unit. Then the MV -algebra $\Gamma(G, u)$ is isomorphic with Chang's algebra C (see Example 2.6).

A sequence $\mathbf{a} = (a_1, a_2, ...)$ of elements of an arbitrary MV-algebra A is said to be *good* iff for each $i = 1, 2...$

$$a_i \oplus a_{i+1} = a_i,$$

and there is an integer n such that $a_r = 0$ for all $r > n$.

Instead of $\mathbf{a} = (a_1, a_2, ..., a_n, 0, 0, ..)$ we shall often write $\mathbf{a} = (a_1, a_2, ..., a_n)$.

For each $a \in A$, the good sequence $(a, 0, ..., 0...)$ will be denoted by (a).

DEFINITION 2.13. For any two good sequences $\mathbf{a} = (a_1, a_2, ..., a_n)$ and $\mathbf{b} = (b_1, b_2, ..., b_m)$ their *sum* $\mathbf{c} = \mathbf{a} + \mathbf{b}$ is defined by $\mathbf{c} = (c_1, c_2, ...)$, where for all $i = 1, 2...$

$$c_i = a_i \oplus (a_{i-1} \odot b_1) \oplus ... \oplus (a_1 \odot b_{i-1}) \oplus b_i.$$

We denote by M_A the set of good sequences of A equipped with the addition.

In [**45**] we have the following results of good sequences:

PROPOSITION 2.52. *Let A be an MV-algebra. Then $(M_A, +)$ is an abelian monoid with the following additional properties:*

(*i*) *(cancellation) For any good sequences* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *if* $\mathbf{a} + \mathbf{b} = \mathbf{a} + \mathbf{c}$ *then* $\mathbf{b} = \mathbf{c}$;
(*ii*) *(zero-law) If* $\mathbf{a} + \mathbf{b} = (0)$ *then* $\mathbf{a} = \mathbf{b} = (0)$.

DEFINITION 2.14. For any two good sequences $\mathbf{a} = (a_1, a_2, ..., a_n)$ and $\mathbf{b} = (b_1, b_2, ..., b_n)$ we write $\mathbf{b} \leq \mathbf{a}$ iff $\mathbf{b}$ and $\mathbf{a}$ satisfy the equivalent conditions:

(*i*) There is a good sequence $\mathbf{c}$ such that $\mathbf{b} + \mathbf{c} = \mathbf{a}$;
(*ii*) $b_i \leq a_i$, for all $i = 1, 2, ...n$.

REMARK 2.20. *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be good sequences. If* $\mathbf{b} \leq \mathbf{a}$ *then there is a unique good sequence* $\mathbf{c}$ *such that* $\mathbf{b} + \mathbf{c} = \mathbf{a}$. *This* $\mathbf{c}$, *denoted* $\mathbf{a} - \mathbf{b}$ *is given by* $\mathbf{c} = (a_1, a_2, ..., a_n, ...) + (b_1^*, b_2^*, ..., b_n^*, ...)$. *In particular, for each* $a \in A$, *we have* $(a^*) = (1) - (a)$.

PROPOSITION 2.53. *Let* $\mathbf{a} = (a_1, a_2, ..., a_n, ...)$ *and* $\mathbf{b} = (b_1, b_2, ..., b_n, ...)$ *be good sequences of an MV-algebra A. The sequences*

$$\mathbf{a} \vee \mathbf{b} = (a_1 \vee b_1, ..., a_n \vee b_n, ...)$$

and

$$\mathbf{a} \wedge \mathbf{b} = (a_1 \wedge b_1, ..., a_n \wedge b_n, ...)$$

are good and are in fact the supremum and infimum of $\mathbf{a}$ *and* $\mathbf{b}$ *with respect the order defined by Definition 2.14.*

REMARK 2.21. *For all* $a, b \in A$, *we have* $((a) + (b)) \wedge (1) = (a \oplus b)$.

From the abelian monoid M_A enriched with the lattice-order we obtain (via Maltzev theorem) an abelian l-group G_A such that M_A is isomorphic, both as a monoid and as a lattice, to positive cone G_A^+. Let us agree to say that a pair of good sequences $(\mathbf{a}, \mathbf{b})$ is *equivalent* to another pair $(\mathbf{a}', \mathbf{b}')$ iff $\mathbf{a} + \mathbf{b}' = \mathbf{a}' + \mathbf{b}$. The equivalence class of the pair $(\mathbf{a}, \mathbf{b})$ shall be denoted by $[\mathbf{a}, \mathbf{b}]$. Let G_A be the set of equivalence classes of pairs of good sequences, where the *zero element* 0, is the equivalence class $[(0), (0)]$, an addition $+$ is defined by

$$[\mathbf{a}, \mathbf{b}] + [\mathbf{c}, \mathbf{d}] = [\mathbf{a} + \mathbf{c}, \mathbf{b} + \mathbf{d}],$$

a subtraction - is defined by

$$-[\mathbf{a}, \mathbf{b}] = [\mathbf{b}, \mathbf{a}].$$

Then $G_A = (G_A, 0, +, -)$ is an abelian group. We shall now equip G_A with a lattice -order. We define

$$[\mathbf{a}, \mathbf{b}] \preceq [\mathbf{c}, \mathbf{d}]$$

iff

$$\mathbf{a} + \mathbf{d} \leq \mathbf{c} + \mathbf{b},$$

where $\leq$ is the partial order of M_A. The supremum ($\curlyvee$) and infimum ($\curlywedge$) are given by:

$$[\mathbf{a}, \mathbf{b}] \curlyvee [\mathbf{c}, \mathbf{d}] = [(\mathbf{a} + \mathbf{d}) \vee (\mathbf{c} + \mathbf{b}), \mathbf{b} + \mathbf{d}]$$

and

$$[\mathbf{a}, \mathbf{b}] \curlywedge [\mathbf{c}, \mathbf{d}] = [(\mathbf{a} + \mathbf{d}) \wedge (\mathbf{c} + \mathbf{b}), \mathbf{b} + \mathbf{d}].$$

The l-group G_A with the above lattice order is called the *Chang l-group* of the MV -algebra A. The element $u_A = [(1), (0)]$ is a *strong unit* of the l-group G_A.

A crucial property of the *lu*-group G_A is given by the following result (for more details, see [**45**]):

THEOREM 2.54. *The correspondence* $a \longmapsto \varphi_A(a) = [(a),(0)]$ *defines an isomorphism from the* MV*-algebra* A *onto the* MV*-algebra* $\Gamma(G_A, u_A) = [0, u_A]$.

Proof. By definition, $[(0),(0)] \preceq [\mathbf{a},\mathbf{b}] \preceq u_A$ iff there is $c \in A$ such that $(\mathbf{a},\mathbf{b})$ is equivalent to $((c),(0))$. Thus, φ_A maps A onto the unit interval $[[(0),(0)], u_A]$ of G_A. Is obviously that this map is one-one. By Remark 2.21, $\varphi_A(a \oplus b) = (\varphi_A(a) + \varphi_A(b)) \wedge u_A$ and by Remark 2.20, $\varphi_A(a^*) = u_A - \varphi_A(a)$; we deduce that φ_A is a homomorphism from the MV-algebra A onto the MV-algebra $\Gamma(G_A, u_A) = [0, u_A]$. ■

Using the good sequences we obtain the Completeness Theorem (for more details, see [**45**]):

THEOREM 2.55. *An equation holds in* $[0,1]$ *if and only if it holds in every* MV*-algebra.*

The natural equivalence between MV-algebras and abelian lu-groups with strong unit was first established in [**105**], building on previous work by Chang [**43**] for the totally ordered case.

We shall prove that Γ is a natural equivalence between the categories $\mathcal{UG}$ and $\mathcal{MV}$.

We give an explicit construction of an adjoint functor of Γ.

Our starting point is the lu-group G_A with order unit u_A.

Let A and B be MV-algebras, $h : A \to B$ a homomorphism. If $a = (a_1, a_2, ...)$ is a good sequence of A then $(h(a_1), h(a_2), ...)$ is a good sequence of B. If $h^* : M_A \to M_B$ is defined by $h^*(a) = (h(a_1), h(a_2), ...)$ for all $a \in M_A$, then we have: $h^*(a+b) = h^*(a) + h^*(b), h^*(a \vee b) = h^*(a) \vee h^*(b), h^*(a \wedge b) = h^*(a) \wedge h^*(b)$. Thus, $h^* : M_A \to M_B$ is both a monoid homomorphism and lattice homomorphism. Let us define the map $h^\# : G_A \to G_B$ by $h^\#([a,b]) = [h^\#(a), h^\#(b)]$ and let u_A and u_B be the strong units of G_A and G_B. Then the map $h^\#$ is a unital l-homomorphism of (G_A, u_A) into (G_B, u_B). For the definition of the functor

$$\Xi : \mathcal{MV} \to \mathcal{UG}$$

(the inverse of the functor Γ which together with Γ determine a categorical equivalence), let us agree to write $\Xi(A) = (G_A, u_A)$ and $\Xi(h) = h^\#$.

In our present notation, Theorem 2.54 states that the map $a \longmapsto \varphi_A(a) = [(a),(0)]$ defines an isomorphism of the MV-algebra A and $\Gamma(\Xi(A))$.

Using the maps φ_A ($A \in \mathcal{MV}$) we obtain:

THEOREM 2.56. *The composite functor* $\Gamma\Xi$ *is naturally equivalent to the identity functor of* $\mathcal{MV}$. *In other words, for all* MV*-algebras* A, B *and homomorphism* $h : A \to B$, *we have a commutative diagram*

$$\begin{array}{ccc} A & \stackrel{h}{\to} & B \\ \downarrow \varphi_A & & \downarrow \varphi_B \\ \Gamma(\Xi(A)) & \underset{\Gamma(\Xi(h))}{\to} & \Gamma(\Xi(B)) \end{array}$$

in the sense that, for each $a \in A$, $\varphi_B(h(a)) = (\Gamma(\Xi(h))(\varphi_A(a))$.

Proof. ([**45**]) For each $a \in A$, $\varphi_B(h(a)) = [(h(a)),(0)]$ and $\varphi_A(a) = [(a),(0)]$. Further, $\Xi(h)([(a),(0)]) = [(h(a)),(0)]$, the latter being an element of $\Gamma(\Xi(B))$. Since $\Gamma(\Xi(h))$ is the restriction of $\Xi(h)$ to $\Gamma(\Xi(B))$, we can write $(\Gamma(\Xi(h))(\varphi_A(a)) = [(h(a)),(0)] = \varphi_B(h(a))$. ■

Now, we prove that the composite functor $\Sigma\Gamma$ is also a naturally equivalent to the identity functor of the category $\mathcal{UG}$.

We first define the dual of the maps φ_A.

In [**45**] it is proved the following (see, Lemma 7.1.3, p. 141 and Corollary 7.1.6, p. 145):

LEMMA 2.57. *Suppose G is an abelian l-group with order unit u, and let $A = \Gamma(G,u) \subseteq G$. For each $0 \leq a \in G$ there is a unique good sequence $g(a) = (a_1, ..., a_n)$ of elements of A such that $a = a_1 + ... + a_n$.*

THEOREM 2.58. *For every $(G,u) \in \mathcal{UG}$ let the map $\psi_{(G,u)} : G \to G_{\Gamma((G,u))}$ be defined by $\psi_{(G,u)}(a) = [g(a^+), g(a^-)]$, for all $a \in G$. It follows that $\psi_{(G,u)}$ is an l-group isomorphism of G onto $G_{\Gamma((G,u))}$ and $\psi_{(G,u)}(u) = [(u), (0)]$.*

From Theorem 2.58, using the maps $\psi_{(G,u)}$ we have the following :

THEOREM 2.59. *The composite functor $\Xi\Gamma$ is naturally equivalent to the identity functor of $\mathcal{UG}$. In other words, for any two abelian l-groups with strong unit (G,u) and (H,v) and unital l-homomorphism $f : (G,u) \to (H,v)$, we have a commutative diagram*

$$\begin{array}{ccc} (G,u) & \stackrel{f}{\to} & (H,v) \\ \downarrow \psi_{(G,u)} & & \downarrow \psi_{(H,v)} \\ \Xi(\Gamma(G,u)) & \underset{\Xi(\Gamma(f))}{\to} & \Xi(\Gamma(G,u)) \end{array}$$

in the sense that, for each $a \in G$, $\psi_{(H,v)}(f(a)) = (\Xi(\Gamma(f))(\psi_{(G,u)}(a))$.

Proof. **([45])** By Lemma 2.57 we can write $g(a^+) = (a_1, ..., a_n)$, for a uniquely determined good sequence $(a_1, ..., a_n) \in M_{\Gamma((G,u))}$. Letting $h = \Gamma(f)$, we then obtain $f(a)^+ = f(a^+) = \sum_{i=1}^{n} f(a_i) = \sum_{i=1}^{n} h(a_i)$, whence $g(f(a)^+) = (h(a_1), ..., h(a_n)) = h^*(g(a^+))$. Similarly, $g(f(a)^-) = h^*(g(a^-))$, whence $\psi_{(H,v)}(f(a)) = [g(f(a)^+), g(f(a)^-)] = [h^*(g(a^+)), h^*(g(a^-))] = h^{\#}([g(a^+), g(a^-)]) = (\Xi(\Gamma(f))([g(a^+), g(a^-)]) = (\Xi(\Gamma(f))(\psi_{(G,u)}(a))$. ■

From the Theorems 2.56 and 2.59 we immediately get:

COROLLARY 2.60. *(Mundici) The functor Γ establishes a categorical equivalence between $\mathcal{UG}$ and $\mathcal{MV}$.*

EXAMPLE 2.20. *Let $(A, \wedge, \vee, ^*, 0, 1)$ be a Boolean algebra; then $(A, \vee, ^*, 0)$ is an $MV-$ algebra (see Example 2.2)*

(*i*). *If $A = \{0,1\} = L_2$, then $G_A = \Xi(A) = (Z, +)$ (because $M_A \approx (N, +)$).*

(*ii*). *If A is finite, then there exists a natural number n such that $A = L_2^n$. So, $\Xi(A) = \Xi(L_2^n) = (\Xi(L_2))^n = Z^n$ with $u = (1, 1, ..., 1)$.*

(*iii*). *If A is infinite, then there exists an infinite set X such that A is a Boolean subalgebra of L_2^X (we can consider for example $X = \{f : A \to L_2 : f$ is morphism of Boolean algebras$\}$). Then, $\Xi(A)$ is isomorphic with an abelian lu-subgroup of $\Xi(L_2^X) = Z^X$. Clearly, the function $u : X \to Z$, $u(x) = 1$, for every $x \in X$ is not a strong unit for Z^X, but if consider $G = \{f \in Z^X :$ there exists an natural number n such that $f \leq nu\}$, then u is a strong unit for G. Thus, $\Xi(A)$ is isomorphic with an abelian lu-subgroup of G.*

EXAMPLE 2.21. *For an MV -chain A we consider G_A as the set of all the order pairs (m, a) with $m \in Z$ and $a \in A$. If on $G_A = Z \times A$ we define:*

$$(m+1, 0) = (m, 1),$$

$$(m, a) + (n, b) = \begin{cases} (m+n, a \oplus b) \text{ if } a \oplus b < 1, \\ (m+n+1, a \odot b) \text{ if } a \oplus b = 1, \end{cases}$$

$$-(m, a) = (-m-1, a^*),$$

then $(G_A, +, (0, 0))$ is an abelian group. Moreoever if we set $(m, a) \leq (n, b)$ iff $m < n$ or $m = n$ and $a \leq b$ (lexicographical order), then G_A becomes an abelian lu-group, $(0, 1)$ is a strong unit and $G_A = \Xi(A)$.

In the sequel G will designate an abelian *lu*-group with strong unit u, and A will designate $[0, u]_G$.

DEFINITION 2.15. For any integer k, let $\pi_k : G \to A$ be defined by

$$\pi_k(g) = ((g - ku) \wedge u) \vee 0.$$

PROPOSITION 2.61. *The maps $\pi_k, k \in Z$, have the following properties for all $f, g \in G$:*

$(mv - c_{32})$ $\pi_{0|A} = 1_A$;
$(mv - c_{33})$ $\pi_k(g) \geq \pi_{k+1}(g)$, *for all* $k \in Z$;
$(mv - c_{34})$ $\pi_k(f \vee g) = \pi_k(f) \vee \pi_k(g)$ *and* $\pi_k(f \wedge g) = \pi_k(f) \wedge \pi_k(g)$, *for all* $k \in Z$, *(hence π_k is an increasing map for all $k \in Z$).*

Proof. $(mv - c_{32})$. If $g \in A$ (that is $0 \leq g \leq u$), then $\pi_0(g) = (g \wedge u) \vee 0 = g \vee 0 = g$, hence $\pi_{0|A} = 1_A$.

$(mv - c_{33})$. From Remark 2.18, we deduce that $ku \leq ku + u = (k+1)u$, so $-(k+1)u \leq -ku$, hence $g-(k+1)u \leq g-ku$. Therefore $\pi_{k+1}(g) = ((g-(k+1)u) \wedge u) \vee 0 \leq ((g - ku) \wedge u) \vee 0 = \pi_k(g)$.

$(mv - c_{34})$. For $f, g \in G$ and $k \in Z$ we have:

$$\pi_k(f \vee g) = ((f \vee g) - ku) \wedge u) \vee 0 =$$

$$= (((f - ku) \vee (g - ku)) \wedge u) \vee 0 = (((f - ku) \wedge u) \vee ((g - ku) \wedge u)) \vee 0 =$$

$$= [((f - ku) \wedge u) \vee 0] \vee [((g - ku) \wedge u)) \vee 0] = \pi_k(f) \vee \pi_k(g)$$

and analogously $\pi_k(f \wedge g) = \pi_k(f) \wedge \pi_k(g)$. ■

REMARK 2.22. *By the proof of Proposition 2.61 we deduce that $mv-c_{32}, mv-c_{33}$ and $mv - c_{34}$ are true in general when G is non-abelian.*

6. MV-algebras and Wajsberg algebras

Mathematicians want to minimize the set of axioms of a certain mathematical theory and maximize the set of consequences of these axioms. In this section we introduce the Wajsberg algebras, which have important consequences each having direct application in fuzzy logic. We also study MV- algebras by giving first a long definition of this algebraic structure.This definition shows some basic properties of this structure. We also prove that there is one-to-one correspondence between MV- algebras and Wajsberg algebras; each MV-algebra can be seen as Wajsberg algebra and conversely. MV- algebras will turn out to be particular residuated lattices.

DEFINITION 2.16. An algebra $(L, \to, ^*, 1)$ of type $(2, 1, 0)$ will be called *Wajsberg algebra* if for every $x, y, z \in L$ the following axioms are verifyed *:*

(W_1) $1 \to x = x$;
(W_2) $(x \to y) \to [(y \to z) \to (x \to z)] = 1$;
(W_3) $(x \to y) \to y = (y \to x) \to x$;
(W_4) $(x^* \to y^*) \to (y \to x) = 1$.

A first example of Wajsberg algebra is offer by a Boolean algebra $(L, \vee, \wedge, ', 0, 1)$, where for $x, y \in L, x \to y = x' \vee y$.

For more information about Wajsberg algebras, I recommend to the reader the paper [**62**] and the book [**22**]

If L is a Wajsberg algebra, on L we define the relation $x \leq y \Leftrightarrow x \to y = 1$; it is immediate that $\leq$ is an order relation on L (called *natural order*) and 1 is the greatest element in L.

THEOREM 2.62. *Let L be a Wajsberg algebra and $x, y, z \in L$. Then*

$(w - c_1)$ *If $x \leq y$, then $y \to z \leq x \to z$;*
$(w - c_2)$ $x \leq y \to x$;
$(w - c_3)$ *If $x \leq y \to z$, then $y \leq x \to z$;*
$(w - c_4)$ $x \to y \leq (z \to x) \to (z \to y)$;
$(w - c_5)$ $x \to (y \to z) = y \to (x \to z)$;
$(w - c_6)$ *If $x \leq y$, then $z \to x \leq z \to y$;*
$(w - c_7)$ $1^* \leq x$;
$(w - c_8)$ $x^* = x \to 1^*$.

Proof. $(w - c_1)$. From W_2 we deduce that $x \to y \leq (y \to z) \to (x \to z)$; since $x \to y = 1$, then $(y \to z) \to (x \to z) = 1$, hence $y \to z \leq x \to z$.

$(w - c_2)$. From $y = 1$ and $w - c_1$ we deduce that $1 \to x \leq y \to x$, hence $x \leq y \to x$.

$(w - c_3)$. If $x \leq y \to z$, then $(y \to z) \to z \leq x \to z$. By W_3 we deduce that $(z \to y) \to y \leq x \to z$. Since $y \leq (z \to y) \to y$ we deduce that $y \leq x \to z$.

$(w - c_4)$. By W_2 we have that $z \to x \leq (x \to y) \to (z \to y)$, so by $w - c_3$ we deduce that $x \to y \leq (z \to x) \to (z \to y)$.

$(w - c_5)$. We have $y \leq (z \to y) \to y = (y \to z) \to z$. By $w - c_4$ we deduce that $(y \to z) \to z \leq (x \to (y \to z)) \to (x \to z)$, hence $y \leq (x \to (y \to z)) \to (x \to z)$, therefore $x \to (y \to z) \leq y \to (x \to z)$.

By a symmetric argument $y \to (x \to z) \leq x \to (y \to z)$. So, it follows the required equality.

$(w - c_6)$. Follows immediate from $w - c_4$.

$(w - c_7)$. We have $x^* \to 1^* \leq 1 \to x = x$, so, $1^* \leq x$.

$(w - c_8)$. We have $x^* \leq (1^*)^* \to x^* \leq x \to 1^*$, by $w - c_4$. On another hand, $x^* \to 1^* \leq 1 \to x = x \Rightarrow x \to 1^* \leq (x^* \to 1^*) \to 1^* = (1^* \to x^*) \to x^* \Rightarrow 1^* \to x^* \leq (x \to 1^*) \to x^*$, by $w - c_3$.

Since $1^* \leq x^*$, by $w - c_6$ we deduce that $1 = (x \to 1^*) \to x^*$, hence $x \to 1^* \leq x^*$, so $x \to 1^* = x^*$. ■

We deduce that 1^* is the lowest element of Wajsberg algebra L relative to natural ordering, that is, $1^* = 0$.

As in the case of residuated lattices, for $x \in L$ we denote $x^{**} = (x^*)^*$.

The following result is straighforward :

PROPOSITION 2.63. *If L is a Wajsberg algebra and $x, y \in L$, then*

$(w - c_9)$ $x^{**} = x$;
$(w - c_{10})$ $x^* \to y^* = y \to x, x^* \to y = y^* \to x$;
$(w - c_{11})$ $x \leq y \Leftrightarrow y^* \leq x^*$.

Proof. $(w - c_9)$. From $(x \to 1^*) \to (x \to 1^*) = 1$ we deduce by $w - c_5$ that $x \to [(x \to 1^*) \to 1^*] = 1$. Thus, $x \leq x^{**}$. By W_4, $(x^* \to 1^*) \to (1 \to x) = 1$, so $x^{**} \to x = 1$. Hence, $x^{**} \leq x$. We obtain that $x^{**} = x$.

$(w - c_{10})$. By W_4 and $w - c_9$ we have $x^* \to y^* \leq y \to x = y^{**} \to x^{**} \leq x^* \to y^*$.

$(w - c_{11})$. If $x \leq y$, then $x \to y = 1$, thus, by $w - c_9$, $x^{**} \to y^{**} = 1$, hence, by W_4, $1 \to (y^* \to x^*) = 1$, which implies $y^* \leq x^*$. By a similar argument, $y^* \leq x^*$ implies $x^{**} \leq y^{**}$, so by $w - c_9$, $x \leq y$.■

PROPOSITION 2.64. *Let L be a Wajsberg algebra. Relative to the natural ordering, L become lattice, where for $x, y \in L$,*

$$(w_\vee) : x \vee y = (x \to y) \to y$$

and

$$(w_\wedge) : x \wedge y = (x^* \vee y^*)^*.$$

Proof. From $w - c_2$ we deduce that $x, y \leq (x \to y) \to y = (y \to x) \to x$. If $z \in L$ is such that $x, y \leq z$ then $x \to z = 1$ and by W_1 we deduce that $(x \to z) \to z = z$. Also, $z \to x \leq y \to x$ hence $(y \to x) \to x \leq (z \to x) \to x = (x \to z) \to z = z$ or $(x \to y) \to y \leq z$, therefore $x \vee y = (x \to y) \to y$.

To prove that $x \wedge y = (x^* \vee y^*)^*$, we observe that from $x^*, y^* \leq x^* \vee y^*$ we deduce that $(x^* \vee y^*)^* \leq x^{**} = x, y^{**} = y$.

Let now $z \in L$ such that $z \leq x, y$. Then $x^*, y^* \leq z^* \Rightarrow x^* \vee y^* \leq z^* \Rightarrow z = z^{**} \leq (x^* \vee y^*)^*$, hence $x \wedge y = (x^* \vee y^*)^*$. ■

COROLLARY 2.65. *If L is a Wajsberg algebra and $x, y \in L$, then*

$(w - c_{12})$ $(x \wedge y)^* = x^* \vee y^*$;
$(w - c_{13})$ $(x \vee y)^* = x^* \wedge y^*$.

In what follows we want to mark some connections between Wajsberg algebras and residuated lattices.

If L is a Wajsberg algebra, for $x, y \in L$ we define $x \odot y = (x \to y^*)^*$.

THEOREM 2.66. *If $(L, \to, ^*, 1)$ is a Wajsberg algebra, then $(L, \vee, \wedge, \odot, \to, 0 = 1^*, 1)$ is a residuated lattice.*

Proof. To prove that the triple $(L, \odot, 1)$ is a commutative monoid, let $x, y, z \in L$. We have $x \odot y = (x \to y^*)^* = (x^{**} \to y^*)^* = (y \to x^{***})^* = (y \to x^*)^* = y \odot x$, hence the operation $\odot$ is commutative.

For the associativity of $\odot$ we have : $x \odot (y \odot z) = x \odot (z \odot y) = x \odot (z \to y^*)^* = [x \to (z \to y^*)^{**}]^* = [x \to (z \to y^*)]^* = [z \to (x \to y^*)]^* = [z \to (x \to y^*)^{**}]^* = z \odot (x \to y^*)^* = z \odot (x \odot y) = (x \odot y) \odot z$.

Also, $x \odot 1 = (x \to 1^*)^* = (x \to 0)^* = x^{**} = x$.

We have to prove $x \odot y \leq z \Leftrightarrow x \leq y \to z$.

Indeed, $x \odot y \leq z \Leftrightarrow (x \to y^*)^* \leq z \Leftrightarrow z^* \leq x \to y^* \Leftrightarrow x \leq z^* \to y^* = y \to z \Leftrightarrow x \leq y \to z$. ■

Thus, all properties valid in any residuated lattice hold in Wajsberg algebras, too.

COROLLARY 2.67. *If L is a Wajsberg algebra and $x, y, z \in L$, then*

$(w - c_{14})$ $(x \vee y) \to z = (x \to z) \wedge (y \to z)$;
$(w - c_{15})$ $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$;
$(w - c_{16})$ $(x \to y) \vee (y \to x) = 1$;
$(w - c_{17})$ $(x \wedge y) \to z = (x \to y) \to (x \to z)$.

Proof. $(w - c_{14})$,$(w - c_{15})$. Follows from Theorems 2.66 and rules of calculus from residuated lattices .

$(w - c_{16})$. We have $(y \to x) \to (x \to y) = [(x \vee y) \to x] \to [(x \vee y) \to y] = [x^* \to (x \vee y)^*] \to [y^* \to (x \vee y)^*] = y^* \to \{[x^* \to (x \vee y)^*] \to (x \vee y)^*\} =$

$= y^* \to [x^* \vee (x \vee y)^*] = [x^* \vee (x \vee y)^*]^* \to y = [x \wedge (y \vee x)] \to y = x \to y$, hence $(x \to y) \vee (y \to x) = [(x \to y) \to (y \to x)] \to (y \to x) = (y \to x) \to (y \to x) = 1$.

$(w - c_{17})$. We have $(x \wedge y) \to z = (x^* \vee y^*)^* \to (z^*)^* = z^* \to (x^* \vee y^*) = z^* \to [(y^* \to x^*) \to x^*] = z^* \to [(x \to y) \to x^*] = (x \to y) \to (z^* \to x^*) = (x \to y) \to (x \to z)$.■

REMARK 2.23. *By Theorem 2.66, any Wajsberg algebra can viewed as a residuated lattice. In general, the converse is not true. For an example of residuated lattice which is not Wajsberg algebra see* [**129**].

We will give necessary and sufficient conditions for a residuated lattice to be Wajsberg algebra.

Define on a Wajsberg algebra L a binary operation $\oplus$, for $x, y \in L$, by $x \oplus y = x^* \to y$. Then we have

$$(mv_{\wedge}) : x \wedge y = (x \oplus y^*) \odot y$$

and

$$(mv_{\vee}) : x \vee y = (x \odot y^*) \oplus y.$$

Indeed, $x \wedge y = (x^* \vee y^*)^* = [(x^* \to y^*) \to y^*]^* = [y \to (x^* \to y^*)^*]^* = y \odot (x^* \to y^*) = y \odot (x \oplus y^*) = (x \oplus y^*) \odot y$ and $x \vee y = (x \to y) \to y = (x \to y^{**})^{**} \to y = (x \odot y^*)^* \to y = (x \odot y^*) \oplus y$.

Is eassy to verify that the following equations under the given notation are satisfied in every Wajsberg algebra:

$x \oplus y = y \oplus x$;	$x \odot y = y \odot x$;
$x \oplus (y \oplus z) = (x \oplus y) \oplus z$;	$x \odot (y \odot z) = (x \odot y) \odot z$;
$x \oplus x^* = 1$;	$x \odot x^* = 0$;
$x \oplus 1 = 1$;	$x \odot 0 = 0$;
$x \oplus 0 = x$;	$x \odot 1 = x$;
$(x \oplus y)^* = x^* \odot y^*$;	$(x \odot y)^* = x^* \oplus y^*$;
$x^{**} = x$;	$1^* = 0$;
$x \vee y = y \vee x$;	$x \wedge y = y \wedge x$;
$x \vee (y \vee z) = (x \vee y) \vee z$;	$x \wedge (y \wedge z) = (x \wedge y) \wedge z$;
$x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z)$;	$x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$,

that is, $(L, \oplus, \odot, ^*, 0, 1)$ is an $MV-$ algebra (see Definition 2.1).

Also the converse is true: given an $MV-$ algebra $(L, \oplus, \odot, ^*, 0, 1)$ we can define a binary operation $\to$ such that the Wajsberg algebra axioms hold.

PROPOSITION 2.68. *Define on an $MV-$ algebra $(L, \oplus, \odot, ^*, 0, 1)$ a binary operation $\to$ by*

$$x \to y = x^* \oplus y \ (x, y \in L).$$

Then we obtain a Wajsberg algebra $(L, \to, ^*, 1)$.

Proof. Let $x, y, z \in L$. We show that $(L, \rightarrow, ^*, 1)$ satisfies the axioms $W_1 - W_4$.

Indeed, $1 \rightarrow x = 1^* \oplus x = 0 \oplus x = x$;

$(x \rightarrow y) \rightarrow [(y \rightarrow z) \rightarrow (x \rightarrow z)] = (x^* \oplus y)^* \oplus [(y^* \oplus z)^* \oplus (x^* \oplus z)] = (x \odot y^*) \oplus [(y \odot z^*) \oplus (x^* \oplus z)] = [(y^* \odot x) \oplus x^*] \oplus [(y \odot z^*) \oplus z] = (y^* \vee x^*) \oplus (y \vee z) \geq y^* \oplus y = 1$

and $(x \rightarrow y) \rightarrow y = (x^* \oplus y)^* \oplus y = (x \odot y^*) \oplus y = x \vee y = y \vee x = (y \odot x^*) \oplus x = (y^* \oplus x)^* \oplus x = (y \rightarrow x) \rightarrow x$;

Finally, $y \rightarrow x = y^* \oplus x = x \oplus y^* = x^{**} \oplus y^* = x^* \rightarrow y^*$.

Thus, $(x^* \rightarrow y^*)^* \oplus (y \rightarrow x) = 1$, so $(x^* \rightarrow y^*) \rightarrow (y \rightarrow x) = 1$. ■

By Theorems 2.66 and 2.68 we deduce

THEOREM 2.69. *There is a one-to-one correspondence between $MV-$ algebras and Wajsberg algebras.*

REMARK 2.24. *By Theorem 2.69 we deduce that an $MV-$ algebra has all the Wajsberg algebra properties and conversely. Moreover, the category of $MV-$ algebras and the category of Wajsberg algebras are equivalent. Still, the Wajsberg algebras are special, since they are the structure that naturally arise from Lukasiewicz logic.*

THEOREM 2.70. *Let $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ be a residuated lattice . Then $(L, \rightarrow, ^*, 1)$ is a Wajsberg algebra iff $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$, for every $x, y \in L$, where $x^* = x \rightarrow 0$.*

Proof. ,, $\Rightarrow$ ". The condition is clearly necessary.

,, $\Leftarrow$ ". From $(x \rightarrow 0) \rightarrow 0 = (0 \rightarrow x) \rightarrow x$ we deduce that $x^{**} = 1 \rightarrow x = x$, hence $x^{**} = x$, for $x \in L$. So, take in consideration the calculus rules $lr - c_1 - lr - c_{20}$ from residuated lattices, we deduce that W_1, W_2 and W_3 holds.

For W_4 : $x^* \rightarrow y^* = (x \rightarrow 0) \rightarrow (y \rightarrow 0) = y \rightarrow [(x \rightarrow 0) \rightarrow 0] = y \rightarrow x^{**} = y \rightarrow x$ and the proof is complete. ■

REMARK 2.25. *Theorem 2.70 states that Wajsberg algebras, or equivalently, $MV-$ algebras, are exactly those residuated lattices where $x \vee y$ and $(x \rightarrow y) \rightarrow y$ coincide.*

CHAPTER 3

BL-algebras

BL-algebras are particular residuated lattices.

The origin of BL-algebras is in Mathematical Logic. BL-algebras have been introduced by Hájek [**75**] in order to investigate many-valued logic by algebraic means. His motivations for introducing BL-algebras were of two kinds.

The first one was providing an algebraic counterpart of a propositional logic, called Basic Logic, which embodies a fragment common to some of the most important many-valued logics, namely Łukasiewicz Logic, Gödel Logic and Product Logic. This Basic Logic (BL for short) is proposed as the most general many-valued logic with truth values in $[0, 1]$ and BL-algebras are the corresponding Lindenbaum-Tarski algebras.

The second one was to provide an algebraic mean for the study of continuous t-norms (triangular norms) on $[0, 1]$. An exhaustive treatment of t-norms can be found in the monograph [**89**].

It turns out that the variety of BL-algebras is generated by the class of algebras of the form $([0, 1], \min, \max, \odot, \rightarrow, 0, 1)$, where $\odot$ is a continuous t-norm and $\rightarrow$ is its residuum [**44**], called usually BL-algebras.

The standard references for the domain of BL-algebras are the monographs [**75**], [**129**].

In this chapter we present some basic definitions and results on BL-algebras and we give more examples.

The MV-center of a BL-algebra, defined by Turunen and Sessa in [**132**], is a very important construction, which associates an MV-algebra with every BL-algebra.

In this way, many properties can be transferred from MV-algebras to BL-algebras and backwards. We shall use more times this construction. We present some results in the more general setting of pseudo BL -algebras [**53**] and new results about the injective BL-algebras: we prove that the complete and divisible MV-algebras are injective objects in the category of BL-algebras.

For a BL-algebra A we denote by Ds(A) the lattice of all deductive systems of A. We put in evidence characterizations for the meet-irreducible elements on Ds(A). Hyperarchimedean BL-algebras, too, are characterized (Corollary 3.55). Also, we prove a Nachbin type theorem for BL-algebras (see Theorem 3.56).

These results are in the general spirit of algebras of logic, as exposed in [**118**].

1. Definitions and first properties. Some examples. Rules of calculus.

DEFINITION 3.1. ([**75**]) A *BL-algebra* is an algebra

$$\mathcal{A} = (A, \wedge, \vee, \odot, \rightarrow, 0, 1)$$

of type $(2, 2, 2, 2, 0, 0)$ satisfying the following:

(BL_1) $(A, \wedge, \vee, 0, 1)$ is a bounded lattice;

(BL_2) $(A, \odot, 1)$ is a commutative monoid;

(BL_3) $\odot$ and $\rightarrow$ form an adjoint pair, i.e. $c \leq a \rightarrow b$ iff $a \odot c \leq b$, for all $a, b, c \in A$;

(BL_4) $a \wedge b = a \odot (a \to b)$;
(BL_5) $(a \to b) \vee (b \to a) = 1$, for all $a, b \in A$.

REMARK 3.1. *$BL-$algebras are exactly the residuated lattices satisfying BL_4, BL_5 (see Definition 1.2).*

In order to simplify the notation, a BL-algebra $\mathcal{A} = (A, \wedge, \vee, \odot, \to, 0, 1)$ will be referred by its support set, A. A BL-algebra is *nontrivial* if $0 \neq 1$.

REMARK 3.2. *For any BL-algebra A, the reduct $L(A) = (A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice. Indeed, let $x, y, z \in A$. First, $x \wedge y, x \wedge z \leq x \wedge (y \vee z)$, therefore, $(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z)$. The converse holds, too as by $BL_4, lr - c_{20}$ and $lr - c_{12}$ (since by Remark 3.1, A is a residuated lattice), we have $x \wedge (y \vee z) = (y \vee z) \odot [(y \vee z) \to x] = \{y \odot [(y \vee z) \to x]\} \vee \{z \odot [(y \vee z) \to x]\} \leq [y \odot (y \to x)] \vee [z \odot (z \to x)] = (y \wedge x) \vee (z \wedge x) = (x \wedge y) \vee (x \wedge z)$.*

A *BL -chain* is a totally ordered BL-algebra, i.e., a BL-algebra such that its lattice order is total.

For any $a \in A$, we define $a^* = a \to 0$ and denote $(a^*)^*$ by a^{**}. Clearly, $0^* = 1$.

We define $a^0 = 1$ and $a^n = a^{n-1} \odot a$ for $n \in N \backslash \{0\}$. The *order* of $a \in A, a \neq 1$, in symbols $ord(a)$ is the smallest $n \in N$ such that $a^n = 0$; if no such n exists, then $ord(a) = \infty$.

A BL-algebra is called *locally finite* if all non unit elements in it have finite order.

EXAMPLE 3.1. *Define on the real unit interval $I = [0, 1]$ the binary operations $\odot$ and $\to$ by*

$$x \odot y = \max\{0, x + y - 1\}$$

$$x \to y = \min\{1, 1 - x + y\}.$$

Then $(I, \leq, \odot, \to, 0, 1)$ is a BL-algebra (called Łukasiewicz structure).

EXAMPLE 3.2. *Define on the real unit interval $I = [0, 1]$*

$$x \odot y = \min\{x, y\}$$

$$x \to y = 1 \text{ iff } x \leq y \text{ and } y \text{ otherwise.}$$

Then $(I, \leq, \odot, \to, 0, 1)$ is a BL-algebra (called Gődel structure).

EXAMPLE 3.3. *Let $\odot$ be the usual multiplication of real numbers on the unit interval $I = [0, 1]$ and $x \to y = 1$ iff $x \leq y$ and y/x otherwise. Then $(I, \leq, \odot, \to, 0, 1)$ is a BL-algebra (called Product structure or Gaines structure).*

REMARK 3.3. *Not every residuated lattice, however, is a BL-algebra (see* [**129**]*, p.16). Consider, for example a residuated lattice defined on the unit interval, for all $x, y, z \in I$, such that*

$$x \odot y = 0, \text{ iff } x + y \leq \frac{1}{2} \text{ and } x \wedge y \text{ elsewhere}$$

$$x \to y = 1 \text{ if } x \leq y \text{ and } \max\{\frac{1}{2} - x, y\} \text{ elsewhere.}$$

Let $0 < y < x$, $x + y < \frac{1}{2}$. Then $y < \frac{1}{2} - x$ and $0 \neq y = x \wedge y$, but $x \odot (x \to y) = x \odot (\frac{1}{2} - x) = 0$. Therefore BL_4 does not hold.

EXAMPLE 3.4. *If* $(A, \wedge, \vee, \rceil, 0, 1)$ *is a Boolean algebra, then* $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ *is a BL-algebra where the operation* $\odot$ *coincide with* $\wedge$ *and* $x \rightarrow y = \rceil x \vee y$, *for all* $x, y \in A$.

EXAMPLE 3.5. *If* $(A, \wedge, \vee, \rightarrow, 0, 1)$ *is a relative Stone lattice (see* [**2**], *p.176), then* $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ *is a BL-algebra where the operation* $\odot$ *coincide with* $\wedge$.

EXAMPLE 3.6. *If* $(A, \oplus, ^*, 0)$ *is an MV-algebra, then* $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ *is a BL-algebra, where for* $x, y \in A$:

$$x \odot y = (x^* \oplus y^*)^*,$$

$$x \rightarrow y = x^* \oplus y, 1 = 0^*,$$

$$x \vee y = (x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x \text{ and } x \wedge y = (x^* \vee y^*)^*.$$

REMARK 3.4. *If in a BL− algebra,* $x^{**} = x$ *for all* $x \in A$, *and for* $x, y \in A$ *we denote* $x \oplus y = (x^* \odot y^*)^*$ *then* $(A, \oplus, ^*, 0)$ *is an MV− algebra.*

REMARK 3.5. *MV−algebras will turn to be particular case of BL−algebras. Indeed, by Theorems 2.66 and 2.69, MV−algebras are residuated lattices where the BL-algebra axioms* BL_4, BL_5 *hold by* $w_\vee, w - c_{16}$ *and* $mv_\wedge$.

EXAMPLE 3.7. *From the logical point of view, the most important example of a BL-algebra is the Lindenbaum-Tarski algebra* L_{BL} *of the propositional Basic Logic BL. The formulas in this logic are built up of denumerable many propositional variables* $v_1, ...v_n$ *with two operations* & *and* $\rightarrow$ *and one constant* $\overline{0}$ *as follows:*

(*i*) *every propositional variable is a formula;*
(*ii*) $\overline{0}$ *is a formula;*
(*iii*) *if* ϕ, ψ *are formulas, then* $\phi\&\psi$ *and* $\phi \rightarrow \psi$ *are formulas.*

Let us denote by Fmla the set of all formulas of BL. Further connectives can be defined:

$$\phi \wedge \psi := \phi\&(\phi \rightarrow \psi),$$

$$\phi \vee \psi := ((\phi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \phi) \rightarrow \phi),$$

$$\rceil\phi := \phi \rightarrow \overline{0},$$

$$\phi \leftrightarrow \psi := (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi),$$

$$1 := \overline{0} \rightarrow \overline{0}.$$

The axioms of a BL are:

(A_1) $(\phi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\phi \rightarrow \chi))$;
(A_2) $(\phi\&\psi) \rightarrow \phi$;
(A_3) $(\phi\&\psi) \rightarrow (\psi\&\phi)$;
(A_4) $(\phi\&(\phi \rightarrow \psi)) \rightarrow (\psi\&(\psi \rightarrow \phi))$;
(A_5) $(\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\phi\&\psi) \rightarrow \chi)$;
(A_6) $((\phi\&\psi) \rightarrow \chi) \rightarrow (\phi \rightarrow (\psi \rightarrow \chi))$;
(A_7) $((\phi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \phi) \rightarrow \chi) \rightarrow \chi)$;
(A_8) $\overline{0} \rightarrow \phi$.

The deduction rule is modus ponens: if ϕ *and* $\phi \rightarrow \psi$ *then* ψ. *We say that* ϕ *is a theorem and we denote by* $\vdash \phi$ *if there is a proof of* ϕ *from* $A_1 - A_8$ *using modus ponens. The completeness theorem for BL says that* $\vdash \phi$ *if and only if* ϕ *is a tautology in every standard BL-algebra* [**44**].

On the set $Fmla$ of all formulas we define the equivalence relation $\equiv$ by:

$$\phi \equiv \psi \text{ iff } \vdash \phi \leftrightarrow \psi.$$

Let us denote by $[\phi]$ the equivalence class of the formula ϕ, and L_{BL} the set of all equivalence classes. We define

$$0 := [\overline{0}],$$
$$1 := [\overline{1}],$$
$$[\phi] \wedge [\psi] := [\phi \wedge \psi],$$
$$[\phi] \vee [\psi] := [\phi \vee \psi],$$
$$[\phi] \odot [\psi] := [\phi \& \psi],$$
$$[\phi] \to [\psi] := [\phi \to \psi].$$

Then $(L_{BL}, \wedge, \vee, \odot, \to, 0, 1)$ is a BL-algebra.

EXAMPLE 3.8. *A product algebra (or P-algebra) ([**75**], [**46**]) is a BL-algebra A satisfying:*

(P_1) $c^{**} \leq (a \odot c \to b \odot c) \to (a \to b)$;
(P_2) $a \wedge a^* = 0$.

*Product algebras are the algebraic counterparts of propositional Product Logic [**75**]. The standard product algebra is the Product structure.*

EXAMPLE 3.9. *A G-algebra ([**75**], Definition 4.2.12) is a BL-algebra A satisfying:*

(G) $a \odot a = a$, *for all* $a \in A$.

G-algebras are the algebraic counterpart of Gödel Logic. The standard G-algebra is the Gödel structure.

EXAMPLE 3.10. *If $(A, \wedge, \vee, \odot, \to, 0, 1)$ is a BL-algebra and X is a nonempty set, then the set A^X becomes a BL-algebra $(A^X, \wedge, \vee, \odot, \to, \underline{0}, \underline{1})$ with the operations defined pointwise. If $f, g \in A^X$, then*

$$(f \wedge g)(x) = f(x) \wedge g(x),$$
$$(f \vee g)(x) = f(x) \vee g(x),$$
$$(f \odot g)(x) = f(x) \odot g(x),$$
$$(f \to g)(x) = f(x) \to g(x)$$

for all $x, y \in X$ and $\underline{0}, \underline{1} : X \to A$ are the constant functions associated with $0, 1 \in A$.

EXAMPLE 3.11. ([**84**])

We give an example of a finite BL-algebra which is not an MV-algebra. Let $A = \{0, a, b, c, 1\}$.

Define on A the following operations:

$\rightarrow$	0	c	a	b	1
0	1	1	1	1	1
c	0	1	1	1	1
a	0	b	1	b	1
b	0	a	a	1	1
1	0	c	a	b	1

,

$\odot$	0	c	a	b	1
0	0	0	0	0	0
c	0	c	c	c	c
a	0	c	a	c	a
b	0	c	c	b	b
1	0	c	a	b	1

.

We have, $0 \leq c \leq a, b \leq 1$, but a, b are incomparable, hence A is not a $BL-$ chain. We remark that $x \odot y = x \wedge y$ for all $x, y \in A$, so $ord(x) = \infty$ for all $x \in A, x \neq 0$. It follows also that $x \odot x = x \wedge x = x$ for all $x \in A$, so A is a G-algebra. It is easy to see that $0^* = 1$ and $x^* = 0$ for all $x \in A, x \neq 0$, so $0^{**} = 0$ and $x^{**} = 1$ for all $x \in A, x \neq 0$. Thus, A is not an $MV-$ algebra.

EXAMPLE 3.12. ([**84**])
We give an example of a finite MV-algebra which is not an MV-chain. The set

$$L_{3\times 2} = \{0, a, b, c, d, 1\} \approx L_3 \times L_2 = \{0, 1, 2\} \times \{0, 1\} =$$

$$= \{(0,0), (0,1), (1,0), (1,1), (2,0), (2,1)\}$$

organized as lattice as in figure

and as $BL-$algebra with the operation $\rightarrow$ and

$$x \odot y = \min\{z : x \leq y \rightarrow z\} = (x \rightarrow y^*)^*, x^* = x \rightarrow 0$$

as in the following tables, is a non-linearly ordered MV -algebra

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	c	c	1	1	1	1
c	b	c	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

,

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

We have in $L_{3\times 2}$ the following operations:

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	a	c	c	1	1
b	b	c	d	1	d	1
c	c	c	1	1	1	1
d	d	1	d	1	d	1
1	1	1	1	1	1	1

,

$*$	0	a	b	c	d	1
	1	d	c	b	a	0

It is easy to see that $0^* = 1, a^* = d, b^* = c, c^* = b, d^* = a, 1^* = 0$ and $x^{**} = x$, for all $x \in A$, hence $L_{3\times 2}$ is an $MV-$ algebra which is not chain.

By Remark 3.1, all properties valid in any residuated lattices hold in any BL-algebras. Using this rules of calculus and axioms BL_4, BL_5 it is eassy to proved that if A is a BL-algebra and $a, a', a_1, ..., a_n, b, b', c, b_i \in A$, $(i \in I)$ we have the following rules of calculus (for more details see [**75**] and [**129**]) :

$(bl - c_1)$ $a \odot b \leq a, b$, hence $a \odot b \leq a \wedge b$ and $a \odot 0 = 0$;
$(bl - c_2)$ $a \leq b$ implies $a \odot c \leq b \odot c$;
$(bl - c_3)$ $a \leq b$ iff $a \to b = 1$;
$(bl - c_4)$ $1 \to a = a, a \to a = 1, a \leq b \to a, a \to 1 = 1$;
$(bl - c_5)$ $a \odot a^* = 0$;
$(bl - c_6)$ $a \odot b = 0$ iff $a \leq b^*$;
$(bl - c_7)$ $a \vee b = 1$ implies $a \odot b = a \wedge b$;
$(bl - c_8)$ $a \to (b \to c) = (a \odot b) \to c = b \to (a \to c)$;
$(bl - c_9)$ $(a \to b) \to (a \to c) = (a \wedge b) \to c$;
$(bl - c_{10})$ $a \to (b \to c) \geq (a \to b) \to (a \to c)$;
$(bl - c_{11})$ $a \leq b$ implies $c \to a \leq c \to b, b \to c \leq a \to c$ and $b^* \leq a^*$;
$(bl - c_{12})$ $a \leq (a \to b) \to b$, $((a \to b) \to b) \to b = a \to b$;
$(bl - c_{13})$ $c \odot (a \vee b) = (c \odot a) \vee (c \odot b)$;
$(bl - c_{14})$ $c \odot (a \wedge b) = (c \odot a) \wedge (c \odot b)$;
$(bl - c_{15})$ $a \vee b = ((a \to b) \to b) \wedge ((b \to a) \to a)$;
$(bl - c_{16})$ $(a \wedge b)^n = a^n \wedge b^n, (a \vee b)^n = a^n \vee b^n$, hence $a \vee b = 1$ implies $a^n \vee b^n = 1$ for any $n \in N$;
$(bl - c_{17})$ $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$;
$(bl - c_{18})$ $(a \wedge b) \to c = (a \to c) \vee (b \to c)$;
$(bl - c_{19})$ $(a \vee b) \to c = (a \to c) \wedge (b \to c)$;
$(bl - c_{20})$ $a \to b \leq (b \to c) \to (a \to c)$;
$(bl - c_{21})$ $a \to b \leq (c \to a) \to (c \to b)$;
$(bl - c_{22})$ $a \to b \leq (a \odot c) \to (b \odot c)$;
$(bl - c_{23})$ $a \odot (b \to c) \leq b \to (a \odot c)$;
$(bl - c_{24})$ $(b \to c) \odot (a \to b) \leq a \to c$;
$(bl - c_{25})$ $(a_1 \to a_2) \odot (a_2 \to a_3) \odot ... \odot (a_{n-1} \to a_n) \leq a_1 \to a_n$;
$(bl - c_{26})$ $a, b \leq c$ and $c \to a = c \to b$ implies $a = b$;
$(bl - c_{27})$ $a \vee (b \odot c) \geq (a \vee b) \odot (a \vee c)$, hence $a^m \vee b^n \geq (a \vee b)^{mn}$, for any $m, n \in N$;
$(bl - c_{28})$ $(a \to b) \odot (a' \to b') \leq (a \vee a') \to (b \vee b')$;
$(bl - c_{29})$ $(a \to b) \odot (a' \to b') \leq (a \wedge a') \to (b \wedge b')$;
$(bl - c_{30})$ $(a \to b) \to c \leq ((b \to a) \to c) \to c$;
$(bl - c_{31})$ $a \odot (\bigwedge_{i \in I} b_i) \leq \bigwedge_{i \in I} (a \odot b_i)$;

$a \odot (\bigvee_{i\in I} b_i) = \bigvee_{i\in I} (a \odot b_i)$;
$a \to (\bigwedge_{i\in I} b_i) = \bigwedge_{i\in I} (a \to b_i)$;
$(\bigvee_{i\in I} b_i) \to a = \bigwedge_{i\in I} (b_i \to a)$;
$\bigvee_{i\in I} (b_i \to a) \leq (\bigwedge_{i\in I} b_i) \to a$;

$\bigvee_{i\in I}(a \to b_i) \leq a \to (\bigvee_{i\in I} b_i)$;
$a \wedge (\bigvee_{i\in I} b_i) = \bigvee_{i\in I} (a \wedge b_i)$; if A is a BL-chain then $a \vee (\bigwedge_{i\in I} b_i) = \bigwedge_{i\in I} (a \vee b_i)$,
(whenever the arbitrary meets and unions exist)

$(bl - c_{32})$ $a \leq a^{**}$, $1^* = 0$, $0^* = 1, a^{***} = a^*, a^{**} \leq a^* \to a$;
$(bl - c_{33})$ $(a \wedge b)^* = a^* \vee b^*$ and $(a \vee b)^* = a^* \wedge b^*$;
$(bl - c_{34})$ $(a \wedge b)^{**} = a^{**} \wedge b^{**}$, $(a \vee b)^{**} = a^{**} \vee b^{**}, (a \odot b)^{**} = a^{**} \odot b^{**}$, $(a \to b)^{**} = a^{**} \to b^{**}$;
$(bl - c_{35})$ $(a^{**} \to a)^* = 0, (a^{**} \to a) \vee a^{**} = 1$;
$(bl - c_{36})$ $a = a^{**} \odot (a^{**} \to a)$;
$(bl - c_{37})$ $a \to b^* = b \to a^* = a^{**} \to b^* = (a \odot b)^*$;
$(bl - c_{38})$ If $a^{**} \leq a^{**} \to a$, then $a^{**} = a$;
$(bl - c_{39})$ $b^* \leq a$ implies $a \to (a \odot b)^{**} = b^{**}$.

Proof. $(bl - c_1)$. See $lr - c_2$.
$(bl - c_2)$. See $lr - c_8$.
$(bl - c_3)$. See $lr - c_4$.
$(bl - c_4)$. See $lr - c_1$.
$(bl - c_5)$. See $lr - c_{15}$.
$(bl - c_6)$. See $lr - c_{15}$.
$(bl - c_7)$. See $lr - c_{28}$.
$(bl - c_8)$. See $lr - c_{13}$.
$(bl - c_9)$. We have $(a \to b) \to (a \to c) \overset{bl-c_8}{=} [(a \to b) \odot a] \to c = (a \wedge b) \to c$.
$(bl - c_{10})$. See $lr - c_{29}$.
$(bl - c_{11})$. See $lr - c_{11}$.
$(bl - c_{12})$. Follows from $a \wedge b \leq b$.
$(bl - c_{13})$. See $lr - c_{20}$.
$(bl - c_{14})$. By $lr - c_{21}$, $c \odot (a \wedge b) \leq (c \odot a) \wedge (c \odot b)$.

Conversely, we prove first that $a \to b = a \to (a \wedge b) : a \wedge b \leq b \Rightarrow a \to (a \wedge b) \leq a \to b$ and $a \to b \leq a \to (a \wedge b) \Leftrightarrow a \odot (a \to b) \leq a \wedge b \Leftrightarrow a \wedge b \leq a \wedge b$.

So, we get $a \to b = a \to (a \wedge b) \overset{bl-c_{22}}{\leq} (c \odot a) \to (c \odot (a \wedge b))$.

Thus $a \to b \leq [(c \odot a) \to (c \odot b)] \to [(c \odot a) \to (c \odot (a \wedge b))]$ and by replacing a by b and b by a we obtain $b \to a \leq [(c \odot b) \to (c \odot a)] \to [(c \odot b) \to (c \odot (b \wedge a))]$. It is eassy to proof that the right term of the last two inequalities are equal (see also, $psbl - c_{23}$) and we denote the common value by x. So, $a \to b \leq x, b \to a \leq x$.

On other side, $(a \to b) \vee (b \to a) \overset{BL_5}{=} 1$, therefore we get $1 \leq x \vee x = x$, hence $x = 1$.

Thus $(c \odot a) \to (c \odot b) \leq (c \odot a) \to (c \odot (a \wedge b)) \Leftrightarrow (c \odot a) \odot [(c \odot a) \to (c \odot b)] \leq c \odot (a \wedge b) \Leftrightarrow (c \odot a) \wedge (c \odot b) \leq c \odot (a \wedge b)$.

$(bl-c_{15})$. Denote $x=((a\to b)\to b)\wedge((b\to a)\to a)$. By BL_4, $a\wedge b=a\odot(a\to b)$ so $a\leq(a\to b)\to b$; from $bl-c_4$, $b\leq(a\to b)\to b$; it follows that $a\vee b\leq(a\to b)\to b$. Analogous, $a\vee b\leq(b\to a)\to a$. Hence $a\vee b\leq((a\to b)\to b)\wedge((b\to a)\to a)$.

We have $x=x\odot 1\overset{BL_5}{=}x\odot[(a\to b)\vee(b\to a)]\overset{bl-c_{13}}{=}[x\odot(a\to b)]\vee[x\odot(b\to a)]$; but $x\odot(a\to b)=[(a\to b)\to b)\wedge((b\to a)\to a]\odot(a\to b)\leq[(a\to b)\to b)]\odot(a\to b)\overset{BL_4}{=}(a\to b)\wedge b\leq b$; similary, $x\odot(b\to a)\leq a$. Hence, $x=[x\odot(a\to b)]\vee[x\odot(b\to a)]\leq b\vee a$. It follows that $a\vee b=x$.

$(bl-c_{16})$. If $a\vee b=1$ then, $a=a\odot 1=a\odot(a\vee b)=(a\odot a)\vee(a\odot b)\leq a^2\vee b$. Hence $a^2\vee b\geq a$. Then $(a^2\vee b)\vee b\geq a\vee b=1$, so $a^2\vee b=1$. Similarly, $b=1\odot b=(a^2\vee b)\odot b=(a^2\odot b)\vee(b\odot b)\leq a^2\vee b^2$. Thus, $a^2\vee b^2\geq b$; hence $a^2\vee(a^2\vee b^2)\geq a^2\vee b=1$, so $a^2\vee b^2=1$.

It follows that $1=a\vee b=a^2\vee b^2=(a^2)^2\vee(b^2)^2=\ldots$. We obtain $a^{2^n}\vee b^{2^n}=1$, for each integer $n\geq 1$. Since $n\leq 2^n$ it follows that $a^n\vee b^n\geq a^{2^n}\vee b^{2^n}=1$, which implies $a^n\vee b^n=1$.

$(bl-c_{17})$. See $lr-c_{22}$.

$(bl-c_{18})$. Let $t\in A$ such that $a\to c, b\to c\leq t$, so $(a\wedge b)\to c$.

Conversely,.....

$(bl-c_{19})$. See $lr-c_{23}$.

$(bl-c_{20})$. See $lr-c_{10}$.

$(bl-c_{21})$. See $lr-c_9$.

$(bl-c_{22})$. See $lr-c_7$.

$(bl-c_{23})$. See $lr-c_{12}$.

$(bl-c_{24})$. We have $(b\to c)\odot(a\to b)\leq a\to c\Leftrightarrow a\odot(b\to c)\odot(a\to b)\leq c\Leftrightarrow(a\wedge b)\odot(b\to c)\leq c$.

But $(a\wedge b)\odot(b\to c)\leq b)\odot(b\to c)=b\wedge c\leq c$.

$(bl-c_{25})$. Similarly with $bl-c_{24}$.

$(bl-c_{26})$. We have $a=a\wedge c=c\odot(c\to a)=c\odot(c\to b)=c\wedge b=b$

$(bl-c_{27})$. See $lr-c_{30}$.

$(bl-c_{28})$. See $lr-c_{31}$.

$(bl-c_{29})$. See $lr-c_{32}$.

$(bl-c_{30})$. We have $[(b\to a)\to c]\odot[(a\to b)\to c]\leq[(b\to a)\to c]\wedge[(a\to b)\to c]\overset{bl-c_{19}}{=}[(b\to a)\vee(a\to b)]\to c=1\to c=c$ so, $(a\to b)\to c\leq((b\to a)\to c)\to c$.

$(bl-c_{31})$. See Theorem 1.3.

$(bl-c_{32})$. See $lr-c_{16}$ and $lr-c_{17}$.

$(bl-c_{33})$. For $(a\vee b)^*=a^*\wedge b^*$, see $lr-c_{28}$. By $lr-c_{27}$, we have $(a\wedge b)^*\geq a^*\vee b^*$.

Conversely, we get that $a\to b=a\to(a\wedge b)\overset{bl-c_{20}}{\leq}(a\wedge b)^*\to a^*$ and $b\to a=b\to(b\wedge a)\overset{bl-c_{20}}{\leq}(a\wedge b)^*\to b^*$, so

$(a\to b)\odot(a\wedge b)^*\leq a^*$ and $(b\to a)\odot(a\wedge b)^*\leq b^*$. It follows that $(a\wedge b)^*=1\odot(a\wedge b)^*=[(a\to b)\vee(b\to a)]\odot(a\wedge b)^*=[(a\to b)\odot(a\wedge b)^*]\vee[(b\to a)\odot(a\wedge b)^*]\leq a^*\vee b^*$.

$(bl-c_{34})$. We prove that $(a\odot b)^{**}=a^{**}\odot b^{**}$.

Since $a\odot b\leq a$, we have that $(a\odot b)^{**}\leq a^{**}$ and $(a\odot b)^{**}=(a\odot b)^{**}\wedge a^{**}=(b\to a^*)^*\wedge a^{**}=a^{**}\odot[a^{**}\to(b\to a^*)^*]=a^{**}\odot[a^{**}\to(b^{**}\to a^*)^*]=a^{**}\odot[(a^{**}\to b^*)\to a^*]=a^{**}\odot[(a^{**}\to b^*)\odot a^{**}]^*=a^{**}\odot(a^{**}\wedge b^*)^*=a^{**}\odot(a^*\vee b^{**})=(a^{**}\odot a^*)\vee(a^{**}\odot b^{**})=0\vee(a^{**}\odot b^{**})=a^{**}\odot b^{**}$.

See also $psbl-c_{52}$.

$(bl-c_{35})$. Since $a^* \leq a^{**} \to a$ we have $(a^{**} \to a)^* \leq a^{**}$. Hence $(a^{**} \to a)^* = a^{**} \wedge (a^{**} \to a)^* = a^{**} \odot [a^{**} \to (a^{**} \to a)^*] = a^{**} \odot [(a^{**} \to a) \odot a^{**}]^* = a^{**} \odot (a^{**} \wedge a)^* = a^{**} \odot a^* = 0$.

We have $[(a^{**} \to a) \to a^{**}] \to a^{**} = [(a^{**} \to a)^{**} \to a^{**}] \to a^{**} = (1 \to a^{**}) \to a^{**} = 1$.

On the other hand, we have $[a^{**} \to (a^{**} \to a)] \to (a^{**} \to a) = [(a^{**} \to (a^{**} \to a)) \odot a^{**}] \to a = [a^{**} \wedge (a^{**} \to a)] \to a = [(a^{**} \to a) \odot ((a^{**} \to a) \to a^{**})] \to a =$

$[(a^{**} \to a) \odot (a^* \to (a^{**} \to a)^*)] \to a = [(a^{**} \to a) \odot a^{**}] \to a = 1$.

So, $(a^{**} \to a) \vee a^{**} = 1$.

$(bl-c_{36})$. We have $a^{**} \odot (a^{**} \to a) = a^{**} \wedge a = a$.

$(bl-c_{37})$. See $lr-c_{19}$.

$(bl-c_{38})$. If $a^{**} \leq a^{**} \to a$, then by $bl-c_{35}$, $a^{**} \to a = (a^{**} \to a) \vee a^{**} = 1$, hence $a^{**} = a$.

$(bl-c_{39})$. Assume that $b^* \leq a$, then by using $bl-c_{37}$, $a \to (a \odot b)^{**} = (a \odot b)^* \to a^* = [(a \to b^*) \odot a]^* = (a \wedge b^*)^* = b^{**}$. ■

REMARK 3.6. *Also, we obtain the rules* $(bl-c_1)-(bl-c_{39})$ *by* $(psbl-c_1)-(psbl-c_{69})$ *if* $x \odot y = y \odot x$, *for all* $x, y \in A$, *that is, the pseudo BL-algebra A is a BL-algebra (see Chapter 5).*

PROPOSITION 3.1. *If in a BL-algebra A,* $z^{**} = z$, *for all* $z \in A$, *then for all* $x, y \in A, x \vee y = (y \to x) \to x$.

Proof. $x \vee y = (x \vee y)^{**} = (x^* \wedge y^*)^* = [x^* \odot (x^* \to y^*)]^* = (x^* \to y^*) \to x^{**} \overset{w-c_{10}}{=} (y \to x) \to x^{**} = (y \to x) \to x$. ■

REMARK 3.7. *By Proposition 3.1, if* $z^{**} = z$, *holds for all* $z \in A$, *then for all* $x, y \in A, (y \to x) \to x = x \vee y = (x \to y) \to y$. *In* [**129**], *MV− algebras where defined in following way: BL− algebras of this kind will turn out to be so called MV− algebras.*

As an immediate consequence of Theorem 2.70 and Proposition 3.1 we obtain (see Remark 1.2):

THEOREM 3.2. *A BL− algebra A is an MV− algebra iff* $x^{**} = x$ *for all* $x \in A$.

For any BL-algebra A, $B(A)$ denotes the Boolean algebra of all complemented elements in $L(A)$ (hence $B(A) = B(L(A))$).

PROPOSITION 3.3. *For* $e \in A$, *the following are equivalent:*

(*i*) $e \in B(A)$;
(*ii*) $e \odot e = e$ *and* $e = e^{**}$;
(*iii*) $e \odot e = e$ *and* $e^* \to e = e$;
(*iv*) $e \vee e^* = 1$.

Proof. $(i) \Rightarrow (ii)$. Suppose that $e \in B(A)$. Then $e \vee a = 1$ and $e \wedge a = 0$, for some $a \in A$. By $bl-c_6$ and $bl-c_7$ we obtain $a \leq e^*$. Moreover $e^* = 1 \odot e^* = (e \vee a) \odot e^* = (e \odot e^*) \vee (a \odot e^*) = 0 \vee (a \odot e^*) = a \odot e^* \leq a$. Hence $e^* \leq a$. Thus, $a = e^*$ is the complement of e. It follows that $e^* \in B(A)$ and, similarly, e^{**} is the complement of e^*. But the complement of e^* is also e. Since $L(A)$ is distributive, we get $e = e^{**}$.

$(ii) \Rightarrow (iii)$. We have that $e \to e^* = e \to (e \to 0) = (e \odot e) \to 0 = e \to 0 = e^*$.

Hence, $e\wedge e^*=e\odot(e\to e^*)=e\odot e^*=0$. Since $e\wedge e^*=e^*\wedge e=e^*\odot(e^*\to e)=0$, by $bl-c_6$ we obtain that $e^*\to e\leq e^{**}=e$. But, by $bl-c_4$, $e\leq e^*\to e$. We have that $e^*\to e=e$.

$(iii)\Rightarrow(iv)$. Applying $bl-c_{15}$, $e\vee e^*=1\Leftrightarrow(e\to e^*)\to e^*=1$ and $(e^*\to e)\to e=1$. By $(iii), e^*\to e=e$, hence $(e^*\to e)\to e=1$. We also have that $e\to e^*=e\to(e\to 0)=(e\odot e)\to 0=e\to 0=e^*$. So, $(e\to e^*)\to e^*=1$.

$(iv)\Rightarrow(i)$. From $e\vee e^*=1$ it follows that, by $bl-c_7, e\wedge e^*=e\odot e^*=0$. Hence e^* is the complement of e. That is, $e\in B(A)$.■

REMARK 3.8. *If $a\in A$, and $e\in B(A)$, then $e\odot a=e\wedge a, a\to e=(a\odot e^*)^*=a^*\vee e$; if $e\leq a\vee a^*$, then $e\odot a\in B(A)$. Indeed, $e\wedge a=e\odot(e\to a)=e\odot e\odot(e\to a)=e\odot(e\wedge a)=(e\odot e)\wedge(e\odot a)=e\wedge(e\odot a)=e\odot a$.*

PROPOSITION 3.4. *For $e\in A$, the following are equivalent:*

(i) $e\in B(A)$;
(ii) $(e\to x)\to e=e$, for every $x\in A$.

Proof. $(i)\Rightarrow(ii)$. If $x\in A$, then from $0\leq x$ we deduce $e^*\leq e\to x$ hence $(e\to x)\to e\leq e^*\to e=e$. Since $e\leq(e\to x)\to e$ we obtain $(e\to x)\to e=e$.

$(ii)\Rightarrow(i)$. If $x\in A$, then from $(e\to x)\to e=e$ we deduce $(e\to x)\odot[(e\to x)\to e]=(e\to x)\odot e$, hence $(e\to x)\wedge e=e\wedge x$. For $x=0$ we obtain that $e^*\wedge e=0$. Also, from hypothesis (for $x=0$) we obtain $e^*\to e=e$. So, from $bl-c_{15}$ we obtain

$$\begin{aligned}
e\vee e^* &= [(e\to e^*)\to e^*]\wedge[(e^*\to e)\to e]\\
&= [(e\to e^*)\to e^*]\wedge(e\to e)\\
&= [(e\to e^*)\to e^*]\wedge 1\\
&= (e\to e^*)\to e^*\\
&= [e\odot(e\to e^*)]^* \text{ (by } bl-c_{37})\\
&= (e\wedge e^*)^*=0^*=1,
\end{aligned}$$

hence $e\in B(A)$.■

REMARK 3.9. *If $L_{3\times 2}$ is the $MV-$ algebra from Example 3.12, we remark that $0\oplus 0=0, a\oplus a=a, c\oplus c=1\neq c, b\oplus b=d\neq b, d\oplus d=d$ and $1\oplus 1=1$, hence $B(A)=\{0,a,d,1\}$.*

LEMMA 3.5. *If $e,f\in B(A)$ and $x,y\in A$, then:*

$(bl-c_{40})$ $e\vee(x\odot y)=(e\vee x)\odot(e\vee y)$;
$(bl-c_{41})$ $e\wedge(x\odot y)=(e\wedge x)\odot(e\wedge y)$;
$(bl-c_{42})$ $e\odot(x\to y)=e\odot[(e\odot x)\to(e\odot y)]$;
$(bl-c_{43})$ $x\odot(e\to f)=x\odot[(x\odot e)\to(x\odot f)]$;
$(bl-c_{44})$ $e\to(x\to y)=(e\to x)\to(e\to y)$.

Proof. $(bl-c_{40})$. We have

$$(e\vee x)\odot(e\vee y)\overset{bl-c_{13}}{=}[(e\vee x)\odot e]\vee[(e\vee x)\odot y]\overset{bl-c_{13}}{=}[(e\vee x)\odot e]\vee[(e\odot y)\vee(x\odot y)]$$

$$=[(e\vee x)\wedge e]\vee[(e\odot y)\vee(x\odot y)]=e\vee(e\odot y)\vee(x\odot y)=e\vee(x\odot y).$$

$(bl-c_{41})$. We have

$$(e\wedge x)\odot(e\wedge y)=(e\odot x)\odot(e\odot y)=(e\odot e)\odot(x\odot y)=e\odot(x\odot y)=e\wedge(x\odot y).$$

$(bl-c_{42})$. By $bl-c_{22}$ we have $x\to y\leq(e\odot x)\to(e\odot y)$, hence $e\odot(x\to y)\leq e\odot[(e\odot x)\to(e\odot y)]$. Conversely, $e\odot[(e\odot x)\to(e\odot y)]\leq e$ and $(e\odot x)\odot[(e\odot x)\to(e\odot y)]\leq e\odot y\leq y$ so $e\odot[(e\odot x)\to(e\odot y)]\leq x\to y$. Hence $e\odot[(e\odot x)\to(e\odot y)]\leq e\odot(x\to y)$.

$(bl-c_{43})$. We have

$$x\odot[(x\odot e)\to(x\odot f)]=x\odot[(x\odot e)\to(x\wedge f)]\overset{bl-c_{31}}{=}$$

$$\overset{bl-c_{31}}{=}x\odot[((x\odot e)\to x)\wedge((x\odot e)\to f)]=$$

$$=x\odot[1\wedge((x\odot e)\to f)]=x\odot((x\odot e)\to f)$$

$$\overset{bl-c_{8}}{=}x\odot[x\to(e\to f)]=x\wedge(e\to f)=x\odot(e\to f).$$

$(bl-c_{44})$. Follows from $bl-c_8$ and $bl-c_9$ since $e\wedge x=e\odot x$.■

LEMMA 3.6. *If a,b,x are elements of A and $a,b\leq x$ then*

$(bl-c_{45})$ $a\odot(x\to b)=b\odot(x\to a)$.

Proof. We have

$$a\odot(x\to b)=(x\wedge a)\odot(x\to b)=[x\odot(x\to a)]\odot(x\to b)$$

$$=[x\odot(x\to b)]\odot(x\to a)=(x\wedge b)\odot(x\to a)=b\odot(x\to a).\blacksquare$$

PROPOSITION 3.7. *For a BL- algebra $(A,\wedge,\vee,\odot,\to,0,1)$ the following are equivalent:*

(i) $(A,\to,1)$ *is a Hilbert algebra;*
(ii) $(A,\wedge,\vee,\to,0,1)$ *is a relative Stone lattice.*

Proof. $(i)\Rightarrow(ii)$. Suppose that $(A,\to,1)$ is a Hilbert algebra (see Definition 1.7), then for every $x,y,z\in A$ we have

$$x\to(y\to z)=(x\to y)\to(x\to z)$$

From $bl-c_8$ and $bl-c_9$ we have

$$x\to(y\to z)=(x\odot y)\to z$$

and

$$(x\to y)\to(x\to z)=(x\wedge y)\to z,$$

so we obtain

$$(x\odot y)\to z=(x\wedge y)\to z$$

hence $x\odot y=x\wedge y$, that is $(A,\wedge,\vee,\to,0,1)$ is a relative Stone lattice.

$(ii)\Rightarrow(i)$. If $(A,\wedge,\vee,\to,0,1)$ is a relative Stone lattice, then $(A,\wedge,\vee,\to,0,1)$ is a Heyting algebra, so by Remark 1.8, $(A,\to,1)$ is a Hilbert algebra. ■

DEFINITION 3.2. Let A and B be $BL-$algebras. A function $f:A\to B$ is a *morphism of $BL-$algebras* iff it satisfies the following conditions, for every $x,y\in A$:

(BL_6) $f(0_A)=0_B$;
(BL_7) $f(x\odot y)=f(x)\odot f(y)$;
(BL_8) $f(x\to y)=f(x)\to f(y)$.

REMARK 3.10. *If $f : A \to B$ is a morphism of $BL-$algebras then for every $x, y \in A$,*

$$f(x^*) = [f(x)]^*, f(1_A) = 1_B,$$
$$f(x \oplus y) = f(x) \oplus f(y), \text{ where } x \oplus y = (x^* \odot y^*)^*,$$
$$\text{if } x \leq y \text{ then } f(x) \leq f(y),$$
$$f(x \wedge y) = f(x) \wedge f(y), f(x \vee y) = f(x) \vee f(y).$$

Indeed, using $BL_6 - BL_8$ we obtain: $f(x^*) = f(x \to 0_A) = f(x) \to f(0_A) = f(x) \to 0_B = [f(x)]^*$; $f(1_A) = f(0_A \to 0_A) = f(0_A) \to f(0_A) = 0_B \to 0_B = 1_B$; $f(x \oplus y) = f((x^* \odot y^*)^*) = [f(x^* \odot y^*)]^* = [f(x^*) \odot f(y^*)]^* = [f(x)^* \odot f(y)^*]^* = f(x) \oplus f(y)$; if $x \leq y$ then $x \to y = 1_A$, thus $f(x \to y) = f(1_A) \Leftrightarrow f(x) \to f(y) = 1_B$, hence $f(x) \leq f(y)$; $f(x \wedge y) = f(x \odot (x \to y)) = f(x) \odot f(x \to y) = f(x) \odot [f(x) \to f(y)] = f(x) \wedge f(y)$; $f(x \vee y) = f(((x \to y) \to y) \wedge ((y \to x) \to x)) = f((x \to y) \to y) \wedge f((y \to x) \to x) = [(\ f(x) \to f(y)) \to f(y)] \wedge [(f(y) \to f(x)) \to f(x)] = f(x) \vee f(y)$.

We deduce that every morphism of BL -algebras is a morphism of $MV-$ algebras (see Definition 2.3).

We shall denote by $\mathcal{BL}$ the category whose objects are nontrivial $BL-$ algebras and whose morphisms are $BL-$ morphisms. Clearly, the category $\mathcal{MV}$ of $MV-$ algebras is a subcategory of $\mathcal{BL}$.

2. Injective objects in the BL-algebras category.

The first aim of this Subsection is to present the MV-center of a BL-algebra, defined by Turunen and Sessa in [**132**]. This is a very important construction, which associates an MV-algebra with every BL-algebra. In this way, many properties can be transferred from MV-algebras to BL-algebras and backwards. We shall use more times this construction. We present some known results, which can be found in [**70**] and we also prove some new ones.

The second one was to present some results about the injective BL-algebras.

We recall that an MV-algebra is called complete if it contains the greatest lower bound and the lowest upper bound of any subset. In [**129**] the injective MV-algebras are characterized: In fact, an MV-algebra A is injective if and only if A is complete and divisible, i.e. for any $a \in A$ and for any natural number $n \geq 1$ there is $x \in A$, called the n-divisor of a, such that $nx = a$ and $a^* \oplus [(n-1)x] = x^*$. It is also known that all injective MV-algebras are either isomorphic to Łukasiewicz structure or, more generally, isomorphic to retracts of power of Łukasiewicz structure.

In [**30**], we prove that the complete and divisible MV-algebras are injective objects in the category of BL-algebras (see also [**63**]).

2.0.1. *MV-center of a BL-algebra.* As we saw in Example 3.6, MV-algebras are BL-algebras, and more, a BL-algebra A is an MV-algebra iff $a^{**} = a$ for every $a \in A$.

The *MV-center* of a A, denoted by $MV(A)$ is defined as

$$MV(A) = \{a \in A : a^{**} = a\} = \{a^* : a \in A\}.$$

Hence, a BL-algebra A is an MV-algebra iff $A = MV(A)$.

By Proposition 3.3 follows that $B(A) \subseteq MV(A)$.

EXAMPLE 3.13. ([**132**]) *If A is a product algebra or a G-algebra, then $MV(A)$ is a Boolean algebra. If A is the Product structure or the Gődel structure, then $MV(A) = \{0, 1\}$.*

EXAMPLE 3.14. *If A is the 5-element BL-algebra from Example 3.11, $MV(A) = \{0, 1\}$.*

PROPOSITION 3.8. ([**132**], *Theorem 2*) *Let A be a BL-algebra and let us define for all $a, b \in A, a^* \oplus b^* = (a \odot b)^*$. Then*

(*i*) *$(MV(A), \oplus, ^*, 0)$ is an MV-algebra;*

(*ii*) *the order $\leq$ of A agrees with the one of $MV(A)$, defined by*

$$a \leq_{MV} b \text{ iff } a^* \oplus b = 1, \text{ for all } a, b \in MV(A);$$

(*iii*) *the residuum $\rightarrow$ of A coincides with the residuum $\rightarrow_{MV}$ in $MV(A)$, defined by*

$$a \rightarrow_{MV} b = a^* \oplus b, \text{ for all } a, b \in MV(A);$$

(*iv*) *the product $\odot_{MV}$ on $MV(A)$ is such that*

$$a \odot_{MV} b = (a \odot b)^{**} = a \odot b, \text{ for all } a, b \in MV(A);$$

(*v*) *$MV(A)$ is the largest $MV-$ subalgebra of A.*

PROPOSITION 3.9. *$B(A) = B(MV(A))$.*

Proof. Applying Proposition 3.3, (*ii*), we get that $B(A) = MV(A) \cap \{a \in A : a \odot a = a\} = B(MV(A))$, following Theorem 2.8. ■

2.1. Reflexive subcategories.

REMARK 3.11. ([**2**], *p.31*) *Since the categories $\mathcal{MV}$ and $\mathcal{BL}$ are equational, then in these categories the monomorphisms are exactly the one-one morphisms.*

DEFINITION 3.3. ([**2**], p.27) A subcategory $\mathcal{B}$ of category $\mathcal{A}$ is *reflective* if there is a functor $\mathcal{R} : \mathcal{A} \rightarrow \mathcal{B}$ called *reflector*, such that for each $A \in Ob(\mathcal{A})$, there exists a morphism $\Phi_{\mathcal{R}}(A) : A \rightarrow \mathcal{R}(A)$ of $\mathcal{A}$ with the following properties:

(R_1) If $f \in Hom_{\mathcal{A}}(A, A')$, then $\Phi_{\mathcal{R}}(A') \circ f = \mathcal{R}(f) \circ \Phi_{\mathcal{R}}(A)$, that is the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ \downarrow_{\Phi_{\mathcal{R}}(A)} & & \downarrow_{\Phi_{\mathcal{R}}(A')} \\ \mathcal{R}(A) & \xrightarrow{\mathcal{R}(f)} & \mathcal{R}(A') \end{array}$$

is commutative,

(R_2) If $B \in Ob(\mathcal{B})$, and $f \in Hom_{\mathcal{A}}(A, B)$, then there exists a unique morphism $f' \in Hom_{\mathcal{B}}(\mathcal{R}(A), B)$, such that $f' \circ \Phi_{\mathcal{R}}(A) = f$, that is the diagram

$$\begin{array}{ccc} A & \xrightarrow{\Phi_{\mathcal{R}}(A)} & \mathcal{R}(A) \\ & \searrow^{f} \quad \swarrow_{f'} & \\ & B & \end{array}$$

is commutative.

THEOREM 3.10. ([**2**], *p.29*) *Suppose $\mathcal{R} : \mathcal{A} \rightarrow \mathcal{B}$ is a reflector. Then $\mathcal{R}$ preserves inductive limits of partially ordered systems.*

THEOREM 3.11. ([**2**], *p.30*) *Suppose $\mathcal{R} : \mathcal{A} \rightarrow \mathcal{B}$ is a reflector which preserves monomorphisms. If B is an injective object in $\mathcal{B}$, then it is also injective in $\mathcal{A}$.*

THEOREM 3.12. *The category $\mathcal{MV}$ of MV-algebras is a reflective subcategory of the category $\mathcal{BL}$ of BL-algebras and the reflector $\mathcal{R} : \mathcal{BL} \to \mathcal{MV}$ preserves monomorphisms.*

Proof. Let $(A, \wedge, \vee, \odot, \to, 0, 1) \in Ob(\mathcal{BL})$ and define

$$\mathcal{R}(A) = MV(A) = \{x^* : x \in A\} = \{x \in A : x^{**} = x\}.$$

By Proposition 3.8, $(\mathcal{R}(A), \wedge, \vee, \oplus, ^*, 0)$ is the greatest MV-subalgebra of A via the operations:

$$x^* \oplus y^* = x^{**} \to y^* = (x \odot y)^* = (x^{**} \odot y^{**})^*,$$

$$x^* \vee y^* = (x^* \to y^*) \to y^* = (y^* \to x^*) \to x^*,$$

and

$$x^* \wedge y^* = (x^{**} \vee y^{**})^*.$$

We define $\Phi_{\mathcal{R}}(A) : A \to \mathcal{R}(A)$ by $\Phi_{\mathcal{R}}(A)(x) = x^{**}$, for every $x \in A$.

By $bl - c_{34}$ we deduce that $\Phi_{\mathcal{R}}(A)$ is a morphism in $\mathcal{BL}$.

If $A, A' \in Ob(\mathcal{BL})$ and $f \in Hom_{\mathcal{BL}}(A, A')$, then $\mathcal{R}(f) : \mathcal{R}(A) \to \mathcal{R}(A')$ defined by $\mathcal{R}(f)(x^*) = f(x^*) = (f(x))^*$ for every $x \in A$ is a morphism in $\mathcal{MV}$.

Indeed, if $x, y \in A$, then $\mathcal{R}(f)(x^* \oplus y^*) = \mathcal{R}(f)((x \odot y)^*) = (f(x \odot y))^* = (f(x) \odot f(y))^* = f(x)^* \oplus f(y)^* = (\mathcal{R}(f)(x^*)) \oplus (\mathcal{R}(f)(y^*))$ and $\mathcal{R}(f)((x^*)^*) = (f(x^*))^* = (f(x)^*)^* = (\mathcal{R}(f)(x))^*$.

So, we obtain a functor $\mathcal{R} : \mathcal{BL} \to \mathcal{MV}$.

To prove $\mathcal{R}$ is a reflector, we consider the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ \downarrow_{\Phi_{\mathcal{R}}(A)} & & \downarrow_{\Phi_{\mathcal{R}}(A')} \\ \mathcal{R}(A) & \xrightarrow{\mathcal{R}(f)} & \mathcal{R}(A') \end{array}$$

with $A, A' \in Ob(\mathcal{BL})$.

If $x \in A$, then $(\Phi_{\mathcal{R}}(A') \circ f)(x) = \Phi_{\mathcal{R}}(A')(f(x)) = (f(x))^{**}$ and $(\mathcal{R}(f) \circ \Phi_{\mathcal{R}}(A))(x) = \mathcal{R}(f)(\Phi_{\mathcal{R}}(A)(x)) = \mathcal{R}(f)(x^{**}) = (f(x^*))^* = (f(x)^*)^* = (f(x))^{**}$, hence $\Phi_{\mathcal{R}}(A') \circ f = \mathcal{R}(f) \circ \Phi_{\mathcal{R}}(A)$, that is the above diagram is commutative.

Let now $A \in Ob(\mathcal{BL})$, $M \in Ob(\mathcal{MV})$ and $f : A \to M$ a morphism in $\mathcal{BL}$.

$$\begin{array}{ccc} A & \xrightarrow{\Phi_{\mathcal{R}}(A)} & \mathcal{R}(A) \\ & \searrow^{f} \quad \swarrow_{f'} & \\ & M & \end{array}$$

For $x \in A$, we define $f'(x^*) = f(x^*) = f(x)^*$ (hence $f' = f_{|\mathcal{R}(A)}$).

For $x, y \in A$, we have $f'(x^* \oplus y^*) = f'((x \odot y)^*) = (f(x \odot y))^* = (f(x) \odot f(y))^* = f(x)^* \oplus f(y)^*$, $f((x^*)^*) = f(x)^{**} = f(x) = (f(x^*))^*$ and $f'(0) = f'(1^*) = f(1)^* = 1^* = 0$, hence f' is an morphism in $\mathcal{MV}$. Since $(f' \circ \Phi_{\mathcal{R}}(A))(x) = f'(\Phi_{\mathcal{R}}(A)(x)) = f'(x^{**}) = f(x)^{**} = f(x)$, we deduce that $f' \circ \Phi_{\mathcal{R}}(A) = f$.

If we have again $f'' : \mathcal{R}(A) \to M$ a morphism in $\mathcal{MV}$ such that $f'' \circ \Phi_{\mathcal{R}}(A) = f$, then for any $x \in A$, $(f'' \circ \Phi_{\mathcal{R}}(A))(x^*) = f(x^*)$, hence $f''(x^*) = f(x^*) = f'(x^*)$, so $f'' = f'$.

Let now $f : A \to A'$ a monomorphism in $\mathcal{BL}$ and $x, y \in A$ such that $\mathcal{R}(f)(x^*) = \mathcal{R}(f)(y^*)$.

Then $f(x^*) = f(y^*)$, hence $x^* = y^*$, that is $\mathcal{R}(f)$ is a monomorphism in $\mathcal{MV}$ (by Remark 3.11). ■

We recall that an *MV-algebra* is called *complete* if it contains the greatest lower bound and the lowest upper bound of any subset.

DEFINITION 3.4. An MV-algebra A is called *divisible* if for any $a \in A$ and for any natural number $n \geq 1$ there is $x \in A$ such that $nx = a$ and $a^* \oplus [(n-1)x] = x^*$.

In [**63**] and [**129**], p.66, it is proved:

THEOREM 3.13. *For any MV-algebra A the next assertions are equivalent:*

(*i*) *A is injective object in the category* $\mathcal{MV}$,
(*ii*) *A is complete and divisible MV-algebra.*

THEOREM 3.14. *If A is a complete and divisible MV-algebra, then A is an injective object in the category* $\mathcal{BL}$.

Proof. By Theorem 3.13, A is an injective object in the category $\mathcal{MV}$. Since $\mathcal{MV}$ is reflective subcategory of $\mathcal{BL}$ and the reflector $R : \mathcal{BL} \to \mathcal{MV}$ preserves monomorphisms (by Theorem 3.12), then by Theorem 3.11 we deduce that A is injective object in the category $\mathcal{BL}$. ■

3. The lattice of deductive systems of a BL-algebra

For a BL-algebra A we denote by Ds(A) the lattice of all deductive systems of A. We put in evidence characterizations for the meet-irreducible elements on Ds(A). Hyperarchimedean BL-algebras, too, are characterized (Corollary 3.55). We also prove some results relative to the lattice of deductive systems of a BL-algebra (Theorem 3.20 characterizes the BL-algebras for which the lattice of deductive systems is a Boolean lattice) and we put in evidence characterizations for prime and completely meet-irreducible deductive systems of a BL-algebra (see Proposition 3.31, Corollary 3.33, Theorem 3.34, Theorem 3.39, Theorem 3.40 and Corollary 3.41).

Also we introduce the notions of archimedean and hyperarchimedean BL-algebra and we prove a Nachbin type theorem for BL-algebras (see Theorem 3.56).

3.1. The lattice of deductive systems of a BL-algebra. As in the case of residuated lattices (see Definition 1.8) we have:

DEFINITION 3.5. A non empty subset $D \subseteq A$ is a *deductive system* of A, **ds** for short, if the following conditions are satisfied:

$(bl - Ds_1)$ $1 \in D$;
$(bl - Ds_2)$ If $x, x \to y \in D$, then $y \in D$.

Clearly $\{1\}$ and A are **ds**; a **ds** of A is called *proper* if $D \neq A$.

REMARK 3.12. *A* ***ds*** *D is proper iff* $0 \notin D$ *iff no element* $a \in A$ *holds* $a, a^* \in D$.

REMARK 3.13. *In* [**130**] *it is proved that a non empty subset* $D \subseteq A$ *is a* ***ds*** *of A, iff D is a filter of A (i.e. for all* $a, b \in A$*:*

$(bl - Ds'_1)$ $a, b \in D$ *implies* $a \odot b \in D$;
$(bl - Ds'_2)$ $a \in D$ *and* $a \leq b$ *implies* $b \in D$).

REMARK 3.14. *A deductive system $D \subseteq A$ is a lattice filter of A. Indeed, let $a, b \in D$. Since $a \to [b \to (a \odot b)] = 1 \in D$, we have $b \to (a \odot b) \in D$, and, moreover, $a \odot b \in D$. Now $a \odot b \leq a \odot 1 = a, a \odot b \leq 1 \odot b = b$, hence $a \odot b \leq a \wedge b$, so $a \wedge b \in D$. Convesely, if $a \wedge b \in D$ then $a, b \in D$ as $a \wedge b \leq a, b$. Thus, D is a lattice filter of A.*

Deductive systems are called also implicative filters in literature.

We denote by $Ds(A)$ the set of all deductive systems of A.

For a nonempty subset $M \subseteq A$ we denote by $[M)$ the *ds of A generated by M* (that is, $[M) = \cap\{D \in Ds(A) : M \subseteq D\}$).

If $M = \{a\}$ with $a \in A$, we denote by $[a)$ the **ds** generated by $\{a\}$ ($[a)$ is called *principal*).

For $D \in Ds(A)$ and $a \in A \backslash D$, we denote by $D(a) = [D \cup \{a\})$.

PROPOSITION 3.15. *(i) If $M \subseteq A$ is a nonempty subset of A, then:*

$$[M) = \{a \in A : x_1 \odot ... \odot x_n \leq a, \text{ for some } x_1, ..., x_n \in M\}.$$

In particular, for $a \in A$,

$$[a) = \{x \in A : x \geq a^n, \text{ for some } n \in N\};$$

(ii) If $D \in Ds(A)$ and $a \in A \backslash D$, then

$$D(a) = \{x \in A : x \geq y \odot a^n, \text{ with } y \in D \text{ and } n \in N\};$$

(iii) If $x, y \in A$, and $x \leq y$, then $[y) \subseteq [x)$;

(iv) If $x, y \in A$, then $[x) \cap [y) = [x \vee y)$.

Proof. $(i), (ii)$. As in the case of residuated lattices (see the proof of Proposition 1.29).

(iii). Let $z \in [y)$. Then there is $n \geq 1$ such that $z \geq y^n \geq x^n$, hence $z \in [x)$.

(iv). As in the case of residuated lattices (see the proof of Proposition 1.32, (iii)). ■

REMARK 3.15. *If $e, f \in B(A)$, then $[e) = \{x \in A : e \leq x\}$ and $[e) = [f)$ iff $e = f$. Indeed, by Proposition 3.15, $[e) = \{x \in A : x \geq e^n = e$, for some $n \in N\} = \{x \in A : e \leq x\}$; if $[e) = [f)$, then $e \in [f)$, so $e \geq f$ and $f \in [e)$, so $f \geq e$. We deduce that $e = f$.*

EXAMPLE 3.15. *Let A be the $BL-$algebra from Example 3.11. Then $[a) = \{a, 1\}, [b) = \{b, 1\}$ and $[c) = \{a, b, c, 1\}$.*

REMARK 3.16. ([**129**], *p.17*) *If $D \in Ds(A)$ and $a \in A$, then $a \in D$ iff $a^n \in D$, for any $n \in N$.*

For D_1, $D_2 \in Ds(A)$ we put $D_1 \wedge D_2 = D_1 \cap D_2$ and $D_1 \vee D_2 = [D_1 \cup D_2) = \{a \in A : a \geq x \odot y$, for some $x \in D_1$ and $y \in D_2\}$.

Then $(Ds(A), \wedge, \vee, \{1\}, A)$ is a *complete Brouwerian lattice.*

PROPOSITION 3.16. *The lattice $(Ds(A), \subseteq)$ is an algebraic lattice (see Definition 1.9).*

Proof. See the proof of Proposition 1.33. ■

LEMMA 3.17. *If $x, y \in A$, then $[x) \vee [y) = [x \odot y)$.*

Proof. Since $x \odot y \leq x, y$, then $[x), [y) \subseteq [x \odot y)$, hence $[x) \vee [y) \subseteq [x \odot y)$.

If $z \in [x \odot y)$, then for some natural number n, $z \geq (x \odot y)^n = x^n \odot y^n \in [x) \vee [y)$, hence $z \in [x) \vee [y)$, that is $[x \odot y) \subseteq [x) \vee [y)$, so $[x) \vee [y) = [x \odot y)$.■

For D_1, $D_2 \in Ds(A)$ we put

$$D_1 \to D_2 = \{a \in A : D_1 \cap [a) \subseteq D_2\}.$$

LEMMA 3.18. *If D_1, $D_2 \in Ds(A)$ then*

(*i*) $D_1 \to D_2 \in Ds(A)$;

(*ii*) *If $D \in Ds(A)$, then $D_1 \cap D \subseteq D_2$ iff $D \subseteq D_1 \to D_2$ (that is, $D_1 \to D_2 = \sup\{D \in Ds(A) : D_1 \cap D \subseteq D_2\}$).*

Proof. See the case of residuated lattices (Lemma 1.35). ■

REMARK 3.17. *From Lemma 3.18 we deduce that $(Ds(A), \vee, \wedge, \to, \{1\})$ is a Heyting algebra; for $D \in Ds(A)$,*

$$D^* = D \to 0 = D \to \{1\} = \{x \in A : [x) \cap D = \{1\}\}$$

and so, for $a \in A$,

$$[a)^* = \{x \in A : [x) \cap [a) = \{1\}\} =$$
$$= \{x \in A : [x \vee a) = \{1\}\} = \{x \in A : x \vee a = 1\}.$$

PROPOSITION 3.19. *If $x, y \in A$, then $[x \odot y)^* = [x)^* \cap [y)^*$.*

Proof. If $a \in [x \odot y)^*$, then $a \vee (x \odot y) = 1$. Since $x \odot y \leq x, y$ then $a \vee x = a \vee y = 1$, hence $a \in [x)^* \cap [y)^*$, that is $[x \odot y)^* \subseteq [x)^* \cap [y)^*$.

Let now $a \in [x)^* \cap [y)^*$, that is $a \vee x = a \vee y = 1$.

By $bl - c_{27}$ we deduce $a \vee (x \odot y) \geq (a \vee x) \odot (a \vee y) = 1$, hence $a \vee (x \odot y) = 1$, that is $a \in [x \odot y)^*$.

It follows that $[x)^* \cap [y)^* \subseteq [x \odot y)^*$, hence $[x \odot y)^* = [x)^* \cap [y)^*$. ■

As in the case of residuated lattices (see Theorem 1.39) we have:

THEOREM 3.20. *If A is a BL-algebra, then the following assertions are equivalent:*

(*i*) *$(Ds(A), \vee, \wedge, ^*, \{1\}, A)$ is a Boolean algebra,*

(*ii*) *Every **ds** of A is principal and for every $x \in A$, there is $n \in N$ such that $x \vee (x^n)^* = 1$.*

3.2. The spectrum of a BL-algebra. For the lattice $Ds(A)$ (which is distributive) we denote by $Spec(A)$ the set of all (finitely) meet-irreducible (hence meet-prime) elements ($Spec(A)$ is called the *spectrum* of A) and by $Irc(A)$ the set of all (completely) meet-irreducible elements of the lattice $Ds(A)$ (see Definition 1.10).

DEFINITION 3.6. ([**129**], p.18) A proper **ds** D of A is called *prime* if, for any $a, b \in A$, the condition $a \vee b \in D$ implies $a \in D$ or $b \in D$.

THEOREM 3.21. *A non-degenerate BL-algebra contains a prime **ds**.*

Proof. See [**129**], p.18, Theorem 1. ■

EXAMPLE 3.16. *Let A be the $BL-$algebra from Example 3.11. Then the **ds** of A are $\{1\}, \{a, 1\}, \{b, 1\}, \{a, b, c, 1\}$ and A. Since $\odot = \wedge$ the **ds** of A coincide with the filters of the associated lattice $L(A)$. It is eassy to see that A has three prime filters $\{a, 1\}, \{b, 1\}, \{a, b, c, 1\}$.*

A congruence relation on a $BL-$ algebra A is an equivalence relation with respect to the operations $\odot, \to, \vee$ and $\wedge$.

In [**129**], (p.21, Propositions 26, 27 and 28) it is proved that there is one-to-one correspondence between deductive systems of A and congruence relations on A.

Indeed, if $\sim$ is a congruence relation on a $BL-$ algebra A then $D = \{a \in A : a \sim 1\}$ is a deductive system of A. Conversly, if D is a deductive system of A then by defining $x \sim_D y$ iff $(x \to y) \odot (y \to x) \in D$, we obtain a congruence relations on A.

Starting from a **ds** D, the quotient algebra A/D becomes a BL-algebra with the natural operations induced from those of A. We let x/D be the congruence class of x modulo $\equiv_D$, $x \in A$.

In [**75**] it is proved the following result:

PROPOSITION 3.22. *Let D be a ds of A and x, $y \in A$.*

(i) $x/D = 1/D$ *iff* $x \in D$;
(ii) $x/D = 0/D$ *iff* $x^* \in D$;
(iii) if D is proper and $x/D = 0/D$ then $x \notin D$;
(iv) $x/D \leq y/D$ *iff* $x \to y \in D$;
*(v) A/D is $BL-$ chain iff D is a prime **ds** of A.*

THEOREM 3.23. ([**53**],[**54**],[**129**]) *For a proper $P \in Ds(A)$ the following are equivalent:*

(i) P is prime;
(ii) For all $a, b \in A$, $a \to b \in P$ or $b \to a \in P$;
(iii) A/P is a chain.

Proof. First we prove the equivalence $(i) \Leftrightarrow (ii)$.

$(i) \Rightarrow (ii)$. Since for all $a, b \in A$, $(a \to b) \vee (b \to a) = 1 \in P$, we have that either $a \to b \in P$ or $b \to a \in P$.

$(ii) \Rightarrow (i)$. Assume either $a \to b \in P$ or $b \to a \in P$, holds for all $a, b \in A$. Let $a \vee b \in P$, and say, $a \to b \in P$. Now $a \vee b \leq (a \to b) \to b \in P$, therefore $b \in P$. Thus, P is prime.

$(i) \Rightarrow (iii)$. If P is prime then by equivalence $(i) \Leftrightarrow (ii)$, for all $a, b \in A$, $a \to b \in P$ or $b \to a \in P$. Thus $a/P \leq b/P$ or $b/P \leq a/P$, so A/P is a chain.

$(i) \Rightarrow (iii)$. If A/P is chain, then for all $a, b \in A$, either $a/P \leq b/P$ or $b/P \leq a/P$, whence either $a \to b \in P$ or $b \to a \in P$. Thus, by equivalence $(i) \Leftrightarrow (ii)$, P is prime. ■

THEOREM 3.24. *A is a $BL-$ chain iff then any proper **ds** of A is a prime **ds** of A.*

Proof. Assume first that A is a $BL-$ chain and let D a proper **ds** of A. Then for all $a, b \in A$, $a \vee b = a$ or $a \vee b = b$. Thus $a \vee b \in D$ iff $a \in D$ or $b \in D$ and so each **ds** D of A is prime. Conversely, if each **ds** D of A is prime, then, in particular, $\{1\}$ is prime and as, for all $a, b \in A$, $(a \to b) \vee (b \to a) = 1$, either $a \to b = 1$ or $b \to a = 1$, that is, either $a \leq b$ or $b \leq a$, whence A is a $BL-$ chain.■

THEOREM 3.25. *If P is a prime **ds** of A and D is a proper **ds** of A such that $P \subseteq D$, then also D is prime.*

Proof. Assume $a, b \in A$ such that $a \vee b \in D$. Since P is prime either $a \to b \in P$ or $b \to a \in P$. Assume that $a \to b \in P$. Then $a \to b \in D$. Since $a \vee b \leq (a \to b) \to b \in D$

we deduce that $b \in D$. Analogously, the condition $b \to a \in P$ implies $a \in D$, thus D is prime. ■

THEOREM 3.26. *The set of proper deductive systems including a given prime* ***ds*** *P of A is totally ordered with respect the inclusion.*

Proof. Let D, D' two proper **ds** containing P such that $D \nsubseteq D'$ and $D' \nsubseteq D$. Then there are two disjoint elements $a, b \in A$ such that $a \in D \backslash D'$ and $b \in D' \backslash D$. Since P is prime either $a \to b \in P$ or $b \to a \in P$. If $a \to b \in P \subseteq D$, then $b \in D$, a contradiction. Similarly, if $b \to a \in P \subseteq D'$, then $a \in D'$, another contradiction. Thus, either $D \subseteq D'$ or $D' \subseteq D$. ■

DEFINITION 3.7. If D is a proper **ds** and there exists another proper **ds** D' such that $D \subseteq D'$ we say that D can be *extended* to D'.

THEOREM 3.27. *Any proper* ***ds*** *D of a non-degenerate $BL-$ algebra A can be extended to a prime* ***ds****.*

Proof. See [**129**], p.19, Theorem 2. ■

As in the case of residuated lattices (see Theorem 1.43) we have:

THEOREM 3.28. *(Prime* ***ds*** *theorem) If $D \in Ds(A)$ and I an ideal of the lattice $L(A)$ such that $D \cap I = \varnothing$, then there is a prime* ***ds*** *P of A such that $D \subseteq P$ and $P \cap I = \varnothing$.*

COROLLARY 3.29. *If $D \in Ds(A)$ is proper and $a \in A \backslash D$, then there is $P \in Spec(A)$ such that $D \subseteq P$ and $a \notin P$. In particular, for $D = \{1\}$ we deduce that for any $a \in A, a \neq 1$, there is $P_a \in Spec(A)$, such that $a \notin P_a$.*

PROPOSITION 3.30. *For a proper $P \in Ds(A)$ the following are equivalent:*

(i) P is prime;
(ii) $P \in Spec(A)$;
(iii) If $a, b \in A$ and $a \vee b = 1$, then $a \in P$ or $b \in P$.

Proof. $(i) \Rightarrow (ii)$. Let $D_1, D_2 \in Ds(A)$ such that $D_1 \cap D_2 = P$.

Since $P \subseteq D_1, P \subseteq D_2$, by Theorem 3.26, $D_1 \subseteq D_2$ or $D_2 \subseteq D_1$, hence $P = D_1$ or $P = D_2$.

$(ii) \Rightarrow (i)$. Let $a, b \in A$, such that $a \vee b \in P$.

Since $P(a) \cap P(b) = (P \vee [a)) \cap (P \vee [b)) = P \vee ([a) \cap [b)) = P \vee [a \vee b) = P$, then $P = P(a)$ or $P = P(b)$, hence $a \in P$ or $b \in P$, that is P is prime.

$(i) \Rightarrow (iii)$. Clearly, since $1 \in P$.

$(iii) \Rightarrow (i)$. Followings by Theorem 3.23, $(ii) \Rightarrow (i)$ (since $(a \to b) \vee (b \to a) = 1$ for every $a, b \in A$). ■

As in the case of residuated lattices we have the following results:

PROPOSITION 3.31. *For a proper $P \in Ds(A)$ the following are equivalent:*

(i) $P \in Spec(A)$;
(ii) For every $x, y \in A \backslash P$ there is $z \in A \backslash P$ such that $x \leq z$ and $y \leq z$.

COROLLARY 3.32. *For a proper $P \in Ds(A)$ the following are equivalent:*

(i) $P \in Spec(A)$;
(ii) If $x, y \in A$ and $[x) \cap [y) \subseteq P$, then $x \in P$ or $y \in P$.

COROLLARY 3.33. *For a proper $P \in Ds(A)$ the following are equivalent:*

(*i*) $P \in Spec(A)$;
(*ii*) *For every* $x, y \in A/P, x \neq 1, y \neq 1$ *there is* $z \in A/P, z \neq 1$ *such that* $x \leq z$, $y \leq z$.

THEOREM 3.34. *For a proper* $P \in Ds(A)$ *the following are equivalent:*
(*i*) $P \in Spec(A)$;
(*ii*) *For every* $D \in Ds(A)$, $D \to P = P$ *or* $D \subseteq P$.

COROLLARY 3.35. *For a BL-algebra* A, $Spec(A) \subseteq D(Ds(A)) \cup Rg(Ds(A))$.

REMARK 3.18. *From Corollary 3.29 we deduce that for every* $D \in Ds(A)$,

$$D = \cap\{P \in Spec(A) : D \subseteq P\} \text{ and } \cap\{P \in Spec(A)\} = \{1\}.$$

Relative to the uniqueness of deductive systems as intersection of primes we have:

THEOREM 3.36. *If every* $D \in Ds(A)$ *has a unique representation as an intersection of elements of* $Spec(A)$, *then* $(Ds(A), \vee, \wedge, ^*, \{1\}, A\}$ *is a Boolean algebra.*

LEMMA 3.37. *If* $D \in Ds(A)$, $D \neq A$ *and* $a \notin D$, *then there exists* $D_a \in Ds(A)$ *maximal with the property that* $D \subseteq D_a$ *and* $a \notin D_a$.

If in Lemma 3.37 we consider $D = \{1\}$ we obtain:

COROLLARY 3.38. *For any* $a \in A, a \neq 1$, *there is an* ***ds*** D_a *maximal with the property that* $a \notin D_a$.

THEOREM 3.39. *For* $D \in Ds(A), D \neq A$ *the following are equivalent:*
(*i*) $D \in Irc(A)$;
(*ii*) *There is* $a \in A$ *such that* D *is maximal relative to* a *(see Definition 1.13).*

THEOREM 3.40. *Let* $D \in Ds(A), D \neq A$ *and* $a \in A \backslash D$. *Then the following are equivalent:*
(*i*) D *is maximal relative to* a;
(*ii*) *For every* $x \in A \backslash D$ *there is* $n \in N$ *such that* $x^n \to a \in D$.

COROLLARY 3.41. *For* $D \in Ds(A), D \neq A$, *the following are equivalent:*
(*i*) $D \in Irc(A)$;
(*ii*) *In the set* $A/D \backslash \{1\}$ *we have an element* $p \neq 1$ *with the property that for every* $x \in A/D \backslash \{1\}$ *there is* $n \in N$ *such that* $x^n \leq p$.

PROPOSITION 3.42. *If* P *is a minimal prime* ***ds*** *(see Definition 1.14), then for any* $a \in P$ *there is* $b \in A \backslash P$ *such that* $a \vee b = 1$.

3.3. Maximal deductive systems; archimedean and hyperarchimedean BL-algebras.

DEFINITION 3.8. An **ds** of a BL-algebra A is *maximal* if it is proper and it is not contained in any other proper **ds**.

We shall denote by $Max(A)$ the set of all the maximal **ds** of A; it is obvious that, $Max(A) \subseteq Spec(A)$.
We have:

THEOREM 3.43. *For* $M \in Ds(A), M \neq A$, *the following conditions are equivalent:*

(*i*) $M \in Max(A)$;
(*ii*) *For every* $x \notin M$ *there is* $n \in N$ *such that* $(x^n)^* \in M$;
(*iii*) A/M *is locally finite.*

Proof. $(i) \Rightarrow (ii)$. Assume that $M \in Max(A)$ and $x \notin M$. Define a subset D of A by $D = \{z \in A : \text{for some } y \in M, n \in N, y \odot x^n \leq z\}$. Obviously, $1 \in D$. If $a, a \to b \in D$ then for some $y, y' \in M, n, m \in N$, holds $y \odot x^n \leq a, y' \odot x^m \leq a \to b$. Since $y \odot y' \in M$ and $(y \odot x^n) \odot (y' \odot x^m) = (y \odot y') \odot x^{n+m} \leq a \odot (a \to b) \leq b$. Thus $b \in D$, so D is a **ds**.

Since, for any $y \in M, y \odot x \leq y$, we have $M \subseteq D$. But as $1 \in M$ and $1 \odot x \leq x$, we also have $x \in D$. Since M is maximal, $D = A$, so $0 \in D$. Then there exists $y \in M, n \in N, y \odot x^n \leq 0 \Leftrightarrow y \leq (x^n)^*$. Hence $(x^n)^* \in M$.

$(ii) \Rightarrow (iii)$. Let $x/M \in A/M$ be such that $x/M \neq 1/M$, so $x \notin M$. Then there is $n \in N$ such that $(x^n)^* \in M$ and therefore $(x^n/M)^* = (x^n)^*/M = 1/M$, so $x^n/M \leq (x^n)^{**}/M = 1^*/M = 0/M$. We deduce that $x^n/M = (x/M)^n = 0/M$, whence A/M is locally finite.

$(iii) \Rightarrow (i)$. Let D be a **ds** such that $M \subset D$. Then there is an element $x \in A$ such that $x \in D$ and $x \notin M$. Then $x/M \neq 1/M$ and therefore $x^n/M = 0/M$ for some n, that is $0 \sim_M x^n$. Since $M \subset D$, then $0 \sim_D x^n$, that is $x^n/D = 0/D$. But $x \in D$ so $x^n \in D$, thus $x^n/D = 1/D$, therefore $0/D = 1/D$, which implies $0 \in D$, so $D = A$, whence M is maximal. ■

In [**129**] it is proved that:

THEOREM 3.44. *If* A *is a locally finite* $BL-$ *algebra, then* $x^{**} = x$ *for any element* $x \in A$.

REMARK 3.19. *By Theorem 3.44 and Theorem 3.2, we see that any locally finite* $BL-$ *algebra is an* $MV-$ *algebra; in particular the quotient algebra* A/M *induced by a maximal* ***ds*** M *of a* $BL-$ *algebra* A *is an* $MV-$ *algebra.*

THEOREM 3.45. *In a non-degenerate* $BL-$*algebra any proper* ***ds*** *can be extended to a maximal, prime* ***ds****.*

Proof. Let $D \in Ds(A)$ be a proper **ds**. By Theorem 3.27, D can be extended to a prime **ds** P. Let the set $\mathcal{F} = \{D' : P \subseteq D', D' \text{ a proper } \mathbf{ds} \text{ on } A\}$. By Theorem 3.26 $\mathcal{F}$ is a totally ordered set, and by Theorem 3.25, $D' \in \mathcal{F}$ is a prime **ds**. Let $M = \cup\{D' : D' \in \mathcal{F}\}$. Obviously, $1 \in M$. If $a, a \to b \in M$, then $a, a \to b \in D'$, for some $D' \in \mathcal{F}$, so $b \in D' \subseteq M$. Therefore, M is a **ds**. Since $0 \notin D'$ for any $D' \in \mathcal{F}$, we deduce that $0 \notin M$. Thus M is a proper **ds** and obviously is prime. The maximality of M is implied by the construction of M. ■

Let us remind that a $BL-$ algebra A is a *subdirect product* of a family $\{A_i\}_{i \in I}$ of $BL-$ algebras if

(*i*) A is a $BL-$ subalgebra of $\prod_{i \in I} A_i$;

(*ii*) for all $j \in I$ the $BL-$ morphism $A \hookrightarrow \prod_{i \in I} A_i \overset{\pi_j}{\to} A_j$ is onto.

A *representation* of A as a *subdirect product* of nontrivial $BL-$algebras $\{A_i\}_{i \in I}$ consists a monomorphism $\alpha : A \to \prod_{i \in I} A_i$ such that $\alpha(A)$ is a subdirect product of the family $\{A_i\}_{i \in I}$.

THEOREM 3.46. ([**75**], *Lemma 2.3.16*) *Every* $BL-$ *algebra is a subdirect product of* $BL-$ *chains.*

Applying a general result of universal algebra ([**18**], Lemma II.8.2, P.57), we get also the following generalization of the above theorem:

THEOREM 3.47. *If $\{D_i\}_{i\in I}$ is a family of filters of A such that $\underset{i\in I}{\cap} D_i = \{1\}$, then the family $\{A/D_i\}_{i\in I}$ determines a subdirect representation of A.*

If $f : A \to B$ is a $BL-$ morphism, then the kernel of f is the set $Ker(f) = \{a \in A : f(a) = 1\}$.

PROPOSITION 3.48. *Let $f : A \to B$ be a $BL-$ morphism. Then the following properties hold:*

(*i*) *for any (proper, prime) **ds** D' of B, the set $f^{-1}(D') = \{a \in A : f(a) \in D'\}$ is a (proper, prime) **ds** of A; in particular, $Ker(f)$ is a proper **ds** of A;*
(*ii*) *if M' is a maximal **ds** of B, then $f^{-1}(M')$ is a maximal **ds** of A;*
(*iii*) *if f is surjective and D is a **ds** of A, then $f(D)$ is a **ds** of B;*
(*iv*) *if f is surjective and M is a maximal **ds** of A such that $f(M)$ is proper, then $f(M)$ is a maximal **ds** of B;*
(*v*) *f is injective iff $Ker(f) = \{1\}$.*

Proof. (*i*). The proof follows directly from the classical ones.

(*ii*). By (*i*) we have that $f^{-1}(M')$ is a proper **ds** of A. To prove that it is maximal we shall apply Theorem 3.43. Let $x \in A$ such that $x \notin f^{-1}(M')$, so $f(x) \notin M'$. Since M' is a maximal ds of B, there is $n \geq 1$ such that $[f(x^n)]^* \in M'$, that is $f((x^n)^*) \in M'$. We deduce that $(x^n)^* \in f^{-1}(M')$.

(*iii*). Obvioulsy, $1 = f(1) \in f(D)$. Let $x, y \in f(D)$, that is there are $a, b \in D$ such that $x = f(a), y = f(b)$. It follows that $a \odot b \in D$, so $x \odot y = f(a) \odot f(b) = f(a \odot b) \in f(D)$.

Let now $x, y \in B, x \leq y$ and $x \in f(D)$. Then, there is $a \in D$ such that $x = f(a)$, and since f is surjective, there exists $b \in A$, such that $y = f(b)$. Then $y = x \vee y = f(a) \vee f(b) = f(a \vee b)$ and $a \vee b \geq a \in D$, so $a \vee b \in D$. Thus, $y \in f(D)$.

(*iv*). Let N be a proper **ds** of B such that $f(M) \subseteq N$. We have $M \subseteq f^{-1}(f(M)) \subseteq f^{-1}(N)$ and since $f^{-1}(N)$ is proper, we must have $M = f^{-1}(N)$. It follows that $f(M) = f(f^{-1}(N)) = N$ since f is surjective.

(*v*). Similarly with the proof of Proposition 2.19. ■

DEFINITION 3.9. As in the case of residuated lattices, the intersection of the maximal **ds** of A is called the *radical of A*. It will be denoted by $Rad(A)$. It is obvious that $Rad(A)$ is **ds**.

EXAMPLE 3.17. *Let A be the $BL-$algebra from Example 3.11. It is eassy to see that $\{a, b, c, 1\}$ is the unique maximal ds of A, hence $Rad(A) = \{a, b, c, 1\}$.*

PROPOSITION 3.49. ([**53**], [**54**]) *$Rad(A) = \{a \in A : (a^n)^* \leq a$, for any $n \in N\}$.*

PROPOSITION 3.50. *For any $a, b \in Rad(A)$, $a^* \odot b^* = 0$.*

Proof. Let $a, b \in Rad(A)$; to prove $a^* \odot b^* = 0$ is equivalent with $(a^* \odot b^*)^* = 1$. Suppose that $(a^* \odot b^*)^* \neq 1$. By Corollary 3.29, there is a prime **ds** P such that $(a^* \odot b^*)^* \notin P$. By $bl - c_{37}$ we have $(a^* \odot b^*)^* = a^* \to b^{**} \notin P$, so by Theorem 3.23, $b^{**} \to a^* \in P$, that is $(b^{**} \odot a)^* \in P$.

By Theorem 3.45 there is a maximal **ds** M such that $P \subseteq M$. Then $b^{**} \odot a \notin M$. By Theorem 3.43, there is $n \in N$ such that $[(b^{**} \odot a)^n]^* = [(b^n)^{**} \odot a^n]^* \in M$; so,

if denote $c = (b^n)^{**} \odot a^n$, we have $c^* \in M$. Since $a, b \in Rad(A)$ then we infer that $a, b \in M$, hence $c = (b^n)^{**} \odot a^n \in M$. Hence c and c^* are in M which contradicts the fact that M is a proper **ds** of A. ■

PROPOSITION 3.51. *Let A be a $BL-$ algebra. Then $B(A) \cap Rad(A) = \{1\}$.*

Proof. See the proof of Proposition 1.64. ■

DEFINITION 3.10. As in the case of residuated lattices, an element $a \in A$ is called *infinitesimal* if $a \neq 1$ and $a^n \geq a^*$, for any $n \in N$.

PROPOSITION 3.52. *For every nonunit element a of A the following are equivalent:*

(i) a is infinitesimal;
(ii) $a \in Rad(A)$.

Proof. $(i) \Rightarrow (ii)$. Let $a \neq 1$ an infinitesimal and suppose $a \notin Rad(A)$. Thus, there is a maximal **ds** M of A such that $a \notin M$. By Theorem 3.43, there is $n \in N$ such that $(a^n)^* \in M$. By hypothesis $a^n \geq a^*$ hence $(a^n)^* \leq a^{**}$, so $a^{**} \in M$, hence $(a^{**})^n = (a^n)^{**} \in M$. If denote $b = (a^n)^*$ we conclude that $b, b^* \in M$ which contradicts the fact that M is a proper **ds**.

$(ii) \Rightarrow (i)$. Let $a \in Rad(A)$; then $(a^n)^* \leq a$ for any $n \in N$. For $n = 1$ we obtain that $a^* \leq a$. Since for any $n \in N, a^n \in Rad(A)$ we deduce that $(a^n)^* \leq a^n$. Since $a^* \odot a^n \leq a^* \odot a = 0$ we obtain that $a^* \odot a^n = 0$ for any $n \in N$, hence by $bl - c_6$, $a^* \leq (a^n)^*$. So, for any $n \in N$, $a^* \leq (a^n)^*$ and $(a^n)^* \leq a^n$, hence $a^* \leq a^n$, that is a is an infinitesimal. ■

REMARK 3.20. *If BL-algebra A is an $MV-$ algebra, an element a is infinitesimal if $a \neq 0$ and $na \leq a^*$, for each integer $n \geq 0$. In* [**45**]*, the set of all infinitesimals in A is denoted by $Infinit(A)$ and it is proved (Proposition 3.6.4, p. 73) the following result: For any $MV-$ algebra $A, Rad(A) = Infinit(A) \cup \{0\}$.*

LEMMA 3.53. *If $a \in A$, $n \in N$ such that $a \vee (a^n)^* = 1$ and $a^n \geq a^*$, then $a = 1$.*

Proof. By $bl - c_{11}$ we obtain $(a^n)^* \leq a^{**}$, so $1 = a \vee (a^n)^* \leq a \vee a^{**} = a^{**}$, hence $a^{**} = 1$, that is $a^* = 0$. Then $a \to (a \to 0) = a \to 0 = 0$. From $bl - c_8$ we deduce that $(a^2)^* = 0$. Recursively we obtain that $(a^n)^* = 0$. Then $a \vee 0 = 1$, hence $a = 1$.■

LEMMA 3.54. *In any BL-algebra A the following are equivalent:*

(i) For every $a \in A, a^n \geq a^*$ for any $n \in N$ implies $a = 1$;
(ii) For every $a, b \in A, a^n \geq b^*$ for any $n \in N$ implies $a \to b = b$ and $b \to a = a$.

Proof. $(i) \Rightarrow (ii)$. Let $a, b \in A$ such that $a^n \geq b^*$ for any $n \in N$. We get $(a \vee b)^* = a^* \wedge b^* \leq b^* \leq a^n \leq (a \vee b)^n$, hence $(a \vee b)^n \geq (a \vee b)^*$ for any $n \in N$. By hypothesis, $a \vee b = 1$. From $bl - c_{15}$ we deduce $(a \to b) \to b = (b \to a) \to a = 1$, hence $a \to b = b$ and $b \to a = a$.

$(ii) \Rightarrow (i)$. Let $a \in A$ such that $a^n \geq a^*$ for any $n \in N$. By hypothesis for $n = 1$ and $b = a$ we obtain $a \geq a^*$, hence $a^* \to a = 1$. But $a^* \to a = a$ (by (ii)), so $a = 1$.■

DEFINITION 3.11. A BL-algebra A is called *archimedean* if the equivalent conditions from Lemma 3.54 are satisfied.

One can easily remark that a BL-algebra is archimedean iff it has no infinitesimals.

DEFINITION 3.12. Let A be a BL-algebra. An element $a \in A$ is called *archimedean* if it satisfies the condition:

$$\textit{there is } n \in N, n \geq 1, \textit{ such that } a \vee (a^n)^* = 1.$$

A BL-algebra A is called *hyperarchimedean* if all its elements are archimedean.

From Lemma 3.53 we deduce:

COROLLARY 3.55. *Every hyperarchimedean BL-algebra is archimedean.*

Now, we have a theorem of Nachbin type (see [**2**], p.73) for BL-algebras:

THEOREM 3.56. *For a BL-algebra A the following are equivalent:*

(i) A is hyperarchimedean;
*(ii) For any **ds** D, the quotient BL-algebra A/D is an archimedean BL-algebra,*
(iii) $Spec(A) = Max(A)$;
*(iv) Any prime **ds** is minimal prime.*

Proof. $(i) \Rightarrow (ii)$. To prove A/D is archimedean, let $x = a/D \in A/D$ such that $x^n \geq x^*$ for any $n \in N$. By hypothesis, there is $m \in$N$, m \geq 1$ such that $a \vee (a^m)^* = 1$. It follows that $x \vee (x^m)^* = 1$ (in A/D). In particular we have $x^m \geq x^*$, so by Lemma 3.53 we deduce that $x = 1$, that is A/D is archimedean.

$(ii) \Rightarrow (iii)$. Since $Max(A) \subseteq Spec(A)$, we only have to prove that any prime **ds** of A is maximal. If $P \in Spec(A)$, then A/P is a chain (see Theorem 3.23). By hypothesis A/P is archimedean. By Theorem 3.43 to prove $P \in Max(A)$ is suffice to prove that A/P is locally finite.

Let $x = a/P \in A/P$, $x \neq 1$. Then there is $n \in N, n \geq 1$, such that $x^n \ngeq x^*$. Since A/P is chain we have $x^n \leq x^*$. Thus $x^{n+1} \leq x \odot x^* = 0$, hence $x^{n+1} = 0$, that is $o(x) < \infty$. It follows that A/P is locally finite.

$(iii) \Rightarrow (iv)$. Let P, Q prime **ds** such that $P \subseteq Q$. By hypothesis, P is maximal, so $P = Q$. Thus Q is minimal prime.

$(iv) \Rightarrow (i)$. Let a be a nonunit element from A. We shall prove that a is an archimedean element. If we denote

$$D = [a)^* = \{x \in A : a \vee x = 1\} \text{ (by Remark 3.17)},$$

then $D \in Ds(A)$. Since $a \neq 1$, then $a \notin D$ and we consider

$$D' = D(a) = \{x \in A : x \geq d \odot a^n \text{ for some } d \in D \text{ and } n \in N\}.$$

If we suppose that D' is a proper **ds** of A, then by Corollary 3.29, there is a prime **ds** P such that $D' \subseteq P$. By hypothesis, P is a minimal prime. Since $a \in P$, using Proposition 3.42 , we infer that there is $x \in A \backslash P$ such that $a \vee x = 1$. It follows that $x \in D \subseteq D' \subseteq P$, hence $x \in P$, so we get a contradiction. Thus D' is not proper, so $0 \in D'$, hence there is $n \in N$ and $d \in D$ such that $d \odot a^n = 0$. Thus $d \leq (a^n)^*$ (by $bl - c_6$). We get $a \vee d \leq a \vee (a^n)^*$. But $a \vee d = 1$ (since $d \in D$), so we obtain that $a \vee (a^n)^* = 1$, that is a is an archimedean element. ■

CHAPTER 4

Pseudo MV-algebras

If G is an lu-group, then the interval $[0, u]$ can be endowed with a structure that leads to a non-commutative generalization of MV- algebras.

In 1999, Georgescu and Iorgulescu (see [**66**], [**68**]) defined pseudo MV- algebras as a non-commutative extensions of MV- algebras. Dvurečenskij extended Mundici's equivalence results. In [**58**], he proved that every pseudo MV- algebra is isomorphic with an interval in an l-group and he established the categorical equivalence between pseudo MV- algebras and l-groups with strong unit.

For a detailed study of pseudo MV- algebras one can see [**68**], [**58**].

For an exhaustive theory of l-groups we refer to [**10**].

In this chapter, we review the basic definition of pseudo MV-algebras with more details and more examples, but we also prove many results about the lattice of ideals.

1. Definitions and first properties. Some examples. Rules of calculus

Since $MV-$ algebras are categorically equivalent to abelian l-groups with strong unit (lu-group), started from arbitrary l-groups and thus obtained the more general notion of *pseudo MV-algebra.*

If we consider that the l-group G from Example 2.3 is not necessarily abelian, then it makes sense to define two negations on the interval $[0, u]$:

$$x^* = u - x$$

and

$$x^{\sim} = -x + u$$

for any $x \in [0, u]$.

This was the starting point of the theory of pseudo MV-algebras [**68**].

We shall present briefly some basic definitions and results (for more details, see [**66**], [**68**]) .

We consider an algebra $\mathcal{A} = (A, \oplus, ^-, ^{\sim}, 0, 1)$ of type $(2, 1, 1, 0, 0)$. We put by definition:

$$y \odot x = (x^- \oplus y^-)^{\sim},$$

and we consider that the operation $\odot$ has priority to the operation $\oplus$.

Definition 4.1. A *pseudo $MV-$ algebra* is an algebra $\mathcal{A} = (A, \oplus, \odot, ^-, ^{\sim}, 0, 1)$ of type $(2, 2, 1, 1, 0, 0)$ satisfying the following equations:

$(psMV_1)$ $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
$(psMV_2)$ $x \oplus 0 = 0 \oplus x = x$;
$(psMV_3)$ $x \oplus 1 = 1 \oplus x = 1$;
$(psMV_4)$ $1^{\sim} = 0, 1^- = 0$;
$(psMV_5)$ $(x^- \oplus y^-)^{\sim} = (x^{\sim} \oplus y^{\sim})^-$;
$(psMV_6)$ $x \oplus x^{\sim} \odot y = y \oplus y^{\sim} \odot x = x \odot y^- \oplus y = y \odot x^- \oplus x$;

$(psMV_7)$ $x \odot (x^- \oplus y) = (x \oplus y^\sim) \odot y$;
$(psMV_8)$ $(x^-)^\sim = x$, for every $x, y, z \in A$.

We denote a pseudo $MV-$ algebra $\mathcal{A} = (A, \oplus, \odot, ^-, ^\sim, 0, 1)$ by its universe A.

We can define two implications corresponding to the two negations:

$$x \to y := x^* \oplus y \text{ and } x \rightsquigarrow y := y \oplus x^\sim$$

for any $x, y \in A$.

If $A' \subseteq A$ we write $A' \leq A$ to indicate that A' is a pseudo $MV-$ subalgebra of A.

EXAMPLE 4.1. *A singleton $\{0\}$ is a trivial example of a pseudo $MV-$ algebra; a pseudo $MV-$ algebra is said nontrivial provided its universe has more that one element.*

EXAMPLE 4.2. *Let us consider an arbitrary l-group $(G, +, -, 0, \leq)$. For each $u \in G$, $u > 0$, let*

$$[0, u] = \{x \in G : 0 \leq x \leq u\}$$

and for each $x, y \in [0, u]$, let $x \oplus y = u \wedge (x + y), x \odot y = (x - u + y) \vee 0, x^- = u - x$ and $x^\sim = -x + u$. Then $([0, u], \oplus, \odot, ^-, ^\sim, 0, u)$ is a pseudo $MV-$ algebra. We remark that the order relation is the restriction of the order relation on G. Moreover, $(x^-)^- = u + x - u$ and $(x^\sim)^\sim = -u + x + u$, for every $x, y \in [0, u]$.

In [**58**] Dvurečenskij proved that every pseudo-MV algebra is isomorphic with an interval in an l-group.

EXAMPLE 4.3. *Clearly, every MV algebra is a pseudo $MV-$ algebra, where the unary operations $^-, ^\sim$ coincide.*

Every *commutative pseudo $MV-$ algebra* (i.e. $\oplus$ is commutative) is an $MV-$ algebra (see Proposition 4.9). Also, every finite pseudo $MV-$ algebra is an MV algebra.

For another classes of pseudo $MV-$ algebras (*local, archimedean*) see [**58**] and [**100**].

THEOREM 4.1. *If $x, y, z \in A$ then the following hold:*

$(psmv - c_1)$ $y \odot x = (x^\sim \oplus y^\sim)^-$;
$(psmv - c_2)$ $x \oplus y = (y^- \odot x^-)^\sim = (y^\sim \odot x^\sim)^-$;
$(psmv - c_3)$ $(x^\sim)^- = x$;
$(psmv - c_4)$ $0^\sim = 0^- = 1$;
$(psmv - c_5)$ $x \odot 1 = 1 \odot x = x, x \odot 0 = 0 \odot x = 0$;
$(psmv - c_6)$ $x \oplus x^\sim = 1, x^- \oplus x = 1$;
$(psmv - c_7)$ $x \odot x^- = 0, x^\sim \odot x = 0$;
$(psmv - c_8)$ $(x \oplus y)^- = y^- \odot x^-, (x \oplus y)^\sim = y^\sim \odot x^\sim$;
$(psmv - c_9)$ $(x \odot y)^- = y^- \oplus x^-, (x \odot y)^\sim = y^\sim \oplus x^\sim$;
$(psmv - c_{10})$ $(x^\sim \odot y) \oplus y^\sim = (y^\sim \odot x) \oplus x^\sim$;
$(psmv - c_{11})$ $x \odot (x^- \oplus y) = y \odot (y^- \oplus x)$;
$(psmv - c_{12})$ $x \odot (y \odot z) = (x \odot y) \odot z$.

Proof. $(psmv - c_1)$. Follows by $psMV_5$ and by definition of $\odot$.
$(psmv - c_2)$. $x \oplus y = [(x \oplus y)^\sim]^- = (y^\sim \odot x^\sim)^-$; analogously, $x \oplus y = (y^- \odot x^-)^\sim$.

$(psmv-c_3)$. In $psMV_5$ we make $x=1$. So, $(1^{\sim}\oplus y^{\sim})^{-}=(1^{-}\oplus y^{-})^{\sim}\Rightarrow (0\oplus y^{\sim})^{-}=(0\oplus y^{-})^{\sim}\Rightarrow (y^{\sim})^{-}=(y^{-})^{\sim}\overset{psMV_8}{=}y$.

$(psmv-c_4)$. $0^{\sim}\overset{psMV_4}{=}(1^{-})^{\sim}\overset{psMV_8}{=}1$ and $0^{-}\overset{psMV_4}{=}(1^{\sim})^{-}\overset{psmv-c_3}{=}1$.

$(psmv-c_5)$. $x\odot 1=(x^{\sim}\oplus 1^{\sim})^{-}=(x^{\sim}\oplus 0)^{-}=(x^{\sim})^{-}=x$; analogously, $1\odot x=x$;

$x\odot 0=(x^{\sim}\oplus 0^{\sim})^{-}=(x^{\sim}\oplus 1)^{-}=1^{-}=0$; analogously, $0\odot x=0$,

$(psmv-c_6)$. In $psMV_6$ we make $x=1$; then $y\oplus y^{\sim}=1$.

In $y\oplus y^{\sim}=1$ we make $y=x^{-}$; then $x^{-}\oplus x=x^{-}\oplus (x^{-})^{\sim}=1$.

$(psmv-c_7)$. $x\odot x^{-}=[(x^{-})^{\sim}\oplus x^{\sim}]^{-}=(x\oplus x^{\sim})^{-}=0$; analogously, $x^{\sim}\odot x=0$.

$(psmv-c_8)$. $(x\oplus y)^{-}=[(x^{-})^{\sim}\oplus (y^{-})^{\sim}]^{-}=y^{-}\odot x^{-}$; analogously, $(x\oplus y)^{\sim}=y^{\sim}\odot x^{\sim}$.

$(psmv-c_9)$. $(x\odot y)^{-}=[(y^{-}\oplus x^{-})^{\sim}]^{-}=y^{-}\oplus x^{-}$; analogously, $(x\odot y)^{\sim}=y^{\sim}\oplus x^{\sim}$.

$(psmv-c_{10})$. In $(x\odot y^{-})\oplus y=(y\odot x^{-})\oplus x$ we make the substitution of x by $x^{\sim}$ and of y by $y^{\sim}$; then $(x^{\sim}\odot y)\oplus y^{\sim}=[x^{\sim}\odot (y^{-})^{\sim}]\oplus y^{\sim}=[y^{\sim}\odot (x^{\sim})^{-}]\oplus x^{\sim}=(y^{\sim}\odot x)\oplus x^{\sim}$.

$(psmv-c_{11})$. $x\odot (x^{-}\oplus y)=[(x^{-}\oplus y)^{\sim}\oplus x^{\sim}]^{-}=[(y^{\sim}\odot x)\oplus x^{\sim}]^{-}=[(x^{\sim}\odot y)\oplus y^{\sim}]^{-}=y\odot (y^{-}\oplus x)$.

$(psmv-c_{12})$. $(x\odot y)\odot z=[(x\odot y)\odot z)^{-}]^{\sim}=[z^{-}\oplus (x\odot y)^{-}]^{\sim}=[z^{-}\oplus (y^{-}\oplus x^{-})]^{\sim}=[(z^{-}\oplus y^{-})\oplus x^{-}]^{\sim}=[(y\odot z)^{-}\oplus x^{-}]^{\sim}=[(x\odot (y\odot z))^{-}]^{\sim}=x\odot (y\odot z)$.■

LEMMA 4.2. *For $x,y\in A$, the following conditions are equivalent:*

(i) $x^{-}\oplus y=1$;
(ii) $y^{\sim}\odot x=0$;
(iii) $y=x\oplus (x^{\sim}\odot y)$;
(iv) $x=x\odot (x^{-}\oplus y)$;
(v) *There is an element $z\in A$ such that $x\oplus z=y$;*
(vi) $x\odot y^{-}=0$;
(vii) $y\oplus x^{\sim}=1$.

Proof. $(i)\Rightarrow (ii)$. $x^{-}\oplus y=1\Rightarrow y^{\sim}\odot x=(x^{-}\oplus y)^{\sim}=1^{\sim}=0$.

$(ii)\Rightarrow (i)$. $y^{\sim}\odot x=0\Rightarrow x^{-}\oplus y=(y^{\sim}\odot x)^{-}=0^{-}=1$.

$(ii)\Rightarrow (iii)$. $y^{\sim}\odot x=0\Rightarrow y=y\oplus (y^{\sim}\odot x)\overset{psMV_6}{=}x\oplus (x^{\sim}\odot y)$.

$(iii)\Rightarrow (v)$. Clearly.

$(v)\Rightarrow (i)$. $x^{-}\oplus y=x^{-}\oplus (x\oplus z)=(x^{-}\oplus x)\oplus z=1\oplus z=1$.

$(i)\Rightarrow (iv)$. Clearly.

$(iv)\Rightarrow (ii)$. $x=x\odot (x^{-}\oplus y)=y\odot (y^{-}\oplus x)\Rightarrow y^{\sim}\odot x=y^{\sim}\odot [y\odot (y^{-}\oplus x)]=(y^{\sim}\odot y)\odot (y^{-}\oplus x)=0$.

$(i)\Rightarrow (vi)$. $x=x\odot 1=x\odot (x^{-}\oplus y)=(x\oplus y^{\sim})\odot y\Rightarrow x\odot y^{-}=[(x\oplus y^{\sim})\odot y]\odot y^{-}=(x\oplus y^{\sim})\odot (y\odot y^{-})=0$.

$(vi)\Rightarrow (i)$. $y=0\oplus y=(x\odot y^{-})\oplus y\overset{psMV_6}{=}x\oplus (x^{\sim}\odot y)$, so $x^{-}\oplus y=x^{-}\oplus [x\oplus (x^{\sim}\odot y)]=(x^{-}\oplus x)\oplus (x^{\sim}\odot y)=1\oplus (x^{\sim}\odot y)=1$.

$(vi)\Rightarrow (vii)$. $x\odot y^{-}=0\Rightarrow (x\odot y^{-})^{\sim}=1\Rightarrow (y^{-})^{\sim}\oplus x^{\sim}=1\Rightarrow y\oplus x^{\sim}=1$.

$(vii)\Rightarrow (ii)$. We have $x=x\odot 1=(y\oplus x^{\sim})\odot x\overset{psMV_7}{=}y\odot (y^{-}\oplus x)$. So, $y^{\sim}\odot x=y^{\sim}\odot [y\odot (y^{-}\oplus x)]=(y^{\sim}\odot y)\odot (y^{-}\oplus x)]=0\odot (y^{-}\oplus x)=0$.■

For any two elements $x,y\in A$ let us agree to write $x\leq y$ iff x and y satisfy the equivalent conditions $(i)-(vii)$ in the above lemma. So, $\leq$ is a partial order relation on A (which is called the *natural order* on A).

PROPOSITION 4.3. *$\leq$ is an order relation on A.*

Proof. We have: $x \leq x \Leftrightarrow x^- \oplus x = 1$, obviously; if $x \leq y$ and $y \leq x$ we have $y = x \oplus (x^\sim \odot y) \overset{psMV_6}{=} y \oplus (y^\sim \odot x) = x$; if $x \leq y$ and $y \leq z$, there exist a, b such that $y = x \oplus a$ and $z = y \oplus b$, so $z = (x \oplus a) \oplus b \Rightarrow x \leq z$. ■

REMARK 4.1. *If $\mathcal{A} = (A, \oplus, \odot, ^-, ^\sim, 0, 1)$ is a pseudo $MV-$ algebra then*

$$(A, \odot, \oplus, ^\sim, ^-, 1, 0)$$

is also a pseudo $MV-$ algebra, called the dual pseudo $MV-$ algebra *of $\mathcal{A}$.*

PROPOSITION 4.4. *The following properties hold:*

$(psmv - c_{13})$ *$x \leq y$ iff $y^- \leq x^-$ iff $y^\sim \leq x^\sim$;*
$(psmv - c_{14})$ *If $x \leq y$, then $x \oplus z \leq y \oplus z$ and $z \oplus x \leq z \oplus y$;*
$(psmv - c_{15})$ *If $x \leq y$, then $x \odot z \leq y \odot z$ and $z \odot x \leq z \odot y$;*
$(psmv - c_{16})$ *$x \odot y \leq z$ iff $y \leq x^- \oplus z$ iff $x \leq z \oplus y^\sim$;*
$(psmv - c_{17})$ *$x \odot y \leq x, x \odot y \leq y, x \leq x \oplus y, y \leq x \oplus y$.*

Proof. $(psmv - c_{13})$. $y^- \leq x^- \Leftrightarrow (x^-)^\sim \odot y^- = 0 \Leftrightarrow x \odot y^- = 0 \Leftrightarrow x \leq y$ and $y^\sim \leq x^\sim \Leftrightarrow (y^\sim)^- \oplus x^\sim = 1 \Leftrightarrow y \oplus x^\sim = 1 \Leftrightarrow x \leq y$.

$(psmv - c_{14})$. If $x \leq y$, then there exists an element $a \in A$ such that $x \oplus a = y$, so, $z \oplus y = z \oplus (x \oplus a) = (z \oplus x) \oplus a \Rightarrow z \oplus x \leq z \oplus y$;

By $x \leq y \Rightarrow y \oplus x^\sim = 1 \Rightarrow y \oplus z \oplus (x \oplus z)^\sim = y \oplus z \oplus (z^\sim \odot x^\sim) = y \oplus x^\sim \oplus (x^{\approx} \odot z) = 1 \oplus (x^{\approx} \odot z) = 1 \Rightarrow x \oplus z \leq y \oplus z$.

$(psmv - c_{15})$. $x \leq y \Rightarrow y^\sim \leq x^\sim \Rightarrow y^\sim \oplus z^\sim \leq x^\sim \oplus z^\sim \Rightarrow z \odot x \leq z \odot y$; analogously the other proof.

$(psmv - c_{16})$. We have $x \odot y \leq z \Leftrightarrow (x \odot y)^- \oplus z = 1 \Leftrightarrow y^- \oplus x^- \oplus z = 1 \Leftrightarrow y \leq x^- \oplus z$ and $x \odot y \leq z \Leftrightarrow z \oplus (x \odot y)^\sim = 1 \Leftrightarrow z \oplus y^\sim \oplus x^\sim = 1 \Leftrightarrow x \leq z \oplus y^\sim$.

$(psmv - c_{17})$. Follows from $psmv - c_{14}$ and $psmv - c_{15}$. ■

PROPOSITION 4.5. *On A, the natural order determines a lattice structure. Specifically, the join $x \vee y$ and the meet $x \wedge y$ of the elements x and y are given by:*

$$x \vee y = x \oplus x^\sim \odot y = y \oplus y^\sim \odot x = x \odot y^- \oplus y = y \odot x^- \oplus x,$$

$$x \wedge y = x \odot (x^- \oplus y) = y \odot (y^- \oplus x) = (x \oplus y^\sim) \odot y = (y \oplus x^\sim) \odot x.$$

Proof. For the join we have $x^- \oplus x \oplus (x^\sim \odot y) = 1 \Rightarrow x \leq x \oplus x^\sim \odot y$ and similarly, $y \leq y \oplus y^\sim \odot x$. Let $x, y \leq z$. We shall prove that $y \oplus y^\sim \odot x \leq z$.

Remark that $[y \oplus (y^\sim \odot x)]^- \oplus z = [(y^\sim \odot x)^- \odot y^-] \oplus z = [(x^- \oplus y) \odot y^-] \oplus y \oplus (z^\sim \odot y) = [y \odot (x^- \oplus y)^-] \oplus x^- \oplus y \oplus (z^\sim \odot y) = [y \odot (x^- \oplus y)^-] \oplus x^- \oplus z \oplus (y^\sim \odot x) = 1$.

For the meet it is obvious that $x \odot (x^- \oplus y) = (x \oplus y^\sim) \odot y \leq x, y$. Let $z \leq x, y$; then $x^-, y^- \leq z^-$, hence $x^- \vee y^- \leq z^-$. It follows that $z = (z^-)^\sim \leq (x^- \vee y^-)^\sim = [x^- \oplus (x^-)^\sim \odot y^-]^\sim =$

$[x^- \oplus (x \odot y^-)]^\sim = [(x \odot y^-)]^\sim \odot (x^-)^\sim = [(y^-)^\sim \oplus x^\sim] \odot x = (y \oplus x^\sim) \odot x = x \odot (x^- \oplus y)$. ■

REMARK 4.2. *Clearly, $x \odot y \leq x \wedge y \leq x, y \leq x \vee y \leq x \oplus y$.*

THEOREM 4.6. *Let I be an arbitrary set. If $x, y, z, (x_i)_{i \in I}$ are elements of A, then the following hold:*

$(psmv - c_{18})$ *$x \oplus \left(\bigwedge_{i \in I} x_i \right) = \bigwedge_{i \in I} (x \oplus x_i)$;*

$(psmv - c_{19})$ $\left(\bigwedge_{i\in I} x_i\right) \oplus x = \bigwedge_{i\in I}(x_i \oplus x)$;

$(psmv - c_{20})$ $x \odot \left(\bigvee_{i\in I} x_i\right) = \bigvee_{i\in I}(x \odot x_i)$;

$(psmv - c_{21})$ $\left(\bigvee_{i\in I} x_i\right) \odot x = \bigvee_{i\in I}(x_i \odot x)$;

$(psmv - c_{22})$ $x \wedge \left(\bigvee_{i\in I} x_i\right) = \bigvee_{i\in I}(x \wedge x_i)$;

$(psmv - c_{23})$ $x \vee \left(\bigwedge_{i\in I} x_i\right) = \bigwedge_{i\in I}(x \vee x_i)$;

$(psmv - c_{24})$ $x \oplus \left(\bigvee_{i\in I} x_i\right) = \bigvee_{i\in I}(x \oplus x_i)$;

$(psmv - c_{25})$ $\left(\bigvee_{i\in I} x_i\right) \oplus x = \bigvee_{i\in I}(x_i \oplus x)$;

$(psmv - c_{26})$ $x \odot \left(\bigwedge_{i\in I} x_i\right) = \bigwedge_{i\in I}(x \odot x_i)$;

$(psmv - c_{27})$ $\left(\bigwedge_{i\in I} x_i\right) \odot x = \bigwedge_{i\in I}(x_i \odot x)$,*if all suprema and infima exist.*

Proof. $(psmv - c_{18})$. Obviously, $x \oplus \left(\bigwedge_{i\in I} x_i\right) \leq x \oplus x_i$, for every $i \in I$.

Let now $y \leq x \oplus x_i$, for every $i \in I$; then $y \leq (x^{\sim})^{-} \oplus x_i$, for every $i \in I$. We deduce that $x^{\sim} \odot y \leq x_i$, for every $i \in I$ and hence $x^{\sim} \odot y \leq \bigwedge_{i\in I} x_i$; it follows that $y \leq (x^{\sim})^{-} \oplus \left(\bigwedge_{i\in I} x_i\right) = x \oplus \left(\bigwedge_{i\in I} x_i\right)$. Therefore we get that $x \oplus \left(\bigwedge_{i\in I} x_i\right) = \bigwedge_{i\in I}(x \oplus x_i)$.

($psmv - c_{19}$). Remark first that $\left(\bigwedge_{i\in I} x_i\right) \oplus x \leq x_i \oplus x$, for every $i \in I$. Let now $y \leq x_i \oplus x$, for every $i \in I$; then $y \leq x_i \oplus (x^{-})^{\sim}$, for every $i \in I$ and $y \odot x^{-} \leq x_i$, for every $i \in I$;it follows that and hence $y \odot x^{-} \leq \bigwedge_{i\in I} x_i$ and hence $y \leq \left(\bigwedge_{i\in I} x_i\right) \oplus (x^{-})^{\sim} = \left(\bigwedge_{i\in I} x_i\right) \oplus x$. Therefore we get that $\left(\bigwedge_{i\in I} x_i\right) \oplus x = \bigwedge_{i\in I}(x_i \oplus x)$.

$(psmv - c_{20})$. Obviously, $x \odot x_i \leq x \odot \left(\bigvee_{i\in I} x_i\right)$, for every $i \in I$. Let $x \odot x_i \leq y$, for every $i \in I$; then $x_i \leq x^{-} \oplus y$, for every $i \in I$ so $\bigvee_{i\in I} x_i \leq x^{-} \oplus y$.

It follows that $x \odot \left(\bigvee_{i\in I} x_i\right) \leq y$. Therefore we get that $x \odot \left(\bigvee_{i\in I} x_i\right) = \bigvee_{i\in I}(x \odot x_i)$.

$(psmv - c_{21})$. Remark first that $x_i \odot x \leq \left(\bigvee_{i\in I} x_i\right) \odot x$, for every $i \in I$. Let $x_i \odot x \leq y$, for every $i \in I$; then $x_i \leq y \oplus x^{\sim}$, for every $i \in I$ and hence $\bigvee_{i\in I} x_i \leq y \oplus x^{\sim}$.

It follows that $\left(\bigvee_{i\in I} x_i\right) \odot x \leq y$ and therefore we get that $\left(\bigvee_{i\in I} x_i\right) \odot x = \bigvee_{i\in I}(x_i \odot x)$.

$(psmv-c_{22})$. We have $x\wedge\left(\bigvee\limits_{i\in I}x_i\right)=\left(\bigvee\limits_{i\in I}x_i\right)\odot\left[\left(\bigvee\limits_{j\in I}x_j\right)^{-}\oplus x\right]=$

$\bigvee\limits_{i\in I}\left[x_i\odot\left(\left(\bigvee\limits_{j\in I}x_j\right)^{-}\oplus x\right)\right]$.

But for any $i\in I$, $x_i\leq\bigvee\limits_{j\in I}x_j\Rightarrow\left(\bigvee\limits_{j\in I}x_j\right)^{-}\leq(x_i)^{-}\Rightarrow\left(\bigvee\limits_{j\in I}x_j\right)^{-}\oplus x\leq(x_i)^{-}\oplus x$ $\Rightarrow x_i\odot\left(\left(\bigvee\limits_{j\in I}x_j\right)^{-}\oplus x\right)\leq x_i\odot((x_i)^{-}\oplus x)=x\wedge x_i\Rightarrow\bigvee\limits_{i\in I}\left[x_i\odot\left(\left(\bigvee\limits_{j\in I}x_j\right)^{-}\oplus x\right)\right]\leq\bigvee\limits_{i\in I}(x\wedge x_i)$.

We obtain $x\wedge\left(\bigvee\limits_{i\in I}x_i\right)\leq\bigvee\limits_{i\in I}(x\wedge x_i)$.

The inequality $\bigvee\limits_{i\in I}(x\wedge x_i)\leq x\wedge\left(\bigvee\limits_{i\in I}x_i\right)$ is obvious.

$(psmv-c_{23})$. We have $x\vee\left(\bigwedge\limits_{i\in I}x_i\right)=\left(\bigwedge\limits_{i\in I}x_i\right)\oplus\left[\left(\bigwedge\limits_{j\in I}x_j\right)^{\sim}\odot x\right]=$

$\bigwedge\limits_{i\in I}\left[x_i\oplus\left(\left(\bigwedge\limits_{j\in I}x_j\right)^{\sim}\odot x\right)\right]$

But for any $i\in I$, $\bigwedge\limits_{j\in I}x_j\leq x_i\Rightarrow(x_i)^{\sim}\leq\left(\bigwedge\limits_{j\in I}x_j\right)^{\sim}\Rightarrow(x_i)^{\sim}\odot x\leq\left(\bigwedge\limits_{j\in I}x_j\right)^{\sim}\odot x\Rightarrow x_i\oplus(x_i)^{\sim}\odot x\leq x_i\oplus\left(\left(\bigwedge\limits_{j\in I}x_j\right)^{\sim}\odot x\right)\Leftrightarrow x\vee x_i\leq x_i\oplus\left(\left(\bigwedge\limits_{j\in I}x_j\right)^{\sim}\odot x\right)$.

Hence, $\bigwedge\limits_{i\in I}(x\vee x_i)\leq\bigwedge\limits_{i\in I}\left[x_i\oplus\left(\left(\bigwedge\limits_{j\in I}x_j\right)^{\sim}\odot x\right)\right]$.

We obtain $\bigwedge\limits_{i\in I}(x\vee x_i)\leq x\vee\left(\bigwedge\limits_{i\in I}x_i\right)$.

The inequality $x\vee\left(\bigwedge\limits_{i\in I}x_i\right)\leq\bigwedge\limits_{i\in I}(x\vee x_i)$ is obvious.

$(psmv-c_{24})$. Obviously, $x\oplus x_i\leq x\oplus\left(\bigvee\limits_{i\in I}x_i\right)$, for every $i\in I$. Let now $x\oplus x_i\leq y$, for every $i\in I$; remark that $x\leq y$.

For every $i\in I$ we have $x^{\sim}\odot(x\oplus x_i)\leq x^{\sim}\odot y$.

On other hand $x^{\sim}\odot(x\oplus x_i)=x^{\sim}\odot((x^{\sim})^{-}\oplus x_i)=x^{\sim}\wedge x_i$, hence $x^{\sim}\wedge x_i\leq x^{\sim}\odot y$. It follows that $x^{\sim}\wedge\left(\bigvee\limits_{i\in I}x_i\right)\overset{psmv-c_{22}}{=}\bigvee\limits_{i\in I}(x^{\sim}\wedge x_i)\leq x^{\sim}\odot y$, hence $x\oplus\left[x^{\sim}\wedge\left(\bigvee\limits_{i\in I}x_i\right)\right]\leq x\oplus(x^{\sim}\odot y)=x\vee y=y$, since $x\leq y$; but $x\oplus\left[x^{\sim}\wedge\left(\bigvee\limits_{i\in I}x_i\right)\right]=(x\oplus x^{\sim})\wedge\left[x\oplus\left(\bigvee\limits_{i\in I}x_i\right)\right]=x\oplus\left(\bigvee\limits_{i\in I}x_i\right)$.

Finally, we obtain $x \oplus \left(\bigvee_{i\in I} x_i\right) \leq y$. Therefore we get that $x \oplus \left(\bigvee_{i\in I} x_i\right) = \bigvee_{i\in I}(x \oplus x_i)$.

$(psmv - c_{25})$. As in the case of $psmv - c_{24}$.

$(psmv - c_{26})$. Obviously, $x \odot \left(\bigwedge_{i\in I} x_i\right) = x \odot x_i$, for every $i \in I$. Let now $y \leq x \odot x_i$, for every $i \in I$; remark that $y \leq x$.

For every $i \in I$ we have $x^- \oplus y \leq x^- \oplus (x \odot x_i)$. But $x^- \oplus (x \odot x_i) = x^- \oplus [(x^-)^\sim \odot x_i] = x^- \vee x_i$, for every $i \in I$.

So, $x^- \oplus y \leq x^- \vee x_i$, for every $i \in I$. It follows that $x^- \oplus y \leq \bigwedge_{i\in I}(x^- \vee x_i) \overset{psmv-c_{23}}{=} x^- \vee \left(\bigwedge_{i\in I} x_i\right)$. Hence $x \odot \left[x^- \vee \left(\bigwedge_{i\in I} x_i\right)\right] \geq x \odot (x^- \oplus y) = x \wedge y = y$, since $y \leq x$; but $x \odot \left[x^- \vee \left(\bigwedge_{i\in I} x_i\right)\right] = (x \odot x^-) \vee \left[x \odot \left(\bigwedge_{i\in I} x_i\right)\right] = x \odot \left(\bigwedge_{i\in I} x_i\right)$, so $x \odot \left(\bigwedge_{i\in I} x_i\right) \geq y$.

Therefore we get that $x \odot \left(\bigwedge_{i\in I} x_i\right) = \bigwedge_{i\in I}(x \odot x_i)$.

$(psmv - c_{27})$. Has a similar proof with $psmv - c_{26}$. ■

By Proposition 4.5 and Theorem 4.6 we deduce:

COROLLARY 4.7. *On A, the natural order determines a distributive bounded lattice structure.*

We shall denote this distributive lattice with 0 and 1 by $L(A)$ (see [**42**], [**45**]).

THEOREM 4.8. *If x, y, z are elements of A, then the following hold:*

$(psmv - c_{28})$ $(x \wedge y)^- = x^- \vee y^-, (x \vee y)^- = x^- \wedge y^-$;
$(psmv - c_{29})$ $(x \wedge y)^\sim = x^\sim \vee y^\sim, (x \vee y)^\sim = x^\sim \wedge y^\sim$;
$(psmv - c_{30})$ $x \odot y^- \wedge y \odot x^- = 0, x^\sim \odot y \wedge y^\sim \odot x = 0$;
$(psmv - c_{31})$ $(y \oplus x^\sim) \vee (x \oplus y^\sim) = 1, (y^- \oplus x) \vee (x^- \oplus y) = 1$;
$(psmv - c_{32})$ $x \vee y = x \odot (x \wedge y)^- \oplus y$;
$(psmv - c_{33})$ $x \wedge y = 0 \Rightarrow x \oplus y = x \vee y$;
$(psmv - c_{34})$ $x \wedge y = 0 \Rightarrow x \wedge (y \oplus z) = x \wedge z$;
$(psmv - c_{35})$ *If* $y \oplus x = z \oplus x$ *and* $x \odot y = x \odot z$ *then* $y = z$;
$(psmv - c_{36})$ *If* $x \oplus y = x \oplus z$ *and* $y \odot x = z \odot x$ *then* $y = z$;
$(psmv - c_{37})$ $x \oplus y = y$ *iff* $x^- \oplus y^- = x^-$;
$(psmv - c_{38})$ $x \oplus y = x$ *iff* $x^\sim \oplus y^\sim = y^\sim$;
$(psmv - c_{39})$ $x \oplus x = x$ *iff* $x \odot x = x$.

Proof. $(psmv - c_{28})$. We have $(x \wedge y)^- = ((x \oplus y^\sim) \odot y)^- = y^- \oplus (x \oplus y^\sim)^- = y^- \oplus (y \odot x^-) = y^- \oplus [(y^-)^\sim \odot x^-] = x^- \vee y^-$ and $(x \vee y)^- = (x \oplus x^\sim \odot y)^- = (x^\sim \odot y)^- \odot x^- = (y^- \oplus x) \odot x^- = (y^- \oplus (x^-)^\sim) \odot x^- = x^- \wedge y^-$.

$(psmv - c_{29})$. We have $(x \wedge y)^\sim = [x \odot (x^- \oplus y)]^\sim = (x^- \oplus y)^\sim \oplus x^\sim = (y^\sim \odot x) \oplus x^\sim = [y^\sim \odot (x^\sim)^-] \oplus x^\sim = x^\sim \vee y^\sim$ and $(x \vee y)^\sim = [(x \odot y^-) \oplus y]^\sim = y^\sim \odot (x \odot y^-)^\sim = y^\sim \odot [(y^-)^\sim \oplus x^\sim] = y^\sim \odot [(y^\sim)^- \oplus x^\sim] = x^\sim \wedge y^\sim$.

$(psmv - c_{30})$. We have $x \odot y^- = 0 \vee x \odot y^- = x \odot x^- \vee x \odot y^- = x \odot (x^- \vee y^-)$.

Similarly, $y \odot x^- = y \odot (x^- \vee y^-)$. Then $x \odot y^- \wedge y \odot x^- = [x \odot (x^- \vee y^-)] \wedge [y \odot (x^- \vee y^-)] = (x \wedge y) \odot (x^- \vee y^-) = (x \wedge y) \odot (x \wedge y)^- = 0$.

The second equality is obtained by replacing x by $x^{\sim}$ and y by $y^{\sim}$ in the first one.

$(psmv-c_{31})$. Follows by $psmv-c_{30}$ applying $^{\sim}$ and $^{-}$, respectively.

$(psmv-c_{32})$. $x\odot(x\wedge y)^{-}\oplus y \overset{psmv-c_{28}}{=} x\odot(x^{-}\vee y^{-})\oplus y=[(x\odot x^{-})\vee(x\odot y^{-})]\oplus y=(x\odot y^{-})\oplus y=x\vee y$.

$(psmv-c_{33})$. If $x\wedge y=0$, then in $psmv-c_{32}$ we obtain $x\vee y=x\odot 0^{-}\oplus y=x\odot 1\oplus y=x\oplus y$.

$(psmv-c_{34})$. First, we remark that $x\leq x\wedge(x\oplus z)=(0\oplus x)\wedge(x\oplus z)=[(x\wedge y)\oplus x]\wedge(x\oplus z)=(x\oplus x)\wedge(y\oplus x)\wedge(x\oplus z)$. Then $x\wedge z\leq x\wedge(y\oplus z)\leq(x\oplus x)\wedge(y\oplus x)\wedge(x\oplus z)\wedge(y\oplus z)=(x\wedge y)\oplus(x\wedge z)=x\wedge z$.

$(psmv-c_{35})$. We get $x^{-}\vee y=x^{-}\oplus[(x^{-})^{\sim}\odot y]=x^{-}\oplus(x\odot y)=x^{-}\oplus(x\odot z)=x^{-}\oplus[(x^{-})^{\sim}\odot z]=x^{-}\vee y$ and $x^{-}\wedge y=[y\oplus(x^{-})^{\sim}]\odot x^{-}=(y\oplus x)\odot x^{-}=(z\oplus x)\odot x^{-}=[z\oplus(x^{-})^{\sim}]\odot x^{-}=x^{-}\wedge z$. Since $(A,\vee,\wedge)$ is a distributive lattice, it follows that $y=z$.

$(psmv-c_{36})$. Similarly proof with $psmv-c_{35}$.

$(psmv-c_{37})$. We have the following implications: $x\oplus y=y\Rightarrow[x\oplus(y^{-})^{\sim}]\odot y^{-}=y\odot y^{-}=0\Rightarrow x\wedge y^{-}=0\Rightarrow x^{-}=x^{-}\oplus(x\wedge y^{-})=(x^{-}\oplus x)\wedge(x^{-}\oplus y^{-})=x^{-}\oplus y^{-}$ and $x^{-}\oplus y^{-}=x^{-}\Rightarrow x\wedge y^{-}=x\odot(x^{-}\oplus y^{-})=x\odot x^{-}=0\Rightarrow y=(x\wedge y^{-})\oplus y=(x\oplus y)\wedge(y^{-}\oplus y)=x\oplus y$.

$(psmv-c_{38})$. Similarly, we have the following implications: $x\oplus y=x\Rightarrow x^{\sim}\wedge y=x^{\sim}\odot((x^{\sim})^{-}\oplus y)=x^{\sim}\odot(x\oplus y)=x^{\sim}\odot x=0\Rightarrow y^{\sim}=(x^{\sim}\wedge y)\oplus y^{\sim}=(x^{\sim}\oplus y^{\sim})\wedge(y\oplus y^{\sim})=x^{\sim}\oplus y^{\sim}$ and $x^{\sim}\oplus y^{\sim}=y^{\sim}\Rightarrow x^{\sim}\wedge y=(x^{\sim}\oplus y^{\sim})\odot y=y^{\sim}\odot y=0\Rightarrow x\oplus(x^{\sim}\wedge y)=x\Rightarrow(x\oplus x^{\sim})\wedge(x\oplus y)=x\Rightarrow x\oplus y=x$.

$(psmv-c_{39})$. By $psmv-c_{37}$, $x\oplus x=x\Leftrightarrow x^{-}\oplus x^{-}=x^{-}\Leftrightarrow(x^{-}\oplus x^{-})^{\sim}=(x^{-})^{\sim}\Leftrightarrow x\odot x=x$. ■

PROPOSITION 4.9. *Every commutative pseudo $MV-$ algebra (i.e. $\oplus$ is commutative) is an $MV-$ algebra.*

Proof. Since $x\oplus y=y\oplus x$, for any $x,y\in A$, it follows that $x\odot y=(y^{-}\oplus x^{-})^{\sim}=(x^{-}\oplus y^{-})^{\sim}=y\odot x$. Hence $x^{-}\oplus x=1=x^{\sim}\oplus x$ and $x\odot x^{-}=0=x\odot x^{\sim}$.

Then by $psmv-c_{35}$ we deduce that $x^{-}=x^{\sim}$, for any $x\in A$, so $(A,\oplus,\odot,^{-},0,1)$ is an $MV-$ algebra. ■

LEMMA 4.10. *If a,b,x are elements of A, then:*

$(psmv-c_{40})$ $[(a\wedge x)\oplus(b\wedge x)]\wedge x=(a\oplus b)\wedge x$;

$(psmv-c_{41})$ $x\wedge a^{-}\geq x\odot(a\wedge x)^{-}$ *and* $a^{\sim}\wedge x\geq(a\wedge x)^{\sim}\odot x$.

Proof. $(psmv-c_{40})$. By $psmv-c_{18}$ and $psmv-c_{19}$ we have

$$[(a\wedge x)\oplus(b\wedge x)]\wedge x=((a\wedge x)\oplus b)\wedge((a\wedge x)\oplus x)\wedge x=$$
$$=((a\wedge x)\oplus b)\wedge x=(a\oplus b)\wedge(x\oplus b)\wedge x=(a\oplus b)\wedge x.$$

$(psmv-c_{41})$. We have

$$x\odot(a\wedge x)^{-}=x\odot(a^{-}\vee x^{-})\overset{psmv-c_{20}}{=}(x\odot a^{-})\vee(x\odot x^{-})$$
$$\overset{psmv-c_{7}}{=}(x\odot a^{-})\vee 0=x\odot a^{-}\leq x\wedge a^{-}$$

and

$$(a\wedge x)^{\sim}\odot x=(a^{\sim}\vee x^{\sim})\odot x\overset{psmv-c_{21}}{=}(a^{\sim}\odot x)\vee(x^{\sim}\odot x)$$
$$\overset{psmv-c_{7}}{=}(a^{\sim}\odot x)\vee 0=a^{\sim}\odot x\leq a^{\sim}\wedge x.$$■

2. Boolean center

For a pseudo $MV-$ algebra A we denote by $B(A)$ the boolean algebra associated with the bounded distributive lattice $L(A)$. Elements of $B(A)$ are called the *boolean elements* of A.

We characterize the elements of $B(A)$ in therms of pseudo $MV-$ algebra operations.

THEOREM 4.11. *For every element e in a pseudo $MV-$ algebra A, the following conditions are equivalent:*

(i) $e \in B(A)$;
(ii) $e \vee e^- = 1$;
(iii) $e \vee e^\sim = 1$;
(iv) $e \wedge e^- = 0$;
(v) $e \wedge e^\sim = 0$;
(vi) $e \oplus e = e$;
(vii) $e \odot e = e$.

Proof. First we prove the equivalences: $(iii) \Leftrightarrow (iv) \Leftrightarrow (vi)$.

$(iii) \Rightarrow (vi)$. $e \vee e^\sim = 1 \Rightarrow e = (e \vee e^\sim) \odot e = (e \odot e) \vee (e^\sim \odot e) = e \odot e$. Then apply $psmv - c_{39}$.

$(vi) \Rightarrow (iv)$. $e \oplus e = e \Rightarrow e \wedge e^- = [e \oplus (e^-)^\sim] \odot e^- = (e \oplus e) \odot e^- = e \odot e^- = 0$.

$(iv) \Rightarrow (iii)$. $e \wedge e^- = 0 \Rightarrow e \vee e^\sim = e^\sim \vee (e^-)^\sim = (e \wedge e^-)^\sim = 0^\sim = 1$.

Hence, $(iii) \Leftrightarrow (iv) \Leftrightarrow (vi)$.

Similarly, $(ii) \Leftrightarrow (v) \Leftrightarrow (vi)$.

We deduce that the equivalent conditions (ii) and (iv) state that e^- is a complement of e, thus, in particular, $(iv) \Rightarrow (i)$, and that the equivalent conditions (iii) and (v) state that $e^\sim$ is also a complement of e.

$(i) \Rightarrow (iv)$. Assume there exists $a \in A$ such that $e \wedge a = 0$ and $e \vee a = 1$. Thus, $e \wedge a = 0 \Rightarrow e^- = e^- \oplus (e \wedge a) = (e^- \oplus e^-) \wedge (e^- \oplus a) = e^- \oplus a \Rightarrow a \leq e^-$ and $e \vee a = 1 \Rightarrow e^- = (e \vee a) \odot e^- = (e \odot e^-) \vee (a \odot e^-) = a \odot e^- \Rightarrow e^- \leq a$. We deduce $a = e^-$, so, $e \wedge e^- = 0$.

$(iv) \Rightarrow (i)$. From $e \wedge e^- = 0$ it follows that, by $psmv - c_{33}, e \vee e^- = e \oplus e^- = 1$. Hence e^- is the complement of e. That is, $e \in B(A)$.

$(vi) \Leftrightarrow (vii)$. See $psmv - c_{39}$.

$(vii) \Leftrightarrow (ii)$. $e \odot e = e \Rightarrow e^- \vee e = e^- \oplus [(e^-)^\sim \odot e] = e^- \oplus (e \odot e) = e^- \oplus e = 1$.

■

REMARK 4.3. *By Theorem 4.11 it follows that for every $e \in B(A)$, $e^- = e^\sim$.*

PROPOSITION 4.12. *If $e \in B(A)$ and $x \in A$, then*

(i) $e \oplus x = e \vee x = x \oplus e$, *for all* $x \in A$;
(ii) $e \odot x = e \wedge x = x \odot e$, *for all* $x \in A$.

Proof. (i). We have that $e \vee x \leq e \oplus x$ and $(e \oplus x) \odot (e \vee x)^- = (e \oplus x) \odot (e^- \wedge x^-) = [(e \oplus x) \odot e^-] \wedge [(e \oplus x) \odot x^-] = [(e \oplus x) \odot e^-] \wedge [(e \oplus (x^-)^\sim) \odot x^-] = [(e \oplus x) \odot e^-] \wedge (e \wedge x^-) \leq e \wedge e^- = 0$, so $e \oplus x \leq e \vee x$; thus, $e \oplus x = e \vee x$.

Analogously, $e \vee x \leq x \oplus e$ and $(e \vee x)^\sim \odot (x \oplus e) = [e^\sim \odot (x \oplus e)] \wedge [x^\sim \odot (x \oplus e)] = [e^\sim \odot (x \oplus e)] \wedge [x^\sim \odot ((x^\sim)^- \oplus e)] = [e^\sim \odot (x \oplus e)] \wedge (x^\sim \wedge e) \leq e^\sim \wedge e = 0$, hence $x \oplus e \leq e \vee x$; thus, $x \oplus e = e \vee x$.

(ii). We have that $x \odot e = (e^- \oplus x^-)^\sim \overset{(i)}{=} (e^- \vee x^-)^\sim \overset{(i)}{=} (x^- \oplus e^-)^\sim = e \odot x$. Since $(e^- \vee x^-)^\sim = e \wedge x$ we obtain that $e \odot x = e \wedge x = x \odot e$. ■

COROLLARY 4.13. *(i) $B(A)$ is subalgebra of the pseudo $MV-$ algebra A. A subalgebra B of A is a boolean algebra iff $B \subseteq B(A)$,*
(ii) A pseudo $MV-$ algebra A is a boolean algebra iff the operation $\oplus$ is idempotent, i.e., the equation $x \oplus x = x$ is satisfied in A.

PROPOSITION 4.14. *For $x \in A$, the following are equivalent:*

(i) there is a natural number $n \geq 1$ such that $nx \in B(A)$;
(ii) there is a natural number $n \geq 1$ such that $x^- \vee nx = 1$;
(iii) there is a natural number $n \geq 1$ such that $x^\sim \vee nx = 1$;
(iv) there is a natural number $n \geq 1$ such that $nx = (n+1)x$.

Proof. First we prove the equivalences: $(i) \Leftrightarrow (iv) \Leftrightarrow (ii)$.

$(i) \Rightarrow (iv)$. Suppose that $nx \in B(A)$. Then $nx \oplus nx = nx$ and $nx = (n+1)x$, since $nx \leq (n+1)x \leq 2nx = nx$.

$(iv) \Rightarrow (ii)$. If $nx = (n+1)x$, then $x^- \vee nx = x^- \odot (nx)^- \oplus (nx) = [(n+1)x]^- \oplus (nx) = (nx)^- \oplus (nx) = 1$.

$(ii) \Rightarrow (i)$. Assume that $x^- \vee nx = 1$; hence $[(n+1)x]^- \oplus (nx) = (nx \oplus x)^- \oplus (nx) = [x^- \odot (nx)^-] \oplus (nx) = x^- \vee nx = 1 \Rightarrow (n+1)x \leq nx \Rightarrow (n+1)x = nx \Rightarrow nx = nx \oplus nx \Rightarrow nx \in B(A)$.

We prove that the equivalences: $(i) \Leftrightarrow (iv) \Leftrightarrow (iii)$.

$(iv) \Rightarrow (iii)$. If $nx = (n+1)x$, then $x^\sim \vee nx = (nx) \oplus [(nx)^\sim \odot x^\sim] = (nx) \oplus (x \oplus nx)^\sim = (nx) \oplus [(n+1)x]^\sim = (nx) \oplus (nx)^\sim = 1$.

$(iii) \Rightarrow (i)$. Assume that $x^\sim \vee nx = 1$; hence $(nx) \oplus [(n+1)x]^\sim = (nx) \oplus [x \oplus (nx)]^\sim = (nx) \oplus [(nx)^\sim \odot x^\sim] = x^\sim \vee nx = 1 \Rightarrow (n+1)x \leq nx \Rightarrow (n+1)x = nx \Rightarrow nx = nx \oplus nx \Rightarrow nx \in B(A)$. ■

COROLLARY 4.15. *If $a \in B(A)$, then for all $x, y \in A$:*

$(psmv - c_{42})$ *$x \wedge a^- = x \odot (a \wedge x)^-$ and $a^\sim \wedge x = (a \wedge x)^\sim \odot x$;*
$(psmv - c_{43})$ *$a \wedge (x \oplus y) = (a \wedge x) \oplus (a \wedge y)$;*
$(psmv - c_{44})$ *$a \vee (x \oplus y) = (a \vee x) \oplus (a \vee y)$.*

Proof. $(psmv - c_{42})$. See the proof of $psmv - c_{41}$.

$(psmv - c_{43})$. We have:

$$(a \wedge x) \oplus (a \wedge y) \overset{psmv-c_{18}}{=} [(a \wedge x) \oplus a] \wedge [(a \wedge x) \oplus y]$$

$$\overset{psmv-c_{18}}{=} [(a \oplus a) \wedge (x \oplus a)] \wedge [(a \wedge x) \oplus y] =$$

$$= a \wedge (x \oplus a) \wedge [(a \oplus y) \wedge (x \oplus y)] = a \wedge (a \oplus y) \wedge (x \oplus y) = a \wedge (x \oplus y).$$

$(psmv - c_{44})$. We have

$$(a \vee x) \oplus (a \vee y) = (a \oplus x) \oplus (a \oplus y) =$$

$$= (a \oplus a) \oplus (x \oplus y) = a \oplus (x \oplus y) = a \vee (x \oplus y).$$ ■

3. Homomorphisms and ideals

DEFINITION 4.2. Let A and B be pseudo $MV-$ algebras. A function $f : A \to B$ is a *morphism of pseudo $MV-$ algebras* if it satisfies the following conditions, for every $x, y \in A$:

$(psMV_9)$ $f(0) = 0$;
$(psMV_{10})$ $f(x \oplus y) = f(x) \oplus f(y)$;
$(psMV_{11})$ $f(x^-) = (f(x))^-$;
$(psMV_{12})$ $f(x^\sim) = (f(x))^\sim$.

REMARK 4.4. *It follows that:*

$$f(1) = 1,$$

$$f(x \odot y) = f(x) \odot f(y),$$

$$f(x \vee y) = f(x) \vee f(y),$$

$$f(x \wedge y) = f(x) \wedge f(y),$$

for every $x, y \in A$.

Proof. We have $f(1) = 1$ since $1 = x \oplus x^\sim$ in A implies $f(1) = f(x \oplus x^\sim) = f(x) \oplus f(x^\sim) = f(x) \oplus (f(x))^\sim = 1$ in B;

$f(x \odot y) = f(x) \odot f(y)$ since $f(x \odot y) = f((y^\sim \oplus x^\sim)^-) = (f(y^\sim \oplus x^\sim))^- = (f(y^\sim) \oplus f(x^\sim))^-) = ((f(x) \odot f(y))^\sim)^-) = f(x) \odot f(y)$.

By Proposition 4.5 we deduce $f(x \vee y) = f(x \oplus (x^\sim \odot y)) = f(x) \oplus f(x^\sim \odot y) = f(x) \oplus [f(x^\sim) \odot f(y)] = f(x) \oplus [f(x)^\sim \odot f(y)] = f(x) \vee f(y)$ and $f(x \wedge y) = f(x \odot (x^- \oplus y)) = f(x) \odot f(x^- \oplus y) = f(x) \odot [f(x^-) \oplus f(y)] = f(x) \odot [f(x)^- \oplus f(y)] = f(x) \wedge f(y)$. ■

We recall that a bijective morphism f of pseudo $MV-$ algebras is called *isomorphism* of pseudo $MV-$ algebras; in this case we write $A \approx B$.

In any pseudo $MV-$ algebra A one can define two *distance functions*:

$$d_-(x, y) := (x \odot y^-) \oplus (y \odot x^-), d_\sim(x, y) := (x^\sim \odot y) \oplus (y^\sim \odot x).$$

PROPOSITION 4.16. *The two distances verify the following properties:*

(i) $d_-(x, y) = (x \odot y^-) \vee (y \odot x^-), d_\sim(x, y) = (x^\sim \odot y) \vee (y^\sim \odot x)$;
(ii) $d_-(x, y) = d_-(y, x)$ *and* $d_\sim(x, y) = d_\sim(y, x)$;
(iii) $d_-(x, z) \leq d_-(x, y) \oplus d_-(y, z) \oplus d_-(x, y)$ *and* $d_\sim(x, z) \leq d_\sim(x, y) \oplus d_\sim(y, z) \oplus d_\sim(x, y)$.

Proof. (i). Follow by $psmv - c_{30}$ and $psmv - c_{33}$.
(ii). Follow by (i) and by commutativity of $\vee$.
(iii). Follow by [**68**], Proposition 1.35, (9) and (10). ■
Fore more details relative to distance functions see [**68**], Proposition 1.35.

DEFINITION 4.3. An *ideal* of a pseudo $MV-$ algebra A is a nonempty subset I of A satisfying the following conditions:

(I_1') $0 \in I$;
(I_2') If $x \in I$, $y \in A$ and $y \leq x$, then $y \in I$;
(I_3') If $x, y \in I$, then $x \oplus y \in I$.

If A is a pseudo $MV-$ algebra, then an ideal I of A is *proper* if $I \neq A$. We denote by $Id(A)$ the set of all ideals of A. The intersection of any family of ideals of A is still an ideal of A.

For every subset $M \subseteq A$, the smallest ideal of A which contains M (i.e., the intersection of all ideals $I \supseteq M$), is said to be the *ideal generated* by M, and we denote by $(M]$ this ideal. If $M = \{a\}$ with $a \in A$, we denote by $(a]$ the ideal generated by $\{a\}$($(a]$ is called *principal*).

As in the case of $MV-$algebras we have:

PROPOSITION 4.17. *If $M \subseteq A$, then*

$$(M] = \{x \in A : x \leq x_1 \oplus ... \oplus x_n \text{ for some } x_1, ..., x_n \in M\}.$$

In particular, for $a \in A$, $(a] = \{x \in A : x \leq na$ for some integer $n \geq 0\}$; if $e \in B(A)$, then $(e] = \{x \in A : x \leq e\}$. Remark that $(0] = \{0\}$ and $(1] = A$. Also, for every ideal I of pseudo MV -algebra A and each $a \in A$ we have $(I \cup \{a\}] = \{x \in A : x \leq (x_1 \oplus n_1 a) \oplus ... \oplus (x_m \oplus n_m a)$ for some $x_1, ..., x_m \in M$ and for some integers $m \geq 1$ and $n_1, ..., n_m \geq 0\}$.

For any ideal I, one can associate two equivalence relations $\equiv_{L(I)}$ and $\equiv_{R(I)}$ on A defined by:

$$x \equiv_{L(I)} y \text{ iff } d_-(x, y) \in I,$$
$$x \equiv_{R(I)} y \text{ iff } d_\sim(x, y) \in I.$$

LEMMA 4.18. *The relations $\equiv_{L(I)}$ and $\equiv_{R(I)}$ are equivalence relations on A.*

Proof. The relation $\equiv_{L(I)}$is reflexive since $x \equiv_{L(I)} x \Leftrightarrow d_-(x, x) = (x \odot x^-) \oplus (x \odot x^-) = 0 \oplus 0 = 0 \in I$, which is true. For symmetry we have $x \equiv_{L(I)} y \Leftrightarrow d_-(x, y) \in I \Leftrightarrow d_-(y, x) \in I \Leftrightarrow y \equiv_{L(I)} x$ (by Proposition 4.16, (ii)).

The relation $\equiv_{L(I)}$is tranzitive since ($x \equiv_{L(I)} y$ and $y \equiv_{L(I)} z$)$\Leftrightarrow$ ($d_-(x, y) \in I$ and $d_-(y, z) \in I$) $\Rightarrow d_-(x, y) \oplus d_-(y, z) \oplus d_-(x, y) \in I \Rightarrow d_-(x, z) \in I$ (by Proposition 4.16, (iii)) $\Leftrightarrow x \equiv_{L(I)} z$.

The proof that $\equiv_{R(I)}$ is an equivalence relation is similar. ■

REMARK 4.5. $I = \{x \in A : x \equiv_{L(I)} 0\} = \{x \in A : x \equiv_{R(I)} 0\}$.

Proof. We have $x = d_-(x, 0) \in I \Leftrightarrow x \equiv_{L(I)} 0$ and $x = d_\sim(x, 0) \in I \Leftrightarrow x \equiv_{R(I)} 0$. ■

The relations $\equiv_{L(I)}$and $\equiv_{R(I)}$being equivalence relations, we can consider the quotient sets $A/\equiv_{L(I)}$and $A/\equiv_{R(I)}$. We denote by $x/\equiv_{L(I)}$and $x/\equiv_{R(I)}$the equivalence classes of an element $x \in A$ and called this classes left and right class of x. We define on the set of classes two binary relations $\leq_{L(I)}$ and $\leq_{R(I)}$ by: $x/\equiv_{L(I)} \leq_{L(I)} y/\equiv_{L(I)}$ iff $x \odot y^- \in I$ and $x/\equiv_{R(I)} \leq_{R(I)} y/\equiv_{R(I)}$ iff $y^\sim \odot x \in I$. It eassy to prove that the relations $\leq_{L(I)}$ and $\leq_{R(I)}$ are partial order relations on the respective sets.

For any ideal I, the map $\phi : A/\equiv_{L(I)} \to A/\equiv_{R(I)}$defined by $\phi(x/\equiv_{L(I)}) = (x^-)/\equiv_{R(I)}$is a bijection between the sets $A/\equiv_{L(I)}$ and $A/\equiv_{R(I)}$.

We shall examine the set $Id(A)$ of ideals of a pseudo $MV-$ algebra A.

It is eassy to prove that $(Id(A), \cap, \vee)$ is a complete Browerian lattice, where meet is the intersection of sets and the join of an arbitrary collection of ideals is the ideal generated by the union (as sets) of these ideals, the order relation being the inclusion of sets.

An ideal with the property that its set of left classes is totally ordered is called *prime*.

REMARK 4.6. *For any ideal I of A, the set of left classes is totally ordered by $\leq_{L(I)}$ iff the set of right classes is totally ordered by $\leq_{R(I)}$. Indeed, $x/\equiv_{L(I)} \leq_{L(I)} y/\equiv_{L(I)}$ iff $x\odot y^{-}\in I$ iff $(x^{-})^{\sim}\odot y^{-}\in I$ iff $(x^{-})/\equiv_{R(I)} \geq_{R(I)} (y^{-})/\equiv_{R(I)}$ iff $\phi(x/\equiv_{L(I)}) \geq_{R(I)} \phi(y/\equiv_{L(I)})$.*

The prime ideals are characterized in the next theorem:

THEOREM 4.19. *For $P\in Id(A)$ the following are equivalent:*

(i) P is prime (that is, $A/\equiv_{L(P)}$ or equivalently $A/\equiv_{R(P)}$, is totally ordered);
(ii) $\{I\in Id(A) : I\supseteq P\}$ is totally ordered under inclusion;
(iii) P is finitely meet-irreducible in $Id(A)$;
(iv) If $a\wedge b\in P$ then $a\in P$ or $b\in P$;
(v) If $a\wedge b=0$ then $a\in P$ or $b\in P$;
(vi) For any $a,b\in A, a\odot b^{-}\in P$ or $b\odot a^{-}\in P$;
(vii) For any $a,b\in A, a^{\sim}\odot b\in P$ or $b^{\sim}\odot a\in P$.

Proof. First we prove the implications $(i)\Rightarrow(ii)\Rightarrow(iii)\Rightarrow(iv)\Rightarrow(v)\Rightarrow(i)$.

$(i)\Rightarrow(ii)$. Suppose that I,J are incomparable ideals containing $P: I\supseteq P, J\supseteq P$ and $I\not\subseteq J, J\not\subseteq I$. Then there exists $i\in I\backslash J$ and $j\in J\backslash I$. Let us consider the left classes $i/\equiv_{L(P)}$ and $j/\equiv_{L(P)}$. By (i), we have $i/\equiv_{L(P)}\leq_{L(P)} j/\equiv_{L(P)}$ or $j/\equiv_{L(P)}\leq_{L(P)} i/\equiv_{L(P)}$, so, $i\odot j^{-}\in P$ or $j\odot i^{-}\in P$. We deduce that $i\odot j^{-}\oplus j = i\vee j\in J$ or $j\odot i^{-}\oplus i = j\vee i\in I$, hence $i\in J$ or $j\in I$, a contradiction.

$(ii)\Rightarrow(iii)$. If $I\cap J=P$ then $P\subseteq I$ and $P\subseteq J$. By (ii), we have $I\subseteq J$ or $J\subseteq I$. Suppose $I\subseteq J$; then $P=I\cap J=I$, so $P=I$.

$(iii)\Rightarrow(iv)$.Since $(a\wedge b]\subseteq P$ we obtain $(P\vee(a])\cap(P\vee(b]) = (P\vee((a]\cap(b])) = (P\vee(a\wedge b]) = P$. By (iii), it follows that $P=P\vee(a]$ or $P=P\vee(b]$ so, $a\in P$ or $b\in P$.

$(iv)\Rightarrow(v)$. If $a\wedge b=0\in P$ then by (iv), $a\in P$ or $b\in P$.

$(v)\Rightarrow(i)$. Let $x/\equiv_{L(P)}, y/\equiv_{L(P)}\in A/\equiv_{L(P)}$; since $a\odot b^{-}\wedge b\odot a^{-}=0\in P$ we deduce by (v) that $a/\equiv_{L(P)}\leq_{L(P)} b/\equiv_{L(P)}$ or $b/\equiv_{L(P)}\leq_{L(P)} a/\equiv_{L(P)}$, so $A/\equiv_{L(P)}$ is totally ordered by $\leq_{L(P)}$.

$(v)\Rightarrow(vi)$. Since by $psmv-c_{30}$, $a\odot b^{-}\wedge b\odot a^{-}=0\in P$ we deduce that $a\odot b^{-}\in P$ or $b\odot a^{-}\in P$.

$(vi)\Rightarrow(iv)$. Suppose that $a\wedge b\in P$, and that $a\odot b^{-}\in P$, hence $(a\odot b^{-})\oplus(a\wedge b)\in P$. But $a\leq[(a\odot b^{-})\oplus a]\wedge(a\vee b) = [(a\odot b^{-})\oplus a]\wedge[(a\odot b^{-})\oplus b] = (a\odot b^{-})\oplus(a\wedge b)$. We get that $a\in P$.

$(v)\Rightarrow(vii)$. Following by $psmv-c_{30}$, $a^{\sim}\odot b\wedge a^{\sim}\odot b=0$, so we deduce that $a^{\sim}\odot b\in P$ or $b^{\sim}\odot a\in P$.

$(vii)\Rightarrow(iv)$. Suppose that $a\wedge b\in P$ and $a^{\sim}\odot b\in P$. Since $b\leq(a\vee b)\oplus[b\oplus(a^{\sim}\odot b)] = [a\oplus(a^{\sim}\odot b)]\wedge[b\oplus(a^{\sim}\odot b)] = (a\wedge b)\oplus(a^{\sim}\odot b)\in P$. We get that $b\in P$. ■

By Theorem 4.19 follows immediately:

COROLLARY 4.20. *If $P,Q\in Id(A), P\subseteq Q$ and P is prime, then Q is prime.*

Proof. As in the case of MV- algebras (see the proof of Theorem 2.27). ■

THEOREM 4.21. *(Prime ideal theorem) Let A be a pseudo $MV-$ algebra, $I\in Id(A)$ and $a\in A\backslash I$. Then there is a prime ideal P of A such that $I\subseteq P$ and $a\notin P$.*

Proof. A routine application of Zorn's Lemma shows that there is an ideal $P \in Id(A)$ which is maximal with respect to the property that $I \subseteq P$ and $a \notin P$. We shall prove that P is a prime ideal. Let $x, y \in A$ and suppose that P is not prime, i.e., $x \odot y^- \notin P$ and $y \odot x^- \notin P$. Then the ideal $(P \cup \{x \odot y^-\}]$ moust contain the element a. By Remark 4.17, $a \leq (s_1 \oplus n_1(x \odot y^-)) \oplus ... \oplus (s_m \oplus n_m(x \odot y^-))$ for some $s_1, ..., s_m \in P$ and for some integers $m \geq 1$ and $n_1, ..., n_m \geq 0$. Similarly, there is $t_1, ..., t_k \in P$ and for some integers $k \geq 1$ and $q_1, ..., q_k \geq 0$. such that $a \leq (t_1 \oplus q_1(y \odot x^-)) \oplus ... \oplus (t_k \oplus q_k(y \odot x^-))$. Let $s = s_1 \oplus ... \oplus s_m$ and $t = t_1 \oplus ... \oplus t_k$; then $s, t \in P$. Let $n = \max\limits_{i=1,m}\{n_i\}$ and $q = \max\limits_{i=1,k}\{q_i\}$. Then $a \leq m(s \oplus n(x \odot y^-))$ and $a \leq k(t \oplus q(y \odot x^-))$. Let now $u = s \oplus t$ and $p = \max\{n, q\}$. Then $u \in P$, $a \leq m(u \oplus p(x \odot y^-))$ and $a \leq k(u \oplus p(y \odot x^-))$. Hence $a \leq [m(u \oplus p(x \odot y^-))] \wedge [k(u \oplus p(y \odot x^-))] \leq k[(m(u \oplus p(x \odot y^-))) \wedge (u \oplus p(y \odot x^-))] \leq mk([(u \oplus p(x \odot y^-))] \wedge [(u \oplus p(y \odot x^-))]) = mk[u \oplus (p(x \odot y^-) \wedge p(y \odot x^-))]$
$= mk[u \oplus 0] = mku \in P$, hence $a \in P$, a contradiction. ■

Remark 4.7. *If pseudo $MV-$ algebra A is an $MV-$ algebra we obtain Theorem 2.29.*

Definition 4.4. An ideal H is *normal* if the following condition holds:

$$(N) \text{ for every } x, y \in A,\ y \odot x^- \in H \text{ iff } x^\sim \odot y \in H.$$

Lemma 4.22. *Let H be a normal ideal. Then*

(*i*) *The condition (N) is equivalent with the condition*

$$(N') : \textit{for any } x \in A,\ H \oplus x = x \oplus H,$$

that is, for each $h \in H$, there exists $h' \in H$ such that $h \oplus x = x \oplus h'$ and for each $h' \in H$, there exists $h \in H$ such that $x \oplus h' = h \oplus x$.

(*ii*) *The axiom (N) implies implies the following equivalences: $h \in H \Leftrightarrow h^= \in H$ and $h \in H \Leftrightarrow h^\approx \in H$.*

Proof. (i). $(N) \Rightarrow (N')$. Let $x \in A$ and $h \in H$. We put $y = h \oplus x$, $y \leq x$. Then $(y \odot x^-) \oplus x = y \vee x = y = x \oplus (x^\sim \odot y)$. Hence $h \oplus x = y = (y \odot x^-) \oplus x = x \oplus (x^\sim \odot y)$. If $y \odot x^- \in H$ we get that $h' = x^\sim \odot y \in H$, so there exists $h' \in H$ such that $h \oplus x = x \oplus h'$. Similarly, for each $h' \in H$, there exists $h \in H$ such that $x \oplus h' = h \oplus x$. Thus (N') holds.

$(N') \Rightarrow (N)$. (Dvurecenskij) Suppose that $y \odot x^- \in H$; then putting $h_1 = y \odot x^-$ we have $x \vee y = (y \odot x^-) \oplus x = h_1 \oplus x = x \oplus (x^\sim \odot y)$ and there exists $h_2 = x^\sim \odot y \in H$ such that $x \vee y = x \oplus h_2$. Then $x^\sim \odot y \leq x^\sim \odot (x \vee y\ ) = x^\sim \odot (x \oplus h_2) = x^\sim \wedge h_2 \leq h_2 \in H$. It follows that $x^\sim \odot y \in H$. Similarly, if we assume that $x^\sim \odot y \in H$ we obtain that $y \odot x^- \in H$. Thus (N) holds.

(ii). In (N) for $y = 1$ we obtain $x^- \in H$ iff $x^\sim \in H$. Take then $x = x^\sim$ and $x = x^-$; we get that $x \in H \Leftrightarrow x^= \in H$ and $x \in H \Leftrightarrow x^\approx \in H$. ■

Remark 4.8. *If $e \in B(A)$, then $(e] = \{x \in A : x \leq ne = e$, for some $n \geq 1\} = \{x \in A : x \leq e\}$ is a normal ideal of A. Indeed, if $x, y \in (e]$ we get $y \odot x^- \in (e] \Leftrightarrow y \odot x^- \leq e \Leftrightarrow y \leq e \oplus (x^-)^\sim = e \oplus x = x \oplus e = (x^\sim)^- \oplus e \Leftrightarrow x^\sim \odot y \leq e \Leftrightarrow x^\sim \odot y \in (e]$.*

Lemma 4.23. *(Dvurecenskij) Let H be a normal ideal of A and $a \in A \backslash H$. Then $(H \cup \{a\}] = \{x \in A : x \leq h \oplus na$ for some $h \in H$ and some integers $n \geq 1\}$.*

Proof. By Proposition 4.17, $(H \cup \{a\}] = \{x \in A : x \leq (h_1 \oplus n_1 a) \oplus ... \oplus (h_m \oplus n_m a)$ for some $h_1, ..., h_m \in H$ and for some integers $m \geq 1$ and $n_1, ..., n_m \geq 0\}$. If $m = 1$, then $x \leq h_1 \oplus n_1 a$. If $m = 2$, then $x \leq (h_1 \oplus n_1 a) \oplus (h_2 \oplus n_2 a) = h_1 \oplus (n_1 a \oplus h_2) \oplus n_2 a = h_1 \oplus (h_2' \oplus n_1 a) \oplus n_2 a = (h_1 \oplus h_2') \oplus (n_1 a \oplus n_2 a) = h_{12} \oplus n_{12} a$ with $h_{12} = h_1 \oplus h_2' \in H$ and $n_{12} = n_1 + n_2$ is a natural number.

By induction we get that $x \leq h_{12...m} \oplus n_{12...m} a$ with $h_{12...m} = h_{12...m-1} \oplus h_m' \in H$ and $n_{12...m} = n_1 + ... + n_m$ is a natural number. ■

The next proposition generalizes a well known property of maximal ideals in boolean algebras and $MV-$ algebras. The idea that I must be a normal ideal to be able to prove one of the implications by ussing the above lemma belongs to A. Dvurecenskij.

PROPOSITION 4.24. *For any proper normal ideal I of a pseudo $MV-$ algebra A, the following conditions are equivalent:*

(i) I is a maximal ideal of A,

(ii) For each $x \in A \Leftrightarrow [(nx)^{\sim} \in I$ or $(mx)^{-} \in I$ for some integers $n, m \geq 1]$.

Proof. $(i) \Rightarrow (ii)$. Suppose that I is a maximal ideal of A and let $x \in A \backslash I$. Then $(\{x\} \cup I] = (I \cup \{x\}] = A$, so for some integers $m, n \geq 1$ and $a, b \in I$ we have $1 = nx \oplus a = b \oplus mx$. Hence, $1 = ((nx)^{\sim})^{-} \oplus a = b \oplus ((mx)^{-})^{\sim}$, so $(nx)^{\sim} \leq a$ and $(mx)^{-} \leq b$. Then we get $(nx)^{\sim} \in I$ or $(mx)^{-} \in I$.

If $x \in I$, then $nx \in I$. Since I is proper, i.e. $1 \notin I$, it follows that $(nx)^{\sim} \notin I$ and $(nx)^{-} \notin I$.

$(ii) \Rightarrow (i)$. Let J be an idealof A such that $I \subset J$. Then for every $x \in J \backslash I$ we have by $(ii), (nx)^{\sim} \in I$ or $(mx)^{-} \in I$ for some integers $n, m \geq 1$, so $(nx)^{\sim} \in J$ or $(mx)^{-} \in J$. Since $x \in J$ we have $nx, mx \in J$, so $nx \oplus (nx)^{\sim} = 1 \in J$ or $(mx)^{-} \oplus mx = 1 \in J$. We deduce that $1 \in J$ and $J = A$, so I is a maximal ideal of A. ■

DEFINITION 4.5. A *congruence* on A is an equivalence relation $\equiv$ on an pseudo $MV-$ algebra A satyisfing the following conditions:

(C_1) if $x \equiv y$ and $a \equiv b$ then $x \oplus a \equiv y \oplus b$ and $a \oplus x \equiv b \oplus y$;

(C_2) if $x \equiv y$ then $x^{-} \equiv y^{-}$ and $x^{\sim} \equiv y^{\sim}$.

If H is a normal ideal then $\equiv_{L(H)} \Leftrightarrow \equiv_{R(H)}$; let $\equiv_H$ denote one of therm. The binary relation $\equiv_H$ is a congruence on A and we have $H = \{x \in A : x \equiv_H 0\} = 0/\equiv_H$. Conversely, if $\equiv$ is a congruence on A, then $0/\equiv = \{x \in A : x \equiv 0\}$ is a normal ideal of A and $x \equiv y$ iff $d_{-}(x, y) \equiv 0$, or equivalently, $x \equiv y$ iff $d_{\sim}(x, y) \equiv 0$.

We deduce that there is a bijection between the set of normal ideals and the set of congruences of a pseudo $MV-$ algebra.

To any normal ideal H of A we shall denote the equivalence class of $x \in A$ with respect to $\equiv_H$ by x/H and the quotient set $A/\equiv_H$ by A/H . We remark that A/H becomes a pseudo $MV-$ algebra with the natural operations induced by those of $A : (x/H) \oplus (y/H) = (x \oplus y)/H; (x/H)^{-} = (x^{-})/H; (x/H)^{\sim} = (x^{\sim})/H$. This pseudo $MV-$ algebra $(A/H, \oplus, ^{-}, ^{\sim}, 0/H = H, 1/H)$ is called the quotient algebra of A by the normal ideal H.

The correspondence $x \to x/H$ defines a homomorphism p_H from A onto the quotint algebra A/H, which is called the natural homomorphism from A onto A/H; $Ker(p_H) = H$ since $x \in Ker(p_H) \Leftrightarrow x/H = p_H(x) = 0/H = H \Leftrightarrow x \equiv_H 0 \Leftrightarrow d_{-}(x, 0) \in H$ and $d_{\sim}(x, 0) \in H \Leftrightarrow x \in H$.

REMARK 4.9. *If A is a pseudo $MV-$ chain, then the set of normal ideals of A is totally ordered by inclusion. Indeed, if I, J are normal ideals of A such that $I \not\subseteq J$ and $J \not\subseteq I$, then there would be elements $a, b \in A$ such that $a \in J \backslash I$ and $b \in I \backslash J$, whence $a \not\leq b$ and $b \not\leq a$, which is impossible.*

REMARK 4.10. *Note that a normal ideal I is prime iff A/I is a linearly ordered pseudo $MV-$ algebra.*

If A is an $MV-$ algebra, then $d_- = d_\sim$, i.e.

$$d_-(x, y) = d_\sim(x, y) = d(x, y) = (x \odot y^-) \oplus (y \odot x^-)$$

and d is the distance function of A.

It is obvious that, in this case, any ideal of A is normal.

CHAPTER 5

Pseudo BL-algebras

In [**53**], [**54**], [**67**], A. Di Nola, G. Georgescu and A. Iorgulescu defined the pseudo BL-algebras as a non-commutative extension of BL- algebras (the class of pseudo BL - algebras contains the pseudo MV-algebras, see [**66**], [**68**]). The corresponding propositional logic was established in [**76**], [**77**].

Apart from their logical interest, pseudo BL-algebras have interesting algebraic properties (see [**37**], [**53**], [**54**], [**70**], [**94**]).

1. Definitions and first properties. Some examples. Rules of calculus

We review the basic definitions of pseudo BL- algebras, with more details and more examples; a lot of identities are true in a pseudo BL-algebra. Also we put in evidence connection between pseudo BL-algebras and pseudo MV-algebras, BL-algebras and Hilbert algebras.

DEFINITION 5.1. A *pseudo $BL-$ algebra* is an algebra

$$(A, \vee, \wedge, \odot, \rightarrow, \rightsquigarrow 0, 1)$$

of type (2,2,2,2,2,0,0) satisfying the following:

$(psBL_1)$ $(A, \vee, \wedge, 0, 1)$ is a bounded lattice;
$(psBL_2)$ $(A, \odot, 1)$ is a monoid;
$(psBL_3)$ $a \odot b \leq c$ iff $a \leq b \rightarrow c$ iff $b \leq a \rightsquigarrow c$ for all $a, b, c \in A$;
$(psBL_4)$ $a \wedge b = (a \rightarrow b) \odot a = a \odot (a \rightsquigarrow b)$;
$(psBL_5)$ $(a \rightarrow b) \vee (b \rightarrow a) = (a \rightsquigarrow b) \vee (b \rightsquigarrow a) = 1$, for all $a, b \in A$.

We shall agree that the operations $\wedge, \vee, \odot$ have priority towards the operations $\rightarrow, \rightsquigarrow$.

EXAMPLE 5.1. *Let $(A, \odot, \oplus, ^-, ^\sim, 0, 1)$ be a pseudo $MV-$ algebra and let $\rightarrow, \rightsquigarrow$ be two implications defined by*

$$x \rightarrow y = y \oplus x^-, x \rightsquigarrow y = x^\sim \oplus y.$$

Then $(A, \vee, \wedge, \odot, \rightarrow, \rightsquigarrow 0, 1)$ is a pseudo $BL-$ algebra.

EXAMPLE 5.2. *Let us consider an arbitrary l-group $(G, \vee, \wedge, +, -, 0, 1)$ and let $u \in G, u \leq 0$. We put by definition:*

$$x \odot y = (x + y) \vee u, x \oplus y = (x - u + y) \wedge 0,$$

$$x^- = u - x, x^\sim = -x + u.$$

Then $A = ([u, 0], \odot, \oplus, ^-, ^\sim, \mathbf{0} = u, \mathbf{1} = 0)$ is a pseudo $MV-$ algebra and we define two implications:

$$x \rightarrow y = (y - x) \wedge 0, x \rightsquigarrow y = (-x + y) \wedge 0.$$

Then $A = ([u, 0], \vee, \wedge, \odot, \rightarrow, \rightsquigarrow, \mathbf{0} = u, \mathbf{1} = 0)$ is a pseudo $BL-$ algebra.

A pseudo $BL-$ algebra is *nontrivial* if $0 \neq 1$. An element $a \in A, a \neq 1$ is called *non-unit*. For any pseudo $BL-$ algebra A, the reduct $L(A) = (A, \vee, \wedge, 0, 1)$ is a bounded distributive lattice. For any $a \in A$, we define

$$a^- = a \to 0 \text{ and } a^\sim = a \rightsquigarrow 0.$$

We shall write $a^=$ instead of $(a^-)^-$ and $a^\approx$ instead of $(a^\sim)^\sim$.

We define $a^0 = 1$ and $a^n = a^{n-1} \odot a$ for $n \geq 1$. The *order* of $a, a \neq 1$, in symbols $ord(a)$ is the smallest $n \geq 1$ such that $a^n = 0$; if no such n exists, then $ord(a) = \infty$.

A pseudo $BL-$ algebra is called *locally finite* if all non unit elements in it are finite order.

Now we are able to make the connections of pseudo $BL-$ algebras with $BL-$algebras and pseudo $MV-$algebras.

DEFINITION 5.2. A pseudo $BL-$ algebra A is *commutative* iff $x \odot y = y \odot x$, for any $x, y \in A$.

PROPOSITION 5.1. *A pseudo BL -algebra A is commutative iff* $x \rightsquigarrow y = x \to y$, *for all* $x, y \in A$. *Any commutative pseudo* $BL-$*algebra A is a* $BL-$*algebra.*

Then we shall say that a pseudo $BL-$ algebra is *proper* if it is not commutative, i.e. if it is not a BL algebra.

PROPOSITION 5.2. *Let* $(A, \vee, \wedge, \odot, \to, \rightsquigarrow 0, 1)$ *be a pseudo* $BL-$ *algebra with the property:*

$$(P) : \textit{for all } x \in A, (x^\sim)^- = x = (x^-)^\sim.$$

Let us define on A a new operation by

$$y \oplus x = (x^- \odot y^-)^\sim = (x^\sim \odot y^\sim)^- = x^\sim \to y = y^- \to x.$$

Then $(A, \odot, \oplus, ^-, ^\sim, 0, 1)$ *is a pseudo* $MV-$ *algebra.*

The next Corollary generalizes the following results from [**75**]: A BL algebra $(A, \vee, \wedge, \odot, \to, 0, 1)$ is an MV algebra iff $x^{**} = x$, for all $x \in A$ (where $x^* = x \to 0$).

COROLLARY 5.3. *A pseudo* $BL-$ *algebra A is a pseudo* $MV-$ *algebra iff A has property* (P).

In [**37**], [**53**], [**54**] it is proved that if A is a pseudo $BL-$ algebra and $a, a_1, ..., a_n, a', b, b', c, b_i \in A$, $(i \in I)$ then we have the following rules of calculus:

$(psbl - c_1)$ $a \odot (a \rightsquigarrow b) \leq b \leq a \rightsquigarrow (a \odot b)$ and $a \odot (a \rightsquigarrow b) \leq a \leq b \rightsquigarrow (b \odot a)$;
$(psbl - c_2)$ $(a \to b) \odot a \leq a \leq b \to (a \odot b)$ and $(a \to b) \odot a \leq b \leq a \to (b \odot a)$;
$(psbl - c_3)$ if $a \leq b$ then $a \odot c \leq b \odot c$ and $c \odot a \leq c \odot b$;
$(psbl - c_4)$ if $a \leq b$ then $c \rightsquigarrow a \leq c \rightsquigarrow b$ and $c \to a \leq c \to b$;
$(psbl - c_5)$ if $a \leq b$ then $b \rightsquigarrow c \leq a \rightsquigarrow c$ and $b \to c \leq a \to c$;
$(psbl - c_6)$ $a \leq b$ iff $a \to b = 1$ iff $a \rightsquigarrow b = 1$;
$(psbl - c_7)$ $a \rightsquigarrow a = a \to a = 1$;
$(psbl - c_8)$ $1 \rightsquigarrow a = 1 \to a = a$;
$(psbl - c_9)$ $b \leq a \rightsquigarrow b$ and $b \leq a \to b$;
$(psbl - c_{10})$ $a \odot b \leq a \wedge b$ and $a \odot b \leq a, b$;
$(psbl - c_{11})$ $a \rightsquigarrow 1 = a \to 1 = 1$;
$(psbl - c_{12})$ $a \rightsquigarrow b \leq (c \odot a) \rightsquigarrow (c \odot b)$;
$(psbl - c_{13})$ $a \to b \leq (a \odot c) \to (b \odot c)$;
$(psbl - c_{14})$ if $a \leq b$ then $a \leq c \rightsquigarrow b$ and $a \leq c \to b$;

$(psbl - c_{15})$ $a \to (b \odot c) \geq b \odot (a \to c)$ and $a \rightsquigarrow (b \odot c) \geq (a \rightsquigarrow b) \odot c$;
$(psbl - c_{16})$ if $a \leq b$ then $b^- \leq a^-$ and $b^\sim \leq a^\sim$;
$(psbl - c_{17})$ $0 \odot a = a \odot 0 = 0$;
$(psbl - c_{18})$ $(a \rightsquigarrow b) \odot (b \rightsquigarrow c) \leq a \rightsquigarrow c$ and $(b \to c) \odot (a \to b) \leq a \to c$;
$(psbl - c_{19})$ $(a_1 \rightsquigarrow a_2) \odot (a_2 \rightsquigarrow a_3) \odot ... \odot (a_{n-1} \rightsquigarrow a_n) \leq a_1 \rightsquigarrow a_n$;
$(a_{n-1} \to a_n) \odot ... \odot (a_2 \to a_3) \odot (a_1 \to a_2) \leq a_1 \to a_n$;
$(psbl - c_{20})$ $a \vee b = ((a \rightsquigarrow b) \to b) \wedge ((b \rightsquigarrow a) \to a)$;
$(psbl - c_{21})$ $a \vee b = ((a \to b) \rightsquigarrow b) \wedge ((b \to a) \rightsquigarrow a)$;
$(psbl - c_{22})$ $a \rightsquigarrow (b \rightsquigarrow c) = (b \odot a) \rightsquigarrow c$ and $a \to (b \to c) = (a \odot b) \to c$; $a \rightsquigarrow b = a \rightsquigarrow (a \wedge b)$ and $a \to b = a \to (a \wedge b)$;
$(psbl - c_{23})$ $(a \rightsquigarrow b) \rightsquigarrow (a \rightsquigarrow c) = (b \rightsquigarrow a) \rightsquigarrow (b \rightsquigarrow c)$ and $(a \to b) \to (a \to c) = (b \to a) \to (b \to c)$;
$(psbl - c_{24})$ $c \odot (a \wedge b) = (c \odot a) \wedge (c \odot b)$ and $(a \wedge b) \odot c = (a \odot c) \wedge (b \odot c)$;
$(psbl - c_{25})$ $(a \rightsquigarrow b) \odot (a' \rightsquigarrow b') \leq (a \vee a') \rightsquigarrow (b \vee b')$;
$(psbl - c_{26})$ $(a \rightsquigarrow b) \odot (a' \rightsquigarrow b') \leq (a \wedge a') \rightsquigarrow (b \wedge b')$;
$(psbl - c_{27})$ $(a \to b) \odot (a' \to b') \leq (a \vee a') \to (b \vee b')$;
$(psbl - c_{28})$ $(a \to b) \odot (a' \to b') \leq (a \wedge a') \to (b \wedge b')$;
$(psbl - c_{29})$ $(a \rightsquigarrow b) \rightsquigarrow c \leq ((b \rightsquigarrow a) \rightsquigarrow c) \rightsquigarrow c$ and $(a \to b) \to c \leq ((b \to a) \to c) \to c$;
$(psbl - c_{30})$ $(a \rightsquigarrow b) \rightsquigarrow c \leq ((b \rightsquigarrow a) \rightsquigarrow c) \to c$ and $(a \to b) \to c \leq ((b \to a) \to c) \rightsquigarrow c$;
$(psbl - c_{31})$ $a \rightsquigarrow b \leq (b \rightsquigarrow c) \to (a \rightsquigarrow c)$ and $a \to b \leq (b \to c) \rightsquigarrow (a \to c)$;
$(psbl - c_{32})$ $a \rightsquigarrow b \leq (c \rightsquigarrow a) \rightsquigarrow (c \rightsquigarrow b)$ and $a \to b \leq (c \to a) \to (c \to b)$;
$(psbl - c_{33})$ if $a \vee b = 1$ then $a \odot b = a \wedge b$;
$(psbl - c_{34})$ if $a \vee b = 1$ then, for each $n \geq 1, a^n \vee b^n = 1$,
$(psbl - c_{35})$ for each $n \in \omega$, $n \geq 1$,

$$(a \to b)^n \vee (b \to a)^n = 1;$$

$$(a \rightsquigarrow b)^n \vee (b \rightsquigarrow a)^n = 1;$$

$(psbl - c_{36})$ $a \wedge (\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (a \wedge b_i)$,
$a \odot (\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (a \odot b_i)$,
$(\bigvee_{i \in I} b_i) \odot a = \bigvee_{i \in I} (b_i \odot a)$,
$a \rightsquigarrow (\bigwedge_{i \in I} b_i) = \bigwedge_{i \in I} (a \rightsquigarrow b_i)$,
$a \to (\bigwedge_{i \in I} b_i) = \bigwedge_{i \in I} (a \to b_i)$,
$(\bigvee_{i \in I} b_i) \rightsquigarrow a = \bigwedge_{i \in I} (b_i \rightsquigarrow a)$,
$(\bigvee_{i \in I} b_i) \to a = \bigwedge_{i \in I} (b_i \to a)$,
(whenever the arbitrary meets and unions exist)
$(psbl - c_{37})$ $1^\sim = 1^- = 0, 0^\sim = 0^- = 1$;
$(psbl - c_{38})$ $a \odot a^\sim = a^- \odot a = 0$;
$(psbl - c_{39})$ $b \leq a^\sim$ iff $a \odot b = 0$;
$(psbl - c_{40})$ $b \leq a^-$ iff $b \odot a = 0$;
$(psbl - c_{41})$ $a \leq a^- \rightsquigarrow b, a \leq a^\sim \to b$;
$(psbl - c_{42})$ $a \leq (a \rightsquigarrow b) \to b, a \leq (a \to b) \rightsquigarrow b$, hence $a \leq (a^\sim)^-, a \leq (a^-)^\sim$;
$(psbl - c_{43})$ $a \rightsquigarrow b \leq b^\sim \to a^\sim, a \to b \leq b^- \rightsquigarrow a^-$;
$(psbl - c_{44})$ $a \to b^\sim = b \rightsquigarrow a^-, a \rightsquigarrow b^- = b \to a^\sim$;

$(psbl-c_{45})$ $a \leq b$ implies $b^{\sim} \leq a^{\sim}$ and $b^{-} \leq a^{-}$;
$(psbl-c_{46})$ $((\ a^{\sim})^{-})^{\sim} = a^{\sim}, ((\ a^{-})^{\sim})^{-} = a^{-}$;
$(psbl-c_{47})$ $a \to a^{\sim} = a \rightsquigarrow a^{-}$;
$(psbl-c_{48})$ $(a \odot b)^{-} = a \to b^{-}, (a \odot b)^{\sim} = b \rightsquigarrow a^{\sim}$;
$(psbl-c_{49})$ $(a \wedge b)^{\sim} = a^{\sim} \vee b^{\sim}, (a \vee b)^{\sim} = a^{\sim} \wedge b^{\sim}$;
$(psbl-c_{50})$ $(a \wedge b)^{-} = a^{-} \vee b^{-}, (a \vee b)^{-} = a^{-} \wedge b^{-}$;
$(psbl-c_{51})$ $(a \wedge b)^{\approx} = a^{\approx} \wedge b^{\approx}, (a \vee b)^{\approx} = a^{\approx} \vee b^{\approx}$;
$(psbl-c_{52})$ $(a \wedge b)^{=} = a^{=} \wedge b^{=}, (a \vee b)^{=} = a^{=} \vee b^{=}$;
$(psbl-c_{53})$ $(a \vee c) \odot (b \vee c \vee a^{\sim}) \leq (a \odot b) \vee c$;
$(psbl-c_{54})$ $(b \vee c \vee a^{-}) \odot (a \vee c) \leq (b \odot a) \vee c$;
$(psbl-c_{55})$ $a \vee (b \odot c) \geq (a \vee b) \odot (a \vee c)$.

Proof. $(psbl-c_1)$. $a \odot (a \rightsquigarrow b) = a \wedge b \leq a, b$; the second inequalities follow by $psBL_3 : b \leq a \rightsquigarrow (a \odot b) \Leftrightarrow a \odot b \leq a \odot b$ and $a \leq b \rightsquigarrow (b \odot a) \Leftrightarrow b \odot a \leq b \odot a$.

$(psbl-c_2)$. Has a similar proof with $psbl-c_1$.

$(psbl-c_3)$. $a \leq b \overset{psbl-c_2}{\leq} c \to (b \odot c)$, so by $psBL_3$, $a \odot c \leq b \odot c$ and $a \leq b \overset{psbl-c_1}{\leq} c \rightsquigarrow (c \odot b) \overset{psBL_3}{\Leftrightarrow} c \odot a \leq c \odot b$.

$(psbl-c_4)$. $c \odot (c \rightsquigarrow a) \overset{psbl-c_1}{\leq} a \leq b$, so by $psBL_3$, $c \rightsquigarrow a \leq c \rightsquigarrow a$; $(c \to a) \odot c \overset{psbl-c_2}{\leq} a \leq b$, hence by $psBL_3$, $c \to a \leq c \to b$.

$(psbl-c_5)$. If $a \leq b$ then we deduce that $a \odot (b \rightsquigarrow c) \overset{psbl-c_3}{\leq} b \odot (b \rightsquigarrow c) \overset{psbl-c_1}{\leq} c$ so $b \rightsquigarrow c \leq a \rightsquigarrow c$, by $psBL_3$; if $a \leq b$ then

$(b \to c) \odot a \overset{psbl-c_3}{\leq} (b \to c) \odot b \overset{psBL_4}{=} b \wedge c \leq b$; hence $b \to c \leq a \to c$,by $psBL_3$.

$(psbl-c_6)$. $a \leq b$ iff $a \odot 1 \leq b$ iff $1 \leq a \rightsquigarrow b$ iff $a \rightsquigarrow b = 1$, by $psBL_2, psBL_3$ and $psBL_1$.

$(psbl-c_7)$. Obviously, by $psbl-c_6$, since $a \leq a$.

$(psbl-c_8)$. $a = 1 \wedge a \overset{psBL_4}{=} 1 \odot (1 \rightsquigarrow a) = 1 \rightsquigarrow a$ and $a = 1 \wedge a \overset{psBL_4}{=} (1 \to a) \odot 1 = 1 \to a$, so $1 \rightsquigarrow a = 1 \to a = a$.

$(psbl-c_9)$. $a \leq 1$ implies by $psbl-c_5$, $1 \rightsquigarrow b \leq a \rightsquigarrow b$ and $1 \to b \leq a \to b$ so $b \leq a \rightsquigarrow b$ and $b \leq a \to b$ by $psbl-c_8$.

$(psbl-c_{10})$. Since $b \overset{psbl-c_9}{\leq} a \to b$, then $a \odot b \overset{psbl-c_3}{\leq} a \odot (a \to b) \overset{psBL_4}{=} a \wedge b \leq a, b$.

$(psbl-c_{11})$. By $psbl-c_6$ since $a \leq 1$.

$(psbl-c_{12})$. $a \rightsquigarrow b \overset{psbl-c_3}{\leq} (c \odot a) \rightsquigarrow (c \odot b) \overset{psBL_3}{\Leftrightarrow} (c \odot a) \odot (a \rightsquigarrow b) \leq c \odot b \overset{psBL_4}{\Leftrightarrow} c \odot (a \wedge b) \leq c \odot b$.

$(psbl-c_{13})$. $a \to b \overset{psbl-c_3}{\leq} (a \odot c) \to (b \odot c) \overset{psBL_3}{\Leftrightarrow} (a \to b) \odot (a \odot c) \leq b \odot c \overset{psBL_4}{\Leftrightarrow} (a \wedge b) \odot c \leq b \odot c$.

$(psbl-c_{14})$. By $psbl-c_9$, $a \leq b \leq c \rightsquigarrow b$ and $a \leq b \leq c \to b$.

$(psbl-c_{15})$. $(a \rightsquigarrow b) \odot c \leq a \rightsquigarrow (b \odot c) \overset{psBL_3}{\Leftrightarrow} a \odot [(a \rightsquigarrow b) \odot c] \leq b \odot c \overset{psBL_4}{\Leftrightarrow} (a \wedge b) \odot c \leq b \odot c$ and $b \odot (a \to c) \leq a \to (b \odot c) \overset{psBL_3}{\Leftrightarrow} [b \odot (a \to c)] \odot a \leq b \odot c \overset{psBL_4}{\Leftrightarrow} b \odot (a \wedge c) \leq b \odot c$.

$(psbl-c_{16})$. Follows from $psbl-c_5$, by taking $c = 0$.

$(psbl-c_{17})$. Clearly, $0 \odot 1 = 1 \odot 0 = 0$. Then since $a \leq 1$, it follows that $a \odot 0 \overset{psbl-c_3}{\leq} 1 \odot 0 = 0$ and $0 \odot a \overset{psbl-c_3}{\leq} 0 \odot 1 = 0$. Hence $a \odot 0 = 0 \odot a = 0$.

$(psbl-c_{18})$. We get $(a\rightsquigarrow b)\odot(b\rightsquigarrow c)\leq a\rightsquigarrow c \overset{psBL_3}{\Leftrightarrow} a\odot[(a\rightsquigarrow b)\odot(b\rightsquigarrow c)]\leq c \overset{psBL_4}{\Leftrightarrow} (a\wedge b)\odot(b\rightsquigarrow c)\leq c$, which is true, since $(a\wedge b)\odot(b\rightsquigarrow c)\leq b\odot(b\rightsquigarrow c) \overset{psBL_4}{=} b\wedge c\leq c$ and $(b\to c)\odot(a\to b)\leq a\to c \overset{psBL_3}{\Leftrightarrow} [(b\to c)\odot(a\to b)]\odot a\leq c \overset{psBL_4}{\Leftrightarrow} (b\to c)\odot(a\wedge b)\leq c$, which is true, since $(b\to c)\odot(a\wedge b)\leq (b\to c)\odot b \overset{psBL_4}{=} b\wedge c\leq c$.

$(psbl-c_{19})$. Have a similar proof with $psbl-c_{17}$ and $psbl-c_{18}$.

$(psbl-c_{20})$. Denote $x=((a\rightsquigarrow b)\to b)\wedge((b\rightsquigarrow a)\to a)$. By $psBL_4$, $a\wedge b=a\odot(a\rightsquigarrow b)$ so $a\leq(a\rightsquigarrow b)\to b$; from $psbl-c_9$, we also have that $b\leq(a\rightsquigarrow b)\to b$; it follows that $a\vee b\leq(a\rightsquigarrow b)\to b$. Analogous, $a\vee b\leq(b\rightsquigarrow a)\to a$. Hence $a\vee b\leq((a\rightsquigarrow b)\to b)\wedge((b\rightsquigarrow a)\to a)$.

We have $x=x\odot 1 \overset{psBL_5}{=} x\odot[(a\rightsquigarrow b)\vee(b\rightsquigarrow a)] \overset{psbl-c_{36}}{=} [x\odot(a\rightsquigarrow b)]\vee[x\odot(b\rightsquigarrow a)]$; but $x\odot(a\rightsquigarrow b)=[((a\rightsquigarrow b)\to b)\wedge((b\rightsquigarrow a)\to a)]\odot(a\rightsquigarrow b)\leq[(a\rightsquigarrow b)\to b]\odot(a\rightsquigarrow b) \overset{psBL_4}{=} (a\rightsquigarrow b)\wedge b\leq b$; similary, $x\odot(b\rightsquigarrow a)\leq a$. Hence, $x=[x\odot(a\rightsquigarrow b)]\vee[x\odot(b\rightsquigarrow a)]\leq b\vee a$. It follows that $a\vee b=x$.

$(psbl-c_{21})$. Has a similar proof with $psbl-c_{20}$.

$(psbl-c_{22})$. We have the following equivalences:

$(b\odot a)\rightsquigarrow c\leq a\rightsquigarrow(b\rightsquigarrow c) \overset{psBL_3}{\Leftrightarrow} a\odot[(b\odot a)\rightsquigarrow c]\leq b\rightsquigarrow c \overset{psBL_3}{\Leftrightarrow} b\odot[a\odot((b\odot a)\rightsquigarrow c)]\leq c \overset{psBL_2}{\Leftrightarrow} (b\odot a)\odot[(b\odot a)\rightsquigarrow c]\leq c \overset{psBL_3}{\Leftrightarrow} (b\odot a)\wedge c\leq(b\odot a)\rightsquigarrow c$;

and $a\rightsquigarrow(b\rightsquigarrow c)\leq(b\odot a)\rightsquigarrow c \overset{psBL_3}{\Leftrightarrow} (b\odot a)\odot[a\rightsquigarrow(b\rightsquigarrow c)]\leq c\Leftrightarrow b\odot[a\odot(a\rightsquigarrow(b\rightsquigarrow c))]\leq c \overset{psBL_4}{\Leftrightarrow} b\odot[a\wedge(b\rightsquigarrow c)]\leq c \overset{psBL_3}{\Leftrightarrow} a\wedge(b\rightsquigarrow c)\leq b\rightsquigarrow c$.

So, $(b\odot a)\rightsquigarrow c=a\rightsquigarrow(b\rightsquigarrow c)$.

The second equality has a similar proof.

$a\wedge b=a\odot(a\rightsquigarrow b)\leq a\wedge b$ implies $a\rightsquigarrow b\leq a\rightsquigarrow(a\wedge b)$; On the other side, $a\wedge b\leq b \overset{psbl-c_5}{\Leftrightarrow} a\rightsquigarrow(a\wedge b)\leq a\rightsquigarrow b$, so $a\rightsquigarrow b=a\rightsquigarrow(a\wedge b)$; Similarly, $a\to b=a\to(a\wedge b)$.

$(psbl-c_{23})$. We have $(a\rightsquigarrow b)\rightsquigarrow(a\rightsquigarrow c) \overset{psbl-c_{22}}{=} [a\odot(a\rightsquigarrow b)]\rightsquigarrow c \overset{psBL_4}{=} (a\wedge b)\rightsquigarrow c=(b\wedge a)\rightsquigarrow c \overset{psBL_4}{=} [b\odot(b\rightsquigarrow a)]\rightsquigarrow c \overset{psbl-c_{22}}{=} (b\rightsquigarrow a)\rightsquigarrow(b\rightsquigarrow c)$ and $(a\to b)\to(a\to c) \overset{psbl-c_{22}}{=} [(a\to b)\odot a]\to c \overset{psBL_4}{=} (a\wedge b)\to c=(b\wedge a)\to c \overset{psBL_4}{=} [(b\to a)\odot b]\to c \overset{psbl-c_{22}}{=} (b\to a)\to(b\to c)$.

$(psbl-c_{24})$. By $psbl-c_{22}$ and $psbl-c_{12}$ we get $a\rightsquigarrow b=a\rightsquigarrow(a\wedge b)\leq(c\odot a)\rightsquigarrow(c\odot(a\wedge b))$.

Thus $a\rightsquigarrow b \overset{psbl-c_{14}}{\leq} [(c\odot a)\rightsquigarrow(c\odot b)]\rightsquigarrow[(c\odot a)\rightsquigarrow(c\odot(a\wedge b))]$ and by replacing a by b and b by a we obtain $b\rightsquigarrow a \overset{psbl-c_{14}}{\leq} [(c\odot b)\rightsquigarrow(c\odot a)]\rightsquigarrow[(c\odot b)\rightsquigarrow(c\odot(b\wedge a))]$. By $psbl-c_{23}$, the right term of the last two inequalities are equal and we denote the common value by x. So, $a\rightsquigarrow b\leq x, b\rightsquigarrow a\leq x$.

On other side, $(a\rightsquigarrow b)\vee(b\rightsquigarrow a) \overset{psBL_5}{=} 1$, therefore we get $1\leq x\vee x=x$, hence $x=1$.

Thus $(c\odot a)\rightsquigarrow(c\odot b) \overset{psbl-c_6}{\leq} (c\odot a)\rightsquigarrow(c\odot(a\wedge b)) \overset{psBL_3}{\Leftrightarrow} (c\odot a)\odot[(c\odot a)\rightsquigarrow(c\odot b)]\leq c\odot(a\wedge b)$.

By $psbl-c_3$, $a\wedge b\leq a, b$ implies $c\odot(a\wedge b)\leq c\odot a, c\odot b$, so $c\odot(a\wedge b)\leq(c\odot a)\wedge(c\odot b)$.

Thus the first equality holds.

The second equality, $(a\wedge b)\odot c=(a\odot c)\wedge(b\odot c)$, has a similar proof.

$(psbl - c_{25})$. The inequalities $a \odot (a \rightsquigarrow b) \odot (a' \rightsquigarrow b') \leq a \odot (a \rightsquigarrow b) = a \wedge b \leq b \vee b'$, and $a' \odot (a \rightsquigarrow b) \odot (a' \rightsquigarrow b') \leq a' \odot (a' \rightsquigarrow b') = a' \wedge b' \leq b \vee b'$ imply $(a \rightsquigarrow b) \odot (a' \rightsquigarrow b') \leq a \rightsquigarrow (b \vee b')$ and $(a \rightsquigarrow b) \odot (a' \rightsquigarrow b') \leq a' \rightsquigarrow (b \vee b')$ therefore $(a \rightsquigarrow b) \odot (a' \rightsquigarrow b') \leq [a \rightsquigarrow (b \vee b')] \wedge [a' \rightsquigarrow (b \vee b')] \overset{psbl-c_{36}}{=} (a \vee a') \rightsquigarrow (b \vee b')$.

$(psbl - c_{26})$. The inequalities $(a \wedge a') \odot (a \rightsquigarrow b) \odot (a' \rightsquigarrow b') \leq a \odot (a \rightsquigarrow b) \leq b$ and $(a \wedge a') \odot (a \rightsquigarrow b) \odot (a' \rightsquigarrow b') \leq a' \odot (a' \rightsquigarrow b') \leq b'$ imply $(a \rightsquigarrow b) \odot (a' \rightsquigarrow b') \leq (a \wedge a') \rightsquigarrow b$ and $(a \rightsquigarrow b) \odot (a' \rightsquigarrow b') \leq (a \wedge a') \rightsquigarrow b'$ therefore

$(a \rightsquigarrow b) \odot (a' \rightsquigarrow b') \leq [(a \wedge a') \rightsquigarrow b] \wedge [(a \wedge a') \rightsquigarrow b'] \overset{psbl-c_{36}}{=} (a \wedge a') \rightsquigarrow (b \wedge b')$.

$(psbl - c_{27})$. Has a similar proof with $psbl - c_{25}$.

$(psbl - c_{28})$. Has a similar proof with $psbl - c_{26}$.

$(psbl - c_{29})$. We have $[(b \rightsquigarrow a) \rightsquigarrow c] \odot [(a \rightsquigarrow b) \rightsquigarrow c] \leq [(b \rightsquigarrow a) \rightsquigarrow c] \wedge [(a \rightsquigarrow b) \rightsquigarrow c] \overset{psbl-c_{36}}{=} [(b \rightsquigarrow a) \vee (a \rightsquigarrow b)] \rightsquigarrow c \overset{psBL_5}{=} 1 \rightsquigarrow c \overset{psbl-c_8}{=} c$. So by $psBL_3$ we deduce $(a \rightsquigarrow b) \rightsquigarrow c \leq ((b \rightsquigarrow a) \rightsquigarrow c) \rightsquigarrow c$. The second inequality has a similar proof.

$(psbl - c_{30})$. Have a similar proof with $psbl - c_{29}$.

$(psbl - c_{31})$. We have $a \odot (a \rightsquigarrow b) \odot (b \rightsquigarrow c) = (a \wedge b) \odot (b \rightsquigarrow c) \leq b \odot (b \rightsquigarrow c) = b \wedge c \leq c$, so $a \odot (a \rightsquigarrow b) \odot (b \rightsquigarrow c) \leq c \overset{psBL_3}{\Leftrightarrow} (a \rightsquigarrow b) \odot (b \rightsquigarrow c) \leq a \rightsquigarrow c \overset{psBL_3}{\Leftrightarrow}$

$a \rightsquigarrow b \leq (b \rightsquigarrow c) \rightarrow (a \rightsquigarrow c)$; since $(b \rightarrow c) \odot (a \rightarrow b) \odot a = (b \rightarrow c) \odot (a \wedge b) \leq (b \rightarrow c) \odot b = b \wedge c \leq c$, then $(b \rightarrow c) \odot (a \rightarrow b) \leq a \rightarrow c$ hence $a \rightarrow b \leq (b \rightarrow c) \rightsquigarrow (a \rightarrow c)$.

$(psbl - c_{32})$. $a \rightsquigarrow b \leq (c \rightsquigarrow a) \rightsquigarrow (c \rightsquigarrow b) \overset{psBL_3}{\Leftrightarrow} (c \rightsquigarrow a) \odot (a \rightsquigarrow b) \leq (c \rightsquigarrow b)$ and the right side of the equivalence is true by $psbl - c_{18}$;

$a \rightarrow b \leq (c \rightarrow a) \rightarrow (c \rightarrow b) \overset{psBL_3}{\Leftrightarrow} (a \rightarrow b) \odot (c \rightarrow a) \leq c \rightarrow b$ and the right side of the equivalence is true by $psbl - c_{18}$.

$(psbl - c_{33})$. By $psbl - c_{20}$, $a \vee b = ((a \rightsquigarrow b) \rightarrow b) \wedge ((b \rightsquigarrow a) \rightarrow a) = 1$, therefore $(a \rightsquigarrow b) \rightarrow b = 1 \overset{psbl-c_6}{\Leftrightarrow} a \rightsquigarrow b \leq b$. Thus $a \wedge b = a \odot (a \rightsquigarrow b) \leq a \odot b$.

By $psbl - c_{10}$ we also have $a \odot b \leq a \wedge b$, so $a \odot b = a \wedge b$.

$(psbl - c_{34})$. If $a \vee b = 1$ then, $a = a \odot 1 = a \odot (a \vee b) \overset{psbl-c_{36}(ii)}{=} (a \odot a) \vee (a \odot b) \leq a^2 \vee b$. Hence $a^2 \vee b \geq a$. Then $(a^2 \vee b) \vee b \geq a \vee b = 1$, so $a^2 \vee b = 1$. Similarly, $b = 1 \odot b = (a^2 \vee b) \odot b \overset{psbl-c_{36}}{=} (a^2 \odot b) \vee (b \odot b) \leq a^2 \vee b^2$. Thus, $a^2 \vee b^2 \geq b$; hence $a^2 \vee (a^2 \vee b^2) \geq a^2 \vee b = 1$, so $a^2 \vee b^2 = 1$.

It follows that $1 = a \vee b = a^2 \vee b^2 = (a^2)^2 \vee (b^2)^2 = \ldots$. We obtain $a^{2^n} \vee b^{2^n} = 1$, for each integer $n \geq 1$. Since $n \leq 2^n$ it follows that $a^n \vee b^n \geq a^{2^n} \vee b^{2^n} = 1$, which implies $a^n \vee b^n = 1$.

$(psbl - c_{35})$. Follows by $psbl - c_{34}$ and $psBL_5$.

$(psbl - c_{36})$. (ii). We prove that $a \odot (\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (a \odot b_i)$ and $(\bigvee_{i \in I} b_i) \odot a = \bigvee_{i \in I} (b_i \odot a)$.

Obviously, $a \odot b_i \leq a \odot (\bigvee_{i \in I} b_i)$ for each $i \in I$. Let $a \odot b_i \leq c, i \in I$. Then by $psBL_3$ we have $b_i \leq a \rightsquigarrow c, i \in I$, so $\bigvee_{i \in I} b_i \leq a \rightsquigarrow c \overset{psBL_3}{\Leftrightarrow} a \odot (\bigvee_{i \in I} b_i) \leq c$. Therefore we get that $a \odot (\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (a \odot b_i)$. Analogous the second equality.

(i). We prove that $a \wedge (\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (a \wedge b_i)$.

We have $a \wedge (\bigvee_{i\in I} b_i) \overset{psBL_4}{=} (\bigvee_{i\in I} b_i) \odot [(\bigvee_{j\in I} b_j) \rightsquigarrow a] = \bigvee_{i\in I}(b_i \odot [(\bigvee_{j\in I} b_j) \rightsquigarrow a])$. But, for any $i \in I$, $b_i \leq \bigvee_{j\in I} b_j$, then by $psbl - c_5$, $(\bigvee_{j\in I} b_j) \rightsquigarrow a \leq b_i \rightsquigarrow a$, so by $psbl - c_3$, $b_i \odot [(\bigvee_{j\in I} b_j) \rightsquigarrow a] \leq b_i \odot (b_i \rightsquigarrow a) \overset{psBL_4}{=} b_i \wedge a$; it follows that $\bigvee_{i\in I}(b_i \odot [(\bigvee_{j\in I} b_j) \rightsquigarrow a]) \leq \bigvee_{i\in I}(a \wedge b_i)$. We deduce that $a \wedge (\bigvee_{i\in I} b_i) \leq \bigvee_{i\in I}(a \wedge b_i)$;

the converse inequality is obvious.

By this rule of calculus we immediately get that: *If* $(A, \vee, \wedge, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ *is a pseudo BL-algebra, then* $L(A) = (A, \vee, \wedge, 0, 1)$ *is a bounded distributive lattice.*

(iii). To prove that $(\bigvee_{i\in I} b_i) \odot a = \bigvee_{i\in I}(b_i \odot a)$, remark first that $b_i \odot a \leq (\bigvee_{i\in I} b_i) \odot a$, for every $i \in I$, by $psbl - c_3$. Let x such that $b_i \odot a \leq x, i \in I$, then $b_i \leq a \rightarrow x, i \in I$; hence $\bigvee_{i\in I} b_i \leq a \rightarrow x \overset{psBL_3}{\Leftrightarrow} (\bigvee_{i\in I} b_i) \odot a \leq x$. Thus $(\bigvee_{i\in I} b_i) \odot a = \bigvee_{i\in I}(b_i \odot a)$.

The rules $(iv). : a \rightsquigarrow (\bigwedge_{i\in I} b_i) = \bigwedge_{i\in I}(a \rightsquigarrow b_i), (v) : a \rightarrow (\bigwedge_{i\in I} b_i) = \bigwedge_{i\in I}(a \rightarrow b_i), (vi) : (\bigvee_{i\in I} b_i) \rightsquigarrow a = \bigwedge_{i\in I}(b_i \rightsquigarrow a), (vii) : (\bigvee_{i\in I} b_i) \rightarrow a = \bigwedge_{i\in I}(b_i \rightarrow a)$

has a similar proof.

For example we proof $(vi) : (\bigvee_{i\in I} b_i) \rightsquigarrow a = \bigwedge_{i\in I}(b_i \rightsquigarrow a) :$

We have the following equivalences for any $x \in A$:

$x \leq (\bigvee_{i\in I} b_i) \rightsquigarrow a \overset{psBL_3}{\Leftrightarrow} (\bigvee_{i\in I} b_i) \odot x \leq a \overset{psbl-c_{36(iii)}}{\Leftrightarrow} \bigvee_{i\in I}(b_i \odot x) \leq a \Leftrightarrow b_i \odot x \leq a$ for any $i \in I \Leftrightarrow x \leq b_i \rightsquigarrow a$ for any $i \in I \Leftrightarrow x \leq \bigwedge_{i\in I}(b_i \rightsquigarrow a)$.

$(psbl - c_{37})$. Obviously by $psbl - c_8$ and $psbl - c_7$.

$(psbl - c_{38})$. $a \odot a^{\sim} = a \odot (a \rightsquigarrow 0) \overset{psBL_4}{=} a \wedge 0 = 0$ and $a^{-} \odot a = (a \rightarrow 0) \odot a \overset{psBL_4}{=} a \wedge 0 = 0$.

$(psbl - c_{39})$. $b \leq a^{\sim} \Leftrightarrow b \leq a \rightsquigarrow 0 \overset{psBL_3}{\Leftrightarrow} a \odot b \leq 0 \Leftrightarrow a \odot b = 0$.

$(psbl - c_{40})$. $b \leq a^{-} \Leftrightarrow b \leq a \rightarrow 0 \overset{psBL_3}{\Leftrightarrow} b \odot a \leq 0 \Leftrightarrow b \odot a = 0$.

$(psbl - c_{41})$. $a \leq a^{-} \rightsquigarrow b \overset{psBL_3}{\Leftrightarrow} a^{-} \odot a \leq b \overset{psbl-c_{38}}{\Leftrightarrow} 0 \leq b$ and $a \leq a^{\sim} \rightarrow b \overset{psBL_3}{\Leftrightarrow} a \odot a^{\sim} \leq b \overset{psbl-c_{38}}{\Leftrightarrow} 0 \leq b$.

$(psbl - c_{42})$. $a \leq (a \rightsquigarrow b) \rightarrow b \overset{psBL_3}{\Leftrightarrow} a \odot (a \rightsquigarrow b) \leq b \overset{psBL_4}{\Leftrightarrow} a \wedge b \leq b$, obviously; for $b = 0$ we obtain $a \leq (a^{\sim})^{-}$.

The other inequality , $a \leq (a \rightarrow b) \rightsquigarrow b$ has a similar proof; for $b = 0$ we obtain $a \leq (a^{-})^{\sim}$.

$(psbl - c_{43})$. By $psbl - c_{31}, a \rightsquigarrow b \leq (b \rightsquigarrow 0) \rightarrow (a \rightsquigarrow 0) = b^{\sim} \rightarrow a^{\sim}$ and $a \rightarrow b \leq (b \rightarrow 0) \rightsquigarrow (a \rightarrow 0) = b^{-} \rightsquigarrow a^{-}$.

$(psbl - c_{44})$. By $psbl - c_{43}$, we have $a \rightarrow b^{\sim} \leq (b^{\sim})^{-} \rightsquigarrow a^{-} \leq b \rightsquigarrow a^{-}$,by $psbl - c_{42}$ and $psbl - c_5$; similarly, $a \rightsquigarrow b^{-} \leq (b^{-})^{\sim} \rightarrow a^{\sim} \leq b \rightarrow a^{\sim}$.

By using these inequalities, we get $b \rightsquigarrow a^{-} \leq a \rightarrow b^{\sim}$ and $b \rightarrow a^{\sim} \leq a \rightsquigarrow b^{-}$. Thus, the equalities hold.

$(psbl - c_{45})$. By $psbl - c_5$, $a \leq b$ implies $b^{\sim} = b \rightsquigarrow 0 \leq a \rightsquigarrow 0 = a^{\sim}$ and $b^{-} = b \rightarrow 0 \leq a \rightarrow 0 = a^{-}$.

$(psbl - c_{46})$. $a \leq (a^{\sim})^{-}$ (by $psbl - c_{42}$) implies by $psbl - c_{45}$ that $((\ a^{\sim})^{-})^{\sim} \leq a^{\sim}$ and $a \leq (a^{-})^{\sim}$ implies $((\ a^{-})^{\sim})^{-} \leq a^{-}$; the converse inequalities follows by $psbl - c_{42}$.

$(psbl - c_{47})$. We have the following equivalences for any $x \in A$:

$x \leq a \to a^{\sim} \Leftrightarrow x \odot a \leq a^{\sim} \Leftrightarrow a \odot (x \odot a) = 0 \Leftrightarrow (a \odot x) \odot a = 0 \Leftrightarrow a \odot x \leq a^{-} \Leftrightarrow x \leq a \rightsquigarrow a^{-}$.

$(psbl - c_{48})$. $(a\odot\ b)^{-} = (a \odot b) \to 0 \overset{psbl-c_{22}}{=} a \to (b \to 0) = a \to b^{-}$ and $(a \odot b)^{\sim} = (a \odot b) \rightsquigarrow 0 \overset{psbl-c_{22}}{=} b \rightsquigarrow (a \rightsquigarrow 0) = b \rightsquigarrow a^{\sim}$.

$(psbl - c_{49})$. We get that $a \rightsquigarrow b = a \rightsquigarrow (a \wedge b) \overset{psbl-c_{43}}{\leq} (a \wedge b)^{\sim} \to a^{\sim}$ and $b \rightsquigarrow a = b \rightsquigarrow (b \wedge a) \overset{psbl-c_{43}}{\leq} (a \wedge b)^{\sim} \to b^{\sim}$.

By $psBL_3$, $(a \rightsquigarrow b) \odot (a \wedge b)^{\sim} \leq a^{\sim}$ and $(b \rightsquigarrow a) \odot (a \wedge b)^{\sim} \leq b^{\sim}$. It follows that $(a \wedge b)^{\sim} = 1 \odot (a \wedge b)^{\sim} \overset{psBL_5}{=} [(a \rightsquigarrow b) \vee (b \rightsquigarrow a)] \odot (a \wedge b)^{\sim} \overset{psbl-c_{36}}{=} [(a \rightsquigarrow b) \odot (a \wedge b)^{\sim}] \vee [(b \rightsquigarrow a) \odot (a \wedge b)^{\sim}] \leq a^{\sim} \vee b^{\sim}$.

The converse inequality follows since $a^{\sim}, b^{\sim} \leq (a \wedge b)^{\sim}$.

The second equality: $(a \vee b)^{\sim} = a^{\sim} \wedge b^{\sim}$, follows by $psbl - c_{36}$, $(a \vee b)^{\sim} = (a \vee b) \rightsquigarrow 0 = (a \rightsquigarrow 0) \vee (b \rightsquigarrow 0) = a^{\sim} \wedge b^{\sim}$.

$(psbl - c_{50})$. Has a similar proof with $psbl - c_{49}$.

$(psbl - c_{51})$.$(a \wedge b)^{\approx} = ((a \wedge b)^{\sim})^{\sim} \overset{psbl-c_{49}}{=} (a^{\sim} \vee b^{\sim})^{\sim} \overset{psbl-c_{49}}{=} a^{\approx} \wedge b^{\approx}$; the second equality follows similary.

$(psbl - c_{52})$. Has a similar proof with $psbl - c_{51}$.

$(psbl - c_{53})$ and $(psbl - c_{54})$ has a similar proof.

$(psbl - c_{54})$ Let $u = (b \odot a) \vee c$ and $v = u \vee a$; hence $u \leq v$.

$v \to u = (u \vee a) \to u \overset{psbl-c_{36}}{=} (u \to u) \wedge (a \to u) = 1 \wedge (a \to u) = a \to u$.

Then $u = v \wedge u = (v \to u) \odot v = (a \to u) \odot (u \vee a)$. But $b \odot a \leq u$ implies $b \leq a \to u$.

Also, we have $a^{-} \leq a \to u$ since $0 \leq u$. It follows that $b \vee a^{-} \leq a \to u$. Since $u \leq a \to u$ then $u \vee b \vee a^{-} \leq a \to u$, so by $psbl - c_3$, $[u \vee b \vee a^{-}] \odot (u \vee a) \leq (a \to u) \odot (u \vee a) = u = (b \odot a) \vee c$.

Since $u \vee b = [(b \odot a) \vee c] \vee b \leq b \vee c \vee b = b \vee c$ and $b \vee c \leq [(b \odot a) \vee c] \vee b = u \vee b$, we obtain $u \vee b = b \vee c$; similarly, $u \vee a = a \vee c$.

Replacing in the previous inequality, we obtain that $(b \vee c \vee a^{-}) \odot (a \vee c) \leq (b \odot a) \vee c$.

$(psbl - c_{55})$. $(a \vee b) \odot (a \vee c) \overset{psbl-c_{36}}{=} [(a \vee b) \odot a] \vee [(a \vee b) \odot c] = (a \odot a) \vee (b \odot a) \vee (a \odot c) \vee (b \odot c) \geq a \vee a \vee a \vee (b \odot c) = a \vee (b \odot c)$. ■

LEMMA 5.4. *For every $a, b, c \in A$, we have:*

$(psbl - c_{56})$ $a \wedge (b \odot c) \geq b \odot (a \wedge c)$;
$(psbl - c_{57})$ $a \wedge (b \odot c) \geq (a \wedge b) \odot c$;
$(psbl - c_{58})$ $a \wedge (b \odot c) \geq (a \wedge b) \odot (a \wedge c)$.

Proof. $(psbl - c_{56})$. From $psbl - c_{15}$ we have $a \to (\ b \odot c) \geq b \odot (a \to c)$. We deduce that $[a \to (\ b \odot c)] \odot a \geq b \odot [(a \to c) \odot a]$, so $a \wedge (b \odot c) \geq b \odot (a \wedge c)$.

$(psbl - c_{57})$. As in the case of $psbl - c_{56}$.

$(psbl - c_{58})$. From $a\wedge\ b \leq a, b$ and $a\wedge\ c \leq a, c$ we deduce $(a\wedge\ b) \odot (a\wedge\ c) \leq b \odot c$ and $(a\wedge\ b) \odot (a\wedge\ c) \leq a^2 \leq a$, hence $(a\wedge\ b) \odot (a\wedge\ c) \leq a \wedge (b \odot c)$.■

LEMMA 5.5. *For every $a, b, c \in A$, we have:*

$(psbl - c_{59})$ $a \to (b \rightsquigarrow c) \leq b \rightsquigarrow (a \to c)$,
$(psbl - c_{60})$ $a \rightsquigarrow (b \to c) \leq b \to (a \rightsquigarrow c)$.

Proof. $(psbl-c_{59})$. We have $b\odot[a\to(b\rightsquigarrow c)]\odot a \overset{psBL_4}{=} b\odot[a\wedge(b\rightsquigarrow c)] \overset{psbl-c_{24}}{=} (b\odot a)\wedge[b\odot(b\rightsquigarrow c)] \overset{psBL_4}{=} (b\odot a)\wedge(b\wedge c) = (b\odot a)\wedge c\leq c$, so $b\odot[a\to(b\rightsquigarrow c)]\leq a\to c$, hence $a\to(b\rightsquigarrow c)\leq b\rightsquigarrow(a\to c)$.

$(psbl-c_{60})$. We have $a\odot[a\rightsquigarrow(b\to c)]\odot b \overset{psBL_4}{=} [a\wedge(b\to c)]\odot b \overset{psbl-c_{24}}{=} (a\odot b)\wedge[(b\to c)\odot b] \overset{psBL_4}{=} (a\odot b)\wedge(b\wedge c) = (a\odot b)\wedge c\leq c$, so $[a\rightsquigarrow(b\to c)]\odot b\leq a\rightsquigarrow c$, hence $a\rightsquigarrow(b\to c)\leq b\to(a\rightsquigarrow c)$.■

COROLLARY 5.6. *For every $a,b,c\in A$, we have:*

$(psbl-c_{61})$ $a\to(b\rightsquigarrow c) = b\rightsquigarrow(a\to c)$;
$(psbl-c_{62})$ $a\to(a\rightsquigarrow c) = a\rightsquigarrow(a\to c)$.

Proof. $(psbl-c_{61})$. From $psbl-c_{59}$ we deduce that $a\to(b\rightsquigarrow c)\leq b\rightsquigarrow(a\to c)$. If in $psbl-c_{60}$ we change a with b we obtain $b\rightsquigarrow(a\to c)\leq a\to(b\rightsquigarrow c)$, that is, $a\to(b\rightsquigarrow c) = b\rightsquigarrow(a\to c)$.

$(psl-c_{62})$. Follow from $psbl-c_{61}$ if consider $a=b$.■

REMARK 5.1. *In particular for $c=0$, from $psbl-c_{61}$ and $psbl-c_{62}$ we deduce $psbl-c_{44}$.*

LEMMA 5.7. *For every $a,b\in A$, we have:*

$(psbl-c_{63})$ $a^{=}\odot b^{=}\leq(a\odot b)^{=}$,
$(psbl-c_{64})$ $a^{\approx}\odot b^{\approx}\leq(a\odot b)^{\approx}$.

Proof. $(psbl-c_{63})$. By $psbl-c_{48}$, $(a\odot b)^{-} = a\to b^{-}$, so $(a\odot b)^{-}\odot a\leq b^{-}$. By $psbl-c_{45}$ we deduce that $b^{=}\leq[(a\odot b)^{-}\odot a]^{-} = (a\odot b)^{-}\to a^{-}$, so $b^{=}\odot(a\odot b)^{-}\leq a^{-}$. Then

$$a^{=}\leq[b^{=}\odot(a\odot b)^{-}]^{-} = b^{=}\to(a\odot b)^{=},$$

that is, $a^{=}\odot b^{=}\leq(a\odot b)^{=}$.

$(psbl-c_{64})$. By $psbl-c_{48}$, $(a\odot b)^{\sim} = b\rightsquigarrow a^{\sim}$, so $b\odot(a\odot b)^{\sim}\leq a^{\sim}$. Then $a^{\approx}\leq[b\odot(a\odot b)^{\sim}]^{\sim} = (a\odot b)^{\sim}\rightsquigarrow b^{\sim}$, so $(a\odot b)^{\sim}\odot a^{\approx}\leq b^{\sim}$. Then

$$b^{\approx}\leq[(a\odot b)^{\sim}\odot a^{\approx}]^{\sim} = a^{\approx}\rightsquigarrow(a\odot b)^{\approx},$$

that is, $a^{\approx}\odot b^{\approx}\leq(a\odot b)^{\approx}$. ■

COROLLARY 5.8. *For every $a\in A$ and $n\geq 1$ we have:*

$(psbl-c_{65})$ $(a^{=})^{n}\leq(a^{n})^{=}$ *and* $(a^{\approx})^{n}\leq(a^{n})^{\approx}$.

LEMMA 5.9. *For every $a,b,c\in A$ we have:*

$(psbl-c_{66})$ $a\to(b\to c)\geq(a\to b)\to(a\to c)$,
$(psbl-c_{67})$ $a\rightsquigarrow(b\rightsquigarrow c)\geq(a\rightsquigarrow b)\rightsquigarrow(a\rightsquigarrow c)$,
$(psbl-c_{68})$ $a\to(b\rightsquigarrow c)\geq(a\to b)\rightsquigarrow(a\to c)$,
$(psbl-c_{69})$ $a\rightsquigarrow(b\to c)\geq(a\rightsquigarrow b)\to(a\rightsquigarrow c)$.

Proof. $(psbl-c_{66})$. By $psbl-c_{22}$ we have $a\to(b\to c) = (a\odot b)\to c$ and $(a\to b)\to(a\to c) = [(a\to b)\odot a]\to c = (a\wedge b)\to c$. Since $a\odot b\leq a\wedge b$ we deduce that $(a\odot b)\to c\geq(a\wedge b)\to c$, that is, $a\to(b\to c)\geq(a\to b)\to(a\to c)$.

$(psbl-c_{67})$. As in the case of $psbl-c_{66}$.

$(psbl-c_{68})$. By $psbl-c_{61}$ we have $a\to(b\rightsquigarrow c) = b\rightsquigarrow(a\to c)$. Since $b\leq a\to b$ we deduce that $b\rightsquigarrow(a\to c)\geq(a\to b)\rightsquigarrow(a\to c)$, that is $a\to(b\rightsquigarrow c)\geq(a\to b)\rightsquigarrow(a\to c)$.

$(psbl - c_{69})$. As in the case of $psbl - c_{68}$. ■

For any pseudo BL - algebra A, let us denote

$$G(A) = \{x \in A : x \odot x = x\},$$

$$M(A) = \{x \in A : x = (x^{\sim})^{-} = (x^{-})^{\sim}\}$$

and let $B(A)$ be the Boolean algebra ([**120**]) of all complemented elements in the distributive lattice $L(A) = (A, \vee, \wedge, 0, 1)$ of a pseudo BL-algebra A (hence $B(A) = B(L(A))$).

PROPOSITION 5.10. ([**54**]) *If A is a pseudo $BL-$ algebra and $a \in G(A), b \in A$, then*

(i) $a \odot b = a \wedge b = b \odot a$,
(ii) $a \wedge a^{\sim} = 0 = a \wedge a^{-}$,
(iii) $a \rightsquigarrow b = a \to b$,
(iv) $a^{\sim} = a^{-}$.

Proof. (i). $a \wedge b = a \odot (a \rightsquigarrow b) = a \odot a \odot (a \rightsquigarrow b) = a \odot (a \wedge b) \overset{psbl-c_{24}}{=} (a \odot a) \wedge (a \odot b) = a \wedge (a \odot b) = a \odot b$.
(ii). Follows by (i) and $psbl - c_{38}$.
(iii). We have the following equivalences for any $x \in A$:
$x \leq a \rightsquigarrow b \Leftrightarrow a \odot x \leq b \Leftrightarrow x \odot a \leq b \Leftrightarrow x \leq a \to b$.
(iv). Follows taking $b = 0$ in (iii). ■

LEMMA 5.11. *If A is a pseudo $BL-$algebra, then*

$$B(A) = M(A) \cap G(A).$$

Proof. Consider $x \in B(A)$; then for some $y \in A$ we have $x \vee y = 1$ and $x \wedge y = 0$. Then $y \odot x = x \wedge y = 0$, so $x \leq y^{\sim}$.

We also have $y^{\sim} = 1 \odot y^{\sim} = (x \vee y) \odot y^{\sim} \overset{psbl-c_{36}}{=} (x \odot y^{\sim}) \vee (y \odot y^{\sim}) \overset{psbl-c_{38}}{=} (x \odot y^{\sim}) \vee 0 = x \odot y^{\sim}$, hence $y^{\sim} \leq x$. Thus $x = y^{\sim}$.

Similarly, $x = y^{-}$. But $x^{-} \wedge y^{-} = 0$ and $x^{-} \vee y^{-} = 1$, i.e. $x^{-} \wedge x = 0$ and $x^{-} \vee x = 1$, and also $x^{\sim} \wedge y^{\sim} = 0$ and $x^{\sim} \vee y^{\sim} = 1$, i.e. $x^{\sim} \wedge x = 0$ and $x^{\sim} \vee x = 1$. Then $x^{-} = x^{\sim}$ is the unique complement of x, since the lattice $L(A)$ is distributive; hence $x^{-} = x^{\sim} \in B(A)$ and $(x^{\sim})^{-}$ is the complement of $x^{\sim}$. But x also is the complement of $x^{\sim}$.

It follows that $x = (x^{\sim})^{-}$ and thus $x \in M(A)$ and $x^{\sim}$ is the complement of x. Then $x \vee x^{\sim} = 1$, hence $x = x \odot 1 = x \odot (x \vee x^{\sim}) \overset{psbl-c_{36}}{=} (x \odot x) \vee (x \odot x^{\sim}) \overset{psbl-c_{38}}{=} (x \odot x) \vee 0 = x \odot x$, and thus $x \in G(A)$.

Conversely, consider $x \in M(A) \cap G(A)$. By Proposition 5.10, (ii), $x \wedge x^{\sim} = 0$, hence $1 = (x \wedge x^{\sim})^{-} \overset{psbl-c_{50}}{=} x^{-} \vee (x^{\sim})^{-} = x^{-} \vee x = x^{\sim} \vee x$, since $x \in M(A)$ and by Proposition 5.10, (iv). It follows that $x \in B(A)$. ■

PROPOSITION 5.12. ([**53**], [**54**]) *If A is a pseudo $BL-$ algebra, then for $e \in A$, the following are equivalent:*

(i) $e \in B(A)$;
(ii) $e \odot e = e$ *and* $e = (e^{\sim})^{-} = (e^{-})^{\sim}$;
(iii) $e \odot e = e$ *and* $e^{-} \to e = e$;
(iii') $e \odot e = e$ *and* $e^{\sim} \rightsquigarrow e = e$;

(iv) $e \vee e^- = 1$;
(iv') $e \vee e^\sim = 1$.

Proof. The equivalence $(i) \Leftrightarrow (ii)$ follows by Lemma 5.11.
The equivalence $(i) \Leftrightarrow (iv)$ and $(i) \Leftrightarrow (iv')$ has a similar proof.
We prove the second equivalence.
The implication $(i) \Rightarrow (iv')$ is obvious.

Conversely, $e \vee e^\sim = 1 \Rightarrow e \odot (e \vee e^\sim) = e \odot 1 = e \overset{psbl-c_{36}}{\Rightarrow} (e \odot e) \vee (e \odot e^\sim) = e \Rightarrow e \odot e = e$.

By Proposition 5.10, (iv), we deduce $e^- = e^\sim$.
On other hand, $e \vee e^\sim = 1$ implies $e \wedge e^\sim = 0$ and $e \vee e^- = 1$ implies $e \wedge e^- = 0$.

Indeed, $e \vee e^\sim = 1 \Rightarrow (e \vee e^\sim)^- = 0 \overset{psbl-c_{50}}{\Rightarrow} e^- \wedge (e^\sim)^- = 0 \Rightarrow e^- \wedge e = 0$. Thus $e \in B(A)$.

The equivalence $(iii) \Leftrightarrow (iv)$ and $(iii') \Leftrightarrow (iv')$ has a similar proof.
We prove the second equivalence.
$(iii') \Rightarrow (iv')$. $e \vee e^\sim \overset{psbl-c_{20}}{=} [(e \rightsquigarrow e^\sim) \to e^\sim] \wedge [(e^\sim \rightsquigarrow e) \to e]$.
$e \vee e^\sim = 1 \Leftrightarrow [(e \rightsquigarrow e^\sim) \to e^\sim = 1$ and $(e^\sim \rightsquigarrow e) \to e = 1]$
$\Leftrightarrow [e \rightsquigarrow e^\sim \leq e^\sim$ and $e^\sim \rightsquigarrow e \leq e] \Leftrightarrow [e \rightsquigarrow e^\sim = e^\sim$ and $e^\sim \rightsquigarrow e = e]$.

But $e \rightsquigarrow e^\sim \overset{psbl-c_{48}}{=} (e \odot e)^\sim$, hence $e \rightsquigarrow e^\sim = e^\sim \Leftrightarrow (e \odot e)^\sim = e^\sim$ (indeed, $e \odot e = e$ implies $(e \odot e)^\sim = e^\sim$).

$(iv') \Rightarrow (iii')$. We have $e \vee e^\sim = 1 \Rightarrow e \odot (e \vee e^\sim) = e \odot 1 = e \overset{psbl-c_{36}}{\Leftrightarrow} (e \odot e) \vee (e \odot e^\sim) = e \Rightarrow e \odot e = e$.
Also, $e \vee e^\sim = 1 \Rightarrow (e^\sim \rightsquigarrow e) \to e = 1 \Leftrightarrow e^\sim \rightsquigarrow e \leq e \Leftrightarrow e^\sim \rightsquigarrow e = e$.■

Remark 5.2. *If $a \in A$, and $e \in B(A)$, then $e \odot a = e \wedge a = a \odot e$ and $e^\sim = e^-$.*

Proposition 5.13. *If $a \in A$, and $e \in B(A)$, then*

$(psbl - c_{70})$ $a \to e = (a \odot e^\sim)^- = a^- \vee e$;
$(psbl - c_{71})$ $a \rightsquigarrow e = (e^- \odot a)^\sim = e \vee a^\sim$.

Proof. We have

$$a \to e = a \to (e^\sim)^- \overset{psbl-c_{48}}{=} (a \odot e^\sim)^- = (a \wedge e^\sim)^- \overset{psbl-c_{50}}{=} a^- \vee (e^\sim)^- = a^- \vee e$$

and

$$a \rightsquigarrow e = a \rightsquigarrow (e^-)^\sim \overset{psbl-c_{48}}{=} (e^- \odot a)^\sim = (e^- \wedge a)^\sim \overset{psbl-c_{49}}{=} (e^-)^\sim \vee a^\sim = e \vee a^\sim.$$■

Proposition 5.14. *Let A be a pseudo $BL-$ algebra. For $e \in A$, the following are equivalent:*

(i) $e \in B(A)$,
(ii) $(e \to x) \to e = (e \rightsquigarrow x) \rightsquigarrow e = e$, *for every* $x \in A$,

Proof.$(i) \Rightarrow (ii)$. If $x \in A$, then from $0 \leq x$ we deduce $e \to 0 \leq e \to x$ and $e \rightsquigarrow 0 \leq e \rightsquigarrow x$, so $e^- \leq e \to x$ and $e^\sim \leq e \rightsquigarrow x$ hence $(e \to x) \to e \leq e^- \to e = e$ and $(e \rightsquigarrow x) \rightsquigarrow e \leq e^\sim \rightsquigarrow e = e$. Since $e \leq (e \to x) \to e$, $e \leq (e \rightsquigarrow x) \rightsquigarrow e$ (by $psbl - c_9$) we obtain $(e \to x) \to e = (e \rightsquigarrow x) \rightsquigarrow e = e$.

$(ii) \Rightarrow (i)$. If $x \in A$, then from $(e \to x) \to e = e$ we deduce $[(e \to x) \to e] \odot (e \to x) = e \odot (e \to x)$, hence $(e \to x) \wedge e = (e \to x) \odot e$ so $(e \to x) \wedge e = e \wedge x$. For $x = 0$ we obtain that $e^- \wedge e = 0$. Analogously, from $(e \rightsquigarrow x) \rightsquigarrow e = e$ we deduce $(e \rightsquigarrow x) \odot [(e \rightsquigarrow x) \rightsquigarrow e] = (e \rightsquigarrow x) \odot e$, hence $(e \rightsquigarrow x) \wedge e = (e \rightsquigarrow x) \odot e$ so

$(e \rightsquigarrow x) \wedge e = e \wedge x$. For $x = 0$ we obtain that $e \wedge e^{\sim} = 0$, so $e^{-} \wedge e = 0 = e \wedge e^{\sim}$. From hypothesis (for $x = 0$) we obtain by Proposition 5.10, $e^{-} \to e = e^{-} \rightsquigarrow e = e^{\sim} \rightsquigarrow e = e^{\sim} \to e = e$ and $e^{-} = e^{\sim}$.

From $psbl - c_{21}$ we obtain

$$\begin{aligned} e^{-} \vee e &= [(e^{-} \to e) \rightsquigarrow e] \wedge [(e \to e^{-}) \rightsquigarrow e^{-}] \\ &= (e \rightsquigarrow e) \wedge [(e \to e^{-}) \rightsquigarrow e^{-}] \\ &= 1 \wedge [(e \to e^{-}) \rightsquigarrow e^{-}] \\ &= (e \to e^{-}) \rightsquigarrow e^{-} = (e \to e^{\sim}) \rightsquigarrow e^{\sim} \\ &= [e \odot (e \to e^{\sim})]^{\sim} \text{ (by } psbl - c_{48}) \\ &= [(e \to e^{-}) \odot e]^{\sim} = (e \wedge e^{-})^{\sim} = 0^{\sim} = 1, \end{aligned}$$

hence $e \in B(A)$.■

LEMMA 5.15. *([**37**]) Let A be a pseudo $BL-$ algebra. If $e, f \in B(A)$ and $x, y \in A$, then:*

$(psbl - c_{72})$ $e \vee (x \odot y) = (e \vee x) \odot (e \vee y)$;
$(psbl - c_{73})$ $e \wedge (x \odot y) = (e \wedge x) \odot (e \wedge y)$;
$(psbl - c_{74})$ $e \odot (x \rightsquigarrow y) = e \odot [(e \odot x) \rightsquigarrow (e \odot y)]$ *and* $(x \to y) \odot e = [(x \odot e) \to (y \odot e)] \odot e$;
$(psbl - c_{75})$ $x \odot (e \rightsquigarrow f) = x \odot [(x \odot e) \rightsquigarrow (x \odot f)]$ *and* $(e \to f) \odot x = [(e \odot x) \to (f \odot x)] \odot x$;
$(psbl - c_{76})$ $e \to (x \to y) = (e \to x) \to (e \to y)$ *and* $e \rightsquigarrow (x \rightsquigarrow y) = (e \rightsquigarrow x) \rightsquigarrow (e \rightsquigarrow y)$.

Proof. $(psbl - c_{72})$. We have

$$(e \vee x) \odot (e \vee y) \overset{psbl-c_{36}}{=} [(e \vee x) \odot e] \vee [(e \vee x) \odot y] = [(e \vee x) \odot e] \vee [(e \odot y) \vee (x \odot y)]$$
$$= [(e \vee x) \wedge e] \vee [(e \odot y) \vee (x \odot y)] = e \vee (e \odot y) \vee (x \odot y) = e \vee (x \odot y).$$

$(psbl - c_{73})$. We have

$$(e \wedge x) \odot (e \wedge y) = (e \odot x) \odot (e \odot y) = (e \odot e) \odot (x \odot y) = e \odot (x \odot y) = e \wedge (x \odot y).$$

$(psbl - c_{74})$. By $psbl - c_{13}$ we have $x \to y \leq (x \odot e) \to (y \odot e)$, hence by $psbl - c_3$, $(x \to y) \odot e \leq [(x \odot e) \to (y \odot e)] \odot e$. Conversely, $[(x \odot e) \to (y \odot e)] \odot e \leq e$ and $[(x \odot e) \to (y \odot e)] \odot (x \odot e) \leq y \odot e \leq y$ so $[(x \odot e) \to (y \odot e)] \odot e \leq x \to y$. Hence $[(x \odot e) \to (y \odot e)] \odot e \leq (x \to y) \wedge e = (x \to y) \odot e$.

By $psbl - c_{12}$ we have $x \rightsquigarrow y \leq (e \odot x) \rightsquigarrow (e \odot y)$, hence by c_3, $e \odot (x \rightsquigarrow y) \leq e \odot [(e \odot x) \rightsquigarrow (e \odot y)]$. Conversely, $e \odot [(e \odot x) \rightsquigarrow (e \odot y)] \leq e$ and $(e \odot x) \odot [(e \odot x) \rightsquigarrow (e \odot y)] \leq e \odot y \leq y$ so $e \odot [(e \odot x) \rightsquigarrow (e \odot y)] \leq x \rightsquigarrow y$.

Hence $e \odot [(e \odot x) \rightsquigarrow (e \odot y)] \leq e \wedge (x \rightsquigarrow y) = e \odot (x \rightsquigarrow y)$.

$(psbl - c_{75})$. We have

$$[(e \odot x) \to (f \odot x)] \odot x = [(e \odot x) \to (f \wedge x)] \odot x$$
$$\overset{psbl-c_{36}}{=} [((e \odot x) \to f) \wedge ((e \odot x) \to x)] \odot x$$
$$= [((e \odot x) \to f) \wedge 1] \odot x = [(e \odot x) \to f] \odot x = [(x \odot e) \to f] \odot x$$
$$\overset{psbl-c_{22}}{=} [x \to (e \to f)] \odot x = x \wedge (e \to f) = x \odot (e \to f).$$

We have

$$x \odot [(x \odot e) \rightsquigarrow (x \odot f)] = x \odot [(x \odot e) \rightsquigarrow (x \wedge f)]$$
$$\overset{psbl-c_{36}}{=} x \odot [((x \odot e) \rightsquigarrow x) \wedge ((x \odot e) \rightsquigarrow f)] = x \odot [1 \wedge ((x \odot e) \rightsquigarrow f)]$$
$$= x \odot [(x \odot e) \rightsquigarrow f] = x \odot [(e \odot x) \rightsquigarrow f] \overset{psbl-c_{22}}{=} x \odot [x \rightsquigarrow (e \rightsquigarrow f)]$$

$$= x \wedge (e \rightsquigarrow f) = x \odot (e \rightsquigarrow f).$$

$(psbl - c_{76})$.We have

$$(e \to x) \to (e \to y) \overset{psbl-c_{22}}{=} [(e \to x) \odot e] \to y = (e \wedge x) \to y = (e \odot x) \to y \overset{psbl-c_{22}}{=} e \to (x \to y)$$

and

$$(e \rightsquigarrow x) \rightsquigarrow (e \rightsquigarrow y) \overset{psbl-c_{22}}{=} [e \odot (e \rightsquigarrow x)] \rightsquigarrow y = (e \wedge x) \rightsquigarrow y = (x \odot e) \rightsquigarrow y \overset{psbl-c_{22}}{=} e \rightsquigarrow (x \rightsquigarrow y).\blacksquare$$

LEMMA 5.16. *If a, b, x are elements of a pseudo $BL-$ algebra A and $a, b \leq x$, then*

$(psbl - c_{77})$ $a \odot (x \rightsquigarrow b) = (x \to a) \odot b$.

Proof. We have

$$a \odot (x \rightsquigarrow b) = (x \wedge a) \odot (x \rightsquigarrow b) = [(x \to a) \odot x] \odot (x \rightsquigarrow b)$$
$$= (x \to a) \odot [x \odot (x \rightsquigarrow b)] = (x \to a) \odot (x \wedge b) = (x \to a) \odot b.\blacksquare$$

PROPOSITION 5.17. *For a pseudo $BL-$ algebra $(A, \vee, \wedge, \odot, \to, \rightsquigarrow, 0, 1)$ the following are equivalent:*

(i) $(A, \to, 1)$ and $(A, \rightsquigarrow, 1)$ are Hilbert algebras;

(ii) $(A, \vee, \wedge, \to, 0, 1)$ and $(A, \vee, \wedge, \rightsquigarrow, 0, 1)$ are relative Stone lattices.

Proof. $(i) \Rightarrow (ii)$. Suppose that $(A, \to, 1)$ and $(A, \rightsquigarrow, 1)$ are Hilbert algebras, then for every $x, y, z \in A$ we have

$$x \to (y \to z) = (x \to y) \to (x \to z)$$

and

$$x \rightsquigarrow (y \rightsquigarrow z) = (x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z).$$

From $psbl - c_{22}$ we have

$$x \to (y \to z) = (x \odot y) \to z \text{ and } x \rightsquigarrow (y \rightsquigarrow z) = (y \odot x) \rightsquigarrow z.$$

But for every $x, y, z \in A$

$$(x \wedge y) \to z = [(x \to y) \odot x] \to z \overset{psbl-c_{22}}{=} (x \to y) \to (x \to z)$$

and

$$(x \wedge y) \rightsquigarrow z = [x \odot (x \rightsquigarrow y)] \rightsquigarrow z \overset{psbl-c_{22}}{=} (x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z),$$

so we obtain

$$(x \odot y) \to z = (x \wedge y) \to z$$
$$(x \odot y) \rightsquigarrow z = (x \wedge y) \rightsquigarrow z$$

hence $x \odot y = x \wedge y$, that is $(A, \vee, \wedge, \to, 0, 1)$ and $(A, \vee, \wedge, \rightsquigarrow, 0, 1)$ are relative Stone lattices.

$(ii) \Rightarrow (i)$. If $(A, \vee, \wedge, \to, 0, 1)$ and $(A, \vee, \wedge, \rightsquigarrow, 0, 1)$ are relative Stone lattices, then $(A, \vee, \wedge, \to, 0, 1)$ and $(A, \vee, \wedge, \rightsquigarrow, 0, 1)$ are Heyting algebras, so $(A, \to, 1)$ and $(A, \rightsquigarrow, 1)$ are Hilbert algebras. $\blacksquare$

DEFINITION 5.3. Let A and B be a pseudo $BL-$ algebras. A function $f : A \to B$ is a *morphism of pseudo $BL-$ algebras* iff it satisfies the following conditions, for every $x, y \in A$:

$(psBL_6)$ $f(0) = 0$;

$(psBL_7)$ $f(x \odot y) = f(x) \odot f(y)$;

$(psBL_8)$ $f(x \to y) = f(x) \to f(y)$;
$(psBL_9)$ $f(x \rightsquigarrow y) = f(x) \rightsquigarrow f(y)$.

REMARK 5.3. *It follows that:*

$$f(1) = 1,$$
$$f(x^-) = [f(x)]^-, f(x^\sim) = [f(x)]^\sim,$$
$$f(x \vee y) = f(x) \vee f(y), f(x \wedge y) = f(x) \wedge f(y),$$

for every $x, y \in A$.

REMARK 5.4. *If* f *is a homomorphism beteewn the pseudo* $BL-$ *algebras* A *and* B*, then* f *is a homomorphism beteewn the lattices* $L(A)$ *and* $L(B)$*, see Remark 5.3.*

2. The lattice of filters of a pseudo BL-algebra

We begin the investigation of filters and congruences. We define the filters of a pseudo BL-algebra A and we denote by F(A)($\mathrm{F_n}$(A)) the lattice of all filters (normal filters) of A; we put in evidence some results about the lattice F(A)($\mathrm{F_n}$(A)). By using the two distance functions we define two binary relations on , $\equiv_{L(F)}$ and $\equiv_{R(F)}$, related to a filter F of A; these two relations are equivalence relations, but they are not congruences. The quotient set A/L(F) and A/R(F) are bounded distributive lattices. We give characterizations for the maximal and prime elements on F(A)($\mathrm{F_n}$(A)) and we prove the prime filter theorem. We characterize the pseudo BL-algebras for which the lattice of filters (normal filters) is a Boolean lattice and the archimedean and hyperarchimedean pseudo BL-algebras. In the end we prove a theorem of Nachbin type for pseudo BL-algebras.

2.1. The lattice of filters (normal filters) of a pseudo BL-algebra. We denote by A a pseudo - BL algebra.

DEFINITION 5.4. A non empty subset $F \subseteq A$ is called a *fillter* of A, if the following conditions are satisfied:

(F_1) If x, $y \in F$, then $x \odot y \in F$;
(F_2) If $x \in F, y \in A, x \leq y$, then $y \in F$.

Clearly $\{1\}$ and A are filters; a filter F of A is called *proper* if $F \neq A$.

REMARK 5.5. *Any filter of the pseudo* $BL-$ *algebra* A *is a filter of the lattice* $L(A)$.

REMARK 5.6. *For a nonempty subset* F *of* A *the following are equivalent:*

(1) F *is a filter;*
(2) $1 \in F$ *and if* $x, x \to y \in F$*, then* $y \in F$;
(2′) $1 \in F$ *and if* $x, x \rightsquigarrow y \in F$*, then* $y \in F$.

Proof. (1) $\Rightarrow$ (2′). $x, x \rightsquigarrow y \in F \Rightarrow x \wedge y \overset{psBL_4}{=} x \odot (x \rightsquigarrow y) \in F$; but $x \wedge y \leq y$, so by F_2 we obtain $y \in F$.

(2′) $\Rightarrow$ (1). We verify the condition of Definition 5.4:

F_1 : If x, $y \in F$ then $y \rightsquigarrow (x \rightsquigarrow x \odot y) \overset{psbl-c_{22}}{=} (x \odot y) \rightsquigarrow (x \odot y) = 1 \in F$, so $x \odot y \in F$.

F_2 : If $x \in F$ and $y \in A, x \leq y$, then $x \rightsquigarrow y = 1 \in F$, so $y \in F$.

Similarly, (1) $\Leftrightarrow$ (2). ■

We denote by $\mathcal{F}(A)$ the set of all filters of A.

For a nonempty subset $X \subseteq A$, the smallest filter of A which contains X, i.e. $\cap\{F \in \mathcal{F}(A) : X \subseteq F\}$, is said to be *the filter of A generated by X* and will be denoted by $[X)$.

If $X = \{a\}$, with $a \in A$, we denote by $[a)$ the *filter generated by* $\{a\}$ ($[a)$ is called *principal*).

For $F \in \mathcal{F}(A)$ and $a \in A\backslash F$, we denote by $F(a) = [F \cup \{a\})$.

PROPOSITION 5.18. ([**53**])

(*i*) *If X is a filter, then $[X) = X$;*

(*ii*) *If $X \subseteq A$ is a nonempty subset of A, then*

$$[X) = \{x \in A : x_1 \odot ... \odot x_n \leq x, \text{ for some } n \geq 1 \text{ and } x_1, ..., x_n \in X\};$$

(*iii*) *In particular, for $a \in A$,*

$$[a) = \{x \in A : x \geq a^n, \text{ for some } n \geq 1\}$$

(let $\mathcal{F}_p(A)$ be the set of all principal filters of A);

(*iv*) *If $F \in \mathcal{F}(A)$ and $a \in A\backslash F$, then*

$$\begin{aligned} F(a) &= \{x \in A : x \geq (f_1 \odot a^{n_1}) \odot ... \odot (f_m \odot a^{n_m}), \\ \text{for some } m &\geq 1, n_1, ..., n_m \geq 0 \text{ and } f_1, ..., f_m \in F\}; \end{aligned}$$

(*v*) *If $x, y \in A$, and $x \leq y$, then $[y) \subseteq [x)$;*

(*vi*) *If $x, y \in A$, then $[x \vee y), = [x \vee y)$;*

(*vii*) *If $x \in A$, then $[x) = \{1\}$ iff $x = 1$.*

Proof. $(i) - (iv), (vi), (vii)$. Is obviously (see also the proofs of Proposition 1.29 and Lemma 1.32).

(vi). Clearly, $x \in [x)$ and $y \in [y)$; since $x, y \leq x \vee y$ we get $x \vee y \in [x)$ and $x \vee y \in [y)$; then $x \vee y \in [x) \cap [y)$, which is s filter. So, $[x \vee y) \subseteq [x) \cap [y)$.

Conversely, suppose that $z \in [x) \cap [y)$ there exist $n, m \geq 1$, such that $z \geq x^n$ and $z \geq y^m$. It follows that $z \geq x^n \vee y^m \geq (x^n \vee y)^m \geq ((x \vee y)^n)^m = (x \vee y)^{nm}$, thus $z \in [x \vee y)$.■

REMARK 5.7. *If $F \in \mathcal{F}(A)$ and $a \in A\backslash F$, then $F(a) = F \vee [a)$.*

Proof. Clearly, $F, [a) \subseteq F(a)$. Let $H \in \mathcal{F}(A)$ such that $F, [a) \subseteq H$ and $x \in F(a)$. By Proposition 5.18, (ii), $x \geq (f_1 \odot a^{n_1}) \odot ... \odot (f_m \odot a^{n_m})$, for some $m \geq 1, n_1, ..., n_m \geq 0$ and $f_1, ..., f_m \in F$. Clearly, $(f_1 \odot a^{n_1}) \odot ... \odot (f_m \odot a^{n_m}) \in H$, hence $x \in H$, so $F(a) \subseteq H$, that is, $F(a) = F \vee (a]$. ■

PROPOSITION 5.19. *If F_1, F_2 are nonempty sets of A such that $1 \in F_1 \cap F_2$, then*

$$\begin{aligned} [F_1 \cup F_2) &= \{x \in A : x \geq (f_1 \odot f_1') \odot .... \odot (f_n \odot f_n'), \\ \text{for some } n &\geq 1, f_1, ..., f_n \in F_1 \text{ and } f_1', ..., f_n' \in F_2\}. \end{aligned}$$

Proof. Let

$$\begin{aligned} H &= \{x \in A : x \geq (f_1 \odot f_1') \odot ... \odot (f_n \odot f_n'), \\ \text{for some } n &\geq 1, f_1, ..., f_n \in F_1 \text{ and } f_1', ..., f_n' \in F_2\}. \end{aligned}$$

We prove that $H \in \mathcal{F}(A)$.

Let $x, y \in A, x \leq y$ and $x \in H$. Since $x \geq (f_1 \odot f_1') \odot ... \odot (f_n \odot f_n')$, for some $n \geq 1, f_1, ..., f_n \in F_1$ and $f_1', ..., f_n' \in F_2$ we have $y \geq (f_1 \odot f_1') \odot ... \odot (f_n \odot f_n')$, so $y \in H$.

For $x, y \in H$ there exist $n, m \geq 1$, $f_1, ..., f_n, g_1, ..., g_m \in F_1$ and $f'_1, ..., f'_n, g'_1, ..., g'_m \in F_2$ such that $x \geq (f_1 \odot f'_1) \odot ... \odot (f_n \odot f'_n)$ and $y \geq (g_1 \odot g'_1) \odot ... \odot (g_m \odot g'_m)$. We deduce $x \odot y \geq (f_1 \odot f'_1) \odot ... \odot (f_n \odot f'_n) \odot (g_1 \odot g'_1) \odot ... \odot (g_m \odot g'_m)$, so $x \odot y \in H$.

Since $1 \in F_1 \cap F_2$, we deduce that $F_1, F_2 \subseteq H$ (since for every $a \in A$, $a = a \odot 1 = 1 \odot a$), hence $F_1 \cup F_2 \subseteq H$, so we deduce that $[F_1 \cup F_2) \subseteq H$.

Let now $F \in \mathcal{F}(A)$ such that $F_1 \cup F_2 \subseteq F$. Then $H \subseteq F$, so $H \subseteq \cap F = [F_1 \cup F_2)$, hence $[F_1 \cup F_2) = H$. ■

COROLLARY 5.20. *If $F_1, F_2 \in \mathcal{F}(A)$, since $1 \in F_1 \cap F_2$, we deduce that*

$$\begin{aligned} [F_1 \cup F_2) &= \{x \in A : x \geq (f_1 \odot f'_1) \odot .... \odot (f_n \odot f'_n), \\ \text{for some } n &\geq 1, f_1, ..., f_n \in F_1 \text{ and } f'_1, ..., f'_n \in F_2\}. \end{aligned}$$

LEMMA 5.21. *If $x, y \in A$, then $[x) \vee [y) = [x \wedge y) = [x \odot y)$.*

Proof. Since $x \odot y \leq x \wedge y \leq x, y$, then $[x), [y) \subseteq [x \wedge y) \subseteq [x \odot y)$, hence $[x) \vee [y) \subseteq [x \wedge y) \subseteq [x \odot y)$.

If $z \in [x \odot y)$, then for some natural number $n \geq 1$, $z \geq (x \odot y)^n \in [x) \vee [y)$ (since $x \in [x), y \in [y)$), hence $z \in [x) \vee [y)$, that is, $[x \odot y) \subseteq [x) \vee [y)$, so $[x) \vee [y) = [x \wedge y) = [x \odot y)$. ■

COROLLARY 5.22. *For every $x, y \in A$, $[x \odot y) = [y \odot x)$.*

COROLLARY 5.23. *For every $x, y \in A$, we have $[x \to y) \vee [x) = [x \rightsquigarrow y) \vee [x)$.*

Proof. From Lemma 5.21 we deduce $[x \to y) \vee [x) = [(x \to y) \odot x) = [x \odot (x \rightsquigarrow y)) = [x \rightsquigarrow y) \vee [x)$. ■

COROLLARY 5.24. *$\mathcal{F}p(A)$ is a bounded sublattice of $\mathcal{F}(A)$.*

Proof. Apply Proposition 5.18, (vi), Lemma 5.21, the fact that $\mathbf{0} = \{1\} = [1) \in \mathcal{F}p(A)$ and $\mathbf{1} = A = [0) \in \mathcal{F}p(A)$. ■

As in the case of $BL-$algebras (Proposition 3.16) we have:

PROPOSITION 5.25. *The lattice $(\mathcal{F}(A), \vee, \wedge)$ is a complete Brouwerian algebraic lattice, the compacts elements being exactly the principal filters of A.*

REMARK 5.8. *The Proposition 5.25 is a generalization of Proposition 2.11 from* [**68**] *(the results of this proposition are mentioned in* [**94**] *without proof).*

For F_1, $F_2 \in \mathcal{F}(A)$ we put

$$F_1 \to F_2 = \{a \in A : F_1 \cap [a) \subseteq F_2\}.$$

LEMMA 5.26. *If F_1, $F_2 \in \mathcal{F}(A)$ then*

(i) $F_1 \to F_2 \in \mathcal{F}(A)$;

(ii) If $F \in \mathcal{F}(A)$, then $F_1 \cap F \subseteq F_2$ iff $F \subseteq F_1 \to F_2$, that is,

$$F_1 \to F_2 = \sup\{F \in \mathcal{F}(A) : F_1 \cap F \subseteq F_2\}.$$

Proof. See the case $BL-$ algebras, Lemma 3.18. ■

In [**95**], for F_1, $F_2 \in \mathcal{F}(A)$, the *relative pseudocomplement of F_1 with respect to F_2* is defined by

$$F_1 * F_2 = \{x \in A : x \vee y \in F_2, \text{ for all } y \in F_1\}.$$

PROPOSITION 5.27. *For all* $F_1, F_2 \in \mathcal{F}(A), F_1 * F_2 = F_1 \to F_2$.

Proof. Let $x \in F_1 * F_2$; then $x \vee y \in F_2$, for all $y \in F_1$. To prove $x \in F_1 \to F_2$, that is, $[x) \cap F_1 \subseteq F_2$, let $z \in [x) \cap F_1$. Thus $z \in F_1$ and $x^n \leq z$ for some $n \geq 1$. Since $z \leq x \to z$ we deduce that $x \to z \in F_1$, hence $x \vee (x \to z) \in F_2$. By $psbl - c_{21}$ we deduce that $[x \to (x \to z)] \rightsquigarrow (x \to z) \in F_2$, that is, $(x^2 \to z) \rightsquigarrow (x \to z) \in F_2$ (1).

Analogously from $z \leq x^{n-1} \to z$ and $z \in F_1$ we deduce that $x^{n-1} \to z \in F_1$, so $x \vee (x^{n-1} \to z) \in F_2$, hence $(x^n \to z) \rightsquigarrow (x^{n-1} \to z) \in F_2$. Since $x^n \leq z$ we deduce that $x^{n-1} \to z \in F_2$.

More generally if $k \geq 1$ we deduce that if $x^k \to z \in F_2$, then $x^{k-1} \to z \in F_2$ (because we obtain that $(x^k \to z) \rightsquigarrow (x^{k-1} \to z) \in F_2$). Recursively we obtain that $x^2 \to z \in F_2$. By (1) we deduce that $x \to z \in F_2$. Since $z \in F_1$ we deduce that $x \vee z \in F_2$, hence $(x \to z) \rightsquigarrow z \in F_2$. Thus $z \in F_2$, hence $[x) \cap F_1 \subseteq F_2$, that is, $x \in F_1 \to F_2$.

Thus $F_1 * F_2 \subseteq F_1 \to F_2$.

Let now $x \in F_1 \to F_2$. Thus $[x) \cap F_1 \subseteq F_2$, so if $y \in F_1$ then $x \vee y \in [x) \cap F_1$, hence $x \vee y \in F_2$. We deduce that $x \in F_1 * F_2$, so $F_1 \to F_2 \subseteq F_1 * F_2$. Since $F_1 * F_2 \subseteq F_1 \to F_2$ we deduce that $F_1 * F_2 = F_1 \to F_2$. ■

COROLLARY 5.28. $(\mathcal{F}(A), \vee, \wedge, \to, \{1\})$ *is a Heyting algebra, where for* $F \in \mathcal{F}(A)$,

$$F^* = F \to \mathbf{0} = F \to \{1\} = \{x \in A : [x) \cap F = \{1\}\},$$

hence for every $x \in F$ *and* $y \in F^*, x \vee y = 1$. *In particular, for every* $a \in A$,

$$[a)^* = \{x \in A : x \vee a = 1\}.$$

Proof. By Lemma 5.26 we deduce that $(\mathcal{F}(A), \vee, \wedge, \to, \{1\}, A)$ is a Heyting algebra. For every $F \in \mathcal{F}(A)$ and $y \in F^*$, then $[y) \cap F = \{1\}$. Since for every $x \in F$, $x \vee y \in [y) \cap F = \{1\}$ we deduce that $x \vee y = 1$. For every $a \in A$,

$$[a)^* = \{x \in A : [x) \cap [a) = \{1\}\} = \{x \in A : [x \vee a) = \{1\}\}$$

$$= \{x \in A : x \vee a = 1\}, \text{ (by Proposition 5.18, } (vii)).\blacksquare$$

COROLLARY 5.29. *If* $a \in A, F = [a)^*$, *then*

$$F(a) = \{x \in A : x \geq f \odot a^n, \text{ for some } n \geq 1 \text{ and } f \in F\}.$$

Proof. By Proposition 5.18, (iv),

$$F(a) = \{x \in A : x \geq (f_1 \odot a^{n_1}) \odot ... \odot (f_m \odot a^{n_m}) \ ,$$

$$\text{for some } f_1, ..., f_m \in F \text{ and } m \geq 1, n_1, ..., n_m \geq 0\}.$$

But for every $f \in F$ we have $a \vee f = 1$. By $psbl - c_{34}$ we obtain $a^n \vee f^n = 1$, for all $n \geq 1$. We have $a^n \vee f \geq a^n \vee f^n$ and $a^n \vee f^n = 1$, so $a^n \vee f = 1$, for all $n \geq 1$.

By $psbl - c_{33}$ we deduce that $f \odot a^n = f \wedge a^n = a^n \wedge f = a^n \odot f$, for every $f \in F$ and $n \geq 1$.

Then $(f_1 \odot a^{n_1}) \odot ... \odot (f_m \odot a^{n_m}) = (f_1 \odot ... \odot f_m) \odot (a^{n_1} \odot ... \odot a^{n_m}) = f \odot a^n$, where $f = f_1 \odot ... \odot f_m \in F$ (since F is a filter) and $n = n_1 + ... + n_m \geq 1$, so $F(a) = \{x \in A : x \geq f \odot a^n$, for some $n \geq 1$ and $f \in F\}$. ■

PROPOSITION 5.30. *If* $x, y \in A$, *then* $[x \odot y)^* = [x)^* \cap [y)^*$.

Proof. See the proof of Proposition 3.19. ■

Let A be a pseudo $BL-$algebra. We define two distance function on A by:

$$d_1(x,y) = (x \to y) \odot (y \to x) \text{ and } d_2(x,y) = (x \rightsquigarrow y) \odot (y \rightsquigarrow x).$$

The two distance functions fulfil some properties (see [**53**]).

PROPOSITION 5.31. ([**53**]) *Let us consider a filter F of A. Define two binary relations on A by:*

$$\equiv_{L(F)}\colon x \equiv_{L(F)} y \text{ iff } d_1(x,y) = (x \to y) \odot (y \to x) \in F,$$

$$\equiv_{R(F)}\colon x \equiv_{R(F)} y \text{ iff } d_2(x,y) = (x \rightsquigarrow y) \odot (y \rightsquigarrow x) \in F.$$

For a given filter F, the relations $\equiv_{L(F)}$ and $\equiv_{R(F)}$ are equivalence relations on A; moreover we have $F = \{x \in A : x \equiv_{L(F)} 1\} = \{x \in A : x \equiv_{R(F)} 1\}$.

We shall denote by $A/L(F)$ ($A/R(F)$, respectively) the quotient set associated with $\equiv_{L(F)}$ ($\equiv_{R(F)}$, respectively); $x/L(F)$ ($x/R(F)$, respectively) will denote the equivalence class of $x \in A$ with respect to $\equiv_{L(F)}$ ($\equiv_{R(F)}$,respectively).

Let us define the binary relation $\leq_{L(F)}$ on $A/L(F)$ by: $x/L(F) \leq_{L(F)} y/L(F)$ $\Leftrightarrow x \to y \in F$.

It is straightforward to prove that $\leq_{L(F)}$ is an order on $A/L(F)$.

Similarly, we define an order $\leq_{R(F)}$ on $A/R(F)$ by: $x/R(F) \leq_{R(F)} y/R(F)$ $\Leftrightarrow x \rightsquigarrow y \in F$.

We have the following result (see [**53**]):

PROPOSITION 5.32. $(A/L(F), \vee, \wedge, 0/L(F), 1/L(F))$ *($(A/R(F), \vee, \wedge, 0/R(F), 1/R(F))$ respectively) is a bounded distributive lattice, such that $\leq_{L(F)}$ ($\leq_{R(F)}$, respectively) is the induced order relation.*

In [**54**], A. Di Nola, G. Georgescu and A. Iorgulescu introduce the normal filters in order to characterize the congruence of a pseudo $BL-$algebra.

DEFINITION 5.5. A filter H of A will be called a *normal filter* iff

$$(N) \text{ for every } x, y \in A, x \to y \in H \text{ iff } x \rightsquigarrow y \in H.$$

REMARK 5.9. *Clearly, $\{1\}$ and A are normal filters. If $f : A \to B$ is a morphism of pseudo $BL-$ algebras, then that $f^{-1}(1_B)$ is a normal filter of A.*

We denote by $\mathcal{F}_n(A)$ the set of all normal filters of A.

PROPOSITION 5.33. *If $a \in G(A)$, then $[a) = \{x \in A : a \leq x\} \in \mathcal{F}_n(A)$.*

Proof. If $x, y \in A$ such that $x \to y \in [a)$, then $a \leq x \to y \Leftrightarrow a \odot x \leq y$ (since $a \odot x = x \odot a$) $\Leftrightarrow x \odot a \leq y \Leftrightarrow a \leq x \rightsquigarrow y \Leftrightarrow x \rightsquigarrow y \in [a)$, that is, $[a) \in \mathcal{F}_n(A)$. ■

REMARK 5.10. *Let $H \in \mathcal{F}_n(A)$. Then*

(i) $x^- \in H$ iff $x^\sim \in H$;

(ii) $x \in H$ implies $x^= , x^{\approx} \in H$.

Proof. (i). Take $y = 0$ in (N).

(ii). Indeed, if $x \in H$, then $(x^\sim)^- \in H$, because $x \leq (x^\sim)^-$, then by (i), $x^{\approx} \in H$. Similarly, $x \in H$ implies $x^= \in H$.■

For a filter H of A and $x \in A$ denote $x \odot H = \{x \odot h : h \in H\}$ and $H \odot x = \{h \odot x : h \in H\}$.

PROPOSITION 5.34. *For a filter H of A the following are equivalent:*

(i) $H \in \mathcal{F}_n(A)$;

(ii) $x \odot H = H \odot x$, *for all* $x \in A$.

Proof. $(i) \Rightarrow (ii)$. Let $x \in A$ and $h \in H$. Consider $y = x \odot h$.

Then $x \odot h = y = x \wedge y = (x \to y) \odot x$. If we denote $x \to y = h'$, we obtain $x \odot h = h' \odot x$.

We prove that $h' = x \to y \in H$.

Since $h \leq x \rightsquigarrow (x \odot h) = x \rightsquigarrow h$ and $h \in H$, we deduce that $x \rightsquigarrow h \in H$ and, therefore, $x \to y = h' \in H$, by (N).

$(ii) \Rightarrow (i)$. Assume $x \rightsquigarrow y \in H$. Thus, $x \wedge y = x \odot (x \rightsquigarrow y) = h \odot x$, for some $h \in H$. We have $x \to y = x \to (x \wedge y) = x \to (h \odot x)$. But $h \leq x \to (h \odot x)$, hence $x \to y \in H$.■

If H is a normal filter in a pseudo $BL-$ algebra A, then $d_1(x, y) = (x \to y) \odot (y \to x) \in H$ iff $d_2(x, y) = (x \rightsquigarrow y) \odot (y \rightsquigarrow x) \in H$, for any $x, y \in A$; hence $\equiv_{L(H)}$ and $\equiv_{R(H)}$ coincide. Denote by $\equiv_H$ this equivalence relation and by x/H be the equivalence class of $x \in A$. Hence $x \equiv_H y$ iff $d_1(x, y) \in H$ iff $d_2(x, y) \in H$

Remark that $x \equiv_H y$ iff $x \to y, y \to x \in H$ iff $x \rightsquigarrow y, y \rightsquigarrow x \in H$.

We have the following results:

PROPOSITION 5.35. ([**54**]) $\equiv_H$ *is a congruence on* A *and* $H = \{x \in A : x \equiv_H 1\} = 1/\equiv_H$.

Conversely,

PROPOSITION 5.36. ([**54**]) *Let* $\equiv$ *be a congruence on* A *and let* $H = \{x \in A : x \equiv 1\}$. *Then*

(i) H *is a normal filter of* A;

(ii) $x \equiv y$ *iff* $d_1(x, y) \equiv 1$ *or, equivalently, iff* $d_2(x, y) \equiv 1$.

PROPOSITION 5.37. ([**54**]) *The congruence relations* $\equiv_H$ *of* A *and normal filters* H *are in one-to-one correspondence.*

Starting from a normal filter H, the quotient algebra A/H becomes a pseudo - BL algebra with the natural operations induced from those of A.

Then the function $p_H : A \to A/H$ defined by $p_H(x) = x/H$, for all $x \in A$ is a homomorphism from the pseudo $BL-$ algebra A, onto the pseudo $BL-$ algebra A/H.

For $x, y \in A$, $x/H \leq y/H$ iff $x \to y \in H$ iff $x \rightsquigarrow y \in H$ and $x/H = \mathbf{1} = 1/H$ iff $x \in H$. If $x \in B(A)$, then $x/H \in B(A/H)$.

PROPOSITION 5.38. *Let* H *be a normal filter of* A *and* $a \in A \backslash H$. *Then*

$$\begin{aligned}[H \cup \{a\}) &= \{x \in A : h \odot a^n \leq x, \text{ for some } n \geq 0 \text{ and } h \in H\} \\ &= \{x \in A : a^n \odot h \leq x, \text{ for some } n \geq 0 \text{ and } h \in H\} \\ &= \{x \in A : a^n \to x \in H, \text{ for some } n \geq 1\} \\ &= \{x \in A : a^n \rightsquigarrow x \in H, \text{ for some } n \geq 1\}.\end{aligned}$$

Proof. For the first two equalities, see the proof of Lemma 4.23, for the case of pseudo MV-algebras.

If $x \in [H \cup \{a\})$ then $a^n \odot h \leq x$, for some $n \geq 0$ and $h \in H$. Thus, $h \leq a^n \to x$ so $a^n \to x \in H$. Conversely, assume that $h = a^n \to x \in H$. for some $n \geq 1$. We

have $(a^n \odot h) \to x = h \to (a^n \to x) = h \to h = 1$, hence $a^n \odot h \leq x$. Therefore $x \in [H \cup \{a\})$. ■

COROLLARY 5.39. ([**94**]) *If F_1 or $F_2 \in \mathcal{F}_n(A)$, then*

$$[F_1 \cup F_2) = \{x \in A : x \geq f_1 \odot f_2, \text{ for some } f_1 \in F_1 \text{ and } f_2 \in F_2\}.$$

Proof. Obviously by Proposition 5.19 and Proposition 5.34. ■

Open problem: Characterize the normal filter generated by a non-empty set.

PROPOSITION 5.40. *If $F_1, F_2 \in \mathcal{F}_n(A)$, then*

(i) $F_1 \wedge F_2 \in \mathcal{F}_n(A)$;
(ii) $F_1 \vee F_2 \in \mathcal{F}_n(A)$.

Proof. *(i)*. Let $x, y \in A$ such that $x \to y \in F_1 \wedge F_2 = F_1 \cap F_2$. Then $x \to y \in F_1, F_2$.

Since F_1, $F_2 \in \mathcal{F}_n(A)$ we deduce that $x \rightsquigarrow y \in F_1, F_2$, hence $x \rightsquigarrow y \in F_1 \cap F_2$.

Analogous $x \rightsquigarrow y \in F_1 \cap F_2$ implies $x \to y \in F_1 \cap F_2$, hence $F_1 \cap F_2 \in \mathcal{F}_n(A)$.

(ii). Let $x, y \in A$ such that $x \to y \in F_1 \vee F_2$. By Corollary 5.39 there exist $f_1 \in F_1, f_2 \in F_2$ such that $f_1 \odot f_2 \leq x \to y \Leftrightarrow (f_1 \odot f_2) \odot x \leq y \Leftrightarrow f_1 \odot (f_2 \odot x) \leq y$.

Since $f_2 \in F_2 \in \mathcal{F}_n(A)$ there exists $f_2' \in F_2$ such that $x \odot f_2' = f_2 \odot x$. We obtain

$$f_1 \odot (x \odot f_2') \leq y \Leftrightarrow (f_1 \odot x) \odot f_2' \leq y.$$

Since $f_1 \in F_1 \in \mathcal{F}_n(A)$ there exists $f_1' \in F_1$ such that $x \odot f_1' = f_1 \odot x$. It follows that

$$(x \odot f_1') \odot f_2' \leq y \Leftrightarrow x \odot (f_1' \odot f_2') \leq y \Leftrightarrow f_1' \odot f_2' \leq x \rightsquigarrow y,$$

so $x \rightsquigarrow y \in F_1 \vee F_2$.

Analogous $x \rightsquigarrow y \in F_1 \vee F_2$ implies $x \to y \in F_1 \vee F_2$, hence $F_1 \vee F_2 \in \mathcal{F}_n(A)$. ■

PROPOSITION 5.41. *If $(F_i)_{i \in I}$ is a family of normal filters of A, then*

(i) $\underset{i \in I}{\wedge} F_i \in \mathcal{F}_n(A)$,
(ii) $\underset{i \in I}{\vee} F_i \in \mathcal{F}_n(A)$.

Proof. *(i)*. Clearly, $\underset{i \in I}{\wedge} F_i = \underset{i \in I}{\cap} F_i \in \mathcal{F}_n(A)$.

(ii). We have $\underset{i \in I}{\vee} F_i = [\underset{i \in I}{\cup} F_i)$, so, to prove that $\underset{i \in I}{\vee} F_i \in \mathcal{F}_n(A)$, let $x, y \in A$ such that $x \to y \in [\underset{i \in I}{\cup} F_i)$. By Proposition 5.18, *(ii)*, there exist $\{i_1, ..., i_m\} \subseteq I$ and $x_j \in F_{i_j}$ $(1 \leq j \leq m)$ such that $x_1 \odot ... \odot x_m \leq x \to y \Leftrightarrow (x_1 \odot ... \odot x_m) \odot x \leq y \Leftrightarrow (x_1 \odot .... \odot x_{m-1}) \odot (x_m \odot x) \leq y$.

Since $F_{i_m} \in \mathcal{F}_n(A)$ and $x_m \in F_{i_m}$, there exists $x_m' \in F_{i_m}$ such that $x_m \odot x = x \odot x_m'$. So, we obtain that

$$(x_1 \odot ... \odot x_{m-1}) \odot (x \odot x_m') \leq y.$$

Successively we obtain $x_j' \in F_{i_j}, 1 \leq j \leq m-1$ such that

$$x \odot (x_1' \odot ... \odot x_m') \leq y \Leftrightarrow x_1' \odot ... \odot x_m' \leq x \rightsquigarrow y,$$

hence $x \rightsquigarrow y \in [\underset{i \in I}{\cup} F_i) = \underset{i \in I}{\vee} F_i$.

Analogous $x \rightsquigarrow y \in \underset{i \in I}{\vee} F_i$ implies $x \to y \in \underset{i \in I}{\vee} F_i$, that is $\underset{i \in I}{\vee} F_i \in \mathcal{F}_n(A)$. ■

COROLLARY 5.42. *$\mathcal{F}_n(A)$ is a complete sublattice of $(\mathcal{F}(A), \vee, \wedge)$.*

DEFINITION 5.6. For a nonempty subset $X \subseteq A$, the smallest normal filter of A which contains X, i.e. $\cap\{F \in \mathcal{F}_n(A) : X \subseteq F\}$, is said to be *the normal filter of A generated by X* and will be denoted by $< X >$. Obviously

$$[X) \subseteq < X >,$$

so, if F is a filter, then $F \subseteq < F >$.

PROPOSITION 5.43. *If $H \in \mathcal{F}_n(A)$ and $a \in G(A)$, then $H(a) \in \mathcal{F}_n(A)$.*

Proof. Since $a \in G(A), a \odot a = a$, so, by Proposition 5.38,

$$H(a) = [H \cup \{a\}) = \{x \in A : h \odot a \leq x, \text{ for } h \in H\}$$
$$= \{x \in A : a \odot h \leq x, \text{ for } h \in H\}.$$

By Proposition 5.10, since $a \in G(A)$, we deduce that $a \odot b = b \odot a$, for all $b \in A$.

To prove $H(a) \in \mathcal{F}_n(A)$ let $x, y \in A$ such that $x \to y \in H(a)$. There exists $h \in H$ such that

$$h \odot a \leq x \to y \Leftrightarrow (h \odot a) \odot x \leq y \Leftrightarrow$$
$$\Leftrightarrow h \odot (a \odot x) \leq y \Leftrightarrow h \odot (x \odot a) \leq y \Leftrightarrow (h \odot x) \odot a \leq y.$$

Since $h \in H \in \mathcal{F}_n(A)$ there exists $h' \in H$ such that $x \odot h' = h \odot x$. We obtain

$$(x \odot h') \odot a \leq y \Leftrightarrow x \odot (h' \odot a) \leq y \Leftrightarrow h' \odot a \leq x \rightsquigarrow y,$$

so $x \rightsquigarrow y \in H(a)$.

Analogous $x \rightsquigarrow y \in H(a)$ implies $x \to y \in H(a)$, hence $H(a) \in \mathcal{F}_n(A)$. ■

PROPOSITION 5.44. *For $a \in A$ and $n \geq 1$, the following assertions are equivalent:*

(i) $a^n \in B(A)$;
(ii) $a \vee (a^n)^- = 1$;
(iii) $a \vee (a^n)^\sim = 1$.

Proof. *(i)* $\Rightarrow$ *(ii)*. Since $a^n \in B(A)$, by Proposition 5.12 we deduce that $a^n \vee (a^n)^- = 1$. But $a^n \leq a$, so $1 = a^n \vee (a^n)^- \leq a \vee (a^n)^-$. We obtain that $a \vee (a^n)^- = 1$.

(ii) $\Rightarrow$ *(i)*. Since $a \vee (a^n)^- = 1 \overset{psbl-c_{34}}{\Rightarrow} a^n \vee [(a^n)^-]^n = 1$. Since $[(a^n)^-]^n \leq (a^n)^-$, we obtain $1 = a^n \vee [(a^n)^-]^n \leq a^n \vee (a^n)^-$, so $a^n \vee (a^n)^- = 1$. By Proposition 5.12 we deduce that $a^n \in B(A)$.

(i) $\Leftrightarrow$ *(iii)*. Analogously. ■

THEOREM 5.45. *The following assertions are equivalent:*

(i) $(\mathcal{F}(A), \vee, \wedge, ^*, \{1\}, A)$ *is a Boolean algebra;*
(ii) *Every filter of A is principal and for every $a \in A$ there exists $n \geq 1$ such that $a^n \in B(A)$.*

Proof. *(i)* $\Rightarrow$ *(ii)*. Let $F \in \mathcal{F}(A)$; since $\mathcal{F}(A)$ is Boolean algebra, then $F \vee F^* = A$. Since $0 \in A$, by Corollary 5.20, there exist $m \geq 1$, $f_1, ..., f_m \in F$, $f_1', ..., f_m' \in F^*$ such that $(f_1 \odot f_1') \odot ... \odot (f_m \odot f_m') = 0$. We consider $f = f_1 \odot ... \odot f_m \in F$, $f' = f_1' \odot .... \odot f_m' \in F^*$ and $a = f^m \in F$, $b = (f')^m \in F^*$

Clearly, $f \leq f_i$ and $f' \leq f_i'$, for every $1 \leq i \leq m$, hence $f \odot f' \leq f_i \odot f_i'$, for every $1 \leq i \leq m$.

We deduce that $(f \odot f')^m \leq (f_1 \odot f_1') \odot ... \odot (f_m \odot f_m') = 0$, hence $(f \odot f')^m = 0$. But for $f \in F$ and $f' \in F^*$ we deduce by Corollary 5.28 that $f \vee f' = 1$, hence, by c_{33},

$f \wedge f' = f \odot f' = f' \odot f$. Then we obtain that $f^m \odot (f')^m = 0 \Leftrightarrow a \odot b = 0 \Leftrightarrow a \wedge b = 0$ (by $psbl - c_{33}$, since $a \in F$ and $b \in F^*$, implies $a \vee b = 1$, so $a \odot b = a \wedge b$). So $a \vee b = 1$ and $a \wedge b = 0$, hence we deduce that b is the complement of a in $L(A)$.

If $x \in F$, since $b \in F^*$ we have $b \vee x = 1$. Since $a = a \wedge 1 = a \wedge (b \vee x) = (a \wedge b) \vee (a \wedge x) = 0 \vee (a \wedge x) = a \wedge x$, we deduce that $a \leq x$, so $x \in [a)$, hence $F \subseteq [a)$.

Since $a \in F$ we deduce that $[a) \subseteq F$, that is, $F = [a)$. Hence every filter of A is principal.

Let now $a \in A$; since $\mathcal{F}(A)$ is Boolean algebra, then $[a) \vee [a)^* = A \Leftrightarrow [a)^*(a) = A \Leftrightarrow \{x \in A : x \geq (f_1 \odot a^{n_1}) \odot ... \odot (f_m \odot a^{n_m})$, for some $m \geq 1, n_1, ..., n_m \geq 0$ and $f_1, ..., f_m \in [a)^*\} = A$. For $0 \in A$ we deduce that there exist $m \geq 1, n_1, ..., n_m \geq 0$ and $f_1, ..., f_m \in [a)^*$ such that $(f_1 \odot a^{n_1}) \odot .... \odot (f_m \odot a^{n_m}) = 0$. By Corollary 5.28, $f_i \vee a = 1$, for every $1 \leq i \leq m$, so $f_i \odot a = a \odot f_i = f_i \wedge a$, for every $1 \leq i \leq m$.

Then we obtain that $(f_1 \odot ... \odot f_m) \odot a^{n_1+...+n_m} = a^{n_1+...+n_m} \odot (f_1 \odot ... \odot f_m) = 0$. If consider $f = f_1 \odot ... \odot f_m \in [a)^*$ and $n = n_1 + ... + n_m$ then $f \odot a^n = a^n \odot f = 0$.

So $f \leq a^n \to 0 = (a^n)^- \Rightarrow a \vee f \leq a \vee (a^n)^-$. But $a \vee f = 1$ (since $f \in [a)^*$), so we obtain that $a \vee (a^n)^- = 1$ and by Proposition 5.44 we deduce that $a^n \in B(A)$.

$(ii) \Rightarrow (i)$. By Corollary 5.28, $\mathcal{F}(A)$ is a Heyting algebra. To prove that $\mathcal{F}(A)$ is a Boolean algebra, we must show that for $F \in \mathcal{F}(A)$, $F^* = \{1\}$ only for $F = A$ ([**2**], p. 175). By hypothesis, every filter of A is principal, so we have $a \in A$ such that $F = [a)$.

Also, by hypothesis, for $a \in A$, there is $n \geq 1$ such that $a^n \in B(A)$, equivalent by Proposition 5.44 with $a \vee (a^n)^- = 1$.

By Corollary 5.28, $(a^n)^- \in [a)^* = \{1\}$, hence $(a^n)^- = 1 \Rightarrow [(a^n)^-]^\sim = 1^\sim = 0$. Since $a^n \leq [(a^n)^-]^\sim = 0$ (by $psbl - c_{42}$) we deduce that $a^n = 0$, so $0 \in F$, hence $F = A$. ■

THEOREM 5.46. *The following assertions are equivalent:*

(*i*) $(\mathcal{F}_n(A), \vee, \wedge, ^*, \{1\}, A)$ *is a Boolean algebra;*
(*ii*) *Every normal filter of* A *is principal and for every* $a \in A$ *there is* $n \geq 1$ *such that* $a^n \in B(A)$.

Proof. $(i) \Rightarrow (ii)$. Let $F \in \mathcal{F}_n(A)$; since $\mathcal{F}(A)$ is Boolean algebra, then $F \vee F^* = A$. So, by Corollary 5.39, for $0 \in A$, there exist $a \in F$, $b \in F^*$ such that $a \odot b = 0$.

Since $b \in F^*$, by Corollary 5.28, it follow that $a \vee b = 1$. By $psbl - c_{33}$ we deduce that $a \wedge b = a \odot b = 0$, that is, b is the complement of a in $L(A)$. If $x \in F$, since $b \in F^*$ we have $b \vee x = 1$. Since $a = a \wedge (b \vee x) = (a \wedge b) \vee (a \wedge x) = 0 \vee (a \wedge x) = a \wedge x$, we deduce that $a \leq x$, that is $F = [a)$. Hence every normal filter of A is principal.

For the last assertions see the proof of Theorem 5.45.

$(ii) \Rightarrow (i)$. See the proof of Theorem 5.45. ■

COROLLARY 5.47. *If pseudo - BL algebra* A *is a BL algebra, then the following assertions are equivalent:*

(*i*) $(\mathcal{F}(A), \vee, \wedge, ^*, \{1\}, A)$ *is a Boolean algebra,*
(*ii*) *Every filter of* A *is principal and for every* $x \in A$, *there is* $n \in \omega$ *such that* $x \vee (x^n)^- = 1$.

3. The spectrum of a pseudo - BL algebra

This section contains characterization for prime and completely inf-irreducible filters (normal filters) of a pseudo BL-algebra.

For the lattice $\mathcal{F}(A)$ (which is distributive) we denote by $Spec(A)$ the set of all meet-irreducible elements (see Definition 1.10) ($Spec(A)$ is called the *spectrum* of A) and by $Irc(A)$ the set of all completely meet-irreducible elements of the lattice $\mathcal{F}(A)$.

DEFINITION 5.7. A proper filter P of A is called *prime* if, for any $x, y \in A$, the condition $x \vee y \in P$ implies $x \in P$ or $y \in P$.

PROPOSITION 5.48. *If P is a proper filter, then the following are equivalent:*

(i) P is prime filter;
(ii) For all $x, y \in A$, $x \to y \in P$ or $y \to x \in P$;
(iii) For all $x, y \in A$, $x \rightsquigarrow y \in P$ or $y \rightsquigarrow x \in P$;
(iv) $A/\equiv_{L(P)}$ is a chain;
(v) $A/\equiv_{R(P)}$ is a chain.

Proof. $(i) \Rightarrow (ii)$. Obviously, since $(x \to y) \vee (y \to x) = 1$.

$(ii) \Rightarrow (i)$. Assume that $x \vee y \in P$ and, for example, $x \to y \in P$. But $x \vee y = [(x \to y) \rightsquigarrow y] \wedge [(y \to x) \rightsquigarrow x] \in P$, so $(x \to y) \rightsquigarrow y \in P$; then $y \in P$.

The rest of the proof is straightforward. ■

PROPOSITION 5.49. *Let H be a normal filter of A. Then H is a prime filter iff A/H is a pseudo - BL chain.*

Proof. Similarly with the proof of Theorem 3.23 for the case of BL-algebras. ■

REMARK 5.11. *If A is a pseudo - BL chain, then the set of normal filters of A is totally ordered by inclusion. Indeed, if H_1, H_2 were normal filters of A such that $H_1 \nsubseteq H_2$ and $H_2 \nsubseteq H_1$, then there would be elements $h_1, h_2 \in A$ such that $h_1 \in H_1 \backslash H_2$ and $h_2 \in H_2 \backslash H_1$. Whence $h_1 \not\leq h_2$ and $h_2 \not\leq h_1$, which is impossible.*

COROLLARY 5.50. *If P is a prime filter and Q is a proper filter such that $P \subseteq Q$, then Q is a prime filter.*

Proof. Follows by Proposition 5.48. ■

REMARK 5.12. *If P is a prime filter of A, then $A \backslash P$ is an ideal in $L(A)$.*

Proof. Since P is proper, $0 \notin P$, hence we have $0 \in A \backslash P$. If $a \leq b$ and $b \in A \backslash P$, then $a \in A \backslash P$, since P is a filter of A. If $a, b \in A \backslash P$, then $a \vee b \in A \backslash P$, since P is a prime filter. ■

THEOREM 5.51. *(Prime filter theorem) If $F \in \mathcal{F}(A)$ and I is an ideal of the lattice $L(A)$ such that $F \cap I = \varnothing$, then there is a prime filter P of A such that $F \subseteq P$ and $P \cap I = \varnothing$.*

Proof. ([**53**]) Let $\mathcal{H} = \{H \in \mathcal{F}(A) : F \subseteq H \text{ and } H \cap I = \varnothing\}$. A routine application of Zorn's Lemma shows that $\mathcal{H}$ has a maximal element, P. Suppose that P is not a prime filter of A. Then there are $a, b \in A$ such that $a \rightsquigarrow b \notin P$ and $b \rightsquigarrow a \notin P$. It follows that the filters $[P \cup \{a \rightsquigarrow b\})$ and $[P \cup \{b \rightsquigarrow a\})$ are not in $\mathcal{H}$. Hence, there are $c \in I \cap [P \cup \{a \rightsquigarrow b\})$ and $d \in I \cap [P \cup \{b \rightsquigarrow a\})$. By Proposition

5.18, (iv), $c \geq (s_1 \odot (a \rightsquigarrow b)^{p_1}) \odot ... \odot (s_m \odot (a \rightsquigarrow b)^{p_m})$,for some $m \geq 1, p_1, ..., p_m \geq 0$ and $s_1, ..., s_m \in P$ and $d \geq (t_1 \odot (b \rightsquigarrow a)^{q_1}) \odot ... \odot (t_n \odot (b \rightsquigarrow a)^{q_n})$,for some $n \geq 1, q_1, ..., q_n \geq 0$ and $t_1, ..., t_n \in P$. Let $s = s_1 \odot ... \odot s_m$ and $t = t_1 \odot ... \odot t_n$; then $s, t \in P$.

Let $p = \max_{i=1,m}\{p_i\}$ and $q = \max_{i=1,n}\{q_i\}$; then $c \geq \prod_{i=1}^{m}(s \odot (a \rightsquigarrow b)^p) = [s \odot (a \rightsquigarrow b)^p]^m$ and $d \geq \prod_{i=1}^{n}(t \odot (b \rightsquigarrow a)^q) = [t \odot (b \rightsquigarrow a)^q]^n$. Let now $u = s \odot t$ and $r = \max\{p, q\}$; then $u \in P$ and $c \geq [u \odot (a \rightsquigarrow b)^r]^m$ and $d \geq [u \odot (b \rightsquigarrow a)^r]^n$.

By ..... we get $x = c \vee d \geq [u \odot (a \rightsquigarrow b)^r]^m \vee [u \odot (b \rightsquigarrow a)^r]^n \geq (([u \odot (a \rightsquigarrow b)^r] \vee [u \odot (b \rightsquigarrow a)^r])^m)^n = ([u \odot (a \rightsquigarrow b)^r] \vee [u \odot (b \rightsquigarrow a)^r])^{mn} = (u \odot [(a \rightsquigarrow b)^r \vee (b \rightsquigarrow a)^r])^{mn} = (u \odot 1)^{mn} = u^{mn} \in P$.

Thus, $x \in P$, but $x \in I$ also, since I is an ideal. We have that $P \cap I \neq \varnothing$, a contradiction. ■

COROLLARY 5.52. *If $F \in \mathcal{F}(A)$ is proper and $a \in A \backslash F$, then there is a prime filter P of A such that $F \subseteq P$ and $a \notin P$. In particular, for $F = \{1\}$ we deduce that for any $a \in A, a \neq 1$, there is a prime filter P_a such that $a \notin P_a$.*

PROPOSITION 5.53. *The set of proper filters including a prime filter P of A is a chain.*

Proof. Let P_1, P_2 be to proper filters of A such that $P \subseteq P_1$ and $P \subseteq P_2$. Assume there exist $x \in P_1 \backslash P_2$ and $y \in P_2 \backslash P_1$; then $x \vee y \in P_1 \cap P_2$. Hence $P_1 \cap P_2$ is a prime filter of A. So, $x \in P_1 \cap P_2$ or $y \in P_1 \cap P_2$. This contradiction shows that $P_1 \subseteq P_2$ or $P_2 \subseteq P_1$. ■

COROLLARY 5.54. *Every proper filter F is the intersection of those filters which contain F. In particular, $\cap Spec(A) = \{1\}$.*

PROPOSITION 5.55. *For a proper filter $P \in \mathcal{F}(A)$ the following are equivalent:*

(i) P is prime;
(ii) $P \in Spec(A)$;
(iii) If $a, b \in A$ and $a \vee b = 1$, then $a \in P$ or $b \in P$.

Proof. $(i) \Rightarrow (ii)$. Let $F_1, F_2 \in \mathcal{F}(A)$ such that $F_1 \cap F_2 = P$.

Since $P \subseteq F_1, P \subseteq F_2$, by Proposition 5.53, $F_1 \subseteq F_2$ or $F_2 \subseteq F_1$, hence $P = F_1$ or $P = F_2$.

$(ii) \Rightarrow (i)$. Let $a, b \in A$, such that $a \vee b \in P$.

Since $P(a) \cap P(b) = (P \vee [a)) \cap (P \vee [b)) = P \vee ([a) \cap [b)) = P \vee [a \vee b) = P$, then $P = P(a)$ or $P = P(b)$, hence $a \in P$ or $b \in P$, that is, P is prime.

$(i) \Rightarrow (iii)$. Clearly, since $1 \in P$.

$(iii) \Rightarrow (i)$. Clearly by Proposition 5.48, (ii) (since $(a \to b) \vee (b \to a) = 1$ for every $a, b \in A$).■

PROPOSITION 5.56. *For a proper filter $P \in \mathcal{F}(A)$ the following are equivalent:*

(i) $P \in Spec(A)$;
(ii) For every $x, y \in A \backslash P$ there is $z \in A \backslash P$ such that $x \leq z$ and $y \leq z$.

Proof. $(i) \Rightarrow (ii)$. Let $P \in Spec(A)$ and $x, y \in A \backslash P$. If by contrary, for every $a \in A$ with $x \leq a$ and $y \leq a$ then $a \in P$, since $x, y \leq x \vee y$ we deduce that $x \vee y \in P$, hence, $x \in P$ or $y \in P$, a contradiction.

$(ii) \Rightarrow (i)$. I suppose by contrary that there exist $F_1, F_2 \in \mathcal{F}(A)$ such that $F_1 \cap F_2 = P$ and $P \neq F_1$, $P \neq F_2$. So, we have $x \in F_1 \backslash P$ and $y \in F_2 \backslash P$. By hypothesis there is $z \in A \backslash P$ such that $x \leq z$ and $y \leq z$.

We deduce $z \in F_1 \cap F_2 = P$, a contradiction. ■

COROLLARY 5.57. *For a proper filter $P \in \mathcal{F}(A)$ the following are equivalent:*

(i) $P \in Spec(A)$;

(ii) If $x, y \in A$ and $[x) \cap [y) \subseteq P$, then $x \in P$ or $y \in P$.

Proof. $(i) \Rightarrow (ii)$. Let $x, y \in A$ such that $[x) \cap [y) \subseteq P$ and suppose by contrary that x, $y \notin P$. Then by Proposition 5.56 there is $z \in A \backslash P$ such that $x \leq z$ and $y \leq z$. Hence $z \in [x) \cap [y) \subseteq P$, so $z \in P$, a contradiction.

$(ii) \Rightarrow (i)$. Let $x, y \in A$ such that $x \vee y \in P$. Then $[x \vee y\ ) \subseteq P$.

Since $[x \vee y) = [x) \cap [y)$ (by Proposition 5.18, (v)) we deduce that $[x) \cap [y) \subseteq P$, hence, by hypothesis, $x \in P$ or $y \in P$, that is, $P \in Spec(A)$. ■

We make the following notation:

$$Spec_n(A) = \{F : F \text{ is a normal prime filter of } A\}.$$

REMARK 5.13. *$Spec_n(A) \subseteq Spec(A)$; if A is a BL algebra, then $Spec_n(A) = Spec(A)$.*

COROLLARY 5.58. *For a proper normal filter $P \in \mathcal{F}_n(A)$ the following are equivalent:*

(i) $P \in Spec_n(A)$;

(ii) For every $x, y \in A/P, x \neq \mathbf{1}, y \neq \mathbf{1}$ there is $z \in A/P, z \neq \mathbf{1}$ such that $x \leq z$, $y \leq z$.

Proof. $(i) \Rightarrow (ii)$. Clearly, by Proposition 5.56, since if $x = a/P$, with $a \in A$, then the condition $x \neq \mathbf{1}$ is equivalent with $a \notin P$.

$(ii) \Rightarrow (i)$. Let $x, y \in A/P$. Then in A/P, $x = a/P \neq \mathbf{1}$ and $y = b/P \neq \mathbf{1}$. By hypothesis there is $z = c/P \neq \mathbf{1}$ (that is, $c \notin P$) such that $x, y \leq z$ equivalent with $a \to c, b \to c \in P$. If consider $d = (b \to c) \rightsquigarrow ((a \to c) \rightsquigarrow c)$, then by $psbl - c_9$, $(a \to c) \rightsquigarrow c \leq d$ and $a \leq (a \to c) \rightsquigarrow c$ (because it is equivalent with,$(a \to c) \odot a = a \wedge c \leq c$). So $a \leq d$. By $psbl- c_{22}$, $d = ((a \to c) \odot (b \to c)) \rightsquigarrow c \overset{psbl-c_5}{\geq} (b \to c) \rightsquigarrow c \geq b$ (because it is equivalent with $(b \to c) \odot b = b \wedge c \leq b$).

We deduce that $a, b \leq d$. Since $c \notin P$, by Remark 5.6 we deduce that $d \notin P$, hence by Proposition 5.56, we deduce that $P \in Spec_n(A)$. ■

REMARK 5.14. *From Corollary 5.54 we deduce that for every $F \in \mathcal{F}(A)$,*

$$F = \cap\{P \in Spec(A) : F \subseteq P\} \text{ and } \cap\{P \in Spec(A)\} = \{1\}.$$

Relative to the uniqueness of filters as intersection of primes we have as in the case of $BL-$algebras:

THEOREM 5.59. *If every $F \in \mathcal{F}(A)$ has a unique representation as an intersection of elements of $Spec(A)$, then $(\mathcal{F}(A), \vee, \wedge, ^*, \{1\}, A)$ is a Boolean algebra.*

Also, as in the case of $BL-$algebras, for pseudo $BL-$algebras we have the following results:

LEMMA 5.60. *If $F \in \mathcal{F}(A)$, $F \neq A$ and $a \notin F$, then there exists $F_a \in \mathcal{F}(A)$ maximal with the property that $F \subseteq F_a$ and $a \notin F_a$.*

COROLLARY 5.61. *For any $a \in A, a \neq 1$, there is a filter F_a maximal relative to a.*

THEOREM 5.62. *For $F \in \mathcal{F}(A), F \neq A$ the following are equivalent:*

(*i*) $F \in Irc(A)$;
(*ii*) *There is $a \in A$ such that F is maximal relative to a.*

THEOREM 5.63. *Let $F \in \mathcal{F}_n(A)$ be a normal filter, $F \neq A$ and $a \in A\backslash F$. Then the following are equivalent:*

(*i*) *F is maximal relative to a,*
(*ii*) *For every $x \in A\backslash F$ there is $n \geq 1$ such that $x^n \to a \in F$,*
(*iii*) *For every $x \in A\backslash F$ there is $n \geq 1$ such that $x^n \rightsquigarrow a \in F$.*

Proof. Since $F \in \mathcal{F}_n(A)$, it is sufficient to prove $(i) \Leftrightarrow (ii)$.

$(i) \Rightarrow (ii)$. Let $x \in A\backslash F$. If $a \notin F(x) = F \vee [x)$, since $F \subset F(x)$ then $F(x) = A$ (by the maximality of F) hence $a \in F(x)$, a contradiction. We deduce that $a \in F(x)$, hence $a \geq f \odot x^n$, with $f \in F$ and $n \geq 1$. Then $f \leq x^n \to a$, hence $x^n \to a \in F$.

$(ii) \Rightarrow (i)$. We suppose by contrary that there is $F' \in \mathcal{F}(A), F' \neq A$ such that $a \notin F'$ and $F \subset F'$. Then there is $x_0 \in F'$ such that $x_0 \notin F$, hence by hypothesis there is $n \geq 1$ such that $x_0^n \to a \in F \subset F'$. Thus from $x_0^n \to a \in F'$ and $x_0^n \in F'$, we deduce that $a \in F'$, by Remark 5.6, a contradiction. ■

COROLLARY 5.64. *For a normal filter $F \in \mathcal{F}_n(A), F \neq A$ the following are equivalent:*

(*i*) $F \in Irc(A)$,
(*ii*) *In the set $A/F\backslash\{\mathbf{1}\}$ we have an element $p \neq \mathbf{1}$ with the property that for every $x \in A/F\backslash\{\mathbf{1}\}$ there is $n \geq 1$ such that $x^n \leq p$.*

Proof.$(i) \Rightarrow (ii)$. By Theorem 5.62, F is maximal relative to an element $a \notin F$; then, if denote $p = a/F \in A/F$, $p \neq \mathbf{1}$ (since $a \notin F$) and for every $x = b/F$, $x \neq \mathbf{1}$ (that is $b \notin F$) by Theorem 5.63 there is $n \geq 1$ such that $b^n \to a \in F$, that is, $x^n \leq p$.

$(ii) \Rightarrow (i)$. Let $p = a/F \in A/F\backslash\{\mathbf{1}\}$, (that is, $a \notin F$) and $x = b/F \in A/F\backslash\{\mathbf{1}\}$, (that is, $b \notin F$). By hypothesis there is $n \geq 1$ such that $x^n \leq p$ equivalent with $b^n \to a \in F$. Then, by Theorem 5.63, we deduce that $F \in Irc(A)$. ■

We recall that a filter P of A is a minimal prime filter if $P \in Spec(A)$ and, whenever $Q \in Spec(A)$ and $Q \subseteq P$, we have $P = Q$.

PROPOSITION 5.65. *If P is a minimal prime filter, then for any $a \in P$ there is $b \in A\backslash P$ such that $a \vee b = 1$.*

Proof. See the proof of Proposition 1.56. ■

REMARK 5.15. *For the case of BL-algebras we have an analogous result (more general; see* [**99**]*, p.54).*

4. Maximal filters. Archimedean and hyperarchimedean pseudo BL-algebras

In this section we introduce the notions of archimedean and hyperarchimedean pseudo - BL algebra and we will prove a theorem of Nachbin type for pseudo - BL algebras.

DEFINITION 5.8. A filter of A is *maximal (ultrafilter)* if it is proper and it is not contained in any other proper filter.

We shall denote by $Max(A)$ the set of all maximal filters of A and by $Max_n(A)$ the set of all maximal normal filters of A; it is obvious that $Max_n(A) \subseteq Max(A) \subseteq Spec(A)$ and $Max_n(A) \subseteq Spec_n(A) \subseteq Spec(A)$.

Indeed, let $M \in Max(A)(Max_n(A))$; because M is a proper filter of A, then by Corollary 5.54, there is a prime filter P of A such that $M \subseteq P$. Since P is proper, it follows that $M = P$. Hence, M is prime (normal prime).

We have:

THEOREM 5.66. *If F is a proper filter of A, then the following are equivalent:*

(*i*) *F is a maximal filter;*
(*ii*) *For any $x \notin F$ there exist $f \in F, n, m \geq 1$ such that $(f \odot x^n)^m = 0$.*

Proof. $(i) \Rightarrow (ii)$. If $x \notin F$, then $[F \cup \{x\}) = A$, hence $0 \in [F \cup \{x\})$. By Proposition 5.18, (iv), there exists $m, n_1, ..., n_m \in \omega$ and $f_1, ..., f_m \in F$ such that $(f_1 \odot x^{n_1}) \odot ... \odot (f_m \odot x^{n_m}) \leq 0$. Thus $(f_1 \odot x^{n_1}) \odot ... \odot (f_m \odot x^{n_m}) = 0$ and if consider $f = f_1 \odot ... \odot f_m \in F$ (since F is a filter) and $n = \max\{n_1, ..., n_m\}$, then $f \odot x^n \leq f_i \odot x^{n_i}$ for every $1 \leq i \leq m$, hence

$$(f \odot x^n)^m \leq (f_1 \odot x^{n_1}) \odot ... \odot (f_m \odot x^{n_m}) = 0,$$

that is, $(f \odot x^n)^m = 0$.

$(ii) \Rightarrow (i)$. Assume there is a proper filter F' such that $F \subset F'$. Then there exists $x \in F'$ such that $x \notin F$. By hypothesis there exist $f \in F, n, m \in \omega$ such that $(f \odot x^n)^m = 0$. But $x, f \in F'$ hence we obtain $0 \in F'$, a contradiction. ■

COROLLARY 5.67. *If H is a normal proper filter of A, then the following are equivalent:*

(*i*) *H is a maximal filter;*
(*ii*) *For any $x \in A, x \notin H$ iff $(x^n)^- \in H$, for some $n \geq 1$;*
(*iii*) *For any $x \in A, x \notin H$ iff $(x^n)^\sim \in H$, for some $n \geq 1$.*

Proof. From Theorem 5.66, for any $x \in A, x \notin H$ iff there exist $f \in H$ and $n, m \geq 1$ such that $(f \odot x^n)^m = 0$. Since H is normal, from $(f \odot x^n)^m = 0$ we deduce that there exist $f', f'' \in H$ such that $f' \odot x^{nm} = 0$ and $x^{nm} \odot f'' = 0$. Thus, $(x^{nm})^-, (x^{nm})^\sim \in H$.■

THEOREM 5.68. *If $H \in \mathcal{F}_n(A), H \neq A$, then the following are equivalent:*

(*i*) *$H \in Max_n(A)$;*
(*ii*) *For any $x \in A, x \notin H$ iff $(x^n)^- \in H$, for some $n \geq 1$;*
(*iii*) *For any $x \in A, x \notin H$ iff $(x^n)^\sim \in H$, for some $n \geq 1$;*
(*iv*) *A/H is locally finite.*

Proof. $(i) \Leftrightarrow (iii)$. Follows by Corollary 5.67.

$(i) \Leftrightarrow (iv)$. It follows by observing that the condition (iii) in Corollary 5.67 can be reformulated in the following way: for any $x \in A, x/H \neq 1/H$ iff $(x^n)^\sim/H = 1/H$, for some $n \geq 1$ iff $(x/H)^n = 0/H$,for some $n \geq 1$. ■

PROPOSITION 5.69. *Let H be a normal proper filter of A. For an element $x \in A$, the following properties are equivalent:*

(*i*) *There exists $h \in H$ such that $x \leq h^\sim$;*

(ii) *There exists* $h \in H$ *such that* $h \odot x = 0$;
(i') *There exists* $h \in H$ *such that* $x \leq h^-$;
(ii') *There exists* $h \in H$ *such that* $x \odot h = 0$.

Proof. $(i) \Leftrightarrow (ii)$ and $(i') \Leftrightarrow (ii')$ follows by $psBL_3$.
$(ii) \Leftrightarrow (ii')$ follows by Proposition 5.34. ■
For a pseudo - BL algebra A we make the following notations:

$$U(A) = \{a \in A : (a^n)^\sim \leq a, \text{ for every } n \in N\}$$

$$V(A) = \{a \in A : (a^n)^- \leq a, \text{ for every } n \in N\}.$$

REMARK 5.16. *If* A *is a* BL *algebra, then* $Max_n(A) = Max(A)$ *and* $U(A) = V(A)$.

DEFINITION 5.9. The intersection of the maximal filters (normal filters) of A is called the *radical (normal radical)* of A and will be denoted by $Rad(A)$ ($Rad_n(A)$). It is obvious that $Rad(A)$ and $Rad_n(A)$ are filters of A and $Rad(A) \subseteq Rad_n(A)$.

PROPOSITION 5.70. $Rad(A) \subseteq U(A) \cap V(A) \subseteq U(A) \cup V(A) \subseteq Rad_n(A)$.

Proof.([**54**]) Let $a \notin U(A)$; there exists $n \in N$ such that $(a^n)^\sim \not\leq a$; it follows that $(a^n)^\sim \rightsquigarrow a \neq 1$. Hence, there exists a prime filter P such that $(a^n)^\sim \rightsquigarrow a \notin P$. But P is prime, hence $a \rightsquigarrow (a^n)^\sim \in P$. By Zorn Lemma, there exists a maximal filter M such that $P \subseteq M$, hence $a \rightsquigarrow (a^n)^\sim \in M$. If $a \in M$, then $a^n \in M$; it follows that $(a^n)^\sim \in M$, too, since $a, a \rightsquigarrow (a^n)^\sim \in M$ imply $(a^n)^\sim \in M$; we thus obtain a contradiction: $0 = a^n \odot (a^n)^\sim \in M$. It follows that $a \notin M$, hence $a \notin \cap Max(A)$. Thus we have proved that $\cap Max(A) \subseteq U(A)$. Analogously, $\cap Max(A) \subseteq V(A)$. Hence we have proved the first inclusion.

Let now $a \notin \cap Max_n(A)$, hence, there exists a normal maximal filter M such that $a \notin M$. Then $(a^n)^\sim \in M$, by Theorem 5.68. If $(a^n)^\sim \leq a$, then $a \in M$ contradiction. Hence $(a^n)^\sim \not\leq a$, so $a \notin U(A)$, hence $U(A) \subseteq \cap Max_n(A)$. Analogously, we have $V(A) \subseteq \cap Max_n(A)$. ■

REMARK 5.17. (i) *If* $M \in Max_n(A)$, *then* $< M >= M$,
(ii) *If* $M \in Max(A) \setminus Max_n(A)$, *then* $< M >= A$.

PROPOSITION 5.71. *For any* $a, b \in Rad(A)$, $a^- \odot b^- = a^\sim \odot b^\sim = 0$.

Proof. Let $a, b \in Rad(A)$; to prove that $a^- \odot b^- = 0$ is equivalent with $(a^- \odot b^-)^\sim = 1$. Suppose that $(a^- \odot b^-)^\sim \neq 1$. By Corollary 5.52, there is a prime filter P such that $(a^- \odot b^-)^\sim \notin P$. By c_{48} we have $(a^- \odot b^-)^\sim = b^- \to (a^-)^\sim \notin P$, so by Proposition 5.48, $(a^-)^\sim \to b^- \in P$, that is, $[(a^-)^\sim \odot b]^- \in P$.

There is a maximal filter M such that $P \subseteq M$. Then $(a^-)^\sim \odot b \notin M$. By Theorem 5.68, there is $n \geq 1$ such that $[((a^-)^\sim \odot b)^n]^- \in M$; so, if denote $c = ((a^-)^\sim \odot b)^n$, we have $c^- \in M$. Since $a, b \in Rad(A)$ then we deduce that $a, b \in M$, hence $(a^-)^\sim, b \in M$, so $c = ((a^-)^\sim \odot b)^n \in M$. Hence c and c^- are in M which contradicts the fact that M is a proper filter of A.

Analogous we deduce that $a^\sim \odot b^\sim = 0$. ■

We recall that a pseudo - BL algebra A is called *semisimple* if the intersection of all congruences of A is the congruence Δ_A (where for all $x, y \in A, x \Delta_A y$ iff $x = y$) and a pseudo - BL algebra is *representable* if it can be represented as a subdirect product of pseudo - BL chains.

REMARK 5.18. *A is semisimple iff $Rad_n(A) = \{1\}$.*

Indeed, since in a pseudo BL-algebra A, the congruences are in bijective corespondence with the normal filters, it follows that A is semisimple iff $\cap Max_n(A) = \{1\}$ (see [**54**]).

PROPOSITION 5.72. *If A is semisimple, then A is representable.*

Proof. Since A is semisimple, we have that $\cap Max_n(A) = \{1\}$. But any normal maximal filter H is a normal prime filter and, hence by Proposition 5.49, A/H is a pseudo BL-chain. Then, by standard techniques of universal algebra, we obtain that A is representable. ■

PROPOSITION 5.73. *If A is semisimple, then for every $a, b \in A$,*

$(psbl - c_{78})$ $(a \to b) \odot b = b \odot (a \rightsquigarrow b)$.

Proof. By Proposition 5.72, it is sufficient to consider the case in which A is a pseudo - BL chain. If $a \leq b$, then $a \to b = a \rightsquigarrow b = 1$, so $(a \to b) \odot b = 1 \odot b = b = b \odot 1 = b \odot (a \rightsquigarrow b)$. If $b \leq a$, the equality $(a \to b) \odot b = b \odot (a \rightsquigarrow b)$ follows from $psbl - c_{77}$. ■

COROLLARY 5.74. *If A is semisimple, then for every $a, b \in A$,*

$$[a \to b) \vee [b) = [a \rightsquigarrow b) \vee [b).$$

DEFINITION 5.10. An element a of A is called *infinitesimal* if $a \neq 1$ and $a^n \geq a^- \vee a^\sim$ for any $n \in N$.

We denote by $In(A)$ the set of all infinitesimals of A.

PROPOSITION 5.75. *For every nonunit element a of A, a is infinitesimal implies $a \in Rad_n(A)$.*

Proof. Let $a \neq 1$ be an infinitesimal and suppose $a \notin Rad_n(A)$. Thus, there is a maximal normal filter M of A such that $a \notin M$. By Theorem 5.68, there is $n \geq 1$ such that $(a^n)^- \in M$. By hypothesis $a^n \geq a^- \vee a^\sim \geq a^-$ hence $(a^n)^- \leq a^=$, so $a^= \in M$. By $psbl - c_{64}$ we deduce that $(a^=)^n \leq (a^n)^=$, hence $(a^n)^= \in M$. If denote $b = (a^n)^-$ we conclude that $b, b^- \in M$, hence $0 = b^- \odot b \in M$, that is, $M = A$, which contradicts the fact that M is a proper filter. ■

PROPOSITION 5.76. *For every nonunit element $a \in A$, $a \in Rad(A)$ implies a is infinitesimal.*

Proof. Let $a \in Rad(A) \subseteq U(A) \cap V(A), a \neq 1$; then $(a^n)^- \leq a$ and $(a^n)^\sim \leq a$ for any $n \in N$. For $n = 1$ we obtain that $a^-, a^\sim \leq a$. Since for any $n \in \omega, a^n \in Rad(A)$ we deduce that $(a^n)^-, (a^n)^\sim \leq a^n$. Since $a^- \odot a^n = a^n \odot a^\sim = 0$ for any $n \geq 1$, then by $psbl - c_{39}$ and $psbl - c_{40}$ we obtain that $a^n \leq (a^-)^\sim$ and $a^n \leq (a^\sim)^-$ for any $n \geq 1$. So, for any $n \geq 1$, $a^- = [(a^-)^\sim]^- \leq (a^n)^-, a^\sim = [(a^\sim)^-]^\sim \leq (a^n)^\sim$ and $(a^n)^-, (a^n)^\sim \leq a^n$, hence $a^-, a^\sim \leq a^n$, which implies $a^n \geq a^- \vee a^\sim$, that is, a is an infinitesimal. For $n = 0$ the inequalities are trivial. ■

COROLLARY 5.77. *$Rad(A) \backslash \{1\} \subseteq In(A) \subseteq Rad_n(A)$.*

COROLLARY 5.78. ([**70**]) *If A is a BL algebra, then $Rad(A) \backslash \{1\} = In(A)$.*

LEMMA 5.79. *If $a \in A$ and $n \in N, n \geq 1$ then the following hold: $a^n \in B(A)$ and $a^n \geq a^- \vee a^\sim$, implies $a = 1$.*

Proof. By Proposition 5.44, $a^n \in B(A) \Leftrightarrow a \vee (a^n)^- = 1 \Leftrightarrow a \vee (a^n)^{\sim} = 1$. By hypothesis, $a^n \geq a^- \vee a^{\sim}$ implies $a^n \geq a^-$ and $a^n \geq a^{\sim}$. By $psbl - c_5$ we obtain $(a^n)^- \leq (a^{\sim})^-$ and $(a^n)^{\sim} \leq (a^-)^{\sim}$ so $1 = a \vee (a^n)^- \leq a \vee (a^{\sim})^- = (a^{\sim})^-$, $1 = a \vee (a^n)^{\sim} \leq a \vee (a^-)^{\sim} = (a^-)^{\sim}$ hence $(a^{\sim})^- = (a^-)^{\sim} = 1$, that is, $a^- = a^{\sim} = 0$.

Then $(a \odot a) \to 0 = a \to (a \to 0) = a \to 0 = a^- = 0$, $(a \odot a) \rightsquigarrow 0 = a \rightsquigarrow (a \rightsquigarrow 0) = a \rightsquigarrow 0 = a^{\sim} = 0$, so we deduce that $(a^2)^- = (a^2)^{\sim} = 0$. Recursively we obtain that $(a^n)^- = (a^n)^{\sim} = 0$. Then $a \vee (a^n)^- = a \vee 0 = 1$, $a \vee (a^n)^{\sim} = a \vee 0 = 1$, hence $a = 1$.■

LEMMA 5.80. *In any pseudo - BL algebra A the following are equivalent:*

(*i*) *For every* $a \in A, a^n \geq a^- \vee a^{\sim}$ *for any* $n \in N$ *implies* $a = 1$;
(*ii*) *For every* $a, b \in A, a^n \geq b^- \vee b^{\sim}$ *for any* $n \in N$ *implies* $a \vee b = 1$.

Proof. $(i) \Rightarrow (ii)$. Let $a, b \in A$ such that $a^n \geq b^- \vee b^{\sim}$ for any $n \in \omega$. We get $(a \vee b)^- = a^- \wedge b^- \leq b^- \leq a^n \leq (a \vee b)^n$, $(a \vee b)^{\sim} = a^{\sim} \wedge b^{\sim} \leq b^{\sim} \leq a^n \leq (a \vee b)^n$, hence $(a \vee b)^n \geq (a \vee b)^- \vee (a \vee b)^{\sim}$, for any $n \in \omega$. By hypothesis, $a \vee b = 1$.

$(ii) \Rightarrow (i)$. Let $a \in A$ such that $a^n \geq a^- \vee a^{\sim}$ for any $n \in \omega$. If consider $b = a$ we obtain $a \vee b = 1 \Leftrightarrow a \vee a = 1 \Leftrightarrow a = 1$.■

DEFINITION 5.11. A pseudo - BL algebra A is called *archimedean* if the equivalent conditions from Lemma 5.80 are satisfied.

One can easily remark that a pseudo - BL algebra is archimedean iff it has no infinitesimals.

DEFINITION 5.12. Let A be a pseudo - BL algebra. An element $a \in A$ is called *archimedean* if it satisfy the condition:

$$\text{there is } n \in N,\ n \geq 1,\ \text{such that } a^n \in B(A),$$

equivalent by Proposition 5.44 with $a \vee (a^n)^- = 1$ and $a \vee (a^n)^{\sim} = 1$. A pseudo - BL algebra A is called *hyperarchimedean* if all its elements are archimedean.

From Lemma 5.79 we deduce:

COROLLARY 5.81. *Every hyperarchimedean pseudo - BL algebra is archimedean.*

We recall a theorem of *Nachbin type* for lattices (see [**2**], p.73):

THEOREM 5.82. *A distributive lattice is relatively complemented iff every prime ideal is maximal.*

Now, we present a theorem of *Nachbin type* for pseudo - BL algebras:

THEOREM 5.83. *For a pseudo - BL algebra A, the following are equivalent:*

(*i*) *A is hyperarchimedean;*
(*ii*) *For any normal filter F, the quotient pseudo - BL algebra A/F is an archimedean pseudo - BL algebra;*
(*iii*) $Spec_n(A) = Max_n(A)$;
(*iv*) *Any prime normal filter is minimal prime.*

Proof. $(i) \Rightarrow (ii)$. To prove A/F is archimedean, let $x = a/F \in A/F$ such that $x^n \geq x^- \vee x^{\sim}$ for any $n \in N$. By hypothesis, there is $m \in N, m \geq 1$ such that $a^m \in B(A)$. It follows that $x^m \in B(A/F)$. In particular we have $x^m \geq x^- \vee x^{\sim}$ so by Lemma 5.79 we deduce that $x = \mathbf{1}$. It follows that A/F is archimedean.

$(ii) \Rightarrow (iii)$. Since $Max_n(A) \subseteq Spec_n(A)$, we only have to prove that any prime normal filter of A is maximal. If $P \in Spec_n(A)$, then A/P is a chain (see Proposition 5.49). By hypothesis A/P is archimedean. By Theorem 5.68 to prove $P \in Max_n(A)$ is suffice to prove that A/P is locally finite.

Let $x = a/P \in A/P$, $x \neq 1$. Then there is $n \in N, n \geq 1$, such that $x^n \not\geq x^- \vee x^\sim$. Since A/P is chain we have $x^n \leq x^- \vee x^\sim$. Thus $x^{n+2} \leq x \odot (x^- \vee x^\sim) \odot x = (x \odot x^- \odot x) \vee (x \odot x^\sim \odot x) = 0 \vee 0 = 0$, hence $x^{n+2} = 0$, that is, $o(x) < \infty$. It follows that A/P is locally finite.

$(iii) \Rightarrow (iv)$. Let P, Q prime normal filter such that $P \subseteq Q$. By hypothesis, P is maximal, so $P = Q$. Thus Q is minimal prime.

$(iv) \Rightarrow (i)$. Let a be a nonunit element from A. We shall prove that a is an archimedean element.

If we denote

$$F = [a)^* = \{x \in A : a \vee x = 1\} \text{ (by Corollary 5.28)},$$

then $F \in \mathcal{F}(A)$. Since $a \neq 1$, then $a \notin F$ and we consider

$$F' = F(a) = \{x \in A : x \geq f \odot a^n, \text{ for some } n \geq 1 \text{ and } f \in F\},$$

(see Corollary 5.29).

If we suppose that F' is a proper filter of A, then by Corollary 5.52, there is a prime filter P such that $F' \subseteq P$, so $a \in P$. But $P \subseteq < P >$, and by Corollary 5.50, $< P >$ is prime; by hypothesis, $< P >$ is a minimal (normal) prime filter. Since $a \in < P >$, by Proposition 5.65, we infer that there is $x \in A \setminus < P >$ such that $a \vee x = 1$. It follows that $x \in [a)^* = F \subseteq F' \subseteq P \subseteq < P >$, hence $x \in < P >$, a contradiction.

Thus F' is not proper, so $0 \in F'$, hence (by Corollary 5.29) there exist $n \geq 1$ and $f \in F$ such that $f \odot a^n = 0$.

Thus $f \leq (a^n)^-$. We get $a \vee f \leq a \vee (a^n)^-$. But $a \vee f = 1$ (since $f \in F$), so we obtain that $a \vee (a^n)^- = 1$, that is a is an archimedean element, by Proposition 5.44. ■

As in the case of above implication $(iv) \Rightarrow (i)$ we have:

COROLLARY 5.84. *Let A be a pseudo - BL algebra. If any prime filter of A is minimal prime, then A is hyperarchimedean.*

THEOREM 5.85. *If pseudo - BL algebra A is a BL algebra, the following are equivalent:*

- (i) *A is hyperarchimedean;*
- (ii) *For any filter F, the quotient BL algebra A/F is an archimedean BL algebra;*
- (iii) *$Spec(A) = Max(A)$;*
- (iv) *Any prime filter is minimal prime.*

REMARK 5.19. *In this case we obtain the Theorem 3.56.*

CHAPTER 6

Localization of BL(MV)-algebras

In the first part of this Chapter we introduce the notions of BL(MV)-algebra of fractions relative to $\wedge$-closed system (Section 1), BL(MV)-algebra of fractions and maximal BL(MV)-algebra of quotients for a BL(MV)-algebra (Section 3). In Section 2 we define the notion of strong multiplier for a BL(MV)-algebra.

In Section 3 it is proved the existence of a maximal BL(MV)-algebra of quotients for a BL(MV)-algebra (Theorem 6.19). We study a maximal BL-algebra of quotients and we give an explicit descriptions of this BL-algebra for some classes of BL-algebras.

In the next Sections (4 and 5) we define the localization (strong localization) BL(MV) - algebra of a BL(MV)- algebra A with respect to a topology $\mathcal{F}$ on A. In Section 6 we prove that the maximal BL(MV) - algebra of quotients Q(A) (defined in Section 3) and the BL(MV) - algebra of fractions relative to an $\wedge-$ closed system (defined in Section 1) are strong BL(MV) - algebra of localization (see Proposition 6.33 and Proposition 6.34).

In Section 7 we define and prove analogous results for lu-groups.

In particular, we take on the task of translating the theory of localization of MV-algebras defined in Sections 5 into the language of localization of abelian lu-groups. Thus, this Section is very much in the spirit of [**3**], in which Ball, Georgescu and Leustean translate the theory of convergence and Cauchy completion of lu-groups into the language of MV-algebras.

Historical remarks: The concept of maximal lattice of quotients for a distributive lattice was defined by J.Schmid in [**121**], [**122**] taking as a guide-line the construction of complete ring of quotients by partial morphisms introduced by G. Findlay and J. Lambek (see [**96**], p.36). The central role in this constructions is played by the concept of multiplier (defined for a distributive lattice by W. H. Cornish in [**47**], [**48**]). J. Schmid used the multipliers in order to give a non–standard construction of the maximal lattice of quotients for a distributive lattice (see [**121**]). A direct treatment of the lattices of quotients can be found in [**122**]. In [**64**], G. Georgescu exhibited the localization lattice $\mathrm{L}_{\mathcal{F}}$ of a distributive lattice L with respect to a topology $\mathcal{F}$ on L in a similar way as for rings (see [**113**]) or monoids (see [**124**]).

For the case of Hilbert and Heyting algebras see [**20**], [**21**] and respectively [**49**].

1. BL(MV)-algebra of fractions relative to an $\wedge-$closed system

Definition 6.1. As in the case of residuated lattices, a nonempty subset S of a $BL-$ algebra A is called an $\wedge-$*closed system* in A if $1 \in S$ and $x, y \in S$ implies $x \wedge y \in S$.

We denote by $\mathcal{S}(A)$ the set of all $\wedge-$closed systems of A (clearly $\{1\}, A \in \mathcal{S}(A)$). For $S \in \mathcal{S}(A)$, on the BL-algebra A we consider the relation θ_S defined by

$$(x, y) \in \theta_S \text{ iff there exists } e \in S \cap B(A) \text{ such that } x \wedge e = y \wedge e.$$

Lemma 6.1. *θ_S is a congruence on A.*

Proof. The reflexivity (since $1 \in S \cap B(A)$) and the symmetry of θ_S are immediate. To prove the transitivity of θ_S, let $(x,y),(y,z) \in \theta_S$. Thus there exists $e, f \in S \cap B(A)$ such that $x \wedge e = y \wedge e$ and $y \wedge f = z \wedge f$.

If denote $g = e \wedge f \in S \cap B(A)$, then

$$g \wedge x = (e \wedge f) \wedge x = (e \wedge x) \wedge f = (y \wedge e) \wedge f = (y \wedge f) \wedge e =$$

$$= (z \wedge f) \wedge e = z \wedge (f \wedge e) = z \wedge g,$$

hence $(x,z) \in \theta_S$.

To prove the compatibility of θ_S with the operations $\wedge, \vee, \odot$ and $\rightarrow$, let $x, y, z, t \in A$ such that $(x,y) \in \theta_S$ and $(z,t) \in \theta_S$. Thus there exists $e, f \in S \cap B(A)$ such that $x \wedge e = y \wedge e$ and $z \wedge f = t \wedge f$; we denote $g = e \wedge f \in S \cap B(A)$.

We obtain:

$$(x \wedge z) \wedge g = (x \wedge z) \wedge (e \wedge f) = (x \wedge e) \wedge (z \wedge f) =$$

$$= (y \wedge e) \wedge (t \wedge f) = (y \wedge t) \wedge g,$$

hence $(x \wedge z, y \wedge t) \in \theta_S$ and

$$(x \vee z) \wedge g = (x \vee z) \wedge (e \wedge f) = [(e \wedge f) \wedge x] \vee [(e \wedge f) \wedge z] =$$

$$= [(e \wedge x) \wedge f] \vee [e \wedge (f \wedge z)] = [(e \wedge y) \wedge f] \vee [e \wedge (f \wedge t)] =$$

$$= [(e \wedge f) \wedge y] \vee [(e \wedge f) \wedge t] = (y \vee t) \wedge (e \wedge f) = (y \vee t) \wedge g,$$

hence $(x \vee z, y \vee t) \in \theta_S$.

By Remark 3.8 we obtain:

$$(x \odot z) \wedge g = (x \odot z) \odot g = (x \odot e) \odot (z \odot f) =$$

$$= (y \odot e) \odot (t \odot f) = (y \odot t) \odot g = (y \odot t) \wedge g,$$

hence $(x \odot z, y \odot t) \in \theta_S$ and by $bl - c_{42}$:

$$(x \rightarrow z) \wedge g = (x \rightarrow z) \odot g = g \odot [(g \odot x) \rightarrow (g \odot z)] =$$

$$= g \odot [(g \odot y) \rightarrow (g \odot t)] = (y \rightarrow t) \odot g = (y \rightarrow t) \wedge g,$$

hence $(x \rightarrow z, y \rightarrow t) \in \theta_S$.■

For x we denote by x/S the equivalence class of x relative to θ_S and by

$$A[S] = A/\theta_S.$$

By $p_S : A \rightarrow A[S]$ we denote the canonical map defined by $p_S(x) = x/S$, for every $x \in A$. Clearly, in $A[S]$, $\mathbf{0} = 0/S$, $\mathbf{1} = 1/S$ and for every $x, y \in A$,

$$x/S \wedge y/S = (x \wedge y)/S,$$

$$x/S \vee y/S = (x \vee y)/S,$$

$$x/S \odot y/S = (x \odot y)/S,$$

$$x/S \rightarrow y/S = (x \rightarrow y)/S.$$

So, p_S is an onto morphism of BL-algebras.

REMARK 6.1. *Since for every $s \in S \cap B(A)$, $s \wedge s = s \wedge 1$ we deduce that $s/S = 1/S = \mathbf{1}$, hence $p_S(S \cap B(A)) = \{\mathbf{1}\}$.*

PROPOSITION 6.2. *If $a \in A$, then $a/S \in B(A[S])$ iff there exists $e \in S \cap B(A)$ such that $e \wedge a \in B(A)$. So, if $e \in B(A)$, then $e/S \in B(A[S])$.*

Proof. For $a \in A$, we have $a/S \in B(A[S]) \Leftrightarrow a/S \odot a/S = a/S$ and $(a/S)^{**} = a/S$.

From $a/S \odot a/S = a/S$ we deduce that $(a \odot a)/S = a/S \Leftrightarrow$ there exists $g \in S \cap B(A)$ such that

$$(a \odot a) \wedge g = a \wedge g \Leftrightarrow (a \odot a) \odot g = a \wedge g \Leftrightarrow$$

$$\Leftrightarrow (a \odot g) \odot (a \odot g) = a \wedge g \Leftrightarrow (a \wedge g) \odot (a \wedge g) = a \wedge g.$$

From $(a/S)^{**} = a/S$ we deduce that exists $f \in S \cap B(A)$ such that $a^{**} \wedge f = a \wedge f$. If denote $e = g \wedge f \in S \cap B(A)$, then

$$(a \wedge e) \odot (a \wedge e) = (a \wedge g \wedge f) \odot (a \wedge g \wedge f) =$$

$$= (a \odot g) \odot f \odot (a \odot g) \odot f = a \odot g \odot f = a \wedge g \wedge f = a \wedge e$$

and

$$a^{**} \wedge e = a^{**} \wedge g \wedge f = (a^{**} \wedge f) \wedge g = (a \wedge f) \wedge g = a \wedge e,$$

hence $a \wedge e \in B(A)$.

If $e \in B(A)$, since $1 \in S \cap B(A)$ and $1 \wedge e = e \in B(A)$ we deduce that $e/S \in B(A[S])$. ■

THEOREM 6.3. *If A' is a BL-algebra and $f : A \to A'$ is a morphism of BL-algebras such that $f(S \cap B(A)) = \{\mathbf{1}\}$, then there exists an unique morphism of BL-algebras $f' : A[S] \to A'$ such that the diagram*

$$\begin{array}{ccc} A & \xrightarrow{p_S} & A[S] \\ & \searrow^{f} \quad \swarrow_{f'} & \\ & A' & \end{array}$$

is commutative (i.e. $f' \circ p_S = f$).

Proof. If $x, y \in A$ and $p_S(x) = p_S(y)$, then $(x, y) \in \theta_S$, hence there exists $e \in S \cap B(A)$ such that $x \wedge e = y \wedge e$. Since f is a morphism of BL-algebras, we obtain that

$$f(x \wedge e) = f(y \wedge e) \Leftrightarrow f(x) \wedge f(e) = f(y) \wedge f(e) \Leftrightarrow$$

$$f(x) \wedge \mathbf{1} = f(y) \wedge \mathbf{1} \Leftrightarrow f(x) = f(y).$$

From this observation we deduce that the map $f' : A[S] \to A'$ defined for $x \in A$ by $f'(x/S) = f(x)$ is correctly defined. Clearly, f' is a morphism of BL-algebras. The unicity of f' follows from the fact that p_S is an onto map.■

REMARK 6.2. *Theorem 6.3 allows us to call $A[S]$ the BL-algebra of fractions relative to the $\wedge-$closed system S.*

REMARK 6.3. *If $BL-$ algebra A is an $MV-$ algebra (i.e. $x^{**} = x$, for all $x \in A$), then $(x/S)^{**} = x^{**}/S = x/S$, so $A[S]$ is an $MV-$ algebra. Called $A[S]$ the MV-algebra of fractions relative to the $\wedge-$closed system S.*

EXAMPLE 6.1. *If A is a $BL-$ algebra and $S = \{1\}$ or S is such that $1 \in S$ and $S \cap (B(A) \backslash \{1\}) = \varnothing$, then for $x, y \in A, (x, y) \in \theta_S \iff x \wedge 1 = y \wedge 1 \iff x = y$, hence in this case $A[S] = A$.*

EXAMPLE 6.2. *If A is a $BL-$ algebra and S is an $\wedge-$closed system such that $0 \in S$ (for example $S = A$ or $S = B(A)$), then for every $x, y \in A$, $(x, y) \in \theta_S$ (since $x \wedge 0 = y \wedge 0$ and $0 \in S \cap B(A)$), hence in this case $A[S] = \mathbf{0}$.*

EXAMPLE 6.3. *We consider $BL-$ algebra $A = \{0, a, b, c, 1\}$ from Example 3.11:*

i). The $\wedge-$closed systems of A which contain 0 are:

$$S = A, B(A) = L_2, \{0, c, 1\}, \{0, c, a, 1\}, \{0, c, b, 1\}, \{0, a, 1\} \text{ and } \{0, b, 1\}.$$

In all these cases $A[S] = \mathbf{0}$ (see Example 6.2).

ii). The $\wedge-$closed systems of A which do not contain 0 are:

$$S = \{1\}, \{a, 1\}, \{b, 1\}, \{c, 1\}, \{a, c, 1\}, \{b, c, 1\} \text{ and } \{a, b, c, 1\}.$$

In all these cases $A[S] = A$ (because $S \cap B(A) = \{1\}$, hence θ_S is the identity; see Example 6.1).

EXAMPLE 6.4. *We consider $MV-$ algebra $L_{3\times 2} = \{0, a, b, c, d, 1\}$ from Example 3.12:*

i). The $\wedge-$closed systems of $L_{3\times 2}$ which contain 0 are:

$$S = L_{3\times 2}, \{0, 1\}, \{0, a, 1\}, \{0, b, 1\}, \{0, c, 1\},$$
$$\{0, d, 1\}, \{0, a, b, 1\}, \{0, a, c, 1\}, \{0, a, d, 1\} = B(L_{3\times 2}),$$
$$\{0, b, c, 1\}, \{0, b, d, 1\}, \{0, a, b, c, 1\}, \{0, a, b, d, 1\}, \{0, b, c, d, 1\}.$$

In all these cases $L_{3\times 2}[S] = \mathbf{0}$ (see Example 6.2).

ii). The $\wedge-$closed systems of $L_{3\times 2}$ which do not contain 0 are:

$$S = \{1\}, \{a, 1\}, \{b, 1\}, \{c, 1\}, \{d, 1\}, \{a, c, 1\}, \{b, c, 1\} \text{ and } \{b, c, d, 1\}.$$

In the cases $S = \{1\}, \{b, 1\}, \{c, 1\}, \{b, c, 1\}$, $L_{3\times 2}[S] = L_{3\times 2}$ (because $S \cap B(L_{3\times 2}) = \{1\}$, hence θ_S is the identity; see Example 6.1). In the cases $S = \{a, 1\}, \{a, c, 1\}$ we obtain

$$0/S = b/S = d/S = \{0, b, d\},$$
$$1/S = a/S = c/S = \{a, c, 1\},$$

so $L_{3\times 2}[S] \approx L_2$, and for $S = \{d, 1\}, \{b, d, 1\}, \{b, c, d, 1\}$ we obtain

$$0/S = a/S = \{0, a\},$$
$$b/S = c/S = \{b, c\},$$
$$d/S = 1/S = \{1, d\}.$$

$L_{3\times 2}[S]$ is not a Boolean algebra because $b/S \oplus b/S = (b \oplus b)/S = d/S \neq b/S$.

EXAMPLE 6.5. *Suppose that A is a boolean algebra. Clearly, A is an MV -algebra. Then every ideal of the underlying lattice $L(A)$ is an ideal of A (every ideal of an MV-algebra A is also an ideal of the underlying lattice $L(A)$ - see* [**45**]*, p. 112). If $\mathcal{P}$ is a prime ideal of A (that is $\mathcal{P} \neq A$ and if $x \wedge y \in \mathcal{P}$ implies $x \in \mathcal{P}$ or $y \in \mathcal{P}$), then $S = A \backslash \mathcal{P}$ is an $\wedge-$closed system. We denote $A[S]$ by $A_{\mathcal{P}}$. The set $M = \{x/S : x \in \mathcal{P}\}$ is a maximal ideal of $A_{\mathcal{P}}$. Indeed, if $x, y \in \mathcal{P}$, then $x/S \oplus y/S = (x \oplus y)/S \in M$ (since $x \oplus y \in \mathcal{P}$). If $x, y \in A$ such that $x \in \mathcal{P}$ and $y/S \leq x/S$ then there exists $e \in S \cap B(A)$ such that $y \wedge e \leq x \wedge e$. Since $x \in \mathcal{P}$, then $y \wedge e \in \mathcal{P}$, hence $y \in \mathcal{P}$ (since $e \notin \mathcal{P}$), so $y/S \in M$. To prove the maximality of M let I an ideal of $A_{\mathcal{P}}$ such that $M \subseteq I$ and $M \neq I$. Then there exists $x/S \in I$ such that $x/S \notin M$, (that is $x \notin \mathcal{P} \Longleftrightarrow x \in S$), hence $x/S = \mathbf{1}$ (see Remark 6.1) so $I = A_{\mathcal{P}}$. Moreover, M is the only maximal ideal of $A_{\mathcal{P}}$ (since if we have another maximal ideal M' of $A_{\mathcal{P}}$, then $M' \nsubseteq M$ hence there exists $x/S \in M'$ such that $x/S \notin M$, so $x/S = \mathbf{1}$ and $M' = A_{\mathcal{P}}$, a contradiction!). In other words $A_{\mathcal{P}}$ is a local MV -algebra. The process of passing from A to $A_{\mathcal{P}}$ is called localization at $\mathcal{P}$.*

2. Strong multipliers on a BL(MV)-algebra

DEFINITION 6.2. Let $(P, \leq)$ be an ordered set and $I \subseteq P$ a non-empty set. I is an *order ideal* (alternative terms include *down-set* or *decreasing set*) if, whenever $x \in I, y \in P$ and $y \leq x$, we have $y \in I$. We denote by $\mathcal{I}(P)$ the set of all order ideals of P; clearly, $\mathcal{I}(P)$ is closed under arbitrary intersections. For a nonempty set $M \subseteq P$ we denote by $< M >_P$ the *order ideal of P generated by M.*

REMARK 6.4. *It is eassy to prove that for a nonempty set $M \subseteq P$,*

$$< M >_P = \{x \in P : \text{there exists } a \in M \text{ such that } x \leq a\}.$$

Let A be a $BL-$ algebra. We denote by $\mathcal{I}(A)$ the set of all *order ideals* of A (see Definition 6.2):

$$\mathcal{I}(A) = \{I \subseteq A : \text{if } x, y \in A, x \leq y \text{ and } y \in I, \text{then } x \in I\},$$

and by $I_d(A)$ the set of all *ideals* of the lattice $L(A)$.

REMARK 6.5. *Clearly, $\mathcal{I}(A) \subseteq I_d(A)$ and if $I_1, I_2 \in \mathcal{I}(A)$, then $I_1 \cap I_2 \in \mathcal{I}(A)$. Also, if $I \in \mathcal{I}(A)$, then $0 \in I$.*

DEFINITION 6.3. By *partial strong multiplier* of A we mean a map $f : I \to A$, where $I \in \mathcal{I}(A)$, which verifies the next conditions:

$(sm - BL_1)$ $f(e \odot x) = e \odot f(x)$, for every $e \in B(A)$ and $x \in I$;
$(sm - BL_2)$ $f(x) \leq x$, for every $x \in I$;
$(sm - BL_3)$ If $e \in I \cap B(A)$, then $f(e) \in B(A)$;
$(sm - BL_4)$ $x \wedge f(e) = e \wedge f(x)$, for every $e \in I \cap B(A)$ and $x \in I$ (*note that $e \odot x \in I$ since $e \odot x \leq e \wedge x \leq x$*).

REMARK 6.6. *If A is an $MV-$ algebra the definition of strong multiplier on A is the same as Definition 6.3 for the case of $BL-$ algebras (we recall that in this case, for $x, y \in A$, $x \odot y = (x^* \oplus y^*)^*$).*

Clearly, $f(0) = 0$.

By $dom(f) \in \mathcal{I}(A)$ we denote the domain of f; if $dom(f) = A$, we called f *total*.

To simplify the language, we will use *strong multiplier* instead of *partial strong multiplier,* using *total* to indicate that the domain of a certain multiplier is A.

EXAMPLE 6.6. *The map $\mathbf{0} : A \to A$ defined by $\mathbf{0}(x) = 0$, for every $x \in A$ is a total strong multiplier of A; indeed if $x \in A$ and $e \in B(A)$, then $\mathbf{0}(e \odot x) = 0 = e \odot 0 = e \odot \mathbf{0}(x)$ and $\mathbf{0}(x) \leq x$. Clearly, if $e \in A \cap B(A) = B(A)$, then $\mathbf{0}(e) = 0 \in B(A)$ and for $x \in A$, $x \wedge \mathbf{0}(e) = e \wedge \mathbf{0}(x) = 0$.*

EXAMPLE 6.7. *The map $\mathbf{1} : A \to A$ defined by $\mathbf{1}(x) = x$, for every $x \in A$ is also a total strong multiplier of A; indeed if $x \in A$ and $e \in B(A)$, then $\mathbf{1}(e \odot x) = e \odot x = e \odot \mathbf{1}(x)$ and $\mathbf{1}(x) = x \leq x$. The conditions $sm - BL_3 - sm - BL_4$ are obviously verified.*

EXAMPLE 6.8. *For $a \in B(A)$ and $I \in \mathcal{I}(A)$, the map $f_a : I \to A$ defined by $f_a(x) = a \wedge x$, for every $x \in I$ is a strong multiplier of A (called principal). Indeed, for $x \in I$ and $e \in B(A)$, we have $f_a(e \odot x) = a \wedge (e \odot x) = a \wedge (e \wedge x) = e \wedge (a \wedge x) = e \odot (a \wedge x) = e \odot f_a(x)$ and clearly $f_a(x) \leq x$. Also, if $e \in I \cap B(A)$, $f_a(e) = e \wedge a \in B(A)$ and $x \wedge (a \wedge e) = e \wedge (a \wedge x)$, for every $x \in I$.*

REMARK 6.7. *The condition $sm-BL_4$ is not a consequence of $sm-BL_1-sm-BL_3$. As example, $f: I \to A, f(x) = x \wedge x^*$ for every $x \in I$, verify $sm-BL_1-sm-BL_3$, but if $e \in I \cap B(A)$ and $x \in I$, then*

$$x \wedge f(e) = x \wedge 0 \neq e \wedge (x \wedge x^*) = e \wedge f(x).$$

REMARK 6.8. *In general, if we consider $a \in A$, then $f_a : I \to A$ verifies only $sm - BL_1, sm - BL_2$ and $sm - BL_4$ but does not verify $sm - BL_3$.*

If $dom(f_a) = A$, we denote f_a by $\overline{f_a}$; clearly, $\overline{f_0} = \mathbf{0}$.
For $I \in \mathcal{I}(A)$, we denote

$$M(I, A) = \{f : I \to A \mid f \text{ is a strong multiplier on } A\}$$

and

$$M(A) = \underset{I \in \mathcal{I}(A)}{\cup} M(I, A).$$

If necessary, we denote $M(A)$ by $M_{\mathcal{BL}}(A)$ to indicate that we work in $BL-$ algebras; for the case of $MV-$ algebras we denote $M(A)$ by $M_{\mathcal{MV}}(A)$.

REMARK 6.9. *From Propositions 3.8 and 3.9 we deduce that for every $I \in \mathcal{I}(A)$ the algebra of multipliers $M_{\mathcal{BL}}(I, A)$ for a $BL-$ algebras is in fact a generalization of the algebra of multipliers $M_{\mathcal{MV}}(I, A)$ for $MV-$ algebras, defined in* [**26**]. *Also, we deduce that if $BL-$ algebra A is an $MV-$ algebra (that is $A = MV(A)$), then $M_{\mathcal{BL}}(I, A) = M_{\mathcal{MV}}(I, A)$ for every $I \in \mathcal{I}(A)$.*

DEFINITION 6.4. *If $I_1, I_2 \in \mathcal{I}(A)$ and $f_i \in M(I_i, A), i = 1, 2$, we define $f_1 \wedge f_2$, $f_1 \vee f_2$, $f_1 \boxdot f_2$, $f_1 \to f_2 : I_1 \cap I_2 \to A$ by*

$$(f_1 \wedge f_2)(x) = f_1(x) \wedge f_2(x),$$

$$(f_1 \vee f_2)(x) = f_1(x) \vee f_2(x),$$

$$(f_1 \boxdot f_2)(x) = f_1(x) \odot [x \to f_2(x)] \overset{bl-c_{45}}{=} f_2(x) \odot [x \to f_1(x)],$$

$$(f_1 \to f_2)(x) = x \odot [f_1(x) \to f_2(x)],$$

for every $x \in I_1 \cap I_2$.

LEMMA 6.4. $f_1 \wedge f_2 \in M(I_1 \cap I_2, A)$.

Proof. If $x \in I_1 \cap I_2$ and $e \in B(A)$, then $(f_1 \wedge f_2)(e \odot x) = f_1(e \odot x) \wedge f_2(e \odot x) = (e \odot f_1(x)) \wedge (e \odot f_2(x)) = (e \wedge f_1(x)) \wedge (e \wedge f_2(x)) = e \wedge [f_1(x) \wedge f_2(x)] = e \odot (f_1 \wedge f_2)(x)$.
Since $f_i \in M(I_i, A), i = 1, 2$, we have $(f_1 \wedge f_2)(x) = f_1(x) \wedge f_2(x) \leq x \wedge x = x$, for every $x \in I_1 \cap I_2$ and if $e \in I_1 \cap I_2 \cap B(A)$, then

$$(f_1 \wedge f_2)(e) = f_1(e) \wedge f_2(e) \in B(A).$$

For $e \in I_1 \cap I_2 \cap B(A)$ and $x \in I_1 \cap I_2$ we have:

$$x \wedge (f_1 \wedge f_2)(e) = x \wedge f_1(e) \wedge f_2(e) = [x \wedge f_1(e)] \wedge [x \wedge f_2(e)] =$$

$$= [e \wedge f_1(x)] \wedge [e \wedge f_2(x)] = e \wedge (f_1 \wedge f_2)(x),$$

that is $f_1 \wedge f_2 \in M(I_1 \cap I_2, A)$. ■

LEMMA 6.5. $f_1 \vee f_2 \in M(I_1 \cap I_2, A)$.

Proof. If $x \in I_1 \cap I_2$ and $e \in B(A)$, then $(f_1 \vee f_2)(e \odot x) = f_1(e \odot x) \vee f_2(e \odot x) = (e \odot f_1(x)) \vee (e \odot f_2(x)) \overset{bl-c_{31}}{=} e \odot [f_1(x) \vee f_2(x)] = e \odot (f_1 \vee f_2)(x)$.

Since $f_i \in M(I_i, A), i = 1, 2$, we have $(f_1 \vee f_2)(x) = f_1(x) \vee f_2(x) \leq x \vee x = x$, for every $x \in I_1 \cap I_2$ and if $e \in I_1 \cap I_2 \cap B(A)$, then

$$(f_1 \vee f_2)(e) = f_1(e) \vee f_2(e) \in B(A).$$

For $e \in I_1 \cap I_2 \cap B(A)$ and $x \in I_1 \cap I_2$ we have:

$$x \wedge (f_1 \vee f_2)(e) = x \wedge [f_1(e) \vee f_2(e)] = [x \wedge f_1(e)] \vee [x \wedge f_2(e)] = [e \wedge f_1(x)] \vee [e \wedge f_2(x)] =$$
$$= e \wedge [f_1(x) \vee f_2(x)] = e \wedge (f_1 \vee f_2)(x),$$

that is $f_1 \vee f_2 \in M(I_1 \cap I_2, A)$. ■

LEMMA 6.6. $f_1 \boxdot f_2 \in M(I_1 \cap I_2, A)$.

Proof. If $x \in I_1 \cap I_2$ and $e \in B(A)$, then

$$(f_1 \boxdot f_2)(e \odot x) = f_1(e \odot x) \odot [(e \odot x) \to f_2(e \odot x)] = [e \odot f_1(x)] \odot [(e \odot x) \to (e \odot f_2(x))] =$$
$$= f_1(x) \odot [e \odot ((e \odot x) \to (e \odot f_2(x)))] \overset{bl-c_{42}}{=} f_1(x) \odot [e \odot (x \to f_2(x))] =$$
$$= e \odot [f_1(x) \odot (x \to f_2(x))] = e \odot (f_1 \boxdot f_2)(x).$$

Clearly, $(f_1 \boxdot f_2)(x) = f_1(x) \odot [x \to f_2(x)] \leq f_1(x) \leq x$, for every $x \in I_1 \cap I_2$ and if $e \in I_1 \cap I_2 \cap B(A)$, then by Remark 3.8 we have

$$(f_1 \boxdot f_2)(e) = f_1(e) \odot [e \to f_2(e)] = f_1(e) \odot (e^* \vee f_2(e)) \in B(A).$$

For $e \in I_1 \cap I_2 \cap B(A)$ and $x \in I_1 \cap I_2$ we have:

$$x \wedge (f_1 \boxdot f_2)(e) = x \wedge [f_1(e) \odot (e \to f_2(e)] = x \odot [f_1(e) \odot (e \to f_2(e)] = f_1(e) \odot [x \odot (e \to f_2(e))]$$
$$\overset{bl-c_{43}}{=} f_1(e) \odot [x \odot ((x \odot e) \to (x \odot f_2(e)))] = (f_1(e) \odot x) \odot [(x \odot e) \to (x \odot f_2(e))] =$$
$$= (e \odot f_1(x)) \odot [(e \odot x) \to (e \odot f_2(x))] = f_1(x) \odot [e \odot ((e \odot x) \to (e \odot f_2(x)))]$$
$$\overset{bl-c_{42}}{=} f_1(x) \odot [e \odot (x \to f_2(x))] = e \odot [f_1(x) \odot (x \to f_2(x))] = e \odot (f_1 \boxdot f_2)(x) = e \wedge (f_1 \boxdot f_2)(x),$$

hence

$$x \wedge (f_1 \boxdot f_2)(e) = e \wedge (f_1 \boxdot f_2)(x),$$

that is $f_1 \boxdot f_2 \in M(I_1 \cap I_2, A)$. ■

LEMMA 6.7. $f_1 \to f_2 \in M(I_1 \cap I_2, A)$.

Proof. If $x \in I_1 \cap I_2$ and $e \in B(A)$, then

$$(f_1 \to f_2)(e \odot x) = (e \odot x) \odot [f_1(e \odot x) \to f_2(e \odot x)] = (e \odot x) \odot [(e \odot f_1(x)) \to (e \odot f_2(x))] =$$
$$= x \odot [e \odot ((e \odot f_1(x)) \to (e \odot f_2(x)))] \overset{bl-c_{42}}{=} x \odot [e \odot (f_1(x) \to f_2(x))] =$$
$$= e \odot [x \odot (f_1(x) \to f_2(x))] = e \odot (f_1 \to f_2)(x).$$

Clearly, $(f_1 \to f_2)(x) = x \odot [f_1(x) \to f_2(x)] \leq x$, for every $x \in I_1 \cap I_2$ and if $e \in I_1 \cap I_2 \cap B(A)$, then by Remark 3.8 we have

$$(f_1 \to f_2)(e) = e \odot [f_1(e) \to f_2(e)] = e \odot [(f_1(e))^* \vee f_2(e)] \in B(A).$$

For $e \in I_1 \cap I_2 \cap B(A)$ and $x \in I_1 \cap I_2$ we have:

$$e \wedge (f_1 \to f_2)(x) = e \wedge [x \odot (f_1(x) \to f_2(x))] =$$
$$= (e \odot x) \odot [f_1(x) \to f_2(x)] = x \odot [e \odot (f_1(x) \to f_2(x))] =$$
$$\overset{bl-c_{42}}{=} x \odot [e \odot ((e \odot f_1(x)) \to (e \odot f_2(x)))] = x \odot [e \odot ((x \odot f_1(e)) \to (x \odot f_2(e)))] =$$

$$= e \odot [x \odot ((x \odot f_1(e)) \to (x \odot f_2(e)))] \overset{bl-c_{43}}{=} e \odot [x \odot (f_1(e) \to f_2(e))] =$$
$$= x \odot [e \odot (f_1(e) \to f_2(e))] = x \odot (f_1 \to f_2)(e) = x \wedge (f_1 \to f_2)(e)$$

hence

$$x \wedge (f_1 \to f_2)(e) = e \wedge (f_1 \to f_2)(x),$$

that is $f_1 \to f_2 \in M(I_1 \cap I_2, A)$. ■

PROPOSITION 6.8. *$(M(A), \wedge, \vee, \boxdot, \to, \mathbf{0}, \mathbf{1})$ is a BL-algebra.*

Proof. We verify the axioms of BL-algebras.
(BL_1). Obviously $(M(A), \wedge, \vee, \mathbf{0}, \mathbf{1})$ is a bounded lattice.
(BL_2). Let $f_i \in M(I_i, A)$ where $I_i \in \mathcal{I}(A)$, $i = 1, 2, 3$.
Clearly, $f_1 \boxdot f_2 \in M(A)$ (see Lemma 6.6).
Thus, for $x \in I_1 \cap I_2 \cap I_3$ we have

$$[f_1 \boxdot (f_2 \boxdot f_3)](x) = ((f_2 \boxdot f_3)(x)) \odot (x \to f_1(x)) =$$
$$= [f_2(x) \odot (x \to f_3(x))] \odot (x \to f_1(x)) = f_2(x) \odot [(x \to f_3(x)) \odot (x \to f_1(x))] =$$
$$= f_2(x) \odot [(x \to f_1(x)) \odot (x \to f_3(x))] = [f_2(x) \odot (x \to f_1(x))] \odot (x \to f_3(x)) =$$
$$= ((f_1 \boxdot f_2)(x)) \odot (x \to f_3(x)) = [(f_1 \boxdot f_2) \boxdot f_3](x),$$

that is the operation $\boxdot$ is associative.
By definition

$$(f_1 \boxdot f_2)(x) = f_1(x) \odot [x \to f_2(x)] \overset{bl-c_{45}}{=} f_2(x) \odot [x \to f_1(x)] = (f_2 \boxdot f_1)(x),$$

that is the operation $\boxdot$ is commutative.
Let $f \in M(I, A)$ with $I \in \mathcal{I}(A)$. If $x \in I$, then

$$(f \boxdot \mathbf{1})(x) = f(x) \odot (x \to \mathbf{1}(x)) = f(x) \odot (x \to x) = f(x) \odot 1 = f(x),$$

and

$$(\mathbf{1} \boxdot f)(x) = \mathbf{1}(x) \odot (x \to f(x)) = x \odot (x \to f(x)) = x \wedge f(x) = f(x),$$

hence

$$f \boxdot \mathbf{1} = \mathbf{1} \boxdot f = f,$$

that is $(M(A), \boxdot, \mathbf{1})$ is a commutative monoid.
(BL_3). Let $f_i \in M(I_i, A)$ where $I_i \in \mathcal{I}(A)$, $i = 1, 2, 3$.
Since $f_1 \leq f_2 \to f_3$ for $x \in I_1 \cap I_2 \cap I_3$ we have

$$f_1(x) \leq (f_2 \to f_3)(x) \Leftrightarrow f_1(x) \leq x \odot [f_2(x) \to f_3(x)].$$

So, by $bl - c_2$,

$$f_1(x) \odot [x \to f_2(x)] \leq x \odot (x \to f_2(x)) \odot (f_2(x) \to f_3(x)) \Leftrightarrow$$
$$f_1(x) \odot [x \to f_2(x)] \leq (x \wedge f_2(x)) \odot (f_2(x) \to f_3(x)) \Leftrightarrow$$
$$f_1(x) \odot [x \to f_2(x)] \leq f_2(x) \odot (f_2(x) \to f_3(x)) \Leftrightarrow$$
$$f_1(x) \odot [x \to f_2(x)] \leq f_2(x) \wedge f_3(x) \leq f_3(x) \Leftrightarrow$$
$$(f_2 \boxdot f_1)(x) \leq f_3(x),$$

for every $x \in I_1 \cap I_2 \cap I_3$, that is

$$f_2 \boxdot f_1 \leq f_3.$$

Conversely if $(f_2 \boxdot f_1)(x) \leq f_3(x)$ we have

$$f_1(x) \odot [x \to f_2(x)] \leq f_3(x),$$

for every $x \in I_1 \cap I_2 \cap I_3$.

Obviously,

$$f_2(x) \odot [x \to f_1(x)] \leq f_1(x) \odot [x \to f_2(x)]$$

(see Lemma 3.6)

$$\Leftrightarrow x \to f_1(x) \leq f_2(x) \to [f_1(x) \odot (x \to f_2(x))] \overset{bl-c_{11}}{\leq} f_2(x) \to f_3(x)$$

$$\overset{bl-c_2}{\Leftrightarrow} x \odot (x \to f_1(x)) \leq x \odot (f_2(x) \to f_3(x))$$

$$\Leftrightarrow x \wedge f_1(x) \leq x \odot (f_2(x) \to f_3(x))$$

$$\Leftrightarrow f_1(x) \leq (f_2 \to f_3)(x).$$

So $f_1 \leq f_2 \to f_3$ iff $f_2 \boxdot f_1 \leq f_3$ for all $f_1, f_2, f_3 \in M(A)$.

(BL_4). Let $f_i \in M(I_i, A)$ where $I_i \in \mathcal{I}(A)$, $i = 1, 2$.

Thus, for $x \in I_1 \cap I_2$ we have

$$[f_1 \boxdot (f_1 \to f_2)](x) = [(f_1 \to f_2)(x)] \odot [x \to f_1(x)]$$

$$= x \odot [f_1(x) \to f_2(x)] \odot [x \to f_1(x)] = (x \odot [x \to f_1(x)]) \odot [f_1(x) \to f_2(x)] =$$

$$= [x \wedge f_1(x)] \odot [f_1(x) \to f_2(x)] = f_1(x) \odot [f_1(x) \to f_2(x)] = f_1(x) \wedge f_2(x) = (f_1 \wedge f_2)(x).$$

So,

$$f_1 \wedge f_2 = f_1 \boxdot (f_1 \to f_2).$$

(BL_5). We have

$$[(f_1 \to f_2) \vee (f_2 \to f_1)](x) = [(f_1 \to f_2)(x)] \vee [(f_2 \to f_1)(x)] =$$

$$= [x \odot (f_1(x) \to f_2(x))] \vee [x \odot (f_2(x) \to f_1(x))] =$$

$$\overset{bl-c_{31}}{=} x \odot [(f_1(x) \to f_2(x)) \vee (f_2(x) \to f_1(x))] \overset{BL_5}{=} x \odot 1 = x = \mathbf{1}(x),$$

hence

$$(f_1 \to f_2) \vee (f_2 \to f_1) = \mathbf{1}. \blacksquare$$

Remark 6.10. *To prove that $(M(A), \wedge, \vee, \boxdot, \to, \mathbf{0}, \mathbf{1})$ is a BL-algebra it is sufficient to ask for multipliers to verify only the axioms $sm - BL_1$ and $sm - BL_2$.*

Proposition 6.9. *If $BL-$ algebra $(A, \wedge, \vee, \odot, \to, 0, 1)$ is an $MV-$ algebra $(A, \oplus, ^*, 0)$ (i.e. $x^{**} = x$, for all $x \in A$), then $BL-$ algebra $(M(A), \wedge, \vee, \boxdot, \to, \mathbf{0}, \mathbf{1})$ is an $MV-$ algebra $(M(A), \boxplus, ^*, \mathbf{0})$. If $I_1, I_2 \in \mathcal{I}(A)$ and $f_i \in M(I_i, A), i = 1, 2$, we have $f_1 \boxplus f_2 : I_1 \cap I_2 \to A$,*

$$(f_1 \boxplus f_2)(x) = (f_1(x) \oplus f_2(x)) \wedge x,$$

for every $x \in I_1 \cap I_2$; for $I \in \mathcal{I}(A)$ and $f \in M(I, A)$ we have $f^* : I \to A$

$$f^*(x) = (f \to \mathbf{0})(x) = x \odot (f(x) \to \mathbf{0}(x)) = x \odot (f(x) \to 0) = x \odot (f(x))^*,$$

for every $x \in I$.

Proof. To prove that $BL-$ algebra $M(A)$ is an $MV-$ algebra let $f \in M(I, A)$ with $I \in \mathcal{I}(A)$.

Then

$$f^{**}(x) = [(f \to \mathbf{0}) \to \mathbf{0}](x) = x \odot [(f \to \mathbf{0})(x)]^* = x \odot [x \odot (f(x))^*]^*$$

$$= x \odot [(x \odot (f(x))^*) \to 0] \overset{bl-c_8}{=} x \odot [x \to (f(x))^{**}] = x \wedge (f(x))^{**} = x \wedge f(x) = f(x),$$

(since $f(x) \in A$ which is an $MV-$ algebra), for all $x \in I$.

So, $f^{**} = f$ and $BL-$ algebra $M(A)$ is an MV -algebra.

We have $f_1 \boxplus f_2 = (f_1^* \boxdot f_2^*)^*$ and $f^* = f \to \mathbf{0}$.

Clearly,

$$(f_1 \boxplus f_2)(x) = x \odot [f_1^*(x) \odot (x \to f_2^*(x))]^*$$

$$= x \odot [x \odot (f_1(x))^* \odot (x \to x \odot (f_2(x))^*)]^* = x \odot [(f_1(x))^* \odot x \odot (x \to x \odot (f_2(x))^*)]^*$$

$$\overset{BL_4}{=} x \odot [(f_1(x))^* \odot (x \wedge x \odot (f_2(x))^*)]^* = x \odot [(f_1(x))^* \odot x \odot (f_2(x))^*]^*$$

$$= x \odot [x \odot (f_1(x))^* \odot (f_2(x))^*]^* \overset{bl-c_{37}}{=} x \odot [x \to ((f_1(x))^* \odot (f_2(x))^*)^*]$$

$$\overset{BL_4}{=} x \wedge (f_1(x) \oplus f_2(x)),$$

for all $x \in I_1 \cap I_2$. Then $(M(A), \boxplus, ^*, \mathbf{0})$ is an MV-algebra.■

LEMMA 6.10. *The map $v_A : B(A) \to M(A)$ defined by $v_A(a) = \overline{f_a}$ for every $a \in B(A)$, is a monomorphism of BL-algebras.*

Proof. Clearly, $v_A(0) = \overline{f_0} = \mathbf{0}$. Let $a, b \in B(A)$ and $x \in A$. We have:

$$(v_A(a) \boxdot v_A(b))(x) = v_A(a)(x) \odot (x \to v_A(b)(x)) = (a \wedge x) \odot (x \to (b \wedge x))$$

$$= (a \odot x) \odot (x \to (b \wedge x)) = a \odot [x \odot (x \to (b \wedge x))] = a \odot [x \wedge (b \wedge x)]$$

$$= a \wedge [x \wedge (b \wedge x)] = a \wedge (b \wedge x) = (a \wedge b) \wedge x = (v_A(a \wedge b))(x) = (v_A(a \odot b))(x),$$

hence

$$v_A(a \odot b) = v_A(a) \boxdot v_A(b).$$

Also,

$$(v_A(a) \to v_A(b))(x) = x \odot [v_A(a)(x) \to v_A(b)(x)] = x \odot [(a \wedge x) \to (b \wedge x)]$$

$$= x \odot [(x \odot a) \to (x \odot b)] \overset{bl-c_{43}}{=} x \odot (a \to b) = x \wedge (a \to b)$$

(since $a \to b \in B(A)$)

$$= v_A(a \to b)(x),$$

hence

$$v_A(a) \to v_A(b) = v_A(a \to b),$$

that is v_A is a morphism of BL-algebras.

To prove the injectivity of v_A let $a, b \in B(A)$ such that $v_A(a) = v_A(b)$. Then $a \wedge x = b \wedge x$, for every $x \in A$, hence for $x = 1$ we obtain that $a \wedge 1 = b \wedge 1 \Rightarrow a = b$.■

DEFINITION 6.5. A nonempty set $I \subseteq A$ is called *regular* if for every $x, y \in A$ such that $x \wedge e = y \wedge e$ for every $e \in I \cap B(A)$, then $x = y$.

For example A is a regular subset of A (since if $x, y \in A$ and $x \wedge e = y \wedge e$ for every $e \in A \cap B(A) = B(A)$, then for $e = 1$ we obtain $x \wedge 1 = y \wedge 1 \Leftrightarrow x = y$).

More generally, every subset of A which contains 1 is regular, hence all the filters of A are regular sets.

We denote

$$\mathcal{R}(A) = \{I \subseteq A : I \text{ is a regular subset of } A\}.$$

REMARK 6.11. *The condition $I \in \mathcal{R}(A)$ is equivalent with the condition: for every $x, y \in A$, if $f_{x|I \cap B(A)} = f_{y|I \cap B(A)}$, then $x = y$.*

LEMMA 6.11. *If $I_1, I_2 \in \mathcal{I}(A) \cap \mathcal{R}(A)$, then $I_1 \cap I_2 \in \mathcal{I}(A) \cap \mathcal{R}(A)$.*

Proof. By Remark 6.5, $I_1 \cap I_2 \in \mathcal{I}(A)$. To prove $I_1 \cap I_2 \in \mathcal{R}(A)$ let $x, y \in A$ such that $x \wedge e = y \wedge e$ for every $e \in (I_1 \cap I_2) \cap B(A)$. If $e_i \in I_i \cap B(A)$, $i = 1, 2$ are arbitrary, then $e_1 \wedge e_2 \in I_1 \cap I_2 \cap B(A)$ so, we have

$$(e_1 \wedge e_2) \wedge x = (e_1 \wedge e_2) \wedge y \Leftrightarrow e_1 \wedge (e_2 \wedge x) = e_1 \wedge (e_2 \wedge y).$$

Since $e_1 \in I_1 \cap B(A)$ are arbitrary and $I_1 \in \mathcal{I}(A) \cap \mathcal{R}(A)$, then we obtain $e_2 \wedge x = e_2 \wedge y$.

Since $e_2 \in I_2 \cap B(A)$ are arbitrary and $I_2 \in \mathcal{I}(A) \cap \mathcal{R}(A)$, we obtain $x = y$, hence $I_1 \cap I_2 \in \mathcal{I}(A) \cap \mathcal{R}(A)$.■

Remark 6.12. *By Lemma 6.11, we deduce that*

$$M_r(A) = \{f \in M(A) : dom(f) \in \mathcal{I}(A) \cap \mathcal{R}(A)\}$$

is a BL-subalgebra of $M(A)$.

Proposition 6.12. $M_r(A)$ *is a Boolean subalgebra of* $M(A)$.

Proof. Let $f : I \to A$ be a strong multiplier on A with $I \in \mathcal{I}(A) \cap \mathcal{R}(A)$. Then

$$e \wedge [f \vee f^*](x) = e \wedge [f(x) \vee (x \odot (f(x))^*)] = [e \wedge f(x)] \vee [e \wedge (x \odot (f(x))^*)]$$

$$\overset{sm-BL_4}{=} [x \odot f(e)] \vee [x \odot e \odot (f(x))^*] \overset{bl-c_{42}}{=} [x \odot f(e)] \vee [x \odot e \odot (e \odot f(x))^*]$$

$$\overset{sm-BL_4}{=} [x \odot f(e)] \vee [x \odot e \odot (x \odot f(e))^*] = [x \odot f(e)] \vee [x \odot e \odot (x \wedge f(e))^*]$$

$$\overset{bl-c_{33}}{=} [x \odot f(e)] \vee [x \odot e \odot (x^* \vee (f(e))^*)] \overset{bl-c_{13}}{=} [x \odot f(e)] \vee [e \odot ((x \odot x^*) \vee (x \odot (f(e))^*))]$$

$$\overset{bl-c_5}{=} [x \odot f(e)] \vee [e \odot (0 \vee (x \odot (f(e))^*))] = [x \odot f(e)] \vee [e \odot x \odot (f(e))^*]$$

$$= [x \odot f(e)] \vee [x \odot (e \odot (f(e))^*)] \overset{bl-c_{13}}{=} x \odot [f(e) \vee (e \odot (f(e))^*)]$$

$$= x \odot [f(e) \vee (e \wedge (f(e))^*)] = x \odot [(f(e) \vee e) \wedge (f(e) \vee (f(e))^*)]$$

$$\overset{sm-BL_3}{=} x \odot (e \wedge 1) = x \odot e = x \wedge e = \mathbf{1}(x) \wedge e,$$

hence $(f \vee f^*)(x) = \mathbf{1}(x)$, since $I \in \mathcal{R}(A)$, hence $f \vee f^* = \mathbf{1}$, that is $M_r(A)$ is a Boolean algebra. ■

Remark 6.13. *The axioms* $smBL_3$, $smBL_4$ *is necessary in the proof of Proposition 6.12.*

Definition 6.6. Given two strong multipliers f_1, f_2 on A, we say that f_2 extends f_1 if $dom(f_1) \subseteq dom(f_2)$ and $f_{2|dom(f_1)} = f_1$; we write $f_1 \leq f_2$ if f_2 extends f_1. A strong multiplier f is called *maximal* if f can not be extended to a strictly larger domain.

Lemma 6.13. (*i*) *If* $f_1, f_2 \in M(A)$, $f \in M_r(A)$ *and* $f \leq f_1, f \leq f_2$, *then* f_1 *and* f_2 *coincide on the* $dom(f_1) \cap dom(f_2)$*;*

(*ii*) *Every strong multiplier* $f \in M_r(A)$ *can be extended to a maximal strong multiplier. More precisely, each principal strong multiplier* f_a *with* $a \in B(A)$ *and* $dom(f_a) \in \mathcal{I}(A) \cap \mathcal{R}(A)$ *can be uniquely extended to a total strong multiplier* $\overline{f_a}$ *and each non-principal strong multiplier can be extended to a maximal non-principal one.*

Proof. (i). Assume, to the contrary that there exists $x \in dom(f_1) \cap dom(f_2)$ such that $f_1(x) \neq f_2(x)$. Since $dom(f) \in \mathcal{R}(A)$, there is $e \in dom(f) \cap B(A)$ such that $e \wedge f_1(x) \neq e \wedge f_2(x)$. But $e \wedge f_i(x) = f_i(e \odot x)$ for $i = 1, 2$, thus $f_1(e \odot x) \neq f_2(e \odot x)$. Since $e \odot x \leq e$, we have $e \odot x \in dom(f)$, contradicting $f \leq f_1, f \leq f_2$.

(ii).We first prove that f_a with $a \in B(A)$ can not be extended to a non-principal strong multiplier. Let $I = dom(f_a) \in \mathcal{I}(A) \cap \mathcal{R}(A)$, $f_a : I \to A$ and suppose by absurdum hypothesis that there exists $I' \in \mathcal{I}(A), I \subseteq I'$ (hence $I' \in \mathcal{I}(A) \cap \mathcal{R}(A)$) and a non-principal strong multiplier $f \in M(I', A)$ which extends f_a. Since f is non-principal, there exists $x_0 \in I', x_0 \notin I$ such that $f(x_0) \neq x_0 \wedge a$ (see Remark 6.11). Since $I \in \mathcal{R}(A)$, there exists $e \in I \cap B(A)$ such that $e \wedge f(x_0) \neq e \wedge (a \wedge x_0) \Leftrightarrow f(e \odot x_0) \neq e \wedge (a \wedge x_0) \Leftrightarrow f(e \odot x_0) \neq a \wedge (e \odot x_0)$.

Denoting $x_1 = e \odot x_0 \in I$ (since $x_1 \leq e$), we obtain that $f(x_1) \neq a \wedge x_1$, which is contradiction (since $f_a \leq f$).

Hence f_a is uniquely extended by $\overline{f_a}$.

Now, let $f \in M_r(A)$ be non-principal and

$$M_f = \{(I, g) : I \in \mathcal{I}(A), g \in M(I, A), dom(f) \subseteq I \text{ and } g_{|dom(f)} = f\}$$

(clearly, if $(I, g) \in M_f$, then $I \in \mathcal{I}(A) \cap \mathcal{R}(A)$).

The set M_f is ordered by $(I_1, g_1) \leq (I_2, g_2)$ iff $I_1 \subseteq I_2$ and $g_{2|I_1} = g_1$. Let

$$\{(I_k, g_k) : k \in K\}$$

be a chain in M_f. Then $I' = \underset{k \in K}{\cup} I_k \in \mathcal{I}(A)$ and $dom(f) \subseteq I'$. So, $g' : I' \to A$ defined by $g'(x) = g_k(x)$ if $x \in I_k$ is correctly defined (since if $x \in I_k \cap I_t$ with $k, t \in K$, then by (i), $g_k(x) = g_t(x)$).

Clearly, $g' \in M(I', A)$ and $g'_{|dom(f)} = f$ (since if $x \in dom(f) \subseteq I'$, then $x \in I'$ and so there exists $k \in K$ such that $x \in I_k$, hence $g'(x) = g_k(x) = f(x)$).

So, (I', g') is an upper bound for the family $\{(I_k, g_k) : k \in K\}$, hence by Zorn's lemma, M_f contains at least one maximal strong multiplier h which extends f. Since f is non-principal and h extends f, h is also non-principal. ■

On the Boolean algebra $M_r(A)$ we consider the relation ρ_A defined by

$(f_1, f_2) \in \rho_A$ iff f_1 and f_2 coincide on the intersection of their domains.

LEMMA 6.14. *ρ_A is a congruence on Boolean algebra $M_r(A)$.*

Proof. The reflexivity and the symmetry of ρ_A are immediately; to prove the transitivity of ρ_A let $(f_1, f_2), (f_2, f_3) \in \rho_A$. Therefore f_1, f_2 and respectively f_2, f_3 coincide on the intersection of their domains. If by contrary, there exists $x_0 \in dom(f_1) \cap dom(f_3)$ such that $f_1(x_0) \neq f_3(x_0)$, since $dom(f_2) \in \mathcal{R}(A)$, there exists $e \in dom(f_2) \cap B(A)$ such that $e \wedge f_1(x_0) \neq e \wedge f_3(x_0) \Leftrightarrow f_1(e \odot x_0) \neq f_3(e \odot x_0)$ which is contradictory, since $e \odot x_0 = e \wedge x_0 \in dom(f_1) \cap dom(f_2) \cap dom(f_3)$.

To prove the compatibility of ρ_A with the operations $\wedge, \vee$ and * on $M_r(A)$, let $(f_1, f_2), (g_1, g_2) \in \rho_A$. So, we have f_1, f_2 and respectively g_1, g_2 coincide on the intersection of their domains.

Let $x \in dom(f_1) \cap dom(f_2) \cap dom(g_1) \cap dom(g_2)$. Then $f_1(x) = f_2(x)$ and $g_1(x) = g_2(x)$, hence

$$(f_1 \wedge g_1)(x) = f_1(x) \wedge g_1(x) = f_2(x) \wedge g_2(x) = (f_2 \wedge g_2)(x),$$
$$(f_1 \vee g_1)(x) = f_1(x) \vee g_1(x) = f_2(x) \vee g_2(x) = (f_2 \vee g_2)(x),$$

and

$f_1^*(x) = (f_1 \to \mathbf{0})(x) = x \odot [f_1(x) \to \mathbf{0}(x)] = x \odot [f_2(x) \to \mathbf{0}(x)] = (f_2 \to \mathbf{0})(x) = f_2^*(x)$,

that is the pairs $(f_1 \wedge g_1, f_2 \wedge g_2), (f_1 \vee g_1, f_2 \vee g_2), (f_1^*, f_2^*)$ coincide on the intersection of their domains, hence ρ_A is compatible with the operations $\wedge, \vee$ and *. ■

For $f \in M_r(A)$ with $I = dom(f) \in \mathcal{I}(A) \cap \mathcal{R}(A)$, we denote by $[f, I]$ the congruence class of f modulo ρ_A and $Q(A) = M_r(A)/\rho_A$.

REMARK 6.14. *From Proposition 6.12 we deduce that $Q(A)$ is a Boolean algebra.*

REMARK 6.15. *If we denote by $\mathcal{F} = \mathcal{I}(A) \cap \mathcal{R}(A)$ and consider the partially ordered systems $\{\delta_{I,J}\}_{I,J \in \mathcal{F}, I \subseteq J}$ (where for $I, J \in \mathcal{F}$, $I \subseteq J, \delta_{I,J} : M(J, A) \to M(I, A)$ is defined by $\delta_{I,J}(f) = f_{|I}$), then by above construction of $Q(A)$ we deduce that $Q(A)$ is the inductive limit*

$$Q(A) = \varinjlim_{I \in \mathcal{F}} M(I, A).$$

LEMMA 6.15. *Let the map $\overline{v_A} : B(A) \to Q(A)$ defined by $\overline{v_A}(a) = [\overline{f_a}, A]$ for every $a \in B(A)$. Then*

(i) $\overline{v_A}$ is an injective morphism of Boolean algebras,
(ii) $\overline{v_A}(B(A)) \in \mathcal{R}(Q(A))$.

Proof. (i). Follows from Lemma 6.10.

(ii). To prove $\overline{v_A}(B(A)) \in \mathcal{R}(Q(A))$, if by contrary there exist $f_1, f_2 \in M_r(A)$ such that $[f_1, dom(f_1)] \neq [f_2, dom(f_2)]$ (that is there exists $x_0 \in dom(f_1) \cap dom(f_2)$ such that $f_1(x_0) \neq f_2(x_0)$) and $[f_1, dom(f_1)] \wedge [\overline{f_a}, A] = [f_2, dom(f_2)] \wedge [\overline{f_a}, A]$ for every $[\overline{f_a}, A] \in \overline{v_A}(B(A)) \cap B(Q(A))$ (that is for every $[\overline{f_a}, A] \in \overline{v_A}(B(A))$ with $a \in B(A)$), then $(f_1 \wedge \overline{f_a})(x) = (f_2 \wedge \overline{f_a})(x)$ for every $x \in dom(f_1) \cap dom(f_2)$ and every $a \in B(A) \Leftrightarrow f_1(x) \wedge a \wedge x = f_2(x) \wedge a \wedge x$ for every $x \in dom(f_1) \cap dom(f_2)$ and every $a \in B(A)$. For $a = 1$ and $x = x_0$ we obtain that $f_1(x_0) \wedge x_0 = f_2(x_0) \wedge x_0 \Leftrightarrow f_1(x_0) = f_2(x_0)$ which is contradictory. ■

REMARK 6.16. *Since for every $a \in B(A)$, $\overline{f_a}$ is the unique maximal strong multiplier on $[\overline{f_a}, A]$ (by Lemma 6.13) we can identify $[\overline{f_a}, A]$ with $\overline{f_a}$. So, since $\overline{v_A}$ is injective map, the elements of $B(A)$ can be identified with the elements of the set $\{\overline{f_a} : a \in B(A)\}$.*

LEMMA 6.16. *In view of the identifications made above, if $[f, dom(f)] \in Q(A)$ (with $f \in M_r(A)$ and $I = dom(f) \in \mathcal{I}(A) \cap \mathcal{R}(A)$), then*

$$I \cap B(A) \subseteq \{a \in B(A) : \overline{f_a} \wedge [f, dom(f)] \in B(A)\}.$$

Proof. Let $a \in I \cap B(A)$. Then for every $x \in I, (\overline{f_a} \wedge f)(x) = \overline{f_a}(x) \wedge f(x) = a \wedge x \wedge f(x) = a \wedge f(x) = a \odot f(x) = f(a \odot x) = x \odot f(a)$ (by BL_{16}) $= x \wedge f(a)$, that is $\overline{f_a} \wedge f$ is principal. ■

REMARK 6.17. *The axiom $smBL_4$ is necessary in the proof of Lemma 6.16.*

3. Maximal BL(MV)-algebra of quotients

DEFINITION 6.7. Let A be a $BL(MV)$-algebra. A $BL(MV)$-algebra F is called *$BL(MV)$-algebra of fractions of A* if:

$(BLfr_1)$ $B(A)$ is a $BL(MV)$-subalgebra of F;

$(BLfr_2)$ For every $a', b', c' \in F, a' \neq b'$, there exists $e \in B(A)$ such that $e \wedge a' \neq e \wedge b'$ and $e \wedge c' \in B(A)$.

So, $BL(MV)$-algebra $B(A)$ is a $BL(MV)$-algebra of fractions of itself (since $1 \in B(A)$).

As a notational convenience, we write $A \preceq F$ to indicate that F is a $BL(MV)$-algebra of fractions of A.

DEFINITION 6.8. *A $BL(MV)-$ algebra A_M* is a *maximal $BL(MV)$-algebra of quotients of A* if $A \preceq A_M$ and for every $BL(MV)$-algebra F with $A \preceq F$ there exists a monomorphism of $BL(MV)$-algebras $i : F \to A_M$.

REMARK 6.18. *Let A be a $BL-$ algebra. If $A \preceq F$, then F is a Boolean algebra, hence A_M is a Boolean algebra. Indeed, if by contrary, then there exists $a' \in F$ such that $a' \neq a' \odot a'$ or $(a')^{**} \neq a'$. If $a' \neq a' \odot a'$, since $A \preceq F$, then there exists $e \in B(A)$ such that $e \wedge a' \in B(A)$ and*

$$e \wedge a' \neq e \wedge (a' \odot a') = (e \wedge a') \odot (e \wedge a'),$$

which is contradictory!.

If $(a')^{**} \neq a'$, since $A \preceq F$, then there exists $f \in B(A)$ such that $f \wedge a' \in B(A)$ and

$$f \wedge a' \neq f \wedge (a')^{**} = (f \wedge a')^{**}$$

which is contradictory!.

REMARK 6.19. *If A is a Boolean algebra, then $B(A) = A$. By Remark 6.18, A_M is a Boolean algebra and the axioms $(sm - BL_1) - (sm - BL_4)$ are equivalent with $(sm - BL_1)$, hence A_M is in this case just the classical Dedekind-MacNeille completion of A (see* [**122**]*, p.687). In contrast to the general situation, the Dedekind-MacNeille completion of a Boolean algebra is again distributive and, in fact, is a Boolean algebra* [**2**]*, p.239.*

LEMMA 6.17. *Let A be a $BL-$ algebra, $A \preceq F$; then for every $a', b' \in F, a' \neq b'$, and any finite sequence $c'_1, ..., c'_n \in F$, there exists $e \in B(A)$ such that $e \wedge a' \neq e \wedge b'$ and $e \wedge c'_i \in B(A)$ for $i = 1, 2, ..., n$ $(n \geq 2)$.*

Proof. Assume lemma holds true for $n-1$. So we may find $f \in B(A)$ such that $f \wedge a' \neq f \wedge b'$ and $f \wedge c'_i \in B(A)$ for $i = 1, 2, ..., n-1$. Since $A \preceq F$, we find $g \in B(A)$ such that $g \wedge (f \wedge a') \neq g \wedge (f \wedge b')$ and $g \wedge c'_n \in B(A)$. The element $e = f \wedge g \in B(A)$ has the required properties. ■

LEMMA 6.18. *Let A be a $BL-$ algebra, $A \preceq F$ and $a' \in F$. Then*

$$I_{a'} = \{e \in B(A) : e \wedge a' \in B(A)\} \in \mathcal{I}(B(A)) \cap \mathcal{R}(A).$$

Proof. Clearly, $I_{a'} \in \mathcal{I}(B(A))$.

To prove $I_{a'} \in \mathcal{R}(A)$, let $x, y \in A$ such that $e \wedge x = e \wedge y$ for every $e \in I_{a'} \cap B(A)$. If by contrary, $x \neq y$, since $A \preceq F$, there exists $e_0 \in B(A)$ such that $e_0 \wedge a' \in B(A)$ (that is $e_0 \in I_{a'}$) and $e_0 \wedge x \neq e_0 \wedge y$, which is contradictory. ■

THEOREM 6.19. *Let A be a $BL-$ algebra. $Q(A)$ is a maximal BL-algebra of quotients of A. If $BL-$ algebra A is an $MV-$ algebra, then $Q(A)$ is a maximal MV-algebra of quotients of A.*

Proof. Let A be a $BL-$ algebra. The facts that $B(A)$ is a BL-subalgebra of $Q(A)$ follows from Lemma 6.15, (i). To prove $BLfr_2$, let $[f, dom(f)], [g, dom(g)]$, $[h, dom(h)] \in Q(A)$ with $f, g, h \in M_r(A)$ such that $[g, dom(g)] \neq [h, dom(h)]$ (that is there exists $x_0 \in dom(g) \cap dom(h)$ such that $g(x_0) \neq h(x_0)$).

Put $I = dom(f) \in \mathcal{I}(A) \cap \mathcal{R}(A)$ and

$$I_{[f,dom(f)]} = \{a \in B(A) : \overline{f_a} \wedge [f, dom(f)] \in B(A)\}$$

(by Lemma 6.15, $\overline{f_a} \in B(M(A))$ if $a \in B(A)$). Then by Lemma 6.16,

$$I \cap B(A) \subseteq I_{[f,dom(f)]}.$$

If suppose that for every $a \in I \cap B(A)$, $\overline{f_a} \wedge [g, dom(g)] = \overline{f_a} \wedge [h, dom(h)]$, then $[\overline{f_a} \wedge g, dom(g)] = [\overline{f_a} \wedge h, dom(h)]$, hence for every $x \in dom(g) \cap dom(h)$ we have $(\overline{f_a} \wedge g)(x) = (\overline{f_a} \wedge h)(x)$ i.e. $a \wedge g(x) = a \wedge h(x)$.

Since $I \in \mathcal{R}(A)$ we deduce that $g(x) = h(x)$ for every $x \in dom(g) \cap dom(h)$ so $[g, dom(g)] = [h, dom(h)]$, which is contradictory.

Hence, if $[g, dom(g)] \neq [h, dom(h)]$, then there exists $a \in I \cap B(A)$, such that $\overline{f_a} \wedge [g, dom(g)] \neq \overline{f_a} \wedge [h, dom(h)]$. But for this $a \in I \cap B(A)$ we have

$$\overline{f_a} \wedge [f, dom(f)] \in B(A)$$

(since by Lemma 6.16, $I \cap B(A) \subseteq I_{[f,dom(f)]}$).

To prove the *maximality* of $Q(A)$, let F be a BL-algebra such that $A \preceq F$; thus $B(A) \subseteq B(F)$

$$\begin{array}{ccc} A & \preceq & F \\ & & \swarrow_i \\ & Q(A) & \end{array}$$

For $a' \in F, I_{a'} = \{e \in B(A) : e \wedge a' \in B(A)\} \in \mathcal{I}(B(A)) \cap \mathcal{R}(A)$ (by Lemma 6.18).

Thus $f_{a'} : I_{a'} \to A$ defined by $f_{a'}(x) = x \wedge a'$ is a strong multiplier. Indeed, if $e \in B(A)$ and $x \in I_{a'}$, then

$$f_{a'}(e \odot x) = (e \odot x) \wedge a' = (e \wedge x) \wedge a' = e \wedge (x \wedge a') = e \odot (x \wedge a') = e \odot f_{a'}(x),$$

and

$$f_{a'}(x) \leq x,$$

hence $sm - BL_1$ and $sm - BL_2$ are verified.

To verify $sm - BL_3$, let $e \in I_{a'} \cap B(A) = I_{a'}$. Thus, $f_{a'}(e) = e \wedge a' \in B(A)$ (since $e \in I_{a'}$).

The condition $sm - BL_4$ is obviously verified, hence $[f_{a'}, I_{a'}] \in Q(A)$.

We define $i : F \to Q(A)$ by $i(a') = [f_{a'}, I_{a'}]$, for every $a' \in F$. Clearly $i(0) = \mathbf{0}$.

For $a', b' \in F$ and $x \in I_{a'} \cap I_{b'}$, we have

$$(i(a') \boxdot i(b'))(x) = (a' \wedge x) \odot [x \to (b' \wedge x)] = (a' \odot x) \odot [x \to (b' \wedge x)] = a' \odot [x \odot (x \to (b' \wedge x))]$$

$$= a' \odot [x \wedge (b' \wedge x)] = a' \odot (b' \wedge x) = a' \odot (b' \odot x) = (a' \odot b') \odot x = (a' \odot b') \wedge x = i(a' \odot b')(x),$$

hence $i(a') \boxdot i(b') = i(a' \odot b')$ and

$$(i(a') \to i(b'))(x) = x \odot [i(a')(x) \to i(b')(x)] =$$

$$= x \odot [(a' \wedge x) \to (b' \wedge x)] = x \odot [(x \odot a') \to (x \odot b')] =$$

$$\overset{bl-c_{43}}{=} x \odot (a' \to b') = x \wedge (a' \to b') = i(a' \to b')(x),$$

hence $i(a') \to i(b') = i(a' \to b')$, that is i is a morphism of BL-algebras.

If $BL-$ algebra A is an $MV-$ algebra, then for $a', b' \in F$ and $x \in I_{a'} \cap I_{b'}$, we have

$$(i(a') \boxplus i(b'))(x) = [(a' \wedge x) \oplus (b' \wedge x)] \wedge x \overset{mv-c_{27}}{=} (a' \oplus b') \wedge x = i(a' \oplus b')(x),$$

hence $i(a') \boxplus i(b') = i(a' \oplus b')$.

Also, for $x \in I_{a'}$ we have

$$(i(a'))^*(x) = x \odot [i(a')(x)]^* = x \odot (a' \wedge x)^* = x \odot (a' \odot x)^* =$$
$$= x \odot [x^* \oplus (a')^*] = x \wedge (a')^* = f_{(a')^*}(x) = i((a')^*)(x),$$

hence

$$i((a')^*) = (i(a'))^*,$$

that is i is a morphism of MV-algebras.

To prove the injectivity of i, let $a', b' \in F$ such that $i(a') = i(b')$. It follows that $[f_{a'}, I_{a'}] = [f_{b'}, I_{b'}]$ so $f_{a'}(x) = f_{b'}(x)$ for every $x \in I_{a'} \cap I_{b'}$. We get $a' \wedge x = b' \wedge x$ for every $x \in I_{a'} \cap I_{b'}$. If $a' \neq b'$, by Lemma 6.17 (since $A \preceq F$), there exists $e \in B(A)$ such that $e \wedge a', e \wedge b' \in B(A)$ and $e \wedge a' \neq e \wedge b'$ which is contradictory (since $e \wedge a', e \wedge b' \in B(A)$ implies $e \in I_{a'} \cap I_{b'}$). ■

PROPOSITION 6.20. *Let A be a BL - algebra. Then the following statements are equivalent:*

(*i*) *Every maximal strong multiplier on A has domain A;*
(*ii*) *For every strong multiplier $f \in M(I, A)$ there is $a \in B(A)$ such that $f = f_a$ (that is $f(x) = a \wedge x$ for every $x \in I$);*
(*iii*) $Q(A) \approx B(A)$.

Proof. $(i) \Rightarrow (ii)$. Assume (i) and for $f \in M(I, A)$ let f' its the maximal extension (by Lemma 6.13). By (i), we have $f' : A \to A$. Put $a = f'(1) \in B(A)$ (by $sm - BL_3$), then for every $x \in I$, $f(x) = f(x) \wedge 1 \overset{sm-BL_4}{=} x \wedge f(1) = x \wedge a = f_a(x)$, that is $f = f_a$.

$(ii) \Rightarrow (iii)$. Follow from Lemma 6.15.

$(iii) \Rightarrow (i)$. Follow from Lemma 6.13 and Lemma 6.15.■

DEFINITION 6.9. If A verify one of conditions of Proposition 6.20, we call A *rationnaly complete.*

REMARK 6.20.
1. *If A is a $BL(MV)-$ algebra with $B(A) = \{0, 1\} = L_2$ and $A \preceq F$ then $F = \{0, 1\}$, hence $Q(A) \approx L_2$. Indeed, if $a, b, c \in F$ with $a \neq b$, then by $BLfr_2$ there exists $e \in B(A)$ such $e \wedge a \neq e \wedge b$ (hence $e \neq 0$) and $e \wedge c \in B(A)$. Clearly, $e = 1$, hence $c \in B(A)$, that is $F = B(A)$. As examples of $BL-$ algebras with this property we have local $BL-$ algebras and $BL-$ chains (see* [**99**]*, p.33).*
2. *More general, if A is a $BL(MV)-$ algebra such that $B(A)$ is finite, if $A \preceq F$ then $F = B(A)$, hence $Q(A) = B(A)$. Indeed, consider $a \in F$. $B(A)$ being finite, there exists a largest element $e_a \in B(A)$ such $e_a \wedge a \in B(A)$. Suppose $e_a \vee a \neq e_a$, then there would exists $e \in B(A)$ such that $e \wedge (e_a \vee a) \neq e \wedge e_a$ and $e \wedge a \in B(A)$. But $e \wedge a \in B(A)$ implies $e \leq e_a$ and thus we obtain $e = e \wedge (e_a \vee a) \neq e \wedge e_a = e$, a contradiction. Hence $e_a \vee a = e_a$, so $a \leq e_a$, consequently $a = a \wedge e_a \in B(A)$, that is $F \subseteq B(A)$. Then $F = B(A)$, hence $Q(A) = B(A)$.*

EXAMPLE 6.9. 1. *We consider BL− algebra* $A = \{0, a, b, c, 1\}$ *from Example 3.11, then* $B(A) = \{0, 1\} = L_2, F = \{0, 1\}$, *hence* $Q(A) \approx L_2$.
2. *If* $L_{3\times 2}$ *is MV− algebra from Example 3.12, then* $B(A) = \{0, a, d, 1\}$ *is finite, so* $F = B(A)$ *and* $Q(A) = B(A) = \{0, a, d, 1\}$.

REMARK 6.21. *$B(A)$ is a BL - subalgebra of A and a BL− subalgebra B of A is a Boolean algebra iff $B \subseteq B(A)$. So, in Sections 2 and 3 of this chapter, we can replace the Boolean algebra $B(A)$ with a Boolean subalgebra $B \subseteq B(A)$ and finally we obtain that $Q(A)$ is just $Q(B)$ = the MacNeille completion of B. In particular for $B = B(A)$ we obtain the results of this chapter. This idea will be used in a new paper.*

4. Topologies on a BL(MV)-algebra

Let A be a $BL-$ algebra.

DEFINITION 6.10. A non-empty set $\mathcal{F}$ of elements $I \in \mathcal{I}(A)$ will be called a *topology* on A if the following properties hold:

(top_1) If $I_1 \in \mathcal{F}, I_2 \in \mathcal{I}(A)$ and $I_1 \subseteq I_2$, then $I_2 \in \mathcal{F}$ (hence $A \in \mathcal{F}$);
(top_2) If $I_1, I_2 \in \mathcal{F}$, then $I_1 \cap I_2 \in \mathcal{F}$.

REMARK 6.22. 1. *$\mathcal{F}$ is a topology on A iff $\mathcal{F}$ is a filter of the lattice of power set of A; for this reason a topology on A is usually called a Gabriel filter on $\mathcal{I}(A)$.*
2. *Clearly, if $\mathcal{F}$ is a topology on A, then $(A, \mathcal{F} \cup \{\emptyset\})$ is a topological space.*

Any intersection of topologies on A is a topology; hence the set $T(A)$ of all topologies of A is a complete lattice with respect to inclusion.

EXAMPLE 6.10. *If $I \in \mathcal{I}(A)$, then the set*

$$\mathcal{F}(I) = \{I' \in \mathcal{I}(A) : I \subseteq I'\}$$

is clearly a topology on A.

REMARK 6.23. *If in particular, A is the BL -algebra from Example 3.11 ($A = \{0, c, a, b, 1\}$), then $\mathcal{I}(A) = \{I \subseteq A$: if $x, y \in A, x \leq y$ and $y \in I$, then $x \in I\} = \{I_1, I_2, I_3, I_4, I_5, I_6\}$ where $I_1 = \{0\}$, $I_2 = \{0, c\}$ $I_3 = \{0, c, a\}$ $I_4 = \{0, c, b\}$, $I_5 = \{0, c, a, b\}$ and $I_6 = A$. So, $\mathcal{F}(I_1) = \mathcal{I}(A)$, $\mathcal{F}(I_2) = \{I_2, I_3, I_4, I_5, I_6\}$, $\mathcal{F}(I_3) = \{I_3, I_5, I_6\}$, $\mathcal{F}(I_4) = \{I_4, I_5, I_6\}$, $\mathcal{F}(I_5) = \{I_5, I_6\}$ and $\mathcal{F}(I_6) = \{I_6\}$.*

REMARK 6.24. *In particular, if $L_{3\times 2}$ is the MV -algebra from Example 3.12 ($L_{3\times 2} = \{0, a, b, c, d, 1\}$), then $\mathcal{I}(A) = \{I \subseteq A$:if $x, y \in A, x \leq y$ and $y \in I$, then $x \in I\} = \{I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8, I_9\}$ where $I_1 = \{0\}$, $I_2 = \{0, a\}$ $I_3 = \{0, b\}$ $I_4 = \{0, a, b\}$, $I_5 = \{0, b, d\}, I_6 = \{0, a, b, c\}, I_7 = \{0, a, b, d\}, I_8 = \{0, a, b, c, d\}$ and $I_9 = L_{3\times 2}$. So, $\mathcal{F}(I_1) = \mathcal{I}(L_{3\times 2}), \mathcal{F}(I_2) = \{I_2, I_4, I_6, I_7, I_8, I_9\}, \mathcal{F}(I_3) = \{I_3, I_4, I_5, I_6, I_7, I_8, I_9\}$, $\mathcal{F}(I_4) = \{I_4, I_6, I_7, I_8, I_9\}$, $\mathcal{F}(I_5) = \{I_5, I_7, I_8, I_9\}$, $\mathcal{F}(I_6) = \{I_6, I_8, I_9\}, \mathcal{F}(I_7) = \{I_7, I_8, I_9\}, \mathcal{F}(I_8) = \{I_8, I_9\}$ and $\mathcal{F}(I_9) = \{I_9\}$.*

EXAMPLE 6.11. *If we denote $\mathcal{R}(A) = \{I \subseteq A : I$ is a regular subset of $A\}$, then $\mathcal{I}(A) \cap \mathcal{R}(A)$ is a topology on A.*

REMARK 6.25. *Clearly, if A is the BL -algebra from Example 3.11, since $B(A) = \{0, 1\} = L_2$ then $I_6 = A$ is the only regular subset of A (I_1, I_2, I_3, I_4, I_5 are non regular because contain 0 and for example we have $0 \wedge c = 0 \wedge a$ and $a \neq c$). So, in this case $\mathcal{F} = \mathcal{I}(A) \cap \mathcal{R}(A) = \{A\}$.*

REMARK 6.26. *If $L_{3\times 2}$ is the MV -algebra from Example 3.12 then $I_9 = L_{3\times 2}$ is the only regular subset of $L_{3\times 2}$ ($I_1, I_{2,}I_3, I_4, I_5, I_{6,}I_7, I_8$ are non regular; for example I_2 is non regular because $0 \wedge a = 0 \wedge c, a \wedge a = a \wedge c$ and $a \neq c$). So, in this case $\mathcal{F} = \mathcal{I}(L_{3\times 2}) \cap \mathcal{R}(L_{3\times 2}) = \{L_{3\times 2}\}$.*

EXAMPLE 6.12. *A nonempty set $I \subseteq A$ will be called dense (see [**64**]) if for every $x \in A$ such that $e \wedge x = 0$ for every $e \in I \cap B(A)$, then $x = 0$. If we denote by $D(A)$ the set of all dense subsets of A, then $R(A) \subseteq D(A)$ and $\mathcal{F} = \mathcal{I}(A) \cap D(A)$ is a topology on A.*

REMARK 6.27. *As above, for $BL-$ algebra $A = \{0, c, a, b, 1\}$, from Example 3.11, $D(A) = \{A\}$ (because $I \in D(A)$ if $1 \in I$).*

EXAMPLE 6.13. *For any $\wedge-$ closed subset S of A we set $\mathcal{F}_S = \{I \in \mathcal{I}(A) : I \cap S \cap B(A) \neq \oslash\}$. Then $\mathcal{F}_S$ is a topology on A . Clearly, if $I \in \mathcal{F}_S$ and $I \subseteq J$ (with $J \in \mathcal{I}(A)$), then $I \cap S \cap B(A) \neq \oslash$, hence $J \cap S \cap B(A) \neq \oslash$, that is $J \in \mathcal{F}_S$. If $I_1, I_2 \in \mathcal{F}_S$ then there exist $s_i \in I_i \cap S \cap B(A), i = 1, 2$. If we set $s = s_1 \wedge s_2$, then $s \in (I_1 \cap I_2) \cap S \cap B(A)$, hence $I_1 \cap I_2 \in \mathcal{F}_S$.*

REMARK 6.28. *In the case $A = \{0, c, a, b, 1\}$, from Example 3.11, since $B(A) = \{0, 1\} = L_2$ then for $S \subseteq A$ an $\wedge-$ closed system, $\mathcal{F}_S = \{I \in \mathcal{I}(A) : I \cap S \cap \{0, 1\} \neq \oslash\}$.*

1. *If S is an $\wedge-$closed system of A such that $0 \in S$ (that is $S = A, B(A) = L_2, \{0, c, 1\}, \{0, c, a, 1\}, \{0, c, b, 1\}, \{0, a, 1\}$ and $\{0, b, 1\}$ then: for $S = A, \mathcal{F}_S = \mathcal{I}(A)$; for $S = B(A) = L_2, \mathcal{F}_S = \mathcal{I}(A)$; Also, for $S = \{0, c, 1\}, \{0, c, a, 1\}, \{0, c, b, 1\}, \{0, a, 1\}$ and $\{0, b, 1\}$ we have $I \cap S \cap B(A) = \{0\} \neq \oslash$ for every $I \in \mathcal{I}(A)$, so $\mathcal{F}_S = \mathcal{I}(A)$.*
2. *If $0 \notin S$ (that is $S = \{1\}, \{a, 1\}, \{b, 1\}, \{c, 1\}\}, \{a, c, 1\}, \{b, c, 1\}$ and $\{a, b, c, 1\}$), then $\mathcal{F}_S = \{A\}$ (because, if $I \in \mathcal{I}(A)$ and $1 \in I$ implies $I = A$).*

REMARK 6.29. *If $L_{3\times 2}$ is the MV -algebra from Example 3.12, since $B(L_{3\times 2}) = \{0, a, d, 1\}$ then for $S \subseteq L_{3\times 2}$ an $\wedge-$ closed system, $\mathcal{F}_S = \{I \in \mathcal{I}(L_{3\times 2}) : I \cap S \cap \{0, a, d, 1\} \neq \oslash\}$.*

1. *If S is an $\wedge-$closed system of $L_{3\times 2}$ such that $0 \in S$:*

$$S = L_{3\times 2}, \{0, 1\}, \{0, a, 1\}, \{0, b, 1\}, \{0, c, 1\}, \{0, d, 1\},$$

$$\{0, a, b, 1\}, \{0, a, c, 1\}, \{0, a, d, 1\} = B(L_{3\times 2}),$$

$$\{0, b, c, 1\}, \{0, b, d, 1\}, \{0, a, b, c, 1\}, \{0, a, b, d, 1\}, \{0, b, c, d, 1\}.$$

   then $\mathcal{F}_S = \mathcal{I}(L_{3\times 2})$.
2. *If $0 \notin S$ but $a \in S$ (that is $S = \{a, 1\}, \{a, c, 1\}$) we have $I \cap S \cap \{0, a, d, 1\} = \{a\} \neq \oslash$ so $\mathcal{F}_S = \{I_2, I_4, I_6, I_7, I_8, I_9\}$.*
3. *If $0 \notin S$ but $d \in S$ (that is $S = \{d, 1\}, \{b, c, d, 1\}$) we have $I \cap S \cap \{0, a, d, 1\} = \{d\} \neq \oslash$ so $\mathcal{F}_S = \{I_5, I_7, I_8, I_9\}$.*
4. *If $0, a, d \notin S$ (that is $S = \{1\}, \{b, 1\}, \{c, 1\}, \{b, c, 1\}$) then $\mathcal{F}_S = \{I_9 = L_{3\times 2}\}$.*

5. Localization BL(MV)-algebras

5.1. $\mathcal{F}$-multipliers and localization BL(MV)-algebras. Let A be a $BL-$ algebra and let $\mathcal{F}$ a topology on A. Let us consider the relation $\theta_{\mathcal{F}}$ of A defined in

the following way:

$(x, y) \in \theta_{\mathcal{F}} \Leftrightarrow$ there exists $I \in \mathcal{F}$ such that $e \wedge x = e \wedge y$ for any $e \in I \cap B(A)$.

LEMMA 6.21. *$\theta_{\mathcal{F}}$ is a congruence on A.*

Proof. The reflexivity and the symmetry of $\theta_{\mathcal{F}}$ are immediately; to prove the transitivity of $\theta_{\mathcal{F}}$ let $(x, y), (y, z) \in \theta_{\mathcal{F}}$. Then there exists $I_1, I_2 \in \mathcal{F}$ such that $e \wedge x = e \wedge y$ for every $e \in I_1 \cap B(A)$, and $f \wedge y = f \wedge z$ for every $f \in I_2 \cap B(A)$. If the set $I = I_1 \cap I_2 \in \mathcal{F}$, then for every $g \in I \cap B(A)$, $g \wedge x = g \wedge z$, hence $(x, z) \in \theta_{\mathcal{F}}$.

To prove the compatibility of $\theta_{\mathcal{F}}$ with the operations $\wedge, \vee, \odot$ and $\rightarrow$, let (x, y), $(z, t) \in \theta_{\mathcal{F}}$, that is there exists $I, J \in \mathcal{F}$ such that $e \wedge x = e \wedge y$ for every $e \in I \cap B(A)$ and $f \wedge z = f \wedge t$ for every $f \in J \cap B(A)$. If denote $K = I \cap J$, then $K \in \mathcal{F}$ and for every $g \in K \cap B(A)$, $g \wedge x = g \wedge y$ and $g \wedge z = g \wedge t$.

We obtain

$$g \wedge (x \wedge z) = (g \wedge x) \wedge (g \wedge z) = (g \wedge y) \wedge (g \wedge t) = g \wedge (y \wedge t),$$
$$g \wedge (x \vee z) = (g \wedge x) \vee (g \wedge z) = (g \wedge y) \vee (g \wedge t) = g \wedge (y \vee t),$$

hence $(x \wedge z, y \wedge t), (x \vee z, y \vee t) \in \theta_{\mathcal{F}}$, that is $\theta_{\mathcal{F}}$ is compatible with the operations $\wedge$ and $\vee$.

By $bl - c_{41}$ we deduce that for every $g \in K \cap B(A)$:

$$g \wedge (x \odot z) \overset{bl-c_{41}}{=} (g \wedge x) \odot (g \wedge z) =$$
$$= (g \wedge y) \odot (g \wedge t) \overset{bl-c_{41}}{=} g \wedge (y \odot t),$$

hence $(x \odot z, y \odot t) \in \theta_{\mathcal{F}}$, that is $\theta_{\mathcal{F}}$ is compatible with the operation $\odot$.

Also, by $bl - c_{42}$ we deduce that for every $g \in K \cap B(A)$:

$$g \wedge (x \rightarrow z) = g \odot (x \rightarrow z) \overset{bl-c_{42}}{=} g \odot [(g \odot x) \rightarrow (g \odot z)] =$$
$$= g \odot [(g \wedge x) \rightarrow (g \wedge z)] = g \odot [(g \wedge y) \rightarrow (g \wedge t)] =$$
$$= g \odot [(g \odot y) \rightarrow (g \odot t)] \overset{bl-c_{42}}{=} g \odot (y \rightarrow t) = g \wedge (y \rightarrow t),$$

hence $(x \rightarrow z, y \rightarrow t) \in \theta_{\mathcal{F}}$, that is $\theta_{\mathcal{F}}$ is compatible with the operation $\rightarrow$, so $\theta_{\mathcal{F}}$ is a congruence on A. ■

We shall denote by $x/\theta_{\mathcal{F}}$ the congruence class of an element $x \in A$ and

$$A/\theta_{\mathcal{F}} = \{x/\theta_{\mathcal{F}} : x \in A\}$$

Then, $A/\theta_{\mathcal{F}}$ is a $BL-$ algebra with the natural defined operations and

$$p_{\mathcal{F}} : A \rightarrow A/\theta_{\mathcal{F}}$$

is the canonical onto morphism of BL-algebras.

PROPOSITION 6.22. *For $a \in A$, $a/\theta_{\mathcal{F}} \in B(A/\theta_{\mathcal{F}})$ iff there exists $I \in \mathcal{F}$ such that $a \wedge e \in B(A)$ for every $e \in I \cap B(A)$. So, if $a \in B(A)$, then $a/\theta_{\mathcal{F}} \in B(A/\theta_{\mathcal{F}})$.*

Proof. For $a \in A$, we have $a/\theta_{\mathcal{F}} \in B(A/\theta_{\mathcal{F}}) \Leftrightarrow a/\theta_{\mathcal{F}} \odot a/\theta_{\mathcal{F}} = a/\theta_{\mathcal{F}}$ and $(a/\theta_{\mathcal{F}})^{**} = a/\theta_{\mathcal{F}} \Leftrightarrow (a \odot a)/\theta_{\mathcal{F}} = a/\theta_{\mathcal{F}}$ and $a^{**}/\theta_{\mathcal{F}} = a/\theta_{\mathcal{F}} \Leftrightarrow$ there exists $J, K \in \mathcal{F}$ such that $(a \odot a) \wedge f = a \wedge f$, for every $f \in J \cap B(A)$ and $a^{**} \wedge g = a \wedge g$, for every $g \in K \cap B(A)$.

From $bl - c_{41}$, we deduce that $(a \wedge f) \odot (a \wedge f) = a \wedge f$, for every $f \in J \cap B(A)$.

If denote $I = J \cap K$, then $I \in \mathcal{F}$ and for every $e \in I \cap B(A)$,

$$(a \wedge e) \odot (a \wedge e) = a \wedge e,$$

and

$$(a \wedge e)^{**} \overset{bl-c_{34}}{=} a^{**} \wedge e^{**} = a^{**} \wedge e = a \wedge e,$$

so, $a \wedge e \in B(A)$ for every $e \in I \cap B(A)$.

So, if $a \in B(A)$, then for every $I \in \mathcal{F}$, $a \wedge e \in B(A)$ for every $e \in I \cap B(A)$, hence $a/\theta_{\mathcal{F}} \in B(A/\theta_{\mathcal{F}})$.■

COROLLARY 6.23. *If $\mathcal{F} = \mathcal{I}(A) \cap \mathcal{R}(A)$, then $a \in B(A)$ iff $a/\theta_{\mathcal{F}} \in B(A/\theta_{\mathcal{F}})$.*

DEFINITION 6.11. Let $\mathcal{F}$ be a topology on A. An *$\mathcal{F}-$ partial multiplier* is a mapping $f : I \to A/\theta_{\mathcal{F}}$, where $I \in \mathcal{F}$ and for every $x \in I$ and $e \in B(A)$ the following axioms are fulfilled:

$(m - BL_1)$ $f(e \odot x) = e/\theta_{\mathcal{F}} \wedge f(x) = e/\theta_{\mathcal{F}} \odot f(x)$;
$(m - BL_2)$ $f(x) \leq x/\theta_{\mathcal{F}}$.

By $dom(f) \in \mathcal{F}$ we denote the domain of f; if $dom(f) = A$, we called f *total*.

To simplify language, we will use *$\mathcal{F}-$ multiplier* instead *partial $\mathcal{F}-$ multiplier,* using *total* to indicate that the domain of a certain $\mathcal{F}-$ multiplier is A.

The maps $\mathbf{0}, \mathbf{1} : A \to A/\theta_{\mathcal{F}}$ defined by $\mathbf{0}(x) = 0/\theta_{\mathcal{F}}$ and $\mathbf{1}(x) = x/\theta_{\mathcal{F}}$ for every $x \in A$ are $\mathcal{F}-$ multipliers in the sense of Definition 6.11.

Also for $a \in B(A)$ and $I \in \mathcal{F}$, $f_a : I \to A/\theta_{\mathcal{F}}$ defined by $f_a(x) = a/\theta_{\mathcal{F}} \wedge x/\theta_{\mathcal{F}}$ for every $x \in I$, is an $\mathcal{F}-$ multiplier. If $dom(f_a) = A$, we denote f_a by $\overline{f_a}$; clearly, $\overline{f_0} = \mathbf{0}$.

We shall denote by $M(I, A/\theta_{\mathcal{F}})$ the set of all $\mathcal{F}-$ multipliers having the domain $I \in \mathcal{F}$ and

$$M(A/\theta_{\mathcal{F}}) = \underset{I \in \mathcal{F}}{\cup} M(I, A/\theta_{\mathcal{F}}).$$

If $I_1, I_2 \in \mathcal{F}$, $I_1 \subseteq I_2$ we have a canonical mapping

$$\varphi_{I_1,I_2} : M(I_2, A/\theta_{\mathcal{F}}) \to M(I_1, A/\theta_{\mathcal{F}})$$

defined by

$$\varphi_{I_1,I_2}(f) = f_{|I_1} \text{ for } f \in M(I_2, A/\theta_{\mathcal{F}}).$$

Let us consider the directed system of sets

$$\left\langle \{M(I, A/\theta_{\mathcal{F}})\}_{I \in \mathcal{F}}, \{\varphi_{I_1,I_2}\}_{I_1,I_2 \in \mathcal{F}, I_1 \subseteq I_2} \right\rangle$$

and denote by $A_{\mathcal{F}}$ the inductive limit (in the category of sets):

$$A_{\mathcal{F}} = \varinjlim_{I \in \mathcal{F}} M(I, A/\theta_{\mathcal{F}}).$$

For any $\mathcal{F}-$ multiplier $f : I \to A/\theta_{\mathcal{F}}$ we shall denote by $\widehat{(I, f)}$ the equivalence class of f in $A_{\mathcal{F}}$.

REMARK 6.30. *If $f_i : I_i \to A/\theta_{\mathcal{F}}$, $i = 1, 2$, are $\mathcal{F}-$multipliers, then $\widehat{(I_1, f_1)} = \widehat{(I_2, f_2)}$ (in $A_{\mathcal{F}}$) iff there exists $I \in \mathcal{F}$, $I \subseteq I_1 \cap I_2$ such that $f_{1|I} = f_{2|I}$.*

Let $f_i : I_i \to A/\theta_{\mathcal{F}}$, (with $I_i \in \mathcal{F}$, $i = 1, 2$), $\mathcal{F}-$multipliers. Let us consider the mappings

$$f_1 \wedge f_2 : I_1 \cap I_2 \to A/\theta_{\mathcal{F}}$$
$$f_1 \vee f_2 : I_1 \cap I_2 \to A/\theta_{\mathcal{F}}$$
$$f_1 \boxdot f_2 : I_1 \cap I_2 \to A/\theta_{\mathcal{F}}$$
$$f_1 \to f_2 : I_1 \cap I_2 \to A/\theta_{\mathcal{F}}$$

defined by

$$(f_1 \wedge f_2)(x) = f_1(x) \wedge f_2(x),$$
$$(f_1 \vee f_2)(x) = f_1(x) \vee f_2(x),$$
$$(f_1 \boxdot f_2)(x) = f_1(x) \odot [x/\theta_{\mathcal{F}} \to f_2(x)] \overset{bl-c_{45}}{=} f_2(x) \odot [x/\theta_{\mathcal{F}} \to f_1(x)],$$
$$(f_1 \to f_2)(x) = x/\theta_{\mathcal{F}} \odot [f_1(x) \to f_2(x)]$$

for any $x \in I_1 \cap I_2$, and let

$$\widehat{(I_1, f_1)} \curlywedge \widehat{(I_2, f_2)} = (I_1 \cap \widehat{I_2, f_1} \wedge f_2),$$
$$\widehat{(I_1, f_1)} \curlyvee \widehat{(I_2, f_2)} = (I_1 \cap \widehat{I_2, f_1} \vee f_2),$$
$$\widehat{(I_1, f_1)} \cdot \widehat{(I_2, f_2)} = (I_1 \cap \widehat{I_2, f_1} \boxdot f_2),$$
$$\widehat{(I_1, f_1)} \longmapsto \widehat{(I_2, f_2)} = (I_1 \cap \widehat{I_2, f_1} \to f_2).$$

Clearly the definitions of the operations $\curlywedge, \curlyvee, \cdot$ and $\longmapsto$ on $A_{\mathcal{F}}$ are correct.

LEMMA 6.24. $f_1 \wedge f_2 \in M(I_1 \cap I_2, A/\theta_{\mathcal{F}})$.

Proof. If $x \in I_1 \cap I_2$ and $e \in B(A)$, then

$$(f_1 \wedge f_2)(e \odot x) = f_1(e \odot x) \wedge f_2(e \odot x) =$$
$$= (e/\theta_{\mathcal{F}} \odot f_1(x)) \wedge (e/\theta_{\mathcal{F}} \odot f_2(x)) =$$
$$= (e/\theta_{\mathcal{F}} \wedge f_1(x)) \wedge (e/\theta_{\mathcal{F}} \wedge f_2(x)) =$$
$$= e/\theta_{\mathcal{F}} \wedge [f_1(x) \wedge f_2(x)] = e/\theta_{\mathcal{F}} \odot (f_1 \wedge f_2)(x).$$

Since $f_i \in M(I_i, A/\theta_{\mathcal{F}}), i = 1, 2$, we have $(f_1 \wedge f_2)(x) = f_1(x) \wedge f_2(x) \leq x/\theta_{\mathcal{F}} \wedge x/\theta_{\mathcal{F}} = x/\theta_{\mathcal{F}}$, for every $x \in I_1 \cap I_2$, that is $f_1 \wedge f_2 \in M(I_1 \cap I_2, A/\theta_{\mathcal{F}})$. ■

LEMMA 6.25. $f_1 \vee f_2 \in M(I_1 \cap I_2, A/\theta_{\mathcal{F}})$.

Proof. If $x \in I_1 \cap I_2$ and $e \in B(A)$, then

$$(f_1 \vee f_2)(e \odot x) = f_1(e \odot x) \vee f_2(e \odot x) =$$
$$= (e/\theta_{\mathcal{F}} \odot f_1(x)) \vee (e/\theta_{\mathcal{F}} \odot f_2(x)) \overset{bl-c_{31}}{=}$$
$$\overset{bl-c_{31}}{=} e/\theta_{\mathcal{F}} \odot [f_1(x) \vee f_2(x)] = e/\theta_{\mathcal{F}} \odot (f_1 \vee f_2)(x).$$

Since $f_i \in M(I_i, A/\theta_{\mathcal{F}}), i = 1, 2$, we have $(f_1 \vee f_2)(x) = f_1(x) \vee f_2(x) \leq x/\theta_{\mathcal{F}} \vee x/\theta_{\mathcal{F}} = x/\theta_{\mathcal{F}}$, for every $x \in I_1 \cap I_2$, that is $f_1 \vee f_2 \in M(I_1 \cap I_2, A/\theta_{\mathcal{F}})$. ■

LEMMA 6.26. $f_1 \boxdot f_2 \in M(I_1 \cap I_2, A/\theta_{\mathcal{F}})$.

Proof. If $x \in I_1 \cap I_2$ and $e \in B(A)$, then

$$(f_1 \boxdot f_2)(e \odot x) = f_1(e \odot x) \odot [(e \odot x)/\theta_{\mathcal{F}} \to f_2(e \odot x)] =$$
$$= [e/\theta_{\mathcal{F}} \odot f_1(x)] \odot [(e \odot x)/\theta_{\mathcal{F}} \to (e/\theta_{\mathcal{F}} \odot f_2(x))] =$$
$$= f_1(x) \odot [e/\theta_{\mathcal{F}} \odot ((e \odot x)/\theta_{\mathcal{F}} \to (e/\theta_{\mathcal{F}} \odot f_2(x)))] =$$
$$\overset{bl-c_{42}}{=} f_1(x) \odot [e/\theta_{\mathcal{F}} \odot (x/\theta_{\mathcal{F}} \to f_2(x))] =$$
$$= e/\theta_{\mathcal{F}} \odot [f_1(x) \odot (x/\theta_{\mathcal{F}} \to f_2(x))] = e/\theta_{\mathcal{F}} \odot (f_1 \boxdot f_2)(x).$$

Clearly, $(f_1 \boxdot f_2)(x) = f_1(x) \odot [x/\theta_{\mathcal{F}} \to f_2(x)] \leq f_1(x) \leq x/\theta_{\mathcal{F}}$, for every $x \in I_1 \cap I_2$, that is $f_1 \boxdot f_2 \in M(I_1 \cap I_2, A/\theta_{\mathcal{F}})$. ■

LEMMA 6.27. $f_1 \to f_2 \in M(I_1 \cap I_2, A/\theta_{\mathcal{F}})$.

Proof. If $x \in I_1 \cap I_2$ and $e \in B(A)$, then

$$\begin{aligned}(f_1 \to f_2)(e \odot x) &= (e \odot x)/\theta_{\mathcal{F}} \odot [f_1(e \odot x) \to f_2(e \odot x)] \\ &= (e \odot x)/\theta_{\mathcal{F}} \odot [(e/\theta_{\mathcal{F}} \odot f_1(x)) \to (e/\theta_{\mathcal{F}} \odot f_2(x))] = \\ &= x/\theta_{\mathcal{F}} \odot [e/\theta_{\mathcal{F}} \odot ((e/\theta_{\mathcal{F}} \odot f_1(x)) \to (e/\theta_{\mathcal{F}} \odot f_2(x)))] = \\ &\overset{bl-c_{42}}{=} x/\theta_{\mathcal{F}} \odot [e/\theta_{\mathcal{F}} \odot (f_1(x) \to f_2(x))] = \\ &= e/\theta_{\mathcal{F}} \odot [x/\theta_{\mathcal{F}} \odot (f_1(x) \to f_2(x))] = e/\theta_{\mathcal{F}} \odot (f_1 \to f_2)(x).\end{aligned}$$

Clearly, $(f_1 \to f_2)(x) = x/\theta_{\mathcal{F}} \odot [f_1(x) \to f_2(x)] \leq x/\theta_{\mathcal{F}}$, for every $x \in I_1 \cap I_2$, that is $f_1 \to f_2 \in M(I_1 \cap I_2, A/\theta_{\mathcal{F}})$. ■

PROPOSITION 6.28. *$(M(A/\theta_{\mathcal{F}}), \wedge, \vee, \boxdot, \to, \mathbf{0}, \mathbf{1})$ is a BL-algebra.*

Proof. We verify the axioms of BL-algebras.
BL_1). Obviously $(M(A/\theta_{\mathcal{F}}), \wedge, \vee, \boxdot, \to, \mathbf{0}, \mathbf{1})$ is a bounded lattice.
BL_2). Let $f_i \in M(I_i, A/\theta_{\mathcal{F}})$ where $I_i \in \mathcal{F}$, $i = 1, 2, 3$.
Clearly, $f_1 \boxdot f_2 \in M(A/\theta_{\mathcal{F}})$ (see Lemma 6.26) and

$$\widehat{(I_1, f_1)} \cdot \widehat{(I_2, f_2)} = (I_1 \cap \widehat{I_2, f_1 \boxdot f_2}) \in A_{\mathcal{F}}.$$

Thus, for $x \in I_1 \cap I_2 \cap I_3$ we have

$$\begin{aligned}[f_1 \boxdot (f_2 \boxdot f_3)](x) &= ((f_2 \boxdot f_3)(x)) \odot (x/\theta_{\mathcal{F}} \to f_1(x)) = \\ &= [f_2(x) \odot (x/\theta_{\mathcal{F}} \to f_3(x))] \odot (x/\theta_{\mathcal{F}} \to f_1(x)) = \\ &= f_2(x) \odot [(x/\theta_{\mathcal{F}} \to f_3(x)) \odot (x/\theta_{\mathcal{F}} \to f_1(x))] = \\ &= f_2(x) \odot [(x/\theta_{\mathcal{F}} \to f_1(x)) \odot (x/\theta_{\mathcal{F}} \to f_3(x))] = \\ &= [f_2(x) \odot (x/\theta_{\mathcal{F}} \to f_1(x))] \odot (x/\theta_{\mathcal{F}} \to f_3(x)) = \\ &= ((f_1 \boxdot f_2)(x)) \odot (x/\theta_{\mathcal{F}} \to f_3(x)) = [(f_1 \boxdot f_2) \boxdot f_3](x),\end{aligned}$$

so

$$f_1 \boxdot (f_2 \boxdot f_3) = (f_1 \boxdot f_2) \boxdot f_3$$

and

$$\widehat{(I_1, f_1)} \cdot [\widehat{(I_2, f_2)} \cdot \widehat{(I_3, f_3)}] = [\widehat{(I_1, f_1)} \cdot \widehat{(I_2, f_2)}] \cdot \widehat{(I_3, f_3)},$$

that is the operation $\boxdot$ is associative on $M(A/\theta_{\mathcal{F}})$ and the operation $\cdot$ is associative on $A_{\mathcal{F}}$.

By definition

$$\begin{aligned}(f_1 \boxdot f_2)(x) &= f_1(x) \odot [x/\theta_{\mathcal{F}} \to f_2(x)] \overset{bl-c_{45}}{=} \\ &\overset{bl-c_{45}}{=} f_2(x) \odot [x/\theta_{\mathcal{F}} \to f_1(x)] = (f_2 \boxdot f_1)(x),\end{aligned}$$

so

$$f_1 \boxdot f_2 = f_2 \boxdot f_1$$

and

$$\widehat{(I_1, f_1)} \cdot \widehat{(I_2, f_2)} = \widehat{(I_2, f_2)} \cdot \widehat{(I_1, f_1)},$$

that is the operation $\boxdot$ is commutative on $M(A/\theta_{\mathcal{F}})$ and the operation $\cdot$ is commutative on $A_{\mathcal{F}}$.

Let $f \in M(I, A/\theta_{\mathcal{F}})$ with $I \in \mathcal{F}$. If $x \in I$, then

$$\begin{aligned}(f \boxdot \mathbf{1})(x) &= f(x) \odot (x/\theta_{\mathcal{F}} \to \mathbf{1}(x)) = \\ &= f(x) \odot (x/\theta_{\mathcal{F}} \to x/\theta_{\mathcal{F}}) = f(x) \odot 1/\theta_{\mathcal{F}} = f(x),\end{aligned}$$

and

$$(\mathbf{1} \boxdot f)(x) = \mathbf{1}(x) \odot (x/\theta_{\mathcal{F}} \to f(x)) =$$
$$= x/\theta_{\mathcal{F}} \odot (x/\theta_{\mathcal{F}} \to f(x)) = x/\theta_{\mathcal{F}} \wedge f(x) = f(x),$$

hence

$$f \boxdot \mathbf{1} = \mathbf{1} \boxdot f = f,$$

that is

$$\widehat{(I,f)} \cdot \widehat{(A,\mathbf{1})} = \widehat{(A,\mathbf{1})} \cdot \widehat{(I,f)} = \widehat{(I,f)},$$

and $(M(A/\theta_{\mathcal{F}}), \boxdot, \mathbf{1})$ is a commutative monoid. Clearly, $(A_{\mathcal{F}}, \cdot, \mathbf{1} = \widehat{(A,\mathbf{1})})$ is a commutative monoid.

BL_3). Let $f_i \in M(I_i, A/\theta_{\mathcal{F}})$ where $I_i \in \mathcal{F}$, $i = 1, 2, 3$.

Since $f_1 \leq f_2 \to f_3$ for $x \in I_1 \cap I_2 \cap I_3$ we have

$$f_1(x) \leq (f_2 \to f_3)(x) \Leftrightarrow f_1(x) \leq x/\theta_{\mathcal{F}} \odot [f_2(x) \to f_3(x)].$$

So, by $bl - c_2$,

$$f_1(x) \odot [x/\theta_{\mathcal{F}} \to f_2(x)] \leq x/\theta_{\mathcal{F}} \odot (x/\theta_{\mathcal{F}} \to f_2(x)) \odot (f_2(x) \to f_3(x)) \Leftrightarrow$$
$$f_1(x) \odot [x/\theta_{\mathcal{F}} \to f_2(x)] \leq (x/\theta_{\mathcal{F}} \wedge f_2(x)) \odot (f_2(x) \to f_3(x)) \Leftrightarrow$$
$$f_1(x) \odot [x/\theta_{\mathcal{F}} \to f_2(x)] \leq f_2(x) \odot (f_2(x) \to f_3(x)) \Leftrightarrow$$
$$f_1(x) \odot [x/\theta_{\mathcal{F}} \to f_2(x)] \leq f_2(x) \wedge f_3(x) \leq f_3(x) \Leftrightarrow$$
$$(f_2 \boxdot f_1)(x) \leq f_3(x),$$

for every $x \in I_1 \cap I_2 \cap I_3$, that is

$$f_2 \boxdot f_1 \leq f_3.$$

Conversely if $(f_2 \boxdot f_1)(x) \leq f_3(x)$ we have

$$f_1(x) \odot [x/\theta_{\mathcal{F}} \to f_2(x)] \leq f_3(x),$$

for every $x \in I_1 \cap I_2 \cap I_3$.

Obviously,

$$f_2(x) \odot [x/\theta_{\mathcal{F}} \to f_1(x)] \leq f_1(x) \odot [x/\theta_{\mathcal{F}} \to f_2(x)] \Leftrightarrow$$
$$\Leftrightarrow x/\theta_{\mathcal{F}} \to f_1(x) \leq f_2(x) \to [f_1(x) \odot (x/\theta_{\mathcal{F}} \to f_2(x))].$$

So,

$$x/\theta_{\mathcal{F}} \to f_1(x) \leq f_2(x) \to [f_1(x) \odot (x/\theta_{\mathcal{F}} \to f_2(x))] \overset{bl-c_{11}}{\leq} f_2(x) \to f_3(x)$$
$$\overset{bl-c_2}{\Rightarrow} x/\theta_{\mathcal{F}} \odot (x/\theta_{\mathcal{F}} \to f_1(x)) \leq x/\theta_{\mathcal{F}} \odot (f_2(x) \to f_3(x))$$
$$\Rightarrow x/\theta_{\mathcal{F}} \wedge f_1(x) \leq x/\theta_{\mathcal{F}} \odot (f_2(x) \to f_3(x))$$
$$\Rightarrow f_1(x) \leq (f_2 \to f_3)(x).$$

So, $f_1 \leq f_2 \to f_3$ iff $f_2 \boxdot f_1 \leq f_3$ for all $f_1, f_2, f_3 \in M(A/\theta_{\mathcal{F}})$ and so

$$\widehat{(I_1,f_1)} \leq \widehat{(I_2,f_2)} \longmapsto \widehat{(I_3,f_3)} \text{ iff } \widehat{(I_2,f_2)} \cdot \widehat{(I_1,f_1)} \leq \widehat{(I_3,f_3)}.$$

BL_4). Let $f_i \in M(I_i, A/\theta_{\mathcal{F}})$ where $I_i \in \mathcal{F}$, $i = 1, 2$.

Thus, for $x \in I_1 \cap I_2$ we have

$$[f_1 \boxdot (f_1 \to f_2)](x) = [(f_1 \to f_2)(x)] \odot [x/\theta_{\mathcal{F}} \to f_1(x)]$$
$$= x/\theta_{\mathcal{F}} \odot [f_1(x) \to f_2(x)] \odot [x/\theta_{\mathcal{F}} \to f_1(x)] =$$
$$= (x/\theta_{\mathcal{F}} \odot [x/\theta_{\mathcal{F}} \to f_1(x)]) \odot [f_1(x) \to f_2(x)] =$$
$$= [x/\theta_{\mathcal{F}} \wedge f_1(x)] \odot [f_1(x) \to f_2(x)] = f_1(x) \odot [f_1(x) \to f_2(x)] =$$

$$= f_1(x) \wedge f_2(x) = (f_1 \wedge f_2)(x).$$

So,

$$f_1 \wedge f_2 = f_1 \boxdot (f_1 \to f_2)$$

and

$$\widehat{(I_1, f_1)} \curlywedge \widehat{(I_2, f_2)} = \widehat{(I_1, f_1)} \cdot [\widehat{(I_1, f_1)} \longmapsto \widehat{(I_2, f_2)}].$$

BL_5). We have

$$[(f_1 \to f_2) \vee (f_2 \to f_1)](x) = [(f_1 \to f_2)(x)] \vee [(f_2 \to f_1)(x)] =$$
$$= [x/\theta_{\mathcal{F}} \odot (f_1(x) \to f_2(x))] \vee [x/\theta_{\mathcal{F}} \odot (f_2(x) \to f_1(x))] =$$
$$\overset{bl-c_{31}}{=} x/\theta_{\mathcal{F}} \odot [(f_1(x) \to f_2(x)) \vee (f_2(x) \to f_1(x))] =$$
$$\overset{BL_5}{=} x/\theta_{\mathcal{F}} \odot 1/\theta_{\mathcal{F}} = x/\theta_{\mathcal{F}} = \mathbf{1}(x),$$

hence

$$(f_1 \to f_2) \vee (f_2 \to f_1) = \mathbf{1}$$

and

$$[\widehat{(I_1, f_1)} \longmapsto \widehat{(I_2, f_2)}] \curlyvee [\widehat{(I_2, f_2)} \longmapsto \widehat{(I_1, f_1)}] = \widehat{(A, \mathbf{1})}.\blacksquare$$

COROLLARY 6.29. *$(A_{\mathcal{F}}, \curlywedge, \curlyvee, \cdot, \longmapsto, \mathbf{0} = \widehat{(A, \mathbf{0})}, \mathbf{1} = \widehat{(A, \mathbf{1})})$ is a BL-algebra (see the proof of Proposition 6.28).*

DEFINITION 6.12. The BL-algebra $A_{\mathcal{F}}$ will be called the *localization BL-algebra of A* with respect to the topology $\mathcal{F}$.

PROPOSITION 6.30. *If $BL-$ algebra $(A, \wedge, \vee, \odot, \to, 0, 1)$ is an $MV-$ algebra $(A, \oplus, ^*, 0)$ (i.e. $x^{**} = x$, for all $x \in A$), then $BL-$ algebra $(M(A/\theta_{\mathcal{F}}), \wedge, \vee, \boxdot, \to, \mathbf{0}, \mathbf{1})$ is an $MV-$ algebra $(M(A/\theta_{\mathcal{F}}), \boxplus, ^*, \mathbf{0})$, where for $f_i : I_i \to A/\theta_{\mathcal{F}}$, (with $I_i \in \mathcal{F}$, $i = 1, 2$), $\mathcal{F}-$multipliers we have the mapping*

$$f_1 \boxplus f_2 : I_1 \cap I_2 \to A/\theta_{\mathcal{F}},$$
$$(f_1 \boxplus f_2)(x) = (f_1(x) \oplus f_2(x)) \wedge x/\theta_{\mathcal{F}}$$

for any $x \in I_1 \cap I_2$, and for any $\mathcal{F}-$multiplier $f : I \to A/\theta_{\mathcal{F}}$ (with $I \in \mathcal{F}$) we have the mapping

$$f^* = f \to \mathbf{0} : I \to A/\theta_{\mathcal{F}},$$
$$f^*(x) = (f \to \mathbf{0})(x) = x/\theta_{\mathcal{F}} \odot (f(x))^*$$

for any $x \in I$.

Proof. To prove that $BL-$ algebra $M(A/\theta_{\mathcal{F}})$ is an $MV-$ algebra let $f \in M(I, A/\theta_{\mathcal{F}})$, where $I \in \mathcal{F}$.

Then

$$f^{**}(x) = [(f \to \mathbf{0}) \to \mathbf{0}](x) = x/\theta_{\mathcal{F}} \odot [(f \to \mathbf{0})(x)]^* = x/\theta_{\mathcal{F}} \odot [x/\theta_{\mathcal{F}} \odot (f(x))^*]^*$$
$$= x/\theta_{\mathcal{F}} \odot [x/\theta_{\mathcal{F}} \odot (f(x))^* \to 0/\theta_{\mathcal{F}})] \overset{bl-c_8}{=} x/\theta_{\mathcal{F}} \odot [x/\theta_{\mathcal{F}} \to (f(x))^{**}] =$$
$$= x/\theta_{\mathcal{F}} \wedge (f(x))^{**} = x/\theta_{\mathcal{F}} \wedge f(x) = f(x),$$

(since A is an $MV-$ algebra then $A/\theta_{\mathcal{F}}$ is an $MV-$ algebra and $f(x) \in A/\theta_{\mathcal{F}}$, for all $x \in I$).

So, $f^{**} = f$ and $BL-$ algebra $M(A/\theta_{\mathcal{F}})$ is an MV -algebra.

We have $f_1 \boxplus f_2 = (f_1^* \boxdot f_2^*)^*$.

Clearly,

$$(f_1 \boxplus f_2)(x) = x/\theta_{\mathcal{F}} \odot [f_1^*(x) \odot (x/\theta_{\mathcal{F}} \to f_2^*(x))]^*$$

$$= x/\theta_{\mathcal{F}} \odot [x/\theta_{\mathcal{F}} \odot (f_1(x))^* \odot (x/\theta_{\mathcal{F}} \to x/\theta_{\mathcal{F}} \odot (f_2(x))^*)]^* =$$
$$= x/\theta_{\mathcal{F}} \odot [(f_1(x))^* \odot x/\theta_{\mathcal{F}} \odot (x/\theta_{\mathcal{F}} \to x/\theta_{\mathcal{F}} \odot (f_2(x))^*)]^*$$
$$\overset{BL_4}{=} x/\theta_{\mathcal{F}} \odot [(f_1(x))^* \odot (x/\theta_{\mathcal{F}} \wedge x/\theta_{\mathcal{F}} \odot (f_2(x))^*)]^* = x/\theta_{\mathcal{F}} \odot [(f_1(x))^* \odot x/\theta_{\mathcal{F}} \odot (f_2(x))^*]^*$$
$$= x/\theta_{\mathcal{F}} \odot [x/\theta_{\mathcal{F}} \odot (f_1(x))^* \odot (f_2(x))^*]^* \overset{bl-c_{37}}{=} x/\theta_{\mathcal{F}} \odot [x/\theta_{\mathcal{F}} \to ((f_1(x))^* \odot (f_2(x))^*)^*]$$
$$\overset{BL_4}{=} x/\theta_{\mathcal{F}} \wedge (f_1(x) \oplus f_2(x)),$$

for all $x \in I_1 \cap I_2$.■

COROLLARY 6.31. *If $BL-$ algebra A is an $MV-$ algebra then $BL-$ algebra $(A_{\mathcal{F}}, \curlywedge, \curlyvee, \cdot, \longmapsto, \mathbf{0} = \widehat{(A, \mathbf{0})}, \mathbf{1} = \widehat{(A, \mathbf{1})})$ is an $MV-$ algebra $(A_{\mathcal{F}}, +, ^*, \mathbf{0} = \widehat{(A, \mathbf{0})})$, where*

$$\widehat{(I_1, f_1)} + \widehat{(I_2, f_2)} = \widehat{(I_1 \cap I_2, f_1 \boxplus f_2)}.$$

and

$$\widehat{(I, f)}^* = \widehat{(I, f^*)}.$$

LEMMA 6.32. *If A be a $BL-$ algebra, the map $v_{\mathcal{F}} : B(A) \to A_{\mathcal{F}}$ defined by $v_{\mathcal{F}}(a) = \widehat{(A, \overline{f_a})}$ for every $a \in B(A)$. Then:*

(i) $v_{\mathcal{F}}$ is a morphism of BL-algebras;

(ii) For $a \in B(A)$, $\widehat{(A, \overline{f_a})} \in B(A_{\mathcal{F}})$;

(iii) $v_{\mathcal{F}}(B(A)) \in \mathcal{R}(A_{\mathcal{F}})$.

Proof. *(i)*. We have $v_{\mathcal{F}}(0) = \widehat{(A, \overline{f_0})} = \widehat{(A, \mathbf{0})} = \mathbf{0}$.

For $a, b \in B(A)$ and $x \in A$ we have

$$(a \wedge x) \odot (x \to (b \wedge x)) = (a \odot x) \odot (x \to (b \wedge x)) =$$
$$= a \odot [x \odot (x \to (b \wedge x))] = a \odot [x \wedge (b \wedge x)]$$
$$= a \wedge [x \wedge (b \wedge x)] = a \wedge (b \wedge x) = (a \wedge b) \wedge x = (a \odot b) \wedge x$$

and

$$x \odot [(a \wedge x) \to (b \wedge x)] = x \odot [(x \odot a) \to (x \odot b)] =$$
$$\overset{bl-c_{43}}{=} x \odot (a \to b) = x \wedge (a \to b),$$

hence

$$v_{\mathcal{F}}(a) \cdot v_{\mathcal{F}}(b) = \widehat{(A, \overline{f_a})} \cdot \widehat{(A, \overline{f_b})} =$$
$$= \widehat{(A, \overline{f_a} \boxdot \overline{f_b})} = \widehat{(A, \overline{f_{a \odot b}})} = v_{\mathcal{F}}(a \odot b)$$

and

$$v_{\mathcal{F}}(a) \longmapsto v_{\mathcal{F}}(b) = \widehat{(A, \overline{f_a})} \longmapsto \widehat{(A, \overline{f_b})} =$$
$$= \widehat{(A, \overline{f_a} \to \overline{f_b})} = \widehat{(A, \overline{f_{a \to b}})} = v_{\mathcal{F}}(a \to b)$$

hence $v_{\mathcal{F}}$ is a morphism of BL-algebras.

(ii). For $a \in B(A)$ we have $a \odot a = a$ and $a^{**} = a$, hence

$$(a \wedge x) \odot [x \to (a \wedge x)] = (a \odot x) \odot [x \to (a \wedge x)] =$$
$$= a \odot [x \odot (x \to (a \wedge x))] = a \odot [x \wedge (a \wedge x)] =$$
$$= a \odot (a \wedge x) = a \wedge (a \wedge x) = (a \wedge x),$$

and

$$x \odot [x \odot (a \wedge x)^*]^* \overset{bl-c_{33}}{=} x \odot [x \odot (a^* \vee x^*)]^* =$$

$$\overset{bl-c_{31}}{=} x\odot[(x\odot a^*)\vee(x\odot x^*)]^* =$$
$$\overset{bl-c_5}{=} x\odot[(x\odot a^*)\vee 0]^* = x\odot(x\odot a^*)^* =$$
$$= x\odot(x\wedge a^*)^* \overset{bl-c_{33}}{=} x\odot(x^*\vee a) =$$
$$\overset{bl-c_{31}}{=} (x\odot x^*)\vee(x\odot a) \overset{bl-c_5}{=} 0\vee(x\odot a) = x\odot a = x\wedge a$$

for every $x\in A$.

Since $A\in\mathcal{F}$ we deduce that

$$(a\wedge x)/\theta_{\mathcal{F}}\odot[x/\theta_{\mathcal{F}}\to(a\wedge x)/\theta_{\mathcal{F}}] = (a\wedge x)/\theta_{\mathcal{F}}$$

and

$$x/\theta_{\mathcal{F}}\odot[x/\theta_{\mathcal{F}}\odot((a\wedge x)/\theta_{\mathcal{F}})^*]^* = (a\wedge x)/\theta_{\mathcal{F}},$$

hence $\overline{f_a}\boxdot\overline{f_a} = \overline{f_a}$ and $(\overline{f_a})^{**} = \overline{f_a}$, that is

$$\widehat{(A,\overline{f_a})}\in B(A_{\mathcal{F}}).$$

(iii). To prove that $v_{\mathcal{F}}(B(A))$ is a regular subset of $A_{\mathcal{F}}$, let $\widehat{(I_i,f_i)}\in A_{\mathcal{F}}$, $I_i\in\mathcal{F}$, $i=1,2$, such that $\widehat{(A,\overline{f_a})}\curlywedge\widehat{(I_1,f_1)} = \widehat{(A,\overline{f_a})}\curlywedge\widehat{(I_2,f_2)}$ for every $a\in B(A)$. By (ii), $\widehat{(A,\overline{f_a})}\in B(A_{\mathcal{F}})$.

Then $(f_1\wedge\overline{f_a})(x) = (f_2\wedge\overline{f_a})(x)$ for every $x\in I_1\cap I_2$ and $a\in B(A)\Leftrightarrow f_1(x)\wedge x/\theta_{\mathcal{F}}\wedge a/\theta_{\mathcal{F}} = f_2(x)\wedge x/\theta_{\mathcal{F}}\wedge a/\theta_{\mathcal{F}}$ for every $x\in I_1\cap I_2$ and $a\in B(A)\Leftrightarrow f_1(x)\wedge a/\theta_{\mathcal{F}} = f_2(x)\wedge a/\theta_{\mathcal{F}}$ for every $x\in I_1\cap I_2$ and $a\in B(A)$.

In particular for $a=1$, $a/\theta_{\mathcal{F}} = \mathbf{1}\in B(A/\theta_{\mathcal{F}})$ we obtain that $f_1(x) = f_2(x)$ for every $x\in I_1\cap I_2$, hence $\widehat{(I_1,f_1)} = \widehat{(I_2,f_2)}$, that is $v_{\mathcal{F}}(B(A))\in\mathcal{R}(A_{\mathcal{F}})$. ■

5.2. Strong $\mathcal{F}$-multipliers and strong localization BL(MV)-algebras. To obtain the maximal $BL(MV)$ -algebra of quotients $Q(A)$ as a localization relative to a topology $\mathcal{F}$ we will develope another theory of $\mathcal{F}-$ multipliers (meaning we add new axioms for $\mathcal{F}$-multipliers).

Let A be a $BL-$ algebra.

DEFINITION 6.13. Let $\mathcal{F}$ be a topology on A. A *strong - $\mathcal{F}-$ multiplier* is a mapping $f: I\to A/\theta_{\mathcal{F}}$ (where $I\in\mathcal{F}$) which verifies the axioms $m-BL_1$ and $m-BL_2$ (see Definition 6.11) and

$(m-BL_3)$ If $e\in I\cap B(A)$, then $f(e)\in B(A/\theta_{\mathcal{F}})$;
$(m-BL_4)$ $(x/\theta_{\mathcal{F}})\wedge f(e) = (e/\theta_{\mathcal{F}})\wedge f(x)$, for every $e\in I\cap B(A)$ and $x\in I$.

If $\mathcal{F}=\{A\}$, then $\theta_{\mathcal{F}}$ is the identity congruence of A so an strong $\mathcal{F}-$ multiplier is a strong total multiplier.

REMARK 6.31. *If $(A,\wedge,\vee,\odot,\to,0,1)$ is a $BL-$ algebra, the maps $\mathbf{0},\mathbf{1}: A\to A/\theta_{\mathcal{F}}$ defined by $\mathbf{0}(x) = 0/\theta_{\mathcal{F}}$ and $\mathbf{1}(x) = x/\theta_{\mathcal{F}}$ for every $x\in A$ are strong - $\mathcal{F}-$ multipliers. We recall that if $f_i: I_i\to A/\theta_{\mathcal{F}}$, (with $I_i\in\mathcal{F}$, $i=1,2$) are $\mathcal{F}-$multipliers we consider the mappings $f_1\wedge f_2, f_1\vee f_2, f_1\boxdot f_2, f_1\to f_2: I_1\cap I_2\to A/\theta_{\mathcal{F}}$ defined by*

$$(f_1\wedge f_2)(x) = f_1(x)\wedge f_2(x),$$
$$(f_1\vee f_2)(x) = f_1(x)\vee f_2(x),$$
$$(f_1\boxdot f_2)(x) = f_1(x)\odot[x/\theta_{\mathcal{F}}\to f_2(x)] \overset{bl-c_{45}}{=} f_2(x)\odot[x/\theta_{\mathcal{F}}\to f_1(x)],$$
$$(f_1\to f_2)(x) = x/\theta_{\mathcal{F}}\odot[f_1(x)\to f_2(x)]$$

for any $x \in I_1 \cap I_2$. If f_1, f_2 are strong - $\mathcal{F}-$ multipliers, then the multipliers $f_1 \wedge f_2, f_1 \vee f_2, f_1 \boxdot f_2, f_1 \to f_2$ are also strong - $\mathcal{F}-$ multipliers. Indeed, if $e \in I_1 \cap I_2 \cap B(A)$, then

$$(f_1 \wedge f_2)(e) = f_1(e) \wedge f_2(e) \in B(A/\theta_{\mathcal{F}}),$$
$$(f_1 \vee f_2)(e) = f_1(e) \vee f_2(e) \in B(A/\theta_{\mathcal{F}}).$$

By Remark 3.8 we have

$$(f_1 \boxdot f_2)(e) = f_1(e) \odot [e/\theta_{\mathcal{F}} \to f_2(e)] =$$
$$= f_1(e) \odot ((e/\theta_{\mathcal{F}})^* \vee f_2(e)) \in B(A/\theta_{\mathcal{F}})$$

and

$$(f_1 \to f_2)(e) = e/\theta_{\mathcal{F}} \odot [f_1(e) \to f_2(e)] = e/\theta_{\mathcal{F}} \odot [(f_1(e))^* \vee f_2(e)] \in B(A/\theta_{\mathcal{F}}).$$

For $e \in I_1 \cap I_2 \cap B(A)$ and $x \in I_1 \cap I_2$ we have:

$$x/\theta_{\mathcal{F}} \wedge (f_1 \wedge f_2)(e) = x/\theta_{\mathcal{F}} \wedge f_1(e) \wedge f_2(e) =$$
$$= [x/\theta_{\mathcal{F}} \wedge f_1(e)] \wedge [x/\theta_{\mathcal{F}} \wedge f_2(e)] =$$
$$= [e/\theta_{\mathcal{F}} \wedge f_1(x)] \wedge [e/\theta_{\mathcal{F}} \wedge f_2(x)] = e/\theta_{\mathcal{F}} \wedge (f_1 \wedge f_2)(x),$$

and

$$x/\theta_{\mathcal{F}} \wedge (f_1 \vee f_2)(e) = x/\theta_{\mathcal{F}} \wedge [f_1(e) \vee f_2(e)] =$$
$$= [x/\theta_{\mathcal{F}} \wedge f_1(e)] \vee [x/\theta_{\mathcal{F}} \wedge f_2(e)] =$$
$$= [e/\theta_{\mathcal{F}} \wedge f_1(x)] \vee [e/\theta_{\mathcal{F}} \wedge f_2(x)] =$$
$$= e/\theta_{\mathcal{F}} \wedge [f_1(x) \vee f_2(x)] = e/\theta_{\mathcal{F}} \wedge (f_1 \vee f_2)(x),$$

and

$$x/\theta_{\mathcal{F}} \wedge (f_1 \boxdot f_2)(e) = x/\theta_{\mathcal{F}} \wedge [f_1(e) \odot (e/\theta_{\mathcal{F}} \to f_2(e))]$$
$$= x/\theta_{\mathcal{F}} \odot [f_1(e) \odot (e/\theta_{\mathcal{F}} \to f_2(e))] = f_1(e) \odot [x/\theta_{\mathcal{F}} \odot (e/\theta_{\mathcal{F}} \to f_2(e))]$$
$$\overset{bl-c_{43}}{=} f_1(e) \odot [x/\theta_{\mathcal{F}} \odot ((x \odot e)/\theta_{\mathcal{F}} \to (x/\theta_{\mathcal{F}} \odot f_2(e)))]$$
$$= (f_1(e) \odot x/\theta_{\mathcal{F}}) \odot ((x \odot e)/\theta_{\mathcal{F}} \to (x/\theta_{\mathcal{F}} \odot f_2(e)))$$
$$= (e/\theta_{\mathcal{F}} \odot f_1(x)) \odot ((e \odot x)/\theta_{\mathcal{F}} \to (e/\theta_{\mathcal{F}} \odot f_2(x)))$$
$$= f_1(x) \odot [e/\theta_{\mathcal{F}} \odot ((e/\theta_{\mathcal{F}} \odot x/\theta_{\mathcal{F}}) \to (e/\theta_{\mathcal{F}} \odot f_2(x)))]$$
$$\overset{bl-c_{42}}{=} f_1(x) \odot [e/\theta_{\mathcal{F}} \odot (x/\theta_{\mathcal{F}} \to f_2(x))] =$$
$$= e/\theta_{\mathcal{F}} \odot [f_1(x) \odot (x/\theta_{\mathcal{F}} \to f_2(x))]$$
$$= e/\theta_{\mathcal{F}} \odot (f_1 \boxdot f_2)(x) = e/\theta_{\mathcal{F}} \wedge (f_1 \boxdot f_2)(x),$$

hence

$$x/\theta_{\mathcal{F}} \wedge (f_1 \boxdot f_2)(e) = e/\theta_{\mathcal{F}} \wedge (f_1 \boxdot f_2)(x).$$

Also:

$$e/\theta_{\mathcal{F}} \wedge (f_1 \to f_2)(x) = e/\theta_{\mathcal{F}} \wedge [x/\theta_{\mathcal{F}} \odot (f_1(x) \to f_2(x))]$$
$$= (e \odot x)/\theta_{\mathcal{F}} \odot [f_1(x) \to f_2(x)] = x/\theta_{\mathcal{F}} \odot [e/\theta_{\mathcal{F}} \odot (f_1(x) \to f_2(x))]$$
$$\overset{bl-c_{42}}{=} x/\theta_{\mathcal{F}} \odot [e/\theta_{\mathcal{F}} \odot ((e/\theta_{\mathcal{F}} \odot f_1(x)) \to (e/\theta_{\mathcal{F}} \odot f_2(x)))]$$
$$= x/\theta_{\mathcal{F}} \odot [e/\theta_{\mathcal{F}} \odot ((x/\theta_{\mathcal{F}} \odot f_1(e)) \to (x/\theta_{\mathcal{F}} \odot f_2(e)))] =$$
$$= e/\theta_{\mathcal{F}} \odot [x/\theta_{\mathcal{F}} \odot ((x/\theta_{\mathcal{F}} \odot f_1(e)) \to (x/\theta_{\mathcal{F}} \odot f_2(e)))] =$$
$$\overset{bl-c_{43}}{=} e/\theta_{\mathcal{F}} \odot [x/\theta_{\mathcal{F}} \odot (f_1(e) \to f_2(e))] =$$
$$= x/\theta_{\mathcal{F}} \odot [e/\theta_{\mathcal{F}} \odot (f_1(e) \to f_2(e))] =$$
$$= x/\theta_{\mathcal{F}} \odot (f_1 \to f_2)(e) = x/\theta_{\mathcal{F}} \wedge (f_1 \to f_2)(e)$$

hence

$$x/\theta_{\mathcal{F}} \wedge (f_1 \to f_2)(e) = e/\theta_{\mathcal{F}} \wedge (f_1 \to f_2)(x).$$

If $BL-$algebra $(A, \wedge, \vee, \odot, \to, 0, 1)$ is an $MV-$ algebra $(A, \oplus, ^*, 0)$ we recall that if $f_i : I_i \to A/\theta_{\mathcal{F}}$, (with $I_i \in \mathcal{F}$, $i = 1, 2$), are $\mathcal{F}-$ multipliers, we consider the mapping $f_1 \boxplus f_2 : I_1 \cap I_2 \to A/\theta_{\mathcal{F}}$ defined by

$$(f_1 \boxplus f_2)(x) = (f_1(x) \oplus f_2(x)) \wedge x/\theta_{\mathcal{F}}$$

for any $x \in I_1 \cap I_2$, and for any $\mathcal{F}-$ multiplier $f : I \to A/\theta_{\mathcal{F}}$ (with $I \in \mathcal{F}$) we consider the mapping $f^* : I \to A/\theta_{\mathcal{F}}$ defined by

$$f^*(x) = x/\theta_{\mathcal{F}} \odot (f(x))^*$$

for any $x \in I$. If f_1, f_2 and f are strong - $\mathcal{F}-$ multipliers, then the multipliers $f_1 \boxplus f_2, f^*$ are also strong - $\mathcal{F}-$ multipliers. Indeed, if $e \in I_1 \cap I_2 \cap B(A)$, then

$$(f_1 \boxplus f_2)(e) = [f_1(e) \oplus f_2(e)] \wedge e/\theta_{\mathcal{F}} \in B(A/\theta_{\mathcal{F}}),$$

and if $e \in I \cap B(A)$, then

$$f^*(e) = e/\theta_{\mathcal{F}} \odot [f(e)]^* \in B(A/\theta_{\mathcal{F}}).$$

For $e \in I_1 \cap I_2 \cap B(A)$ and $x \in I_1 \cap I_2$ we have:

$$x/\theta_{\mathcal{F}} \wedge (f_1 \boxplus f_2)(e) = x/\theta_{\mathcal{F}} \wedge [(f_1(e) \oplus f_2(e)) \wedge e/\theta_{\mathcal{F}}] = (f_1(e) \oplus f_2(e)) \wedge x/\theta_{\mathcal{F}} \wedge e/\theta_{\mathcal{F}}$$
$$\overset{mv-c_{30}}{=} (f_1(e) \oplus f_2(e)) \wedge x/\theta_{\mathcal{F}},$$

and

$$e/\theta_{\mathcal{F}} \wedge (f_1 \boxplus f_2)(x) = e/\theta_{\mathcal{F}} \wedge [(f_1(x) \oplus f_2(x)) \wedge x/\theta_{\mathcal{F}}] =$$
$$= e/\theta_{\mathcal{F}} \odot [(f_1(x) \oplus f_2(x)) \wedge x/\theta_{\mathcal{F}}]$$
$$\overset{mv-c_{20}}{=} [e/\theta_{\mathcal{F}} \odot (f_1(x) \oplus f_2(x))] \wedge (e \odot x)/\theta_{\mathcal{F}} \overset{mv-c_{30}}{=}$$
$$\overset{mv-c_{30}}{=} [(e/\theta_{\mathcal{F}} \odot f_1(x)) \oplus (e/\theta_{\mathcal{F}} \odot f_2(x))] \wedge (e \odot x)/\theta_{\mathcal{F}}$$
$$= [x/\theta_{\mathcal{F}} \odot f_1(e) \oplus x/\theta_{\mathcal{F}} \odot f_2(e)] \wedge (e \odot x)/\theta_{\mathcal{F}} =$$
$$= [(f_1(e) \wedge x/\theta_{\mathcal{F}}) \oplus (f_2(e) \wedge x/\theta_{\mathcal{F}})] \wedge (e \wedge x)/\theta_{\mathcal{F}}$$
$$= [[(f_1(e) \wedge x/\theta_{\mathcal{F}}) \oplus (f_2(e) \wedge x/\theta_{\mathcal{F}})] \wedge x/\theta_{\mathcal{F}}] \wedge e/\theta_{\mathcal{F}}$$
$$\overset{mv-c_{27}}{=} ((f_1(e) \oplus f_2(e)) \wedge x/\theta_{\mathcal{F}}) \wedge e/\theta_{\mathcal{F}} \overset{mv-c_{30}}{=} (f_1(e) \oplus f_2(e)) \wedge x/\theta_{\mathcal{F}}$$

hence

$$x/\theta_{\mathcal{F}} \wedge (f_1 \boxplus f_2)(e) = e/\theta_{\mathcal{F}} \wedge (f_1 \boxplus f_2)(x).$$

Since $f \in M(I, A/\theta_{\mathcal{F}})$, for $e \in I \cap B(A)$ and $x \in I$ we have:

$$x/\theta_{\mathcal{F}} \wedge f(e) = e/\theta_{\mathcal{F}} \wedge f(x) \Rightarrow (x/\theta_{\mathcal{F}})^* \vee (f(e))^* = (e/\theta_{\mathcal{F}})^* \vee (f(x))^*$$
$$\Rightarrow (x/\theta_{\mathcal{F}})^* \oplus (f(e))^* = (e/\theta_{\mathcal{F}})^* \oplus (f(x))^*$$
$$\Rightarrow e/\theta_{\mathcal{F}} \odot x/\theta_{\mathcal{F}} \odot [(x/\theta_{\mathcal{F}})^* \oplus (f(e))^*] = x/\theta_{\mathcal{F}} \odot e/\theta_{\mathcal{F}} \odot [(e/\theta_{\mathcal{F}})^* \oplus (f(x))^*] \Rightarrow$$
$$\Rightarrow e/\theta_{\mathcal{F}} \odot [x/\theta_{\mathcal{F}} \wedge (f(e))^*] = x/\theta_{\mathcal{F}} \odot [e/\theta_{\mathcal{F}} \wedge (f(x))^*]$$
$$\Rightarrow e/\theta_{\mathcal{F}} \odot x/\theta_{\mathcal{F}} \odot (f(e))^* = x/\theta_{\mathcal{F}} \odot e/\theta_{\mathcal{F}} \odot (f(x))^*$$
$$\Rightarrow x/\theta_{\mathcal{F}} \odot [e/\theta_{\mathcal{F}} \odot (f(e))^*] = e/\theta_{\mathcal{F}} \odot [x/\theta_{\mathcal{F}} \odot (f(x))^*]$$
$$\Rightarrow x/\theta_{\mathcal{F}} \wedge [e/\theta_{\mathcal{F}} \odot (f(e))^*] = e/\theta_{\mathcal{F}} \wedge [x/\theta_{\mathcal{F}} \odot (f(x))^*] \Rightarrow x/\theta_{\mathcal{F}} \wedge f^*(e) = e/\theta_{\mathcal{F}} \wedge f^*(x).$$

REMARK 6.32. *Analogous as in the case of $\mathcal{F}-$ multipliers if we work with strong-$\mathcal{F}-$ multipliers we obtain a $BL-$ subalgebra of $A_{\mathcal{F}}$ denoted by $s - A_{\mathcal{F}}$ which will be called the strong-localization $BL-$ algebra of A with respect to the topology $\mathcal{F}$.*

6. Applications

If A is a BL -algebra, in the following we describe the localization BL-algebra $A_{\mathcal{F}}$ in some special instances.

1. If $I \in \mathcal{I}(A)$ and $\mathcal{F}$ is the topology

$$\mathcal{F}(I) = \{I' \in \mathcal{I}(A) : I \subseteq I'\}$$

(see Example 6.10), then $A_{\mathcal{F}}$ is isomorphic with $M(I, A/\theta_{\mathcal{F}})$ and $v_{\mathcal{F}} : B(A) \to A_{\mathcal{F}}$ is defined by $v_{\mathcal{F}}(a) = \overline{f_{a|I}}$ for every $a \in B(A)$.

If I is a regular subset of A, then $\theta_{\mathcal{F}}$ is the identity, hence $A_{\mathcal{F}}$ is isomorphic with $M(I, A)$.

If, for example, $I = A = [0, 1]$ (see Example 3.1) then $A_{\mathcal{F}}$ is not a Boolean algebra. We recall that $B(A) = \{0, 1\}$. Indeed if consider $f : [0, 1] \to [0, 1], f(x) = x \wedge x^*$ for every $x \in [0, 1]$, then f is not a Boolean element in $M(I, A)$(For $x = \frac{3}{4}$, then $(f \boxdot f)(\frac{3}{4}) = f(\frac{3}{4}) \odot [\frac{3}{4} \to f(\frac{3}{4})] = (\frac{3}{4} \wedge \frac{1}{4}) \odot [\frac{3}{4} \to (\frac{3}{4} \wedge \frac{1}{4})] = \frac{1}{4} \odot [\frac{3}{4} \to \frac{1}{4}] = \frac{1}{4} \odot \frac{1}{2} = 0 \neq \frac{3}{4} \wedge \frac{1}{4} = \frac{1}{4} = f(\frac{3}{4})$, hence f is not a Boolean element in $M(I, A)$). Also, f is not a principal multiplier. Indeed, if by contrary then there exist $a \in [0, 1]$ such that $x \wedge x^* = a \wedge x$ for every $x \in [0, 1]$ then:

1. if $a = 0$, then for $x = \frac{1}{2}, x^* = \frac{1}{2}$ and $x \wedge x^* = \frac{1}{2} \neq 0 \wedge \frac{1}{2} = 0$,
2. if $a = 1$, then for $x = 1$, $1 \wedge 1^* = 1 \wedge 0 = 0 \neq 1 \wedge 1 = 1$,
3. if $a \in (0, \frac{1}{2})$, then for $x = \frac{1}{2}, x^* = \frac{1}{2}$ and $x \wedge x^* = \frac{1}{2} \neq a \wedge \frac{1}{2} = a$,
4. if $a \in [\frac{1}{2}, 1)$, then for $x = \frac{3}{4}, x^* = \frac{1}{4}$ and $x \wedge x^* = \frac{3}{4} \wedge \frac{1}{4} \neq a \wedge \frac{3}{4}$.

REMARK 6.33. *If consider $BL-$algebra $A = \{0, c, a, b, 1\}$ from Example 3.11, then*

1. *If $I = \{0\}$, then $\mathcal{F}(\{0\}) = \mathcal{I}(A)$ (see Remark 6.23), so $A_{\mathcal{F}} \approx M(I, A/\theta_{\mathcal{F}}) = M(\{0\}, A/\theta_{\mathcal{F}}) = \mathbf{0}$.*
2. *If $I = A$, then $\mathcal{F}(A) = \{A\}$ and $\theta_{\mathcal{F}}$ is the identity, so $A_{\mathcal{F}} \approx M(A, A)$. Since $B(A) = L_2 = \{0, 1\}$, then $f \in M(A, A)$ iff $f(x) \leq x$ for every $x \in A$ (because the condition $sm - BL_1$ is verifyed for $e = 0, 1$). So, $f(0) = 0, f(a) \leq a$ implies $f(a) \in \{0, c, a\}, f(b) \leq b$ implies $f(b) \in \{0, c, b\}, f(c) \leq c$ implies $f(c) \in \{0, c\}$ and $f(1) \leq 1$ implies $f(1) \in \{0, c, a, b, 1\}$. So, if consider $f \in A_{\mathcal{F}} = M(A, A)$ such that $f(a) = c$, then $f^{**}(a) = a \odot [a \odot c^*]^* = a \odot [a \odot 0]^* = a \odot 0^* = a \odot 1 = a \neq c = f(a)$, hence f is not an boolean element in $A_{\mathcal{F}}$ (hence in this case $A_{\mathcal{F}}$ is not a Boolean algebra). Also, f is not a principal multiplier (because $B(A) = \{0, 1\}$ hence the only principal multipliers are $f_0 = \mathbf{0}$ and $f_1 = \mathbf{1}$).*
3. *If for example $I = I_3 = \{0, c, a\}$, $\mathcal{F}(I) = \{I_3, I_5, I_6\}$. Since $0 \in I_3, I_5, I_6$ and $0 \wedge x = 0 \wedge y$, then $(x, y) \in \theta_{\mathcal{F}}$ for every $x, y \in A$, hence in this case $A_{\mathcal{F}} \approx M(I, \mathbf{0}) = 0$. Analogously for $I = I_2, I_4, I_5$.*

REMARK 6.34. *We obtain analogous results if we consider $MV-$ algebra $L_{3\times 2}$ from Example 3.12.*

2. If $\mathcal{F} = \mathcal{I}(A) \cap \mathcal{R}(A)$ is the topology of regular ordered ideals (see Example 6.11), then $\theta_{\mathcal{F}}$ is the identity congruence of A and we obtain the Definition 6.3 for strong multipliers on A, so

$$s - A_{\mathcal{F}} = \varinjlim_{I \in \mathcal{F}} M(I, A),$$

where $M(I,A)$ is the set of multipliers of A having the domain I (in the sense of Definition 6.3).

In this situation we obtain:

PROPOSITION 6.33. *In the case $\mathcal{F}=\mathcal{I}(A)\cap\mathcal{R}(A)$, $s-A_{\mathcal{F}}$ is exactly the maximal BL-algebra $Q(A)$ of quotients of A introduced in* [**33**] *and which is a Boolean algebra.*

REMARK 6.35. *If $BL-$ algebra A is an $MV-$ algebra, $s-A_{\mathcal{F}}$ is exactly the maximal MV-algebra $Q(A)$ of quotients of A introduced in* [**26**].

REMARK 6.36. *If consider in particular BL- algebra $A=\{0,c,a,b,1\}$ from Example 3.11, then $\mathcal{F}=\{A\}$ (see Remark 6.25), hence $s-A_{\mathcal{F}}\approx M(A,A)$. Consider $f\in M(A,A)$. Clearly, $f(0)=0$ and by $sm-BL_3$ we obtain that $f(1)\in\{0,1\}$. If $f(1)=0$, then by $sm-BL_4$ we deduce that for every $x\in A$, $x\wedge f(1)=1\wedge f(x)\Leftrightarrow x\wedge f(1)=f(x)\Leftrightarrow f(x)=0\Leftrightarrow f=\mathbf{0}$. If $f(1)=1, f(x)=x=\mathbf{1}(x)$, hence $f=\mathbf{1}$. So, in this case $s-A_{\mathcal{F}}\approx M(A,A)=L_2$.*

3. Denoting by $\mathcal{D}$ the topology of dense ordered ideals of A (that is $\mathcal{D}=\mathcal{I}(A)\cap D(A)$ - see Example 6.12), then (since $\mathcal{R}(A)\subseteq D(A)$) there exists a morphism of BL -algebras $\alpha: Q(A)\to s-A_{\mathcal{D}}$ such that the diagrame

$$\begin{array}{ccc} B(A) & \xrightarrow{\overline{v_A}} & Q(A) \\ \searrow & & \swarrow \\ {\scriptstyle v_D} & & {\scriptstyle \alpha} \\ & s-A_{\mathcal{D}} & \end{array}$$

is commutative (i.e. $\alpha\circ\overline{v_A}=v_{\mathcal{D}}$). Indeed, if $[f,I]\in Q(A)$ (with $I\in\mathcal{I}(A)\cap\mathcal{R}(A)$ and $f: I\to A$ is a strong multiplier in the sense of Definition 6.3) we denote by $f_{\mathcal{D}}$ the strong - $\mathcal{D}-$multiplier $f_{\mathcal{D}}: I\to A/\theta_{\mathcal{D}}$ defined by $f_{\mathcal{D}}(x)=f(x)/\theta_{\mathcal{D}}$ for every $x\in I$. Thus, α is defined by $\alpha([f,I])=[f_{\mathcal{D}},I]$.

4. Let $S\subseteq A$ an $\wedge-$closed system of $BL(MV)-$algebra A.

PROPOSITION 6.34. *If $\mathcal{F}_S$ is the topology associated with an $\wedge-$closed system $S\subseteq A$ (see Example 6.13), then the $BL(MV)$-algebra $s-A_{\mathcal{F}_S}$ is isomorphic with $B(A[S])$.*

Proof. Let A be a $BL(MV)$-algebra. For $x,y\in A$ we have $(x,y)\in\theta_{\mathcal{F}_S}\Leftrightarrow$ there exists $I\in\mathcal{F}_S$ (hence $I\cap S\cap B(A)\neq\oslash$) such that $x\wedge e=y\wedge e$ for any $e\in I\cap B(A)$. Since $I\cap S\cap B(A)\neq\oslash$ there exists $e_0\in I\cap S\cap B(A)$ such that $x\wedge e_0=y\wedge e_0$, hence $(x,y)\in\theta_S$. So, $\theta_{\mathcal{F}_S}\subseteq\theta_S$.

If $(x,y)\in\theta_S$, there exists $e_0\in S\cap B(A)$ such that $x\wedge e_0=y\wedge e_0$. If we set $I=(e_0]=\{a\in A: a\leq e_0\}$, then $I\in\mathcal{I}(A)$; since $e_0\in I\cap S\cap B(A)$, then $I\cap S\cap B(A)\neq\oslash$, that is $I\in\mathcal{F}_S$. For every $e\in I\cap B(A)$, $e\leq e_0$, hence $e=e\wedge e_0$ and $x\wedge e=x\wedge(e_0\wedge e)=(x\wedge e_0)\wedge e=(y\wedge e_0)\wedge e=y\wedge(e_0\wedge e)=y\wedge e$, hence $(x,y)\in\theta_{\mathcal{F}_S}$, that is $\theta_{\mathcal{F}_S}=\theta_S$.

Then $A[S]=A/\theta_S$; therefore a strong $\mathcal{F}_S-$multiplier can be considered in this case (see $m-BL_1, m-BL_2, m-BL_3, m-BL_4$) as a mapping $f: I\to A[S]$ $(I\in\mathcal{F}_S)$ having the properties $f(e\odot x)=e/S\odot f(x)$ and $f(x)\leq x/S$, for every $x\in I$, and if $e\in I\cap B(A)$, then $f(e)\in B(A[S])$ and for every $e\in I\cap B(A)$ and $x\in I$,

$$(e/S)\wedge f(x)=(x/S)\wedge f(e)$$

(x/S denotes the congruence class of x relative to θ_S).

We recall that for $x \in A$, $x/S \in B(A[S])$ iff there is $e_0 \in S \cap B(A)$ such that $e_0 \wedge x \in B(A)$. In particular if $e \in B(A)$, then $e/S \in B(A[S])$.

If $\widehat{(I_1, f_1)}, \widehat{(I_2, f_2)} \in s - A_{\mathcal{F}_S} = \varinjlim_{I \in \mathcal{F}_S} M(I, A[S])$, and $\widehat{(I_1, f_1)} = \widehat{(I_2, f_2)}$ then there exists $I \in \mathcal{F}_S$ such that $I \subseteq I_1 \cap I_2$ and $f_{1|I} = f_{2|I}$. Since $I, I_1, I_2 \in \mathcal{F}_S$, there exist $e \in I \cap S \cap B(A), e_1 \in I_1 \cap S \cap B(A)$ and $e_2 \in I_2 \cap S \cap B(A)$. We shall prove that $f_1(e_1) = f_2(e_2)$. If denote $f = e \wedge e_1 \wedge e_2$, then $f \in I \cap S \cap B(A)$, and $f \leq e_1, e_2$. Since $e_1 \wedge f = e_2 \wedge f$ then $f_1(e_1 \wedge f) = f_1(e_2 \wedge f) = f_2(e_2 \wedge f) \Leftrightarrow f_1(e_1) \wedge f/S = f_2(e_2) \wedge f/S \Leftrightarrow f_1(e_1) \wedge 1 = f_2(e_2) \wedge 1$ (since $f \in S \Rightarrow f/S = 1$) $\Leftrightarrow f_1(e_1) = f_2(e_2)$. In a similar way can show that $f_1(s_1) = f_2(s_2)$ for any $s_1, s_2 \in I \cap S \cap B(A)$.

In accordance with these considerations we can define the mapping:

$$\alpha : s - A_{\mathcal{F}_S} = \varinjlim_{I \in \mathcal{F}_S} M(I, A[S]) \to B(A[S])$$

by putting

$$\alpha(\widehat{(I, f)}) = f(s) \in B(A[S])$$

where $s \in I \cap S \cap B(A)$.

This mapping is a morphism of BL-algebras, if A is a $BL-$algebra.

Indeed, $\alpha(\mathbf{0}) = \alpha(\widehat{(A, \mathbf{0})}) = \mathbf{0}(e) = 0/S = \mathbf{0}$ for every $e \in S \cap B(A)$. For every $\widehat{(I_i, f_i)} \in s - A_{\mathcal{F}_S}, i = 1, 2$ we have:

$$\begin{gathered}\alpha[\widehat{(I_1, f_1)} \cdot \widehat{(I_2, f_2)}] = \alpha[\widehat{(I_1 \cap I_2, f_1 \boxdot f_2)}.] = \\ = (f_1 \boxdot f_2)(e) = f_1(e) \odot [e/S \to f_2(e)] = \\ = f_1(e) \odot [1 \to f_2(e)] = f_1(e) \odot f_2(e) = \\ = \alpha[\widehat{(I_1, f_1)}] \odot \alpha[\widehat{(I_2, f_2)}]\end{gathered}$$

and

$$\begin{gathered}\alpha[\widehat{(I_1, f_1)} \longmapsto \widehat{(I_2, f_2)}] = \alpha[\widehat{(I_1 \cap I_2, f_1 \to f_2)}.] = \\ = (f_1 \to f_2)(e) = e/S \odot [f_1(e) \to f_2(e)] = \\ = 1 \odot [f_1(e) \to f_2(e)] = f_1(e) \to f_2(e) = \\ = \alpha[\widehat{(I_1, f_1)}] \to \alpha[\widehat{(I_2, f_2)}]\end{gathered}$$

(with $e \in I_1 \cap I_2 \cap S \cap B(A)$).

Clearly, if A is an $MV-$algebra this mapping is a morphism of MV-algebras.

Indeed, $\alpha(\mathbf{0}) = \alpha(\widehat{(A, \mathbf{0})}) = \mathbf{0}(e) = 0/S = \mathbf{0}$ for every $e \in S \cap B(A)$. If $\widehat{(I, f)} \in s - A_{\mathcal{F}_S}$, we have $\alpha(\widehat{(I, f)^*}) = \alpha(\widehat{(I, f^*)}) = f^*(e) = (e/S) \odot [f(e)]^* = 1 \odot (f(e))^* = (f(e))^* = (\alpha(\widehat{(I, f)}))^*$ (with $e \in I \cap S \cap B(A)$). Also, for every $\widehat{(I_i, f_i)} \in s - A_{\mathcal{F}_S}, i = 1, 2$ we have: $\alpha[\widehat{(I_1, f_1)} + \widehat{(I_2, f_2)}] = \alpha[\widehat{(I_1 \cap I_2, f_1 \boxplus f_2)}.] = (f_1 \boxplus f_2)(e) = (f_1(e) \oplus f_2(e)) \wedge (e/S) = f_1(e) \oplus f_2(e) = \alpha[\widehat{(I_1, f_1)}] \oplus \alpha[\widehat{(I_2, f_2)}]$ (with $e \in I_1 \cap I_2 \cap S \cap B(A)$).

We shall prove that α is injective and surjective. To prove the injectivity of α let $\widehat{(I_1, f_1)}, \widehat{(I_2, f_2)} \in s - A_{\mathcal{F}_S}$ such that $\alpha(\widehat{(I_1, f_1)}) = \alpha(\widehat{(I_2, f_2)})$. Then for any $e_1 \in I_1 \cap S \cap B(A)$, $e_2 \in I_2 \cap S \cap B(A)$ we have $f_1(e_1) = f_2(e_2)$. If $f_1(e_1) = x/S, f_2(e_2) = y/S$ with $x, y \in A$, since $x/S = y/S$, there exists $e \in S \cap B(A)$ such that $x \wedge e = y \wedge e$.

If we consider $e' = e \wedge e_1 \wedge e_2 \in I_1 \cap I_2 \cap S \cap B(A)$, we have $x \wedge e' = y \wedge e'$ and $e' \leq e_1, e_2$. It follows that $f_1(e') = f_1(e' \wedge e_1) = f_1(e_1) \wedge (e'/S) = x/S \wedge 1 = x/S = y/S = f_2(e_2) = f_2(e_2) \wedge (e'/S) = f_2(e_2 \wedge e') = f_2(e')$. If denote $I = (e']$ then we

obtained that $I \in \mathcal{F}_S$, $I \subseteq I_1 \cap I_2$ and $f_{1|I} = f_{2|I}$, hence $\widehat{(I_1, f_1)} = \widehat{(I_2, f_2)}$, that is α is injective.

To prove the surjectivity of α, let $a/S \in B(A[S])$ (hence there exists $e_0 \in S \cap B(A)$ such that $a \wedge e_0 \in B(A)$). We consider $I_0 = (e_0] = \{x \in A : x \leq e_0\}$ (since $e_0 \in I_0 \cap S \cap B(A)$, then $I_0 \in \mathcal{F}_S$) and define $f_a : I_0 \to A[S]$ by putting $f_a(x) = x/S \wedge a/S = (x \wedge a)/S$ for every $x \in I_0$.

We shall prove that f_a is an strong $\mathcal{F}_S-$multiplier. Indeed, if $e \in B(A)$ and $x \in I_0$, since $e/S \in B(A[S])$, then

$$f_a(e \odot x) = f_a(e \wedge x) = (e/S) \wedge (x/S) \wedge (a/S)$$
$$= (e/S) \wedge ((x/S) \wedge (a/S)) = (e/S) \wedge f_a(x) = (e/S) \odot f_a(x);$$

Clearly, $f_a(x) \leq x/S$. Also, if $e \in I_0 \cap B(A)$, then $f_a(e) = e/S \wedge a/S \in B(A[S])$.

Clearly if for every $e \in I_0 \cap B(A)$ and $x \in I_0$,

$$(e/S) \wedge f_a(x) = (x/S) \wedge f_a(e),$$

hence f_a is a strong-$\mathcal{F}_S-$multiplier and we shall prove that $\alpha(\widehat{(I_0, f_a)}) = a/S$.

Indeed, since $e_0 \in S$ we have $\alpha(\widehat{(I_0, f_a)}) = f_a(e_0) = (e_0 \wedge a)/S = (e_0/S) \wedge (a/S) = 1 \wedge (a/S) = a/S$.

So, we proved that α is an isomorphism of $BL(MV)$-algebras. ■

REMARK 6.37. *In the proof of Proposition 6.34 the axiom $m - BL_4$ is not necessarily.*

REMARK 6.38. *If A is $BL-$ algebra $A = \{0, c, a, b, 1\}$, since $B(A) = \{0, 1\} = L_2$ then for $S \subseteq A$ an $\wedge-$ closed system, $\mathcal{F}_S = \{I \in \mathcal{I}(A) : I \cap S \cap \{0, 1\} \neq \oslash\}$ and $s - A_{\mathcal{F}_S}$ is isomorphic with $B(A[S])$.*

1. *If S is an $\wedge-$closed systems of A such that $0 \in S$, then $\mathcal{F}_S = \mathcal{I}(A)$ (see Remark 6.28) and $s - A_{\mathcal{F}_S} = A_{\mathcal{I}(A)} \approx B(A[S]) = B(\mathbf{0}) = \mathbf{0}$.*
2. *If $0 \notin S$, $\mathcal{F}_S = A$ (see Remark 6.28) and $s - A_{\mathcal{F}_S} = A_A \approx B(A[S]) = B(A) = \{0, 1\} = L_2$.*

REMARK 6.39. *If $L_{3\times 2}$ is $MV-$ algebra from Example 3.12, since $B(L_{3\times 2}) = \{0, a, d, 1\}$ then for $S \subseteq A$ an $\wedge-$ closed system, $\mathcal{F}_S = \{I \in \mathcal{I}(L_{3\times 2}) : I \cap S \cap \{0, a, d, 1\} \neq \oslash\}$ and $s - (L_{3\times 2})_{\mathcal{F}_S}$ is isomorphic with $B(L_{3\times 2}[S])$.*

1. *If S is an $\wedge-$closed system of $L_{3\times 2}$ such that $0 \in S$, then $\mathcal{F}_S = \mathcal{I}(L_{3\times 2})$ (see Remark 6.29) and $s - (L_{3\times 2})_{\mathcal{F}_S} = (L_{3\times 2})_{\mathcal{I}(L_{3\times 2})} \approx B(L_{3\times 2}[S]) = B(\mathbf{0}) = \mathbf{0}$.*
2. *If $0, a, d \notin S$, $\mathcal{F}_S = L_{3\times 2}$ (see Remark 6.29) and $s-(L_{3\times 2})_{\mathcal{F}_S} \approx B(L_{3\times 2}[S]) = B(L_{3\times 2}) = \{0, a, d, 1\}$.*
3. *If $0 \notin S$ but $a \in S$ then $\mathcal{F}_S = \{I_2, I_4, I_6, I_7, I_8, I_9\}$ (see Remark 6.29) and $s - (L_{3\times 2})_{\mathcal{F}_S} \approx B(L_{3\times 2}[S]) \approx B(L_2) = L_2$.*
4. *If $0 \notin S$ but $d \in S$ then $\mathcal{F}_S = \{I_5, I_7, I_8, I_9\}$ (see Remark 6.29) and $s - (L_{3\times 2})_{\mathcal{F}_S} \approx B(L_{3\times 2}[S]) = \{0/S, 1/S\} \approx L_2$.*

7. Localization of abelian lu-groups

MV- algebras can be studied within the context of abelian lattice-ordered groups with strong units (abelian lu-groups), and this viewpoint plays a crucial role in this section. This viewpoint is made possible by the fundamental result of Mundici (Theorem 2.60) [**105**] that the category of MV-algebras is equivalent to the category of abelian lu-groups ([**3**], [**45**],

[**105**]). The starting point of the above mentioned categorical equivalence is the remark that any interval in the positive cone of an abelian l-group can be endowed with a structure of MV- algebra.

In particular, we take on the task of translating the theory of localization of $MV-$ algebras defined in Sections 5 and 6 into the language of abelian lu-groups. This Section is very much in the spirit of [**3**], in which Ball, Georgescu and Leustean translate the theory of convergence and Cauchy completion of abelian lu-groups into the language of $MV-$ algebras.

PROPOSITION 6.35. *(i) If $H \in \mathcal{I}(G)$, then $\overline{H} = H \cap A \in \mathcal{I}(A)$,*
(ii) If $H \in \mathcal{I}(A)$, then $H_I = \{g \in G : \pi_k(g) \in I$ for all $k \geq 0\}$ is the order ideal of G generated by I in G (that is $H_I \in \mathcal{I}(G)$ and $H_I =< I >_G$). Moreover, $\overline{H_I} = H_I \cap A = I$,
(iii) For every $K \in \mathcal{I}(G), K = H_I$ where $I = K \cap A \in \mathcal{I}(A)$,
(iii) There is a bijective correspondence between $\mathcal{I}(G)$ and $\mathcal{I}(A)$.

Proof. (i). Let $x, y \in A$ such that $x \leq y$ and $y \in \overline{H}$. Thus $y \in H$, hence $x \in H$. Since $x \in A$ we deduce that $x \in H \cap A = \overline{H}$, hence $\overline{H} = H \cap A \in \mathcal{I}(A)$.

(ii). Let $I \in \mathcal{I}(A)$; to prove $H_I \in \mathcal{I}(G)$ let $x, y \in G$ such that $x \leq y$ and $y \in H_I$. By $mv - c_{34}$ we deduce that for every $k \geq 0$, $\pi_k(x) \leq \pi_k(y)$, hence $\pi_k(x) \in I$ for every $k \geq 0$, that is $x \in H_I$. We recall that

$$< I >_G = \bigcap_{H' \in \mathcal{I}(G), I \subseteq H'} H'$$

For $x \in I$ and $k \geq 0$, by $mv - c_{33}$ we deduce that $\pi_k(x) \leq \pi_0(x) = x$, hence $\pi_k(x) \in I$, that is $I \subseteq H_I$. Since $H_I \in \mathcal{I}(G)$ and $I \subseteq H_I$ we deduce that $< I >_G \subseteq H_I$. Let now $x \in H_I, k \geq 0$ and $H' \in \mathcal{I}(G)$ such that $I \subseteq H'$. Thus $\pi_k(x) \in I \subseteq H'$, hence $\pi_k(x) \in H'$. In particular for $k = 0$ we deduce that $x = \pi_0(x) \in H'$, hence $H_I \subseteq H'$. We deduce that $H_I \subseteq \cap H' =< I >_G$, that is $H_I =< I >_G$. To prove the equality $\overline{H_I} = I$ (where $\overline{H_I} = H_I \cap A$) let $x \in H_I \cap A$. Then $x \in H_I$, hence in particular for $k = 0, x = \pi_0(x) \in I$, that is $H_I \cap A \subseteq I$. If $x \in I$ and $k \geq 0$, then by $mv - c_{33}$ we deduce that $\pi_k(x) \leq \pi_0(x) = x$, hence $\pi_k(x) \in I$, so $x \in H_I$, hence $I \subseteq H_I \cap A$ that is $H_I \cap A = I$.

(iii). Follow from (ii).

(iv). Is straightforward by $(i) - (iii)$. ■

DEFINITION 6.14. Let (G, u) be an abelian lu-group. A nonempty set $\mathcal{F}$ of elements $I \in \mathcal{I}(G)$ will be called a *topology* on G (or a *Gabriel filter* on $\mathcal{I}(G)$) if the following properties hold:

(top_1') If $I_1 \in \mathcal{F}, I_2 \in \mathcal{I}(G)$ and $I_1 \subseteq I_2$, then $I_2 \in \mathcal{F}$ (hence $G \in \mathcal{F}$),
(top_2') If $I_1, I_2 \in \mathcal{F}$, then $I_1 \cap I_2 \in \mathcal{F}$.

For an abelian *lu*-group (G, u) we define the *boolean center* $B(G, u)$ of G by

$$B(G, u) = B(A)$$

(where $A = \Gamma(G, u)$). Hence

$$B(G, u) = \{x \in [0, u] : (x + x) \wedge u = x\}.$$

Clearly, $0, u \in B(G)$ and by [**45**], Corollary 7.1.6, we deduce that $B(G, u) \approx B(G_A, u_A) = B(\Xi(A))$.

Clearly, in an lu-group G is possible to have more strong units. So, is necessary to write for example (G,u) to mention that $u\in G$ is a strong unit. Although, if theare is no confusion, to simplify the language, we will use $B(G)$ instead $B(G,u)$ (for example G instead (G,u), $B(G)$ instead $B(G,u)$)

We recall that for every $MV-$ algebra A, $B(A)$ is a subalgebra of A, see Corollary 2.10.

REMARK 6.40. *If A,B are $MV-$ algebras, $\varphi:A\to B$ is an isomorphism of $MV-$ algebras and $\mathcal{F}$ isa topology on A, then $\varphi(\mathcal{F})=\{\varphi(I):I\in\mathcal{F}\}$ is a topology on B and $A_{\mathcal{F}}\approx B_{\varphi(\mathcal{F})}$.*

EXAMPLE 6.14. *If $H\in\mathcal{I}(G)$, then the set*

$$\mathcal{F}(H)=\{H'\in\mathcal{I}(G):H\subseteq H'\}$$

is a topology on G.

EXAMPLE 6.15. *A non-empty set $H\subseteq G$ will be called regular if for every $x,y\in G$ such that $e\wedge x=e\wedge y$ for every $e\in H\cap B(G)$, we have $x=y$. If we denote $\mathcal{R}(G)=\{H\subseteq G:H$ is a regular subset of $G\}$, then $\mathcal{I}(G)\cap\mathcal{R}(G)$ is a topology on G.*

EXAMPLE 6.16. *A subset $S\subseteq G$ is called $\wedge-$ closed if $u\in S$ and if $x,y\in S$ implies $x\wedge y\in S$. For any $\wedge-$ closed subset S of G we set $\mathcal{F}_S=\{H\in\mathcal{I}(G):H\cap S\cap B(G)\neq\emptyset\}$. Then $\mathcal{F}_S$ is a topology on G . Clearly, if $H\in\mathcal{F}_S$ and $H\subseteq H'$ (with $H\in\mathcal{I}(G)$), then $H\cap S\cap B(G)\neq\emptyset$, hence $H'\cap S\cap B(G)\neq\emptyset$, that is $H'\in\mathcal{F}_S$. If $H_1,H_2\in\mathcal{F}_S$ then there exist $s_i\in H_i\cap S\cap B(G), i=1,2$. If we set $s=s_1\wedge s_2$, then $s\in(H_1\cap H_2)\cap S\cap B(G)$, hence $H_1\cap H_2\in\mathcal{F}_S$.*

PROPOSITION 6.36. *Let (G,u) be an abelian lu-group and $A=\Gamma(G,u)=[0,u]$.*

(i) If $\mathcal{F}$ is a topology on G, then $\mathcal{F}_{\mathcal{A}}=\{H\cap A:H\in\mathcal{F}\}$ is a topology on A,

(ii) If $\mathcal{F}$ is a topology on A, then $\mathcal{F}_{\mathcal{G}}=\{H_I:I\in\mathcal{F}\}$ is a topology on G (where H_I is defined by Proposition 6.35, (ii)); if denote $\mathcal{F}_{\mathcal{G}}\cap A=\{H\cap A:H\in\mathcal{F}_{\mathcal{G}}\}$, then $\mathcal{F}_{\mathcal{G}}\cap A=\mathcal{F}_{\mathcal{A}}$,

(iii) There is a bijective correspondence between the topologies on G and the topologies on A.

Proof. (i). Let $\mathcal{F}$ be a topology on $G, H\in\mathcal{F}$ and $K\in\mathcal{I}(A)$ such that $\overline{H}=H\cap A\subseteq K$. By Proposition 6.35, (ii) , there is $H_K\in\mathcal{I}(G)$ such that $K=\overline{H_K}=H_K\cap A$, hence $H\cap A\subseteq H_K\cap A$. We want to prove the inclusion $H\subseteq H_K$. Indeed, if $x\in H$ and $k\geq 0$, then by $mv-c_{33},\pi_k(x)\leq\pi_0(x)=x$, hence $\pi_k(x)\in H$. We deduce that $\pi_k(x)\in H\cap A\subseteq K$, hence $\pi_k(x)\in K$, that is $x\in H_K$. So, $H\subseteq H_K$. Since $H\in\mathcal{F}$ and $\mathcal{F}$ is a topology on G we deduce that $H_K\in\mathcal{F}$, hence $K\in\mathcal{F}_{\mathcal{A}}$.

Clearly, if $H_1,H_2\in\mathcal{F}$, then $\overline{H_1}\cap\overline{H_2}=(H_1\cap A)\cap(H_2\cap A)=(H_1\cap H_2)\cap A\in\mathcal{F}_{\mathcal{A}}$ (since $H_1\cap H_2\in\mathcal{F}$).

(ii). Let $\mathcal{F}$ a topology on $A, I\in\mathcal{F}$ and $K\in\mathcal{I}(G)$ such that $H_I\subseteq K$. Then $I=A\cap H_I\subseteq A\cap K$, hence $A\cap K\in\mathcal{F}$. Since $K=H_{A\cap K}$ (by Proposition 6.35, (iii)) and $A\cap K\in\mathcal{F}$ we deduce that $K\in\mathcal{F}_{\mathcal{G}}$. Let now $I_i\in\mathcal{F}$, $i=1,2$.

Then $H_{I_i}\cap A=I_i$ $(i=1,2)$, hence $(H_{I_1}\cap H_{I_2})\cap A=I_1\cap I_2$, that is $H_{I_1}\cap H_{I_2}=H_{I_1\cap I_2}\in\mathcal{F}_{\mathcal{G}}$ (since $H_1\cap H_2\in\mathcal{F}$).

(iii). Is straightforward by $(i)-(ii)$. ■

In the sequel (G,u) is an abelian *lu*-group, $A=\Gamma(G,u)=[0,u]$ and $\mathcal{F}$ is a topology on G.

Now we are in the situation to define the notion of abelian lu-group of localization of G with respect to the topology $\mathcal{F}$.

By Proposition 6.36, (i), $\mathcal{F}_{\mathcal{A}} = \{H \cap A : H \in \mathcal{F}\}$ is a topology on A. As in Section 5 we can construct the $MV-$ algebra of localization of A with respect to the topology $\mathcal{F}_{\mathcal{A}}$, denoted by $A_{\mathcal{F}_{\mathcal{A}}}$.

DEFINITION 6.15. We denote the abelian lu-group $\Xi(A_{\mathcal{F}_{\mathcal{A}}})$ by $G_{\mathcal{F}}$ and will be called the *localization abelian lu-group of G with respect to the topology $\mathcal{F}$.*

Let now A be an $MV-$ algebra and $\mathcal{F}$ a topology on A. We consider $\Xi(A) = (G_A, u_A)$ and the isomorphism of $MV-$ algebras $\varphi_A : A \to B = [0, u_A] = \Gamma(G_A, u_A)$. By Remark 6.40, $\varphi_A(\mathcal{F}) = \{\varphi_A(I) : I \in \mathcal{F}\}$ is a topology on B and $A_{\mathcal{F}} \approx B_{\varphi_A(\mathcal{F})}$. Then $\Xi(A_{\mathcal{F}}) \approx \Xi(B_{\varphi_A(\mathcal{F})}) = \Xi(A)_{\varphi_A(\mathcal{F})}$ (see Definition 6.15).

So, we obtain:

THEOREM 6.37. *Let A be an $MV-$ algebra and $\mathcal{F}$ a topology on A. Then*

$$\Xi(A)_{\varphi_A(\mathcal{F})} \approx \Xi(A_{\mathcal{F}}).$$

If A is an $MV-$ algebra and $S \subseteq A$ is a $\wedge-$ closed system, then in Section 1 we have defined the notion of $MV-$ *algebra of fraction relative to* S (denoted by $A[S]$). Also in Section 3 we have defined for an $MV-$ algebra A, a *maximal $MV-$ algebra of quotients of* A (denoted by A_M) and we construct the *maximal $MV-$ algebra of quotients of* A, denoted by $Q(A)$.

We shall now define the analogous notions for abelian lu-groups using the functor Ξ.

We continue the running assumption that (G, u) is an abelian *lu*-group with unit interval $A = [0, u]$.

If $S \subseteq G$ is an $\wedge-$ closed system in G (that is $u \in S$ and $x, y \in S$ implies $x \wedge y \in S$), then $\overline{S} = S \cap A$ is an $\wedge-$ closed system in A. So, we consider the $MV-$ algebra of fractions relative to $\overline{S}$ (denoted by $A[\overline{S}]$).

DEFINITION 6.16. We denote the abelian lu-group $\Xi(A[\overline{S}])$ by $G[S]$ and will be called the *abelian lu-group of fraction of G relative to the $\wedge-$ closed system S.* Also, we denote the abelian lu-group $\Xi(Q(A))$ by $Q(G)$ and will be called the *maximal abelian lu-group of quotients of G.*

EXAMPLE 6.17. *For the case of $G[S]$:*

1. *For $G = (Z, +)$ with $u = 1$, and $S = Z$, then $A = \Gamma(Z, 1) = \{0, 1\}$, $\overline{S} = Z \cap \{0, 1\} = \{0, 1\}$ hence $A[\overline{S}] = \mathbf{0}$ (since $0 \in \overline{S}$), so $G[S] = \mathbf{0}$. Analogous for the case of $(Q, +), (R, +)$ with $u = 1$ and $(Z, +)$ with $u = n$, and for the case $S = B(G)$.*
2. *For $G = (R, +)$ with $u = 1$, and $S = \{1\}$, then $A = [0, 1]$, $\overline{S} = S \cap A = \{1\}$, hence $A[\overline{S}] = A$ (see Example 6.1). So $G[S] = \Xi(A[\overline{S}]) = \Xi(A) = \Xi([0, 1]) = Z \times [0, 1]$ (because $[0, 1]$ is chain - see Example 2.21). Analogous for the case of $(Q, +), G[S] = \Xi(A[\overline{S}]) = \Xi(A) = \Xi(Q \cap [0, 1]) = Z \times ([0, 1] \cap Q)$ and for the case of $(Z, +)$ with $u = 1$ we obtain $G[S] = (Z, +)$.*

EXAMPLE 6.18. *For the case of $Q(G)$:*

1. *For $G = (Z, +)$ with $u = 1$, then $A = L_2, Q(G) = \Xi(L_2) = (Z, +)$. Analogous for the case of $(Q, +), (R, +)$ with $u = 1$ and $(Z, +)$ with $u = n$.*

2. *If consider the abelian lu-group* $G = Z \times_{lex} Z$ *with* $u = (1, 0)$, *then* $\Gamma(G, u) = C$ *(see Example 2.6). Since* C *is a chain, then* $B(C) = L_2$, *so* $Q(G) = \Xi(B(C)) = \Xi(L_2) = (Z, +)$ *with* $u = 1$.
3. *If* G *is an abelian lu-group and* $A = [0, u]$ *is such that* $B(A)$ *is finite* ($|B(A)| = 2^n$), *then* $Q(G) = \Xi(Q(A)) = \Xi(B(A)) = Z^n$ *(see Example 2.20).*

As in the case of $MV-$ algebras in the following we describe for an abelian *lu*-group (G, u) the localization abelian *lu*-group $G_{\mathcal{F}}$ in some special instances.

We recall that for the next two examples we work with strong-$\mathcal{F}-$ multipliers (see Definition 6.13).

1. If $\mathcal{F} = \mathcal{I}(G) \cap \mathcal{R}(G)$, then $\mathcal{F}_{\mathcal{A}} = \mathcal{I}(A) \cap \mathcal{R}(A)$ (where we recall that A is the $MV-$ algebra $[0, u]$) and $\mathcal{F}_{\mathcal{A}} = \mathcal{F} \cap A = \{H \cap A : H \in \mathcal{F}\}$. Then $s - A_{\mathcal{F}_{\mathcal{A}}} = Q(A)$ so

$$G_{\mathcal{F}} = \Xi(s - A_{\mathcal{F}_{\mathcal{A}}}) = \Xi(Q(A)) = Q(G)$$

(that is $G_{\mathcal{F}}$ is the maximal abelian *lu*-group of quotients of G).

Since $Q(A)$ is a Boolean algebra, to describe $Q(G) = \Xi(Q(A))$ we can use Example 2.20.

2. If $S \subseteq G$ is an $\wedge-$ closed system of G and $\mathcal{F}_{\mathcal{S}}$ is a topology $\mathcal{F}_{\mathcal{S}} = \{H \in \mathcal{I}(G) : H \cap S \cap B(G) \neq \varnothing\}$, then $\overline{S} = S \cap A$ is an $\wedge-$ closed system of A and $\mathcal{F}_{\overline{\mathcal{S}}} = \{I \in \mathcal{I}(A) : I \cap S \cap B(A) \neq \varnothing\}$ (since $B(G) = B(A)$).

Thus by Proposition 6.34, $s - A_{\mathcal{F}_{\overline{\mathcal{S}}}} \approx B(A[\overline{S}])$, hence

$$G_{\mathcal{F}_{\mathcal{S}}} = \Xi(s - A_{\mathcal{F}_{\overline{\mathcal{S}}}}) \approx \Xi(B(A[\overline{S}])) \approx \Xi(B(\Xi(A[\overline{S}]))) = B(G[S]).$$

CHAPTER 7

Localization of Pseudo MV - algebras

In this chapter, by A we denote a pseudo MV-algebra. We define the localization (strong localization) pseudo MV - algebra of a pseudo MV- algebra A with respect to a topology $\mathcal{F}$ on A. If pseudo MV-algebra A is an MV- algebra we deduce in particular the localization of MV - algebras obtained in Chapter 6.

We introduce the notions of pseudo MV-algebra of fractions relative to an $\wedge$-closed system, pseudo MV-algebra of fractions and maximal pseudo MV-algebra of quotients for a pseudo MV-algebra, taking as a guide-line the case of MV-algebras.

We prove the existence of a maximal pseudo MV- algebra of quotients for a pseudo MV-algebra (Theorem 7.26) and we give explicit descriptions of this pseudo MV-algebra for some classes of pseudo MV-algebras. Also, we prove that the maximal pseudo MV - algebra of quotients Q(A) and the pseudo MV - algebra of fractions relative to an $\wedge-$ closed system are strong pseudo MV - algebra of localization (see Proposition 7.27 and Proposition 7.31).

Following the categorical equivalence between the category of unital l-groups with a strong unit u (lu$-$groups) and the category of pseudo MV- algebras, we define and prove the analogous notions and results for lu-groups. Using this categorical equivalence we take on the task of translating the theory of localization pseudo MV- algebras into the language of localization lu$-$groups.

1. $\mathcal{F}$-multipliers and localization of pseudo MV - algebras

We recall that by $I_d(A)$ we denote the set of all ideals of the lattice $L(A)$ and by $\mathcal{I}(A)$ the set of all order ideals of a pseudo MV-algebra A (see Definition 6.2) :

$$\mathcal{I}(A) = \{I \subseteq A : \text{if } x, y \in A, x \leq y \text{ and } y \in I, \text{then } x \in I\}.$$

Remark 7.1. *Clearly, $I_d(A) \subseteq \mathcal{I}(A)$ and if $I_1, I_2 \in \mathcal{I}(A)$, then $I_1 \cap I_2 \in \mathcal{I}(A)$. Also, if $I \in \mathcal{I}(A)$, then $0 \in I$.*

Let A be a pseudo $MV-$ algebra. A non-empty set $\mathcal{F}$ of elements $I \in \mathcal{I}(A)$ will be called a *topology* on A if verifies the properties of Definition 6.10. $\mathcal{F}$ is a topology on A iff $\mathcal{F}$ is a filter of the lattice of power set of A; for this reason a topology on A is usually called a *Gabriel filter* on $\mathcal{I}(A)$.

Example 7.1. *If $I \in \mathcal{I}(A)$, then the set*

$$\mathcal{F}(I) = \{I' \in \mathcal{I}(A) : I \subseteq I'\}$$

is clearly a topology on A.

Example 7.2. *If we denote $\mathcal{R}(A) = \{I \subseteq A : I$ is a regular subset (see Definition 6.5) of $A\}$ then $\mathcal{I}(A) \cap \mathcal{R}(A)$ is a topology on A.*

Example 7.3. *For any $\wedge-$ closed subset S of A (see Definition 6.1) we set $\mathcal{F}_S = \{I \in \mathcal{I}(A) : I \cap S \cap B(A) \neq \emptyset\}$. Then $\mathcal{F}_S$ is a topology on A.*

Let $\mathcal{F}$ be a topology on a pseudo MV -algebra A. Let us consider the relation $\theta_{\mathcal{F}}$ of A defined in the following way:

$(x,y) \in \theta_{\mathcal{F}} \Leftrightarrow$ there exists $I \in \mathcal{F}$ such that $e \wedge x = e \wedge y$ for any $e \in I \cap B(A)$.

LEMMA 7.1. *$\theta_{\mathcal{F}}$ is a congruence on A.*

Proof. The reflexivity and the symmetry of $\theta_{\mathcal{F}}$ are immediate; in order to prove the transitivity of $\theta_{\mathcal{F}}$ let $(x,y),(y,z) \in \theta_{\mathcal{F}}$. Then there exists $I_1, I_2 \in \mathcal{F}$ such that $e \wedge x = e \wedge y$ for every $e \in I_1 \cap B(A)$, and $f \wedge y = f \wedge z$ for every $f \in I_2 \cap B(A)$. If we set $I = I_1 \cap I_2 \in \mathcal{F}$, then for every $g \in I \cap B(A)$, $g \wedge x = g \wedge z$, hence $(x,z) \in \theta_{\mathcal{F}}$.

To prove the compatibility of $\theta_{\mathcal{F}}$ with the operations $\oplus, ^-$ and $^\sim$, let $(x,y),(z,t) \in \theta_{\mathcal{F}}$, that is there exists $I, J \in \mathcal{F}$ such that $e \wedge x = e \wedge y$ for every $e \in I \cap B(A)$, and $f \wedge z = f \wedge t$ for every $f \in J \cap B(A)$. If we denote $K = I \cap J$, then $K \in \mathcal{F}$ and for every $g \in K \cap B(A)$, $g \wedge x = g \wedge y$ and $g \wedge z = g \wedge t$.

By $psmv - c_{43}$ we deduce that for every $g \in K \cap B(A)$:

$$g \wedge (x \oplus z) = (g \wedge x) \oplus (g \wedge z) = (g \wedge y) \oplus (g \wedge t) = g \wedge (y \oplus t),$$

hence $(x \oplus z, y \oplus t) \in \theta_{\mathcal{F}}$, that is $\theta_{\mathcal{F}}$ is compatible with the operation $\oplus$.

Also, since $e \wedge x = e \wedge y$ for every $e \in I \cap B(A)$ we deduce that $x^- \vee e^- = y^- \vee e^-$, hence

$$e \odot (x^- \vee e^-) = e \odot (y^- \vee e^-) \Leftrightarrow e \odot (e^- \oplus x^-) = e \odot (e^- \oplus y^-)$$

(since $e^- \in B(A)) \Leftrightarrow e \wedge x^- = e \wedge y^-$ for every $e \in I \cap B(A)$, and $x^\sim \vee e^\sim = y^\sim \vee e^\sim \Leftrightarrow (x^\sim \vee e^\sim) \odot e = (y^\sim \vee e^\sim) \odot e \Leftrightarrow (x^\sim \oplus e^\sim) \odot e = (y^\sim \oplus e^\sim) \odot e$ (since $e^\sim \in B(A)) \Leftrightarrow x^\sim \wedge e = y^\sim \wedge e$ for every $e \in I \cap B(A)$, hence $(x^-, y^-),(x^\sim, y^\sim), \in \theta_{\mathcal{F}}$ that is $\theta_{\mathcal{F}}$ is compatible with the operations $^-$ and $^\sim$, so $\theta_{\mathcal{F}}$ is a congruence on A.■

We shall denote by $x/\theta_{\mathcal{F}}$ the congruence class of an element $x \in A$ and by

$$A/\theta_{\mathcal{F}} = \{x/\theta_{\mathcal{F}} : x \in A\}$$

Then, $A/\theta_{\mathcal{F}}$ is a pseudo $MV-$ algebra with the natural defined operations and

$$p_{\mathcal{F}} : A \to A/\theta_{\mathcal{F}}$$

is the canonical onto morphism of pseudo MV-algebras.

PROPOSITION 7.2. *For $a \in A, a/\theta_{\mathcal{F}} \in B(A/\theta_{\mathcal{F}})$ iff there exists $I \in \mathcal{F}$ such that $a \wedge e \in B(A)$ for every $e \in I \cap B(A)$. So, if $a \in B(A)$, then $a/\theta_{\mathcal{F}} \in B(A/\theta_{\mathcal{F}})$.*

Proof. For $a \in A$, we have $a/\theta_{\mathcal{F}} \in B(A/\theta_{\mathcal{F}}) \Leftrightarrow a/\theta_{\mathcal{F}} \oplus a/\theta_{\mathcal{F}} = a/\theta_{\mathcal{F}} \Leftrightarrow (a \oplus a)/\theta_{\mathcal{F}} = a/\theta_{\mathcal{F}} \Leftrightarrow$ there exists $I \in \mathcal{F}$ such that $(a \oplus a) \wedge e = a \wedge e$ for every $e \in I \cap B(A) \overset{psmv-c_{43}}{\Leftrightarrow} (a \wedge e) \oplus (a \wedge e) = a \wedge e$ for every $e \in I \cap B(A) \Leftrightarrow a \wedge e \in B(A)$ for every $e \in I \cap B(A)$. So, if $a \in B(A)$, then for every $I \in \mathcal{F}$, $a \wedge e \in B(A)$ for every $e \in I \cap B(A)$, hence $a/\theta_{\mathcal{F}} \in B(A/\theta_{\mathcal{F}})$.■

COROLLARY 7.3. *If $\mathcal{F} = \mathcal{I}(A) \cap \mathcal{R}(A)$, then $a \in B(A)$ iff $a/\theta_{\mathcal{F}} \in B(A/\theta_{\mathcal{F}})$.*

DEFINITION 7.1. Let A be a pseudo $MV-$ algebra and $\mathcal{F}$ be a topology on A. An *partial $\mathcal{F}-$ multiplier* is a mapping $f : I \to A/\theta_{\mathcal{F}}$, where $I \in \mathcal{F}$ and for every $x \in I$ and $e \in B(A)$ the following axioms are fulfilled:

$(m - psMV_1)$ $f(e \odot x) = e/\theta_{\mathcal{F}} \wedge f(x) = e/\theta_{\mathcal{F}} \odot f(x)$;
$(m - psMV_2)$ $f(x) \leq x/\theta_{\mathcal{F}}$.

By $dom(f) \in \mathcal{F}$ we denote the domain of f; if $dom(f) = A$, we called f *total*.

To simplify language, we will use $\mathcal{F}-$ *multiplier* instead *partial* $\mathcal{F}-$ *multiplier,* using *total* to indicate that the domain of a certain multiplier is A.

The maps $\mathbf{0}, \mathbf{1} : A \to A/\theta_{\mathcal{F}}$ defined by $\mathbf{0}(x) = 0/\theta_{\mathcal{F}}$ and $\mathbf{1}(x) = x/\theta_{\mathcal{F}}$ for every $x \in A$ are $\mathcal{F}$-multipliers in the sense of Definition 7.1.

Also, for $a \in B(A)$ and $I \in \mathcal{F}$, $f_a : I \to A/\theta_{\mathcal{F}}$ defined by $f_a(x) = a/\theta_{\mathcal{F}} \wedge x/\theta_{\mathcal{F}}$ for every $x \in I$, is an $\mathcal{F}-$ multiplier. If $dom(f_a) = A$, we denote f_a by $\overline{f_a}$; clearly, $\overline{f_0} = \mathbf{0}$.

We shall denote by $M(I, A/\theta_{\mathcal{F}})$ the set of all the $\mathcal{F}-$ multipliers having the domain $I \in \mathcal{F}$ and

$$M(A/\theta_{\mathcal{F}}) = \underset{I \in \mathcal{F}}{\cup} M(I, A/\theta_{\mathcal{F}}).$$

If $I_1, I_2 \in \mathcal{F}$, $I_1 \subseteq I_2$ we have a canonical mapping

$$\varphi_{I_1,I_2} : M(I_2, A/\theta_{\mathcal{F}}) \to M(I_1, A/\theta_{\mathcal{F}})$$

defined by

$$\varphi_{I_1,I_2}(f) = f_{|I_1} \text{ for } f \in M(I_2, A/\theta_{\mathcal{F}}).$$

Let us consider the directed system of sets

$$\left\langle \{M(I, A/\theta_{\mathcal{F}})\}_{I \in \mathcal{F}}, \{\varphi_{I_1,I_2}\}_{I_1,I_2 \in \mathcal{F}, I_1 \subseteq I_2} \right\rangle$$

and denote by $A_{\mathcal{F}}$ the inductive limit (in the category of sets):

$$A_{\mathcal{F}} = \varinjlim_{I \in \mathcal{F}} M(I, A/\theta_{\mathcal{F}}).$$

For any $\mathcal{F}-$ multiplier $f : I \to A/\theta_{\mathcal{F}}$ we shall denote by $\widehat{(I, f)}$ the equivalence class of f in $A_{\mathcal{F}}$.

Remark 7.2. *If $f_i : I_i \to A/\theta_{\mathcal{F}}$, $i = 1, 2$, are $\mathcal{F}-$ multipliers, then $\widehat{(I_1, f_1)} = \widehat{(I_2, f_2)}$ (in $A_{\mathcal{F}}$) iff there exists $I \in \mathcal{F}$, $I \subseteq I_1 \cap I_2$ such that $f_{1|I} = f_{2|I}$.*

Let $f_i : I_i \to A/\theta_{\mathcal{F}}$, (with $I_i \in \mathcal{F}$, $i = 1, 2$), $\mathcal{F}-$multipliers. Let us consider the mappings

$$f_1 \boxplus f_2 : I_1 \cap I_2 \to A/\theta_{\mathcal{F}}$$

defined by

$$(f_1 \boxplus f_2)(x) = (f_1(x) \oplus f_2(x)) \wedge x/\theta_{\mathcal{F}}$$

for any $x \in I_1 \cap I_2$, and let $\widehat{(I_1, f_1)} + \widehat{(I_2, f_2)} = (I_1 \cap \widehat{I_2, f_1} \boxplus f_2)$.

Also, for any $\mathcal{F}-$ multiplier $f : I \to A/\theta_{\mathcal{F}}$ (with $I \in \mathcal{F}$) let us consider the mapping

$$f^-, f^\sim : I \to A/\theta_{\mathcal{F}}$$

defined by

$$f^-(x) = x/\theta_{\mathcal{F}} \odot (f(x))^-$$

and

$$f^\sim(x) = (f(x))^\sim \odot x/\theta_{\mathcal{F}}$$

for any $x \in I$ and let $\widehat{(I, f)}^- = \widehat{(I, f^-)}$ respectively $\widehat{(I, f)}^\sim = \widehat{(I, f^\sim)}$.

Clearly the definitions of the operations $+$, $^-$ and $^\sim$ on $A_{\mathcal{F}}$ are correct.

Lemma 7.4. $f_1 \boxplus f_2 \in M(I_1 \cap I_2, A/\theta_{\mathcal{F}})$.

Proof. If $x \in I_1 \cap I_2$ and $e \in B(A)$, then

$$(f_1 \boxplus f_2)(e \odot x) = [f_1(e \odot x) \oplus f_2(e \odot x)] \wedge (e \odot x)/\theta_{\mathcal{F}} =$$
$$= [(e/\theta_{\mathcal{F}} \odot f_1(x)) \oplus (e/\theta_{\mathcal{F}} \odot f_2(x))] \wedge (e/\theta_{\mathcal{F}} \odot x/\theta_{\mathcal{F}}) =$$
$$= [(e/\theta_{\mathcal{F}} \wedge f_1(x)) \oplus (e/\theta_{\mathcal{F}} \wedge f_2(x))] \wedge (e/\theta_{\mathcal{F}} \wedge x/\theta_{\mathcal{F}}) \overset{psmv-c_{43}}{=}$$
$$\overset{psmv-c_{43}}{=} [e/\theta_{\mathcal{F}} \wedge (f_1(x) \oplus f_2(x))] \wedge (e/\theta_{\mathcal{F}} \wedge x/\theta_{\mathcal{F}}) =$$
$$= e/\theta_{\mathcal{F}} \wedge [(f_1(x) \oplus f_2(x)) \wedge x/\theta_{\mathcal{F}}] = e/\theta_{\mathcal{F}} \odot (f_1 \boxplus f_2)(x).$$

Clearly, $(f_1 \boxplus f_2)(x) \leq x/\theta_{\mathcal{F}}$ for every $x \in I_1 \cap I_2$, that is, $f_1 \boxplus f_2 \in M(I_1 \cap I_2, A/\theta_{\mathcal{F}})$. ■

LEMMA 7.5. $f^-, f^\sim \in M(I, A/\theta_{\mathcal{F}})$.

Proof. If $x \in I$ and $e \in B(A)$, then

$$f^-(e \odot x) = (e \odot x)/\theta_{\mathcal{F}} \odot (f(e \odot x))^- = e/\theta_{\mathcal{F}} \odot x/\theta_{\mathcal{F}} \odot (e/\theta_{\mathcal{F}} \odot f(x))^- =$$
$$= e/\theta_{\mathcal{F}} \odot x/\theta_{\mathcal{F}} \odot [(e/\theta_{\mathcal{F}})^- \oplus (f(x))^-] = x/\theta_{\mathcal{F}} \odot (e/\theta_{\mathcal{F}} \odot ((e/\theta_{\mathcal{F}})^- \oplus (f(x))^-)) =$$
$$= x/\theta_{\mathcal{F}} \odot (e/\theta_{\mathcal{F}} \wedge (f(x))^-) = x/\theta_{\mathcal{F}} \odot (e/\theta_{\mathcal{F}} \odot (f(x))^-) =$$
$$= e/\theta_{\mathcal{F}} \odot (x/\theta_{\mathcal{F}} \odot (f(x))^-) = e/\theta_{\mathcal{F}} \odot f^-(x)$$

and

$$f^\sim(e \odot x) = (f(e \odot x))^\sim \odot (e \odot x)/\theta_{\mathcal{F}} = (e/\theta_{\mathcal{F}} \odot f(x))^\sim \odot (e \odot x)/\theta_{\mathcal{F}} =$$
$$= ((f(x))^\sim \oplus (e/\theta_{\mathcal{F}})^\sim) \odot e/\theta_{\mathcal{F}} \odot x/\theta_{\mathcal{F}} = ((f(x))^\sim \wedge e/\theta_{\mathcal{F}}) \odot x/\theta_{\mathcal{F}} =$$
$$((f(x))^\sim \odot e/\theta_{\mathcal{F}}) \odot x/\theta_{\mathcal{F}} = ((f(x))^\sim \odot x/\theta_{\mathcal{F}}) \odot e/\theta_{\mathcal{F}} =$$
$$= f^\sim(x) \odot e/\theta_{\mathcal{F}} = e/\theta_{\mathcal{F}} \odot f^\sim(x)$$

Clearly, $f^-(x) \leq x/\theta_{\mathcal{F}}$ and $f^\sim(x) \leq x/\theta_{\mathcal{F}}$ for every $x \in I$, that is $f^-, f^\sim \in M(I, A/\theta_{\mathcal{F}})$.■

LEMMA 7.6. *Let $f, g \in M(A/\theta_{\mathcal{F}})$ with $f \in M(I, A/\theta_{\mathcal{F}})$ and $g \in M(J, A/\theta_{\mathcal{F}})$, $I, J \in \mathcal{F}$. Then for every $x \in I \cap J$:*

$$(f \boxdot g)(x) = (f(x) \oplus (x/\theta_{\mathcal{F}})^\sim) \odot g(x) = f(x) \odot ((x/\theta_{\mathcal{F}})^- \oplus g(x)).$$

Proof. For $x \in I \cap J$ we denote $a = f(x), b = g(x)$; clearly $a, b \leq x/\theta_{\mathcal{F}}$. So:

$$(f \boxdot g)(x) = (g^- \boxplus f^-)^\sim(x) = [(g^-(x) \oplus f^-(x)) \wedge x/\theta_{\mathcal{F}}]^\sim \odot x/\theta_{\mathcal{F}} =$$
$$= [(x/\theta_{\mathcal{F}} \odot (g(x))^- \oplus x/\theta_{\mathcal{F}} \odot (f(x))^-) \wedge x/\theta_{\mathcal{F}}]^\sim \odot x/\theta_{\mathcal{F}} =$$
$$= [(x/\theta_{\mathcal{F}} \odot b^- \oplus x/\theta_{\mathcal{F}} \odot a^-) \wedge x/\theta_{\mathcal{F}}]^\sim \odot x/\theta_{\mathcal{F}} =$$
$$= [(x/\theta_{\mathcal{F}} \odot b^- \oplus x/\theta_{\mathcal{F}} \odot a^-)^\sim \vee (x/\theta_{\mathcal{F}})^\sim] \odot x/\theta_{\mathcal{F}} =$$
$$\overset{psmv-c_{30}}{=} [(x/\theta_{\mathcal{F}} \odot b^- \oplus x/\theta_{\mathcal{F}} \odot a^-)^\sim \odot x/\theta_{\mathcal{F}}] \vee ((x/\theta_{\mathcal{F}})^\sim \odot x/\theta_{\mathcal{F}}) =$$
$$= [(x/\theta_{\mathcal{F}} \odot b^- \oplus x/\theta_{\mathcal{F}} \odot a^-)^\sim \odot x/\theta_{\mathcal{F}}] \vee 0/\theta_{\mathcal{F}} =$$
$$= (x/\theta_{\mathcal{F}} \odot b^- \oplus x/\theta_{\mathcal{F}} \odot a^-)^\sim \odot x/\theta_{\mathcal{F}} = (x/\theta_{\mathcal{F}} \odot a^-)^\sim \odot (x/\theta_{\mathcal{F}} \odot b^-)^\sim \odot x/\theta_{\mathcal{F}} =$$

$$= ((a^-)^\sim \oplus (x/\theta_{\mathcal{F}})^\sim) \odot ((b^-)^\sim \oplus (x/\theta_{\mathcal{F}})^\sim) \odot x/\theta_{\mathcal{F}} =$$
$$= (a \oplus (x/\theta_{\mathcal{F}})^\sim) \odot (b \oplus (x/\theta_{\mathcal{F}})^\sim) \odot x/\theta_{\mathcal{F}} = (a \oplus (x/\theta_{\mathcal{F}})^\sim) \odot (b \wedge x/\theta_{\mathcal{F}}) =$$
$$= (a \oplus (x/\theta_{\mathcal{F}})^\sim) \odot b = (f(x) \oplus (x/\theta_{\mathcal{F}})^\sim) \odot g(x).$$

Now we shall prove that $(f(x) \oplus (x/\theta_{\mathcal{F}})^\sim) \odot g(x) = f(x) \odot ((x/\theta_{\mathcal{F}})^- \oplus g(x))$.

Indeed,

$$(f(x)\oplus(x/\theta_{\mathcal{F}})^{\sim})\odot g(x)=(f(x)\oplus(x/\theta_{\mathcal{F}})^{\sim})\odot(g(x)\wedge x/\theta_{\mathcal{F}})=$$
$$=(f(x)\oplus(x/\theta_{\mathcal{F}})^{\sim})\odot[x/\theta_{\mathcal{F}}\odot((x/\theta_{\mathcal{F}})^{-}\oplus g(x))]=$$
$$=[(f(x)\oplus(x/\theta_{\mathcal{F}})^{\sim})\odot x/\theta_{\mathcal{F}}]\odot((x/\theta_{\mathcal{F}})^{-}\oplus g(x))=$$
$$=(f(x)\wedge x/\theta_{\mathcal{F}})\odot((x/\theta_{\mathcal{F}})^{-}\oplus g(x))=f(x)\odot((x/\theta_{\mathcal{F}})^{-}\oplus g(x)).\blacksquare$$

REMARK 7.3. *For two elements* $\widehat{(I_1,f_1)},\widehat{(I_2,f_2)}$ *in* $A_{\mathcal{F}}$ *we have*

$$\widehat{(I_1,f_1)}\cdot\widehat{(I_2,f_2)}=(I_1\cap\widehat{I_2,f_1}\boxdot f_2)$$

where $f_1\boxdot f_2$ *are characterized as in Lemma 7.6.*

PROPOSITION 7.7. $(M(A/\theta_{\mathcal{F}}),\boxplus,\boxdot,^{-},^{\sim},\mathbf{0},\mathbf{1})$ *is a pseudo* $MV-$ *algebra.*

Proof. We verify the axioms of pseudo $MV-$ algebras.

$(psMV_1)$. Let $f_i\in M(I_i,A/\theta_{\mathcal{F}})$ where $I_i\in\mathcal{F}$, $i=1,2,3$ and denote $I=I_1\cap I_2\cap I_3\in\mathcal{F}$.

Also, denote $f=f_1\boxplus(f_2\boxplus f_3)$, $g=(f_1\boxplus f_2)\boxplus f_3$ and for $x\in I$, $a=f_1(x),b=f_2(x),c=f_3(x)$.

Clearly $a,b,c\leq x/\theta_{\mathcal{F}}$. Thus, for $x\in I$:

$$f(x)=(f_1(x)\oplus(f_2\boxplus f_3)(x))\wedge x/\theta_{\mathcal{F}}=$$
$$=(f_1(x)\oplus((f_2(x)\oplus f_3(x))\wedge x/\theta_{\mathcal{F}}))\wedge x/\theta_{\mathcal{F}}=$$
$$=(a\oplus((b\oplus c)\wedge x/\theta_{\mathcal{F}}))\wedge x/\theta_{\mathcal{F}}=((a\wedge x/\theta_{\mathcal{F}})\oplus((b\oplus c)\wedge x/\theta_{\mathcal{F}}))\wedge x/\theta_{\mathcal{F}}$$
$$\overset{psmv-c_{40}}{=}(a\oplus(b\oplus c))\wedge x/\theta_{\mathcal{F}}.$$

Analogously, $g(x)=((a\oplus b)\oplus c)\wedge x/\theta_{\mathcal{F}}$, hence $f=g$, so

$$\widehat{(I_1,f_1)}+[\widehat{(I_2,f_2)}+\widehat{(I_3,f_3)}]=[\widehat{(I_1,f_1)}+\widehat{(I_2,f_2)}]+\widehat{(I_3,f_3)},$$

that is the operation $+$ is associative on $A_{\mathcal{F}}$.

$(psMV_2)$. Let $f\in M(I,A/\theta_{\mathcal{F}})$ with $I\in\mathcal{F}$. If $x\in I$, then

$$(f\boxplus\mathbf{0})(x)=(f(x)\oplus\mathbf{0}(x))\wedge x/\theta_{\mathcal{F}}=f(x)\wedge x/\theta_{\mathcal{F}}=f(x),$$
$$(\mathbf{0}\boxplus f)(x)=(\mathbf{0}(x)\oplus f(x))\wedge x/\theta_{\mathcal{F}}=f(x)\wedge x/\theta_{\mathcal{F}}=f(x),$$

hence $f\boxplus\mathbf{0}=\mathbf{0}\boxplus f=f$, so

$$\widehat{(I,f)}+\widehat{(A,\mathbf{0})}=\widehat{(A,\mathbf{0})}+\widehat{(I,f)}=\widehat{(I,f)}.$$

$(psMV_3)$. For $f\in M(I,A/\theta_{\mathcal{F}})$ (with $I\in\mathcal{F}$) and $x\in I$, we have:

$$(f\boxplus\mathbf{1})(x)=(f(x)\oplus\mathbf{1}(x))\wedge x/\theta_{\mathcal{F}}=(f(x)\oplus x/\theta_{\mathcal{F}})\wedge x/\theta_{\mathcal{F}}=x/\theta_{\mathcal{F}}=\mathbf{1}(x),$$
$$(\mathbf{1}\boxplus f)(x)=(\mathbf{1}(x)\oplus f(x))\wedge x/\theta_{\mathcal{F}}=(x/\theta_{\mathcal{F}}\oplus f(x))\wedge x/\theta_{\mathcal{F}}=x/\theta_{\mathcal{F}}=\mathbf{1}(x),$$

hence $f\boxplus\mathbf{1}=\mathbf{1}\boxplus f=\mathbf{1}$, so

$$\widehat{(I,f)}+\widehat{(A,\mathbf{1})}=\widehat{(A,\mathbf{1})}+\widehat{(I,f)}=\widehat{(A,\mathbf{1})}.$$

$(psMV_4)$. For $x\in A$, we have

$$\mathbf{1}^{-}(x)=x/\theta_{\mathcal{F}}\odot(\mathbf{1}(x))^{-}=x/\theta_{\mathcal{F}}\odot(x/\theta_{\mathcal{F}})^{-}=0/\theta_{\mathcal{F}}=\mathbf{0}(x),$$
$$\mathbf{1}^{\sim}(x)=(\mathbf{1}(x))^{\sim}\odot x/\theta_{\mathcal{F}}=(x/\theta_{\mathcal{F}})^{\sim}\odot x/\theta_{\mathcal{F}}=0/\theta_{\mathcal{F}}=\mathbf{0}(x).$$

So, $\mathbf{1}^{\sim}=\mathbf{0}$, and $\mathbf{1}^{-}=\mathbf{0}$, that is

$$\widehat{(A,\mathbf{1})}^{\sim}=\widehat{(A,\mathbf{0})}$$

$$\widehat{(A,\mathbf{1})}^- = \widehat{(A,\mathbf{0})}.$$

($psMV_5$). Let $f \in M(I, A/\theta_{\mathcal{F}}), g \in M(J, A/\theta_{\mathcal{F}})$ (with $I, J \in \mathcal{F}$) and $x \in I \cap J$. If denote $a = f(x), b = g(x)$, then $a, b \leq x/\theta_{\mathcal{F}}$ and from Lemma 7.6, $(g^- \boxplus f^-)^\sim = (f(x) \oplus (x/\theta_{\mathcal{F}})^\sim) \odot g(x) = f(x) \odot ((x/\theta_{\mathcal{F}})^- \oplus g(x))$.
We have:

$$\begin{aligned}
(g^\sim \boxplus f^\sim)^-(x) &= x/\theta_{\mathcal{F}} \odot [((g(x))^\sim \odot x/\theta_{\mathcal{F}} \oplus (f(x))^\sim \odot x/\theta_{\mathcal{F}}) \wedge x/\theta_{\mathcal{F}}]^- = \\
&= x/\theta_{\mathcal{F}} \odot [(b^\sim \odot x/\theta_{\mathcal{F}} \oplus a^\sim \odot x/\theta_{\mathcal{F}}) \wedge x/\theta_{\mathcal{F}}]^- = \\
&= x/\theta_{\mathcal{F}} \odot [(b^\sim \odot x/\theta_{\mathcal{F}} \oplus a^\sim \odot x/\theta_{\mathcal{F}})^- \vee (x/\theta_{\mathcal{F}})^-] \overset{psmv-c_{20}}{=} \\
&= [x/\theta_{\mathcal{F}} \odot (b^\sim \odot x/\theta_{\mathcal{F}} \oplus a^\sim \odot x/\theta_{\mathcal{F}})^-] \vee (x/\theta_{\mathcal{F}} \odot (x/\theta_{\mathcal{F}})^-) = \\
&= [x/\theta_{\mathcal{F}} \odot (b^\sim \odot x/\theta_{\mathcal{F}} \oplus a^\sim \odot x/\theta_{\mathcal{F}})^-] \vee 0/\theta_{\mathcal{F}} = \\
&= x/\theta_{\mathcal{F}} \odot (a^\sim \odot x/\theta_{\mathcal{F}})^- \odot (b^\sim \odot x/\theta_{\mathcal{F}})^- = \\
&= x/\theta_{\mathcal{F}} \odot ((x/\theta_{\mathcal{F}})^- \oplus a) \odot ((x/\theta_{\mathcal{F}})^- \oplus b) = \\
&= (x/\theta_{\mathcal{F}} \wedge a) \odot ((x/\theta_{\mathcal{F}})^- \oplus b) = a \odot ((x/\theta_{\mathcal{F}})^- \oplus b) = \\
&= f(x) \odot ((x/\theta_{\mathcal{F}})^- \oplus g(x)),
\end{aligned}$$

and by Lemma 7.6, we deduce that

$$(g^- \boxplus f^-)^\sim = (g^\sim \boxplus f^\sim)^-,$$

so

$$(\widehat{(J,g)}^- + \widehat{(I,f)}^-)^\sim = (\widehat{(J,g)}^\sim + \widehat{(I,f)}^\sim)^-.$$

($psMV_6$). Let $f \in M(I, A/\theta_{\mathcal{F}}), g \in M(J, A/\theta_{\mathcal{F}})$ (with $I, J \in \mathcal{F}$) and $x \in I \cap J$. We have:

$$\begin{aligned}
(f \boxplus f^\sim \boxdot g)(x) &= [f(x) \oplus ((f(x))^\sim \odot x/\theta_{\mathcal{F}} \oplus (x/\theta_{\mathcal{F}})^\sim) \odot g(x)] \wedge x/\theta_{\mathcal{F}} = \\
&= [f(x) \oplus ((f(x))^\sim \odot ((x/\theta_{\mathcal{F}})^\sim)^- \oplus (x/\theta_{\mathcal{F}})^\sim) \odot g(x)] \wedge x/\theta_{\mathcal{F}} = \\
&= [f(x) \oplus ((f(x))^\sim \vee (x/\theta_{\mathcal{F}})^\sim) \odot g(x)] \wedge x/\theta_{\mathcal{F}} = \\
&= [f(x) \oplus (f(x) \wedge x/\theta_{\mathcal{F}})^\sim \odot g(x)] \wedge x/\theta_{\mathcal{F}} = [f(x) \oplus (f(x))^\sim \odot g(x)] \wedge x/\theta_{\mathcal{F}} = \\
&= [f(x) \vee g(x)] \wedge x/\theta_{\mathcal{F}} = f(x) \vee g(x);
\end{aligned}$$

Analogously, $(g \boxplus g^\sim \boxdot f)(x) = g(x) \vee f(x)$;

$$\begin{aligned}
(f \boxdot g^- \boxplus g)(x) &= [f(x) \odot ((x/\theta_{\mathcal{F}})^- \oplus x/\theta_{\mathcal{F}} \odot (g(x))^-) \oplus g(x)] \wedge x/\theta_{\mathcal{F}} = \\
&= [f(x) \odot ((x/\theta_{\mathcal{F}})^- \oplus ((x/\theta_{\mathcal{F}})^-)^\sim \odot (g(x))^-) \oplus g(x)] \wedge x/\theta_{\mathcal{F}} = \\
&= [f(x) \odot ((x/\theta_{\mathcal{F}})^- \vee (g(x))^-) \oplus g(x)] \wedge x/\theta_{\mathcal{F}} = \\
&= [f(x) \odot (x/\theta_{\mathcal{F}} \wedge g(x))^- \oplus g(x)] \wedge x/\theta_{\mathcal{F}} = [f(x) \odot (g(x))^- \oplus g(x)] \wedge x/\theta_{\mathcal{F}} = \\
&= [f(x) \vee g(x)] \wedge x/\theta_{\mathcal{F}} = f(x) \vee g(x);
\end{aligned}$$

Analogously, $(g \boxdot f^- \boxplus f)(x) = g(x) \vee f(x)$;
So,

$$f \boxplus f^\sim \boxdot g = g \boxplus g^\sim \boxdot f = f \boxdot g^- \boxplus g = g \boxdot f^- \boxplus f,$$

that is

$$\begin{aligned}
&\widehat{(I,f)} + \widehat{(I,f)}^\sim \cdot \widehat{(J,g)} = \widehat{(J,g)} + \widehat{(J,g)}^\sim \cdot \widehat{(I,f)} = \\
&= \widehat{(I,f)} \cdot \widehat{(J,g)}^- + \widehat{(J,g)} = \widehat{(J,g)} \cdot \widehat{(I,f)}^- + \widehat{(I,f)}.
\end{aligned}$$

($psMV_7$). Let $f \in M(I, A/\theta_{\mathcal{F}}), g \in M(J, A/\theta_{\mathcal{F}})$ where $I, J \in \mathcal{F}$. Thus, for $x \in I \cap J$:

$$(f \boxdot (f^- \boxplus g))(x) = f(x) \odot [(x/\theta_{\mathcal{F}})^- \oplus (x/\theta_{\mathcal{F}} \odot (f(x))^- \oplus g(x)) \wedge x/\theta_{\mathcal{F}}] =$$

$$= f(x) \odot ([(x/\theta_{\mathcal{F}})^- \oplus (x/\theta_{\mathcal{F}} \odot (f(x))^- \oplus g(x))] \wedge [(x/\theta_{\mathcal{F}})^- \oplus x/\theta_{\mathcal{F}}]) =$$
$$= f(x) \odot ([(x/\theta_{\mathcal{F}})^- \oplus (x/\theta_{\mathcal{F}} \odot (f(x))^- \oplus g(x))] \wedge 1/\theta_{\mathcal{F}}) =$$
$$= f(x) \odot [((x/\theta_{\mathcal{F}})^- \oplus x/\theta_{\mathcal{F}} \odot (f(x))^-) \oplus g(x)] =$$
$$= f(x) \odot [((x/\theta_{\mathcal{F}})^- \oplus ((x/\theta_{\mathcal{F}})^-)^\sim \odot (f(x))^-) \oplus g(x)] =$$
$$= f(x) \odot [((x/\theta_{\mathcal{F}})^- \vee (f(x))^-) \oplus g(x)] = f(x) \odot [(x/\theta_{\mathcal{F}} \wedge f(x))^- \oplus g(x)] =$$
$$= f(x) \odot [(f(x))^- \oplus g(x)] = f(x) \wedge g(x),$$

and

$$((f \boxplus g^\sim) \boxdot g)(x) = [(f(x) \oplus (g(x))^\sim \odot x/\theta_{\mathcal{F}}) \wedge x/\theta_{\mathcal{F}} \oplus (x/\theta_{\mathcal{F}})^\sim] \odot g(x) =$$
$$= [((f(x) \oplus (g(x))^\sim \odot x/\theta_{\mathcal{F}}) \oplus (x/\theta_{\mathcal{F}})^\sim) \wedge (x/\theta_{\mathcal{F}} \oplus (x/\theta_{\mathcal{F}})^\sim)] \odot g(x) =$$
$$= [((f(x) \oplus (g(x))^\sim \odot x/\theta_{\mathcal{F}}) \oplus (x/\theta_{\mathcal{F}})^\sim) \wedge 1/\theta_{\mathcal{F}}] \odot g(x) =$$
$$= [f(x) \oplus ((g(x))^\sim \odot x/\theta_{\mathcal{F}} \oplus (x/\theta_{\mathcal{F}})^\sim)] \odot g(x) =$$
$$= [f(x) \oplus ((g(x))^\sim \odot ((x/\theta_{\mathcal{F}})^\sim)^- \oplus (x/\theta_{\mathcal{F}})^\sim)] \odot g(x) =$$
$$= [f(x) \oplus ((g(x))^\sim \vee (x/\theta_{\mathcal{F}})^\sim)] \odot g(x) = [f(x) \oplus (g(x) \wedge x/\theta_{\mathcal{F}})^\sim] \odot g(x) =$$
$$= [f(x) \oplus (g(x))^\sim] \odot g(x) = f(x) \wedge g(x),$$

so,

$$f \boxdot (f^- \boxplus g) = (f \boxplus g^\sim) \boxdot g,$$

that is

$$\widehat{(I,f)} \cdot [\widehat{(I,f)}^- + \widehat{(J,g)}] = [\widehat{(I,f)} + \widehat{(J,g)}^\sim] \cdot \widehat{(J,g)}.$$

($psMV_8$). For $f \in M(I, A/\theta_{\mathcal{F}})$ (with $I \in \mathcal{F}$) and $x \in I$, we have:

$$(f^-)^\sim(x) = (x/\theta_{\mathcal{F}} \odot (f(x))^-)^\sim \odot x/\theta_{\mathcal{F}} = [((f(x))^-)^\sim \oplus (x/\theta_{\mathcal{F}})^\sim] \odot x/\theta_{\mathcal{F}} =$$
$$= (f(x) \oplus (x/\theta_{\mathcal{F}})^\sim) \odot x/\theta_{\mathcal{F}} = f(x) \wedge x/\theta_{\mathcal{F}} = f(x).$$

So, $(f^-)^\sim = f$, that is

$$[\widehat{(I,f)}^-]^\sim = \widehat{(I,f)}.\blacksquare$$

COROLLARY 7.8. *$(A_{\mathcal{F}}, +, \cdot, ^-, ^\sim, \mathbf{0} = \widehat{(A,\mathbf{0})}, \mathbf{1} = \widehat{(A,\mathbf{1})})$ is a pseudo $MV-$ algebra.*

REMARK 7.4. *If pseudo MV -algebra $(A, \oplus, \odot, ^-, ^\sim, 0, 1)$ is an MV -algebra (i.e $x \oplus y = y \oplus x$ for all $x, y \in A$), then pseudo MV -algebra $(M(A/\theta_{\mathcal{F}}), \boxplus, \boxdot, ^-, ^\sim, \mathbf{0}, \mathbf{1})$ is an MV -algebra $(M(A/\theta_{\mathcal{F}}), \boxplus, ^- \overset{not}{=} *, \mathbf{0})$. Indeed if $I_1, I_2 \in \mathcal{F}$ and $f_i \in M(I_i, A/\theta_{\mathcal{F}})$, $i = 1, 2$ we have*

$$(f_1 \boxplus f_2)(x) = [f_1(x) \oplus f_2(x)] \wedge x/\theta_{\mathcal{F}} = [f_2(x) \oplus f_1(x)] \wedge x/\theta_{\mathcal{F}} = (f_2 \boxplus f_1)(x),$$

for all $x \in I_1 \cap I_2$, then $f_1 \boxplus f_2 = f_2 \boxplus f_1$, so pseudo MV -algebra $(M(A/\theta_{\mathcal{F}}), \boxplus, \boxdot, ^-, ^\sim, \mathbf{0}, \mathbf{1})$ is commutative, so is an MV -algebra.

LEMMA 7.9. *Let $f_1, f_2 \in M(A/\theta_{\mathcal{F}})$ with $f_i \in M(I_i, A/\theta_{\mathcal{F}})$, $(I_i \in \mathcal{F})$, $i = 1, 2$. Then for every $x \in I_1 \cap I_2$:*

(i) $(f_1 \wedge f_2)(x) = f_1(x) \wedge f_2(x)$;
(ii) $(f_1 \vee f_2)(x) = f_1(x) \vee f_2(x)$.

Proof. We recall that in pseudo $MV-$ algebra $M(A/\theta_{\mathcal{F}})$ we have:

$$f \wedge g = f \boxdot (f^- \boxplus g) = g \boxdot (g^- \boxplus f) = (f \boxplus g^{\sim}) \boxdot g = (g \boxplus f^{\sim}) \boxdot f,$$

and

$$f \vee g = f \boxplus f^{\sim} \boxdot g = g \boxplus g^{\sim} \boxdot f = f \boxdot g^- \boxplus g = g \boxdot f^- \boxplus f.$$

So: (i). Follow immediately from Proposition 7.7, $psMV_7$).

(ii). Follow immediately from Proposition 7.7, $psMV_6$).■

COROLLARY 7.10. *$(A_{\mathcal{F}}, +, \cdot, ^-, ^{\sim}, \mathbf{0} = \widehat{(\mathbf{A}, \mathbf{0})}, \mathbf{1} = \widehat{(\mathbf{A}, \mathbf{1})})$ is a pseudo $MV-$ algebra, where $\mathbf{0} = \widehat{(A, \mathbf{0})}$ and $\mathbf{1} = \mathbf{0}^- = \widehat{(A, \mathbf{1})}$. Also, for two elements $\widehat{(I_1, f_1)}, \widehat{(I_2, f_2)}$ in $A_{\mathcal{F}}$ we have*

$$\widehat{(I_1, f_1)} \curlywedge \widehat{(I_2, f_2)} = (I_1 \cap \widehat{I_2, f_1} \wedge f_2),$$
$$\widehat{(I_1, f_1)} \curlyvee \widehat{(I_2, f_2)} = (I_1 \cap \widehat{I_2, f_1} \vee f_2),$$

where $f_1 \wedge f_2, f_1 \vee f_2$ are characterized as in Lemma 7.9. If pseudo $MV-$ algebra A is an $MV-$ algebra, then $(A_{\mathcal{F}}, +, \cdot, ^-, ^{\sim}, \mathbf{0} = \widehat{(A, \mathbf{0})}, \mathbf{1} = \widehat{(A, \mathbf{1})})$ is an $MV-$ algebra $(A_{\mathcal{F}}, +, ^- \overset{not}{=} *, \mathbf{0} = \widehat{(A, \mathbf{0})})$.

DEFINITION 7.2. The pseudo $MV-$ algebra $A_{\mathcal{F}}$ will be called the *localization pseudo $MV-$ algebra of A with respect to the topology $\mathcal{F}$.*

Clearly, the localization pseudo $MV-$ algebra is a non-commutative generalization of localization $MV-$ algebra obtained in Chapter 6.

We also have for pseudo $MV-$ algebras the next analogous result as for $MV-$ algebras:

LEMMA 7.11. *Let the map $v_{\mathcal{F}} : B(A) \to A_{\mathcal{F}}$ defined by $v_{\mathcal{F}}(a) = \widehat{(A, \overline{f_a})}$ for every $a \in B(A)$. Then:*

(i) $v_{\mathcal{F}}$ is a morphism of pseudo $MV-$ algebras;

(ii) For $a \in B(A)$, $\widehat{(A, \overline{f_a})} \in B(A_{\mathcal{F}})$;

(iii) $v_{\mathcal{F}}(B(A)) \in \mathcal{R}(A_{\mathcal{F}})$.

Proof. (i). We have $v_{\mathcal{F}}(0) = \widehat{(A, \overline{f_0})} = \widehat{(A, \mathbf{0})} = \mathbf{0}$.

For $a, b \in B(A)$, we have $v_{\mathcal{F}}(a) + v_{\mathcal{F}}(b) = \widehat{(A, \overline{f_a})} + \widehat{(A, \overline{f_b})} = (A, \widehat{\overline{f_a} \boxplus \overline{f_b}}) \overset{psmv-c_{40}}{=} (\widehat{A, \overline{f_{a \oplus b}}}) = v_{\mathcal{F}}(a \oplus b)$ and for $x \in A$, since

$$(\overline{f_a})^-(x) = x/\theta_{\mathcal{F}} \odot [(a \wedge x)/\theta_{\mathcal{F}}]^- = x/\theta_{\mathcal{F}} \odot ((x/\theta_{\mathcal{F}})^- \vee (a/\theta_{\mathcal{F}})^-)$$
$$= x/\theta_{\mathcal{F}} \odot ((x/\theta_{\mathcal{F}})^- \oplus (a/\theta_{\mathcal{F}})^-) = x/\theta_{\mathcal{F}} \wedge (a/\theta_{\mathcal{F}})^- = \overline{f_{a^-}}(x),$$

that is $(\overline{f_a})^- = \overline{f_{a^-}}$ we deduce that

$$v_{\mathcal{F}}(a^-) = (\widehat{A, \overline{f_{a^-}}}) = \widehat{(A, \overline{f_a})}^- = (v_{\mathcal{F}}(a))^-,$$

and

$$(\overline{f_a})^{\sim}(x) = [(a \wedge x)/\theta_{\mathcal{F}}]^{\sim} \odot x/\theta_{\mathcal{F}} = ((a/\theta_{\mathcal{F}})^{\sim} \vee (x/\theta_{\mathcal{F}})^{\sim}) \odot x/\theta_{\mathcal{F}}$$
$$= ((a/\theta_{\mathcal{F}})^{\sim} \oplus (x/\theta_{\mathcal{F}})^{\sim}) \odot x/\theta_{\mathcal{F}} = (a/\theta_{\mathcal{F}})^{\sim} \wedge (x/\theta_{\mathcal{F}}) = \overline{f_{a^{\sim}}}(x),$$

that is $(\overline{f_a})^{\sim} = \overline{f_{a^{\sim}}}$ we deduce that

$$v_{\mathcal{F}}(a^{\sim}) = (\widehat{A, \overline{f_{a^{\sim}}}}) = \widehat{(A, \overline{f_a})}^{\sim} = (v_{\mathcal{F}}(a))^{\sim},$$

hence $v_{\mathcal{F}}$ is a morphism of pseudo $MV-$ algebras.

(ii). For $a \in B(A)$ we have $a \oplus a = a$, hence by $psmv - c_{43}$, $((a \wedge x) \oplus (a \wedge x)) \wedge x = a \wedge x$ for every $x \in A$.

Since $A \in \mathcal{F}$ we deduce that $((a \wedge x)/\theta_{\mathcal{F}} \oplus (a \wedge x)/\theta_{\mathcal{F}}) \wedge x/\theta_{\mathcal{F}} = (a \wedge x)/\theta_{\mathcal{F}}$ hence $\overline{f_a} \boxplus \overline{f_a} = \overline{f_a}$, that is

$$\widehat{(A, \overline{f_a})} \in B(A_{\mathcal{F}}).$$

(iii). To prove that $v_{\mathcal{F}}(B(A))$ is a regular subset of $A_{\mathcal{F}}$, let $\widehat{(I_i, f_i)} \in A_{\mathcal{F}}$, $I_i \in \mathcal{F}$, $i = 1, 2$, such that $\widehat{(A, \overline{f_a})} \curlywedge \widehat{(I_1, f_1)} = \widehat{(A, \overline{f_a})} \curlywedge \widehat{(I_2, f_2)}$ for every $a \in B(A)$. By (ii), $\widehat{(A, \overline{f_a})} \in B(A_{\mathcal{F}})$. Then $(f_1 \wedge \overline{f_a})(x) = (f_2 \wedge \overline{f_a})(x)$ for every $x \in I_1 \cap I_2$ and $a \in B(A) \Leftrightarrow f_1(x) \wedge x/\theta_{\mathcal{F}} \wedge a/\theta_{\mathcal{F}} = f_2(x) \wedge x/\theta_{\mathcal{F}} \wedge a/\theta_{\mathcal{F}}$ for every $x \in I_1 \cap I_2$ and $a \in B(A) \Leftrightarrow f_1(x) \wedge a/\theta_{\mathcal{F}} = f_2(x) \wedge a/\theta_{\mathcal{F}}$ for every $x \in I_1 \cap I_2$ and $a \in B(A)$.

In particular for $a = 1$, $a/\theta_{\mathcal{F}} = \mathbf{1} \in B(A/\theta_{\mathcal{F}})$ we obtain that $f_1(x) = f_2(x)$ for every $x \in I_1 \cap I_2$, hence $\widehat{(I_1, f_1)} = \widehat{(I_2, f_2)}$, that is $v_{\mathcal{F}}(B(A)) \in \mathcal{R}(A_{\mathcal{F}})$. ■

2. Applications

In the following we describe the localization pseudo $MV-$ algebra $A_{\mathcal{F}}$ in some special instances.

2.1. Application 1. If $I \in \mathcal{I}(A)$ and $\mathcal{F}$ is the topology

$$\mathcal{F}(I) = \{I' \in \mathcal{I}(A) : I \subseteq I'\}$$

(see Example 7.1), then $A_{\mathcal{F}}$ is isomorphic with $M(I, A/\theta_{\mathcal{F}})$ and $v_{\mathcal{F}} : B(A) \to A_{\mathcal{F}}$ is defined by $v_{\mathcal{F}}(a) = \overline{f_{a|I}}$ for every $a \in B(A)$.

2.2. Application 2: Maximal pseudo MV-algebra of quotients. As for $MV-$ algebras we have:

Definition 7.3. By a *partial strong multiplier* of a pseudo MV-algebra A we mean a map $f : I \to A$, where $I \in \mathcal{I}(A)$, which verify the following conditions:

$(sm - psMV_1)$ $f(e \odot x) = e \odot f(x)$, for every $e \in B(A)$ and $x \in I$;
$(sm - psMV_2)$ $f(x) \leq x$, for every $x \in I$;
$(sm - psMV_3)$ If $e \in I \cap B(A)$, then $f(e) \in B(A)$;
$(sm - psMV_4)$ $x \wedge f(e) = e \wedge f(x)$, for every $e \in I \cap B(A)$ and $x \in I$ (note that $e \odot x \in I$ as $e \odot x \leq e \wedge x \leq x$).

Remark 7.5. *The condition $sm - psMV_4$ is not a consequence of $sm - psMV_1$, $sm - psMV_2$ and $sm - psMV_3$. As example, $f : I \to A$, where $I \in \mathcal{I}(A)$, $f(x) = x \wedge x^-$ for every $x \in I$, verify $sm - psMV_1$, $sm - psMV_1$ and $sm - psMV_3$. Indeed, for $x \in I$ and $e \in B(A)$, we have*

$$f(e \odot x) = (e \odot x) \wedge (e \odot x)^- = (e \wedge x) \wedge (e \wedge x)^- = x \wedge [e \wedge (e \wedge x)^-] =$$

$$= x \wedge [e \wedge (e^- \vee x^-)] = x \wedge [e \odot (e^- \vee x^-)] = x \wedge [e \odot (e^- \oplus x^-)] =$$

$$= x \wedge (e \wedge x^-) = e \wedge (x \wedge x^-) = e \wedge f(x) = e \odot f(x).$$

Clearly, $f(x) \leq x$ for every $x \in I$ and for $e \in I \cap B(A)$, $f(e) = e \wedge e^- = 0 \in B(A)$. But if $e \in I \cap B(A)$ and $x \in I$, then

$$x \wedge f(e) = x \wedge 0 = 0 \neq e \wedge (x \wedge x^-).$$

By $dom(f) \in \mathcal{I}(A)$ we denote the domain of f; if $dom(f) = A$, we called f *total*.

To simplify language, we will use *strong multiplier* instead *partial strong multiplier* using *total* to indicate that the domain of a certain multiplier is A.

We also have for strong multipliers on a pseudo MV-algebra the next analogous examples as for MV-algebras:

EXAMPLE 7.4. *The map $\mathbf{0} : A \to A$ defined by $\mathbf{0}(x) = 0$, for every $x \in A$ is a total strong multiplier of A; indeed if $x \in A$ and $e \in B(A)$, then $\mathbf{0}(e \odot x) = 0 = e \odot 0 = e \odot \mathbf{0}(x)$ and $\mathbf{0}(x) \leq x$. Clearly, if $e \in A \cap B(A) = B(A)$, then $\mathbf{0}(e) = 0 \in B(A)$ and for $x \in A, x \wedge \mathbf{0}(e) = e \wedge \mathbf{0}(x) = 0$.*

EXAMPLE 7.5. *The map $\mathbf{1} : A \to A$ defined by $\mathbf{1}(x) = x$, for every $x \in A$ is also a total strong multiplier of A; indeed if $x \in A$ and $e \in B(A)$, then $\mathbf{1}(e \odot x) = e \odot x = e \odot \mathbf{1}(x)$ and $\mathbf{1}(x) = x \leq x$. The condition $sm - psMV_3$ and $sm - psMV_4$ is obviously verified.*

EXAMPLE 7.6. *For $a \in B(A)$ and $I \in \mathcal{I}(A)$, the map $f_a : I \to A$ defined by $f_a(x) = a \wedge x$, for every $x \in I$ is a strong multiplier of A (called principal). Indeed, for $x \in I$ and $e \in B(A)$, we have $f_a(e \odot x) = a \wedge (e \odot x) = a \wedge (e \wedge x) = e \wedge (a \wedge x) = e \odot (a \wedge x) = e \odot f_a(x)$ and clearly $f_a(x) \leq x$. Also, if $e \in I \cap B(A)$, $f_a(e) = e \wedge a \in B(A)$ and $x \wedge (a \wedge e) = e \wedge (a \wedge x)$, for every $x \in I$.*

REMARK 7.6. *In general, if we consider any $a \in A$, then $f_a : I \to A$ verifies only $sm - psMV_1, sm - psMV_2$ and $sm - psMV_4$ but does not verify $sm - psMV_3$.*

If $dom(f_a) = A$, we denote f_a by $\overline{f_a}$; clearly, $\overline{f_0} = \mathbf{0}$.

For $I \in \mathcal{I}(A)$, we denote

$$M(I, A) = \{f : I \to A \mid f \text{ is a strong multiplier on } A\}$$

and

$$M(A) = \bigcup_{I \in \mathcal{I}(A)} M(I, A).$$

If $I_1, I_2 \in \mathcal{I}(A)$ and $f_i \in M(I_i, A), i = 1, 2$, we define $f_1 \boxplus f_2 : I_1 \cap I_2 \to A$ by

$$(f_1 \boxplus f_2)(x) = (f_1(x) \oplus f_2(x)) \wedge x,$$

for every $x \in I_1 \cap I_2$.

LEMMA 7.12. $f_1 \boxplus f_2 \in M(I_1 \cap I_2, A)$.

Proof. If $x \in I_1 \cap I_2$ and $e \in B(A)$, then

$$(f_1 \boxplus f_2)(e \odot x) = [f_1(e \odot x) \oplus f_2(e \odot x)] \wedge (e \odot x) =$$

$$= [(e \odot f_1(x)) \oplus (e \odot f_2(x))] \wedge (e \wedge x) = [(e \wedge f_1(x)) \oplus (e \wedge f_2(x))] \wedge (e \wedge x) \overset{psmv-c_{43}}{=}$$

$$\overset{psmv-c_{43}}{=} [e \wedge (f_1(x) \oplus f_2(x))] \wedge (e \wedge x) = e \wedge [(f_1(x) \oplus f_2(x)) \wedge x] = e \odot (f_1 \boxplus f_2)(x).$$

Clearly, $(f_1 \boxplus f_2)(x) \leq x$ for every $x \in I_1 \cap I_2$ and if $e \in I_1 \cap I_2 \cap B(A)$ then

$$(f_1 \boxplus f_2)(e) = [f_1(e) \oplus f_2(e)] \wedge e \in B(A).$$

For $e \in I_1 \cap I_2 \cap B(A)$ and $x \in I_1 \cap I_2$ we have:

$$x \wedge (f_1 \boxplus f_2)(e) = x \wedge [(f_1(e) \oplus f_2(e)) \wedge e] = (f_1(e) \oplus f_2(e)) \wedge x \wedge e,$$

and

$$e \wedge (f_1 \boxplus f_2)(x) = e \wedge [(f_1(x) \oplus f_2(x)) \wedge x] = e \odot [(f_1(x) \oplus f_2(x)) \wedge x]$$

$$\overset{psmv-c_{26}}{=} [e \odot (f_1(x) \oplus f_2(x))] \wedge (e \odot x) \overset{psmv-c_{43}}{=} [(e \odot f_1(x)) \oplus (e \odot f_2(x))] \wedge (e \odot x)$$
$$= [x \odot f_1(e) \oplus x \odot f_2(e)] \wedge (e \odot x) = [(f_1(e) \wedge x) \oplus (f_2(e) \wedge x)] \wedge (e \wedge x)$$
$$= [[(f_1(e) \wedge x) \oplus (f_2(e) \wedge x)] \wedge x] \wedge e \overset{psmv-c_{40}}{=} ((f_1(e) \oplus f_2(e)) \wedge x) \wedge e$$
$$= (f_1(e) \oplus f_2(e)) \wedge x \wedge e,$$

hence

$$x \wedge (f_1 \boxplus f_2)(e) = e \wedge (f_1 \boxplus f_2)(x),$$

that is $f_1 \boxplus f_2 \in M(I_1 \cap I_2, A)$.■

For $I \in \mathcal{I}(A)$ and $f \in M(I, A)$ we define $f^-, f^\sim : I \to A$ by

$$f^-(x) = x \odot (f(x))^-,$$

and

$$f^\sim(x) = (f(x))^\sim \odot x,$$

for every $x \in I$.

LEMMA 7.13. *$f^-, f^\sim \in M(I, A)$.*

Proof. If $x \in I$ and $e \in B(A)$, then

$$f^-(e \odot x) = (e \odot x) \odot (f(e \odot x))^- = e \odot x \odot (e \odot f(x))^-$$
$$= e \odot x \odot (e^- \oplus (f(x))^-) = x \odot (e \odot (e^- \oplus (f(x))^-)) =$$
$$= x \odot (e \wedge (f(x))^-) = x \odot (e \odot (f(x))^-) = e \odot (x \odot (f(x))^-) = e \odot f^-(x)$$

and

$$f^\sim(e \odot x) = (f(e \odot x))^\sim \odot (e \odot x) = (e \odot f(x))^\sim \odot (e \odot x) =$$
$$= ((f(x))^\sim \oplus e^\sim) \odot e \odot x = ((f(x))^\sim \wedge e) \odot x = ((f(x))^\sim \odot e) \odot x =$$
$$= ((f(x))^\sim \odot x) \odot e = f^\sim(x) \odot e = e \odot f^\sim(x).$$

Clearly, $f^-(x) \leq x$ and $f^\sim(x) \leq x$ for every $x \in I$.

Clearly, if $e \in I \cap B(A)$, then

$$f^-(e) = e \odot [f(e)]^- \in B(A)$$

and

$$f^\sim(e) = [f(e)]^\sim \odot e \in B(A)$$

Since $f \in M(I, A)$, for $e \in I \cap B(A)$ and $x \in I$ we have:

$$x \wedge f(e) = e \wedge f(x) \Rightarrow x^- \vee (f(e))^- = e^- \vee (f(x))^- \Rightarrow x^- \oplus (f(e))^- = e^- \oplus (f(x))^-$$
$$\Rightarrow e \odot x \odot [x^- \oplus (f(e))^-] = x \odot e \odot [e^- \oplus (f(x))^-] \Rightarrow e \odot [x \wedge (f(e))^-] = x \odot [e \wedge (f(x))^-]$$
$$\Rightarrow e \odot x \odot (f(e))^- = x \odot e \odot (f(x))^- \Rightarrow x \odot [e \odot (f(e))^-] = e \odot [x \odot (f(x))^-]$$
$$\Rightarrow x \wedge [e \odot (f(e))^-] = e \wedge [x \odot (f(x))^-] \Rightarrow x \wedge f^-(e) = e \wedge f^-(x),$$

and

$$x \wedge f(e) = e \wedge f(x) \Rightarrow x^\sim \vee (f(e))^\sim = e^\sim \vee (f(x))^\sim \Rightarrow (f(e))^\sim \oplus x^\sim = (f(x))^\sim \oplus e^\sim$$
$$\Rightarrow [(f(e))^\sim \oplus x^\sim] \odot x \odot e = [(f(x))^\sim \oplus e^\sim] \odot e \odot x \Rightarrow [(f(e))^\sim \wedge x] \odot e = [(f(x))^\sim \wedge e] \odot x$$
$$\Rightarrow (f(e))^\sim \odot x \odot e = (f(x))^\sim \odot e \odot x \Rightarrow [(f(e))^\sim \odot e] \odot x = [(f(x))^\sim \odot x] \odot e$$
$$\Rightarrow [(f(e))^\sim \odot e] \wedge x = [(f(x))^\sim \odot x] \wedge e \Rightarrow x \wedge f^\sim(e) = e \wedge f^\sim(x),$$

hence f^- and $f^\sim$ verify $sm - psMV_4$, that is $f^-, f^\sim \in M(I, A)$.■

For $f \in M(I_1, A)$ and $g \in M(I_2, A)$ with $I_1, I_2 \in \mathcal{I}(A)$ we define $f \boxdot g$ on $I_1 \cap I_2$ by

$$f \boxdot g = (g^- \boxplus f^-)^\sim.$$

LEMMA 7.14. *For every $x \in I_1 \cap I_2$:*

$$(f \boxdot g)(x) = (f(x) \oplus x^\sim) \odot g(x) = f(x) \odot (x^- \oplus g(x)).$$

Proof. For $x \in I_1 \cap I_2$ we denote $a = f(x), b = g(x)$; clearly $a, b \leq x$.

So:

$$(f \boxdot g)(x) = [(g^-(x) \oplus f^-(x)) \wedge x]^\sim \odot x = [(x \odot (g(x))^- \oplus x \odot (f(x))^-) \wedge x]^\sim \odot x =$$

$$= [(x \odot b^- \oplus x \odot a^-) \wedge x]^\sim \odot x = [(x \odot b^- \oplus x \odot a^-)^\sim \vee x^\sim] \odot x \overset{psmv-c_{21}}{=}$$

$$= [(x \odot b^- \oplus x \odot a^-)^\sim \odot x] \vee (x^\sim \odot x) = [(x \odot b^- \oplus x \odot a^-)^\sim \odot x] \vee 0 =$$

$$= (x \odot b^- \oplus x \odot a^-)^\sim \odot x = (x \odot a^-)^\sim \odot (x \odot b^-)^\sim \odot x = ((a^-)^\sim \oplus x^\sim) \odot ((b^-)^\sim \oplus x^\sim) \odot x =$$

$$= (a \oplus x^\sim) \odot (b \oplus x^\sim) \odot x = (a \oplus x^\sim) \odot (b \wedge x) = (a \oplus x^\sim) \odot b = (f(x) \oplus x^\sim) \odot g(x).$$

Now we shall prove that $(f(x) \oplus x^\sim) \odot g(x) = f(x) \odot (x^- \oplus g(x))$.

Indeed,

$$(f(x) \oplus x^\sim) \odot g(x) = (f(x) \oplus x^\sim) \odot (g(x) \wedge x) = (f(x) \oplus x^\sim) \odot [x \odot (x^- \oplus g(x))] =$$

$$= [(f(x) \oplus x^\sim) \odot x] \odot (x^- \oplus g(x)) = (f(x) \wedge x) \odot (x^- \oplus g(x)) = f(x) \odot (x^- \oplus g(x)).\blacksquare$$

PROPOSITION 7.15. $(M(A), \boxplus, \boxdot, ^-, ^\sim, \mathbf{0}, \mathbf{1})$ *is a pseudo $MV-$ algebra.*

Proof. We verify the axioms of pseudo $MV-$ algebras.

$(psMV_1)$. Let $f_i \in M(I_i, A)$ where $I_i \in \mathcal{I}(A)$, $i = 1, 2, 3$ and denote $I = I_1 \cap I_2 \cap I_3 \in \mathcal{I}(A)$.

Also, denote $f = f_1 \boxplus (f_2 \boxplus f_3)$, $g = (f_1 \boxplus f_2) \boxplus f_3$ and for $x \in I$, $a = f_1(x), b = f_2(x), c = f_3(x)$.

Clearly $a, b, c \leq x$. Thus, for $x \in I$:

$$f(x) = (f_1(x) \oplus (f_2 \boxplus f_3)(x)) \wedge x = (f_1(x) \oplus ((f_2(x) \oplus f_3(x)) \wedge x)) \wedge x =$$

$$= (a \oplus ((b \oplus c) \wedge x)) \wedge x = ((a \wedge x) \oplus ((b \oplus c) \wedge x)) \wedge x \overset{psmv-c_{40}}{=} (a \oplus (b \oplus c)) \wedge x.$$

Analogously, $g(x) = ((a \oplus b) \oplus c) \wedge x$, hence $f = g$, that is the operation $\boxplus$ is associative.

$(psMV_2)$. Let $f \in M(I, A)$ with $I \in \mathcal{I}(A)$. If $x \in I$, then

$$(f \boxplus \mathbf{0})(x) = (f(x) \oplus \mathbf{0}(x)) \wedge x = f(x) \wedge x = f(x),$$

and

$$(\mathbf{0} \boxplus f)(x) = (\mathbf{0}(x) \oplus f(x)) \wedge x = f(x) \wedge x = f(x),$$

hence $f \boxplus \mathbf{0} = \mathbf{0} \boxplus f = f$.

$(psMV_3)$. For $f \in M(I, A)$ (with $I \in \mathcal{I}(A)$) and $x \in I$, we have:

$$(f \boxplus \mathbf{1})(x) = (f(x) \oplus \mathbf{1}(x)) \wedge x = (f(x) \oplus x) \wedge x = x = \mathbf{1}(x),$$

and

$$(\mathbf{1} \boxplus f)(x) = (\mathbf{1}(x) \oplus f(x)) \wedge x = (x \oplus f(x)) \wedge x = x = \mathbf{1}(x),$$

hence $f \boxplus \mathbf{1} = \mathbf{1} \boxplus f = \mathbf{1}$.

$(psMV_4)$. For $x \in A$, we have

$$\mathbf{1}^-(x) = x \odot (\mathbf{1}(x))^- = x \odot x^- = 0 = \mathbf{0}(x),$$

and

$$\mathbf{1}^{\sim}(x) = (\mathbf{1}(x))^{\sim} \odot x = x^{\sim} \odot x = 0 = \mathbf{0}(x).$$

So, $\mathbf{1}^{\sim} = \mathbf{0}$, and $\mathbf{1}^{-} = \mathbf{0}$.

$(psMV_5)$. Let $f \in M(I, A), g \in M(J, A)$ (with $I, J \in \mathcal{I}(A)$) and $x \in I \cap J$.
If denote $a = f(x), b = g(x)$, then $a, b \leq x$ and from Lemma 7.14,
$(g^{-} \boxplus f^{-})^{\sim} = (f(x) \oplus x^{\sim}) \odot g(x) = f(x) \odot (x^{-} \oplus g(x))$.
We have:

$$(g^{\sim} \boxplus f^{\sim})^{-}(x) = x \odot [((g(x))^{\sim} \odot x \oplus (f(x))^{\sim} \odot x) \wedge x]^{-} =$$
$$= x \odot [(b^{\sim} \odot x \oplus a^{\sim} \odot x) \wedge x]^{-} = x \odot [(b^{\sim} \odot x \oplus a^{\sim} \odot x)^{-} \vee x^{-}]$$
$$\overset{psmv-c_{20}}{=} [x \odot (b^{\sim} \odot x \oplus a^{\sim} \odot x)^{-}] \vee (x \odot x^{-}) = [x \odot (b^{\sim} \odot x \oplus a^{\sim} \odot x)^{-}] \vee 0 =$$
$$= x \odot (a^{\sim} \odot x)^{-} \odot (b^{\sim} \odot x)^{-} = x \odot (x^{-} \oplus a) \odot (x^{-} \oplus b) =$$
$$= (x \wedge a) \odot (x^{-} \oplus b) = a \odot (x^{-} \oplus b) = f(x) \odot (x^{-} \oplus g(x)),$$

and by Lemma 7.14, we deduce that

$$(g^{-} \boxplus f^{-})^{\sim} = (g^{\sim} \boxplus f^{\sim})^{-}.$$

$(psMV_6)$. Let $f \in M(I, A), g \in M(J, A)$ (with $I, J \in \mathcal{I}(A)$) and $x \in I \cap J$.
We have:

$$(f \boxplus f^{\sim} \boxdot g)(x) = [f(x) \oplus ((f(x))^{\sim} \odot x \oplus x^{\sim}) \odot g(x)] \wedge x =$$
$$= [f(x) \oplus ((f(x))^{\sim} \odot (x^{\sim})^{-} \oplus x^{\sim}) \odot g(x)] \wedge x =$$
$$= [f(x) \oplus ((f(x))^{\sim} \vee x^{\sim}) \odot g(x)] \wedge x =$$
$$= [f(x) \oplus (f(x) \wedge x)^{\sim} \odot g(x)] \wedge x = [f(x) \oplus (f(x))^{\sim} \odot g(x)] \wedge x =$$
$$= [f(x) \vee g(x)] \wedge x = f(x) \vee g(x);$$

Analogously, $(g \boxplus g^{\sim} \boxdot f)(x) = g(x) \vee f(x)$;

$$(f \boxdot g^{-} \boxplus g)(x) = [f(x) \odot (x^{-} \oplus x \odot (g(x))^{-}) \oplus g(x)] \wedge x =$$
$$= [f(x) \odot (x^{-} \oplus (x^{-})^{\sim} \odot (g(x))^{-}) \oplus g(x)] \wedge x =$$
$$= [f(x) \odot (x^{-} \vee (g(x))^{-}) \oplus g(x)] \wedge x = [f(x) \odot (x \wedge g(x))^{-} \oplus g(x)] \wedge x =$$
$$= [f(x) \odot (g(x))^{-} \oplus g(x)] \wedge x = [f(x) \vee g(x)] \wedge x = f(x) \vee g(x);$$

Analogously, $(g \boxdot f^{-} \boxplus f)(x) = g(x) \vee f(x)$;
So,

$$f \boxplus f^{\sim} \boxdot g = g \boxplus g^{\sim} \boxdot f = f \boxdot g^{-} \boxplus g = g \boxdot f^{-} \boxplus f.$$

$(psMV_7)$. Let $f \in M(I, A), g \in M(J, A)$ where $I, J \in \mathcal{I}(A)$. Thus, for $x \in I \cap J$:

$$(f \boxdot (f^{-} \boxplus g))(x) = f(x) \odot [x^{-} \oplus [(x \odot (f(x))^{-} \oplus g(x)) \wedge x]] =$$
$$= f(x) \odot ([x^{-} \oplus (x \odot (f(x))^{-} \oplus g(x))] \wedge [x^{-} \oplus x]) =$$
$$= f(x) \odot ([x^{-} \oplus (x \odot (f(x))^{-} \oplus g(x))] \wedge 1) =$$
$$= f(x) \odot [(x^{-} \oplus x \odot (f(x))^{-}) \oplus g(x)] =$$
$$= f(x) \odot [(x^{-} \oplus (x^{-})^{\sim} \odot (f(x))^{-}) \oplus g(x)] =$$
$$= f(x) \odot [(x^{-} \vee (f(x))^{-}) \oplus g(x)] = f(x) \odot [(x \wedge f(x))^{-} \oplus g(x)] =$$
$$= f(x) \odot [(f(x))^{-} \oplus g(x)] = f(x) \wedge g(x).$$

And

$$((f \boxplus g^{\sim}) \boxdot g)(x) = [(f(x) \oplus (g(x))^{\sim} \odot x) \wedge x \oplus x^{\sim}] \odot g(x) =$$
$$= [((f(x) \oplus (g(x))^{\sim} \odot x) \oplus x^{\sim}) \wedge (x \oplus x^{\sim})] \odot g(x) =$$

$$= [((f(x) \oplus (g(x))^{\sim} \odot x) \oplus x^{\sim}) \wedge 1] \odot g(x) =$$
$$= [f(x) \oplus ((g(x))^{\sim} \odot x \oplus x^{\sim})] \odot g(x) = [f(x) \oplus ((g(x))^{\sim} \odot (x^{\sim})^{-} \oplus x^{\sim})] \odot g(x) =$$
$$= [f(x) \oplus ((g(x))^{\sim} \vee x^{\sim})] \odot g(x) = [f(x) \oplus (g(x) \wedge x)^{\sim}] \odot g(x) =$$
$$= [f(x) \oplus (g(x))^{\sim}] \odot g(x) = f(x) \wedge g(x).$$

So,

$$f \boxdot (f^{-} \boxplus g) = (f \boxplus g^{\sim}) \boxdot g.$$

$psMV_8$). For $f \in M(I, A)$ (with $I \in \mathcal{I}(A)$) and $x \in I$, we have:

$$(f^{-})^{\sim}(x) = (x \odot (f(x))^{-})^{\sim} \odot x = [((f(x))^{-})^{\sim} \oplus x^{\sim}] \odot x =$$
$$= (f(x) \oplus x^{\sim}) \odot x = f(x) \wedge x = f(x).$$

So, $(f^{-})^{\sim} = f$.■

REMARK 7.7. *To prove that $(M(A), \boxplus, \boxdot, ^{-}, ^{\sim}, \mathbf{0}, \mathbf{1})$ is a pseudo MV-algebra is suffice to ask for multipliers only the axioms $sm - psMV_1$ and $sm - psMV_2$.*

REMARK 7.8. *If pseudo MV -algebra $(A, \oplus, \odot, ^{-}, ^{\sim}, 0, 1)$ is an MV -algebra (i.e $x \oplus y = y \oplus x$ for all $x, y \in A$), then pseudo MV -algebra $(M(A), \boxplus, \boxdot, ^{-}, ^{\sim}, \mathbf{0}, \mathbf{1})$ is an MV -algebra $(M(A), \boxplus, ^{-} \overset{not}{=} *, \mathbf{0})$. Indeed if $I_1, I_2 \in \mathcal{I}(A)$ and $f_i \in M(I_i, A)$, $i = 1, 2$ we have*

$$(f_1 \boxplus f_2)(x) = [f_1(x) \oplus f_2(x)] \wedge x = [f_2(x) \oplus f_1(x)] \wedge x = (f_2 \boxplus f_1)(x),$$

for all $x \in I_1 \cap I_2$, then $f_1 \boxplus f_2 = f_2 \boxplus f_1$, so pseudo MV -algebra $(M(A), \boxplus, \boxdot, ^{-}, ^{\sim}, \mathbf{0}, \mathbf{1})$ is commutative, so is an MV -algebra.

LEMMA 7.16. *Let $f, g \in M(A)$. Then for every $x \in dom(f) \cap dom(g)$:*

(i) $(f \wedge g)(x) = f(x) \wedge g(x)$;
(ii) $(f \vee g)(x) = f(x) \vee g(x)$.

Proof. We recall that in pseudo $MV-$ algebra $M(A)$ we have:

$$f \wedge g = f \boxdot (f^{-} \boxplus g) = g \boxdot (g^{-} \boxplus f) = (f \boxplus g^{\sim}) \boxdot g = (g \boxplus f^{\sim}) \boxdot f,$$
$$f \vee g = f \boxplus f^{\sim} \boxdot g = g \boxplus g^{\sim} \boxdot f = f \boxdot g^{-} \boxplus g = g \boxdot f^{-} \boxplus f.$$

So: (i). Follow immediately from Proposition 7.15, $psMV_7$).
(ii). Follow immediately from Proposition 7.15, $psMV_6$).■

LEMMA 7.17. *Let the map $v_A : B(A) \to M(A)$ defined by $v_A(a) = \overline{f_a}$ for every $a \in B(A)$. Then v_A is an injective morphism of pseudo $MV-$ algebras.*

Proof. Clearly, $v_A(0) = \overline{f_0} = \mathbf{0}$. Let $a, b \in B(A)$. We have:

$$(v_A(a) \boxplus v_A(b))(x) = (v_A(a)(x) \oplus v_A(b)(x)) \wedge x = ((a \wedge x) \oplus (b \wedge x)) \wedge x$$
$$\overset{psmv-c_{40}}{=} (a \oplus b) \wedge x = (v_A(a \oplus b))(x),$$

hence

$$v_A(a \oplus b) = v_A(a) \boxplus v_A(b).$$

Also,

$(v_A(a))^{-}(x) = x \odot (v_A(a)(x))^{-} = x \odot (a \wedge x)^{-} = x \odot (a^{-} \vee x^{-}) = x \odot (x^{-} \oplus a^{-}) =$
(since $a^{-} \in B(A)$)

$$= x \wedge a^{-} = v_A(a^{-})(x),$$

hence

$$v_A(a^-) = (v_A(a))^-,$$

and

$(v_A(a))^\sim(x) = (v_A(a)(x))^\sim \odot x = (a \wedge x)^\sim \odot x = (a^\sim \vee x^\sim) \odot x = (a^\sim \oplus x^\sim) \odot x =$
(since $a^\sim \in B(A)$)

$$= a^\sim \wedge x = v_A(a^\sim)(x),$$

hence

$$v_A(a^\sim) = (v_A(a))^\sim,$$

that is v_A is a morphism of pseudo-MV algebras.

To prove the injectivity of v_A let $a, b \in B(A)$ such that $v_A(a) = v_A(b)$. Then $a \wedge x = b \wedge x$, for every $x \in A$, hence for $x = 1$ we obtain that $a \wedge 1 = b \wedge 1 \Rightarrow a = b$.■

We denote $\mathcal{R}(A) = \{I \subseteq A : I$ is a regular subset of pseudo $MV-$algebra $A\}$ (see Definition 6.5).

REMARK 7.9. *The condition $I \in \mathcal{R}(A)$ is equivalent with the condition: for every $x, y \in A$, if $f_{x|I\cap B(A)} = f_{y|I\cap B(A)}$, then $x = y$.*

LEMMA 7.18. *If $I_1, I_2 \in \mathcal{I}(A) \cap \mathcal{R}(A)$, then $I_1 \cap I_2 \in \mathcal{I}(A) \cap \mathcal{R}(A)$.*

REMARK 7.10. *By Lemma 7.18, we deduce that*

$$M_r(A) = \{f \in M(A) : dom(f) \in \mathcal{I}(A) \cap \mathcal{R}(A)\}$$

is a pseudo-MV subalgebra of $M(A)$.

PROPOSITION 7.19. *$M_r(A)$ is a Boolean subalgebra of $M(A)$.*

Proof. Let $f : I \to A$ be a strong multiplier on A with $I \in \mathcal{I}(A) \cap \mathcal{R}(A)$. Then

$$e \wedge [f \boxplus f](x) = e \wedge [f(x) \oplus f(x)] \wedge x =$$

$$= [e \wedge (f(x) \oplus f(x))] \wedge x \overset{psmv-c_{40}}{=} [(e \wedge f(x)) \oplus (e \wedge f(x))] \wedge x \overset{sm-psMV_4}{=}$$

$$\overset{sm-psMV_4}{=} [(x \wedge f(e)) \oplus (x \wedge f(e))] \wedge x = [(x \oplus x) \wedge f(e)] \wedge x =$$

$$= (x \oplus x) \wedge x \wedge f(e) = x \wedge f(e) = e \wedge f(x),$$

so $(f(x) \oplus f(x)) \wedge x = f(x)$ (since $I \in \mathcal{R}(A)$), hence $f \boxplus f = f$, that is $M_r(A)$ is a Boolean subalgebra of $M(A)$. ■

DEFINITION 7.4. *Given two strong multipliers f_1, f_2 on A, we say that f_2 extends f_1 if $dom(f_1) \subseteq dom(f_2)$ and $f_{2|dom(f_1)} = f_1$; we write $f_1 \leq f_2$ if f_2 extended f_1. A strong multiplier f is called maximal if f can not be extended to a strictly larger domain.*

LEMMA 7.20. (*i*) *If $f_1, f_2 \in M(A)$, $f \in M_r(A)$ and $f \leq f_1, f \leq f_2$, then f_1 and f_2 coincide on $dom(f_1) \cap dom(f_2)$;*

(*ii*) *Every strong multiplier $f \in M_r(A)$ can be extended to a maximal strong multiplier. Moreover, each principal strong multiplier f_a with $a \in B(A)$ and $dom(f_a) \in \mathcal{I}(A) \cap \mathcal{R}(A)$ can be uniquely extended to a total strong multiplier $\overline{f_a}$ and each non-principal strong multiplier can be extended to a maximal non-principal one.*

Proof. See the proof of Lemma 6.13. ■

On the boolean algebra $M_r(A)$ we consider the relation ρ_A defined by

$(f_1, f_2) \in \rho_A$ iff f_1 and f_2 coincide on the intersection of their domains.

LEMMA 7.21. *ρ_A is a congruence on $M_r(A)$.*

Proof. The reflexivity and the symmetry of ρ_A are immediately; to prove the transitivity of ρ_A let $(f_1, f_2), (f_2, f_3) \in \rho_A$. Therefore f_1, f_2 and respectively f_2, f_3 coincide on the intersection of their domains. If by contrary, there exists $x_0 \in dom(f_1) \cap dom(f_3)$ such that $f_1(x_0) \neq f_3(x_0)$, since $dom(f_2) \in \mathcal{R}(A)$, there exists $e \in dom(f_2) \cap B(A)$ such that $e \wedge f_1(x_0) \neq e \wedge f_3(x_0) \Leftrightarrow e \odot f_1(x_0) \neq e \odot f_3(x_0) \Leftrightarrow f_1(e \odot x_0) \neq f_3(e \odot x_0)$ which is contradictory, since $e \odot x_0 \in dom(f_1) \cap dom(f_2) \cap dom(f_3)$.

To prove the compatibility of ρ_A with the operations $\boxplus, ^-$ and $^\sim$ on $M_r(A)$, let $(f_1, f_2), (g_1, g_2) \in \rho_A$. So, we have f_1, f_2 and respectively g_1, g_2 coincide on the intersection of their domains.

To prove $(f_1 \boxplus g_1, f_2 \boxplus g_2) \in \rho_A$ let $x \in dom(f_1) \cap dom(f_2) \cap dom(g_1) \cap dom(g_2)$. Then $f_1(x) = f_2(x)$ and $g_1(x) = g_2(x)$, hence $(f_1 \boxplus g_1)(x) = [f_1(x) \oplus g_1(x)] \wedge x = [f_2(x) \oplus g_2(x)] \wedge x = (f_2 \boxplus g_2)(x)$, that is $f_1 \boxplus g_1, f_2 \boxplus g_2$ coincide on the intersection of their domains, hence ρ_A is compatible with the operation $\boxplus$.

If $x \in dom(f_1) \cap dom(f_2)$ then $f_1(x) = f_2(x)$ and $f_1^-(x) = x \odot (f_1(x))^- = x \odot (f_2(x))^- = f_2^-(x)$, and $f_1^\sim(x) = (f_1(x))^\sim \odot x = (f_2(x))^\sim \odot x = f_2^\sim(x)$, hence f_1^-, f_2^- and $f_1^\sim$, $f_2^\sim$ coincide on the intersection of their domains, hence ρ_A is compatible with the operations $^-$ and $^\sim$. ■

REMARK 7.11. *We denote by $Q(A)$ the quotient pseudo $MV-$ algebra $M_r(A)/\rho_A$; this algebra will have a very important role for this paper (see Theorem 7.26).*

For $f \in M_r(A)$ with $I = dom(f) \in \mathcal{I}(A) \cap \mathcal{R}(A)$, we denote by $[f, I]$ the congruence class of f modulo ρ_A.

LEMMA 7.22. *Let the map $\overline{v_A} : B(A) \to Q(A)$ defined by $\overline{v_A}(a) = [\overline{f_a}, A]$ for every $a \in B(A)$. Then*

(i) $\overline{v_A}$ is an injective morphism of Boolean algebras,
(ii) $\overline{v_A}(B(A)) \in \mathcal{R}(Q(A))$.

Proof. (i). Follow from Lemma 7.17.

(iii). To prove $\overline{v_A}(B(A)) \in \mathcal{R}(Q(A))$, if by contrary there exist $f_1, f_2 \in M_r(A)$ such that $[f_1, dom(f_1)] \neq [f_2, dom(f_2)]$ (that is there exists $x_0 \in dom(f_1) \cap dom(f_2)$ such that $f_1(x_0) \neq f_2(x_0)$) and $[f_1, dom(f_1)] \wedge [\overline{f_a}, A] = [f_2, dom(f_2)] \wedge [\overline{f_a}, A]$ for every $[\overline{f_a}, A] \in \overline{v_A}(B(A)) \cap B(Q(A))$ (that is by (ii) for every $[\overline{f_a}, A] \in \overline{v_A}(B(A))$ with $a \in B(A)$), then $(f_1 \wedge \overline{f_a})(x) = (f_2 \wedge \overline{f_a})(x)$ for every $x \in dom(f_1) \cap dom(f_2)$ and every $a \in B(A) \Leftrightarrow f_1(x) \wedge a \wedge x = f_2(x) \wedge a \wedge x$ for every $x \in dom(f_1) \cap dom(f_2)$ and every $a \in B(A)$. For $a = 1$ and $x = x_0$ we obtain that $f_1(x_0) \wedge x_0 = f_2(x_0) \wedge x_0 \Leftrightarrow f_1(x_0) = f_2(x_0)$ which is contradictory. ■

REMARK 7.12. *Since for every $a \in B(A)$, $\overline{f_a}$ is the unique maximal strong multiplier on $[\overline{f_a}, A]$ (by Lemma 7.20) we can identify $[\overline{f_a}, A]$ with $\overline{f_a}$. So, since $\overline{v_A}$ is injective map, the elements of $B(A)$ can be identified with the elements of the set $\{\overline{f_a} : a \in B(A)\}$.*

LEMMA 7.23. *In view of the identifications made above, if* $[f, dom(f)] \in Q(A)$ *(with* $f \in M_r(A)$ *and* $I = dom(f) \in \mathcal{I}(A) \cap \mathcal{R}(A)$*), then*

$$I \cap B(A) \subseteq \{a \in B(A) : \overline{f_a} \wedge [f, dom(f)] \in B(A)\}.$$

Proof. Let $a \in I \cap B(A)$. Since for every $x \in I, (\overline{f_a} \wedge f)(x) = \overline{f_a}(x) \wedge f(x) = a \wedge x \wedge f(x) = a \wedge f(x) = a \odot f(x) = f(a \odot x) = x \odot f(a)$ (by $psMV_{19}$) $= x \wedge f(a)$, we deduce that $\overline{f_a} \wedge f$ is principal. ■

DEFINITION 7.5. Let A be a pseudo $MV-$ algebra. A pseudo $MV-$ algebra F is called *pseudo* $MV-$ *algebra of fractions of* A if:

($psMVfr_1$) $B(A)$ is a pseudo $MV-$ subalgebra of F (that is $B(A) \leq F$);

($psMVfr_2$) For every $a', b', c' \in F, a' \neq b'$, there exists $e \in B(A)$ such that $e \wedge a' \neq e \wedge b'$ and $e \wedge c' \in B(A)$.

So, pseudo $MV-$ algebra $B(A)$ is a pseudo $MV-$ algebra of fractions of itself (since $1 \in B(A)$).

As a notational convenience, we write $A \preceq F$ to indicate that F is a pseudo $MV-$ algebra of fractions for A.

DEFINITION 7.6. A pseudo $MV-$algebra A_M is a *maximal pseudo* $MV-$ *algebra of quotients* of A if $A \preceq A_M$ and for every pseudo $MV-$ algebra F with $A \preceq F$ there exists an injective morphism of pseudo $MV-$ algebras $i : F \to A_M$.

REMARK 7.13. *If* $A \preceq F$*, then* F *is a Boolean algebra hence* A_M *is a Boolean algebra. Indeed, if* $a' \in F$ *such that* $a' \neq a' \oplus a'$*, then there exists* $e \in B(A)$ *such that* $e \wedge a' \in B(A)$ *and* $e \wedge a' \neq e \wedge (a' \oplus a') \overset{psmv-c_{43}}{=} (e \wedge a') \oplus (e \wedge a')$*, which is contradictory!*

LEMMA 7.24. *Let* $A \preceq F$ *; then for every* $a', b' \in F, a' \neq b'$*, and any finite sequence* $c'_1, ..., c'_n \in F$*, there exists* $e \in B(A)$ *such that* $e \wedge a' \neq e \wedge b'$ *and* $e \wedge c'_i \in B(A)$ *for* $i = 1, 2, ..., n$ $(n \geq 2)$.

LEMMA 7.25. *Let* $A \preceq F$ *and* $a' \in F$*. Then*

$$I_{a'} = \{e \in B(A) : e \wedge a' \in B(A)\} \in \mathcal{I}(B(A)) \cap \mathcal{R}(A).$$

THEOREM 7.26. $Q(A)$ *is a maximal pseudo* $MV-$ *algebra of quotients of* A*.*

Proof. The fact that $B(A)$ is a pseudo $MV-$ subalgebra of $Q(A)$ follows from Lemma 7.22, (i). To prove $psMVfr_2$ see the proof of Theorem 6.19).

To prove the maximality of $Q(A)$, let F be a pseudo $MV-$algebra such that $A \preceq F$; thus $B(A) \subseteq B(F)$

$$\begin{array}{ccc} A & \preceq & F \\ & & \swarrow_i \\ & Q(A) & \end{array}$$

For $a' \in F, I_{a'} = \{e \in B(A) : e \wedge a' \in B(A)\} \in \mathcal{I}(B(A)) \cap \mathcal{R}(A)$ (by Lemma 7.25).

Thus $f_{a'} : I_{a'} \to A$ defined by $f_{a'}(x) = x \wedge a'$ is a strong multiplier (see the proof of Theorem 6.19).

We define $i : F \to Q(A)$, by $i(a') = [f_{a'}, I_{a'}]$, for every $a' \in F$. Clearly $i(0) = \mathbf{0}$. For $a', b' \in F$ and $x \in I_{a'} \cap I_{b'}$, we have

$$(i(a') \boxplus i(b'))(x) = [(a' \wedge x) \oplus (b' \wedge x)] \wedge x \overset{psmv-c_{40}}{=}$$

$$= (a' \oplus b') \wedge x = i(a' \oplus b')(x),$$

hence $i(a') \boxplus i(b') = i(a' \oplus b')$.

Also, for $x \in I_{a'}$ we have

$$(i(a'))^-(x) = x \odot [i(a')(x)]^- = x \odot (a' \wedge x)^- =$$
$$= x \odot (a' \odot x)^- = x \odot [x^- \oplus (a')^-] = x \wedge (a')^- =$$
$$= f_{(a')^-}(x) = i((a')^-)(x),$$

and

$$(i(a'))^\sim(x) = [i(a')(x)]^\sim \odot x = (a' \wedge x)^\sim \odot x =$$
$$= (x \odot a')^\sim \odot x = [(a')^\sim \oplus x^\sim] \odot x = (a')^\sim \wedge x =$$
$$= f_{(a')^\sim}(x) = i((a')^\sim)(x),$$

hence

$$i((a')^-) = (i(a'))^-,$$

and

$$i((a')^\sim) = (i(a'))^\sim,$$

that is i is a morphism of pseudo $MV-$ algebras.

To prove the injectivity of i, let $a', b' \in F$ such that $i(a') = i(b')$. It follow that $[f_{a'}, I_{a'}] = [f_{b'}, I_{b'}]$ so $f_{a'}(x) = f_{b'}(x)$ for every $x \in I_{a'} \cap I_{b'}$. We get $a' \wedge x = b' \wedge x$ for every $x \in I_{a'} \cap I_{b'}$. If $a' \neq b'$, by Lemma 7.24 (since $A \preceq F$), there exists $e \in B(A)$ such that $e \wedge a', e \wedge b' \in B(A)$ and $e \wedge a' \neq e \wedge b'$ which is contradictory (since $e \wedge a', e \wedge b' \in B(A)$ implies $e \in I_{a'} \cap I_{b'}$).■

REMARK 7.14.
1. *If A is a pseudo $MV-$ algebra with $B(A) = \{0,1\} = L_2$ and $A \preceq F$ then $F = \{0,1\}$, hence $Q(A) \approx L_2$. Indeed, if $a, b, c \in F$ with $a \neq b$, then by $psMVfr_2$ there exists $e \in B(A)$ such $e \wedge a \neq e \wedge b$ (hence $e \neq 0$) and $e \wedge c \in B(A)$. Clearly, $e = 1$, hence $c \in B(A)$, that is $F = B(A)$. As examples of pseudo $MV-$ algebras with this property we have local pseudo $MV-$ algebras and pseudo $MV-$ chain.*
2. *If A is an MV-algebra, then $Q(A)$ is the maximal MV-algebra of quotients obtained in Section 3 for $MV-$ algebras.*
3. *If A is an Boolean algebra, then $B(A) = A$ and $Q(A)$ is the classical Dedekind-MacNeille completion of A (see* [**122**]*, p.687).*
4. *In* [**58**] *is proved that:*
   (i) Any archimedean pseudo $MV-$ algebra is commutative i.e. an MV algebra.
   (ii) A pseudo $MV-$ algebra has the Dedekind-MacNeille completion as a pseudo $MV-$ algebra iff A is archimedean.

As in the case of MV and $BL-$ algebras, to obtain the maximal pseudo MV -algebra of quotients $Q(A)$ as a localization relative to a topology $\mathcal{F}$ we develope another theory of multipliers (meaning we add new axioms for $\mathcal{F}$-multipliers).

DEFINITION 7.7. Let $\mathcal{F}$ be a topology on a pseudo MV-algebra A. A *strong - $\mathcal{F}-$ multiplier* is a mapping $f : I \to A/\theta_{\mathcal{F}}$ (where $I \in \mathcal{F}$) which verifies the axioms $m - psMV_1$ and $m - psMV_2$ (see Definition 7.1) and

$(m - psMV_3)$ If $e \in I \cap B(A)$, then $f(e) \in B(A/\theta_{\mathcal{F}})$;

$(m - psMV_4)$ $(x/\theta_{\mathcal{F}}) \wedge f(e) = (e/\theta_{\mathcal{F}}) \wedge f(x)$, for every $e \in I \cap B(A)$ and $x \in I$.

REMARK 7.15. *If $\mathcal{F} = \{A\}$, then $\theta_{\mathcal{F}}$ is the identity congruence of A so an strong $\mathcal{F}-$ multiplier is a strong total multiplier.*

REMARK 7.16. *The maps $\mathbf{0}, \mathbf{1} : A \to A/\theta_{\mathcal{F}}$ defined by $\mathbf{0}(x) = 0/\theta_{\mathcal{F}}$ and $\mathbf{1}(x) = x/\theta_{\mathcal{F}}$ for every $x \in A$ are strong - $\mathcal{F}-$ multipliers. We recall that if $f_i : I_i \to A/\theta_{\mathcal{F}}$, (with $I_i \in \mathcal{F}$, $i = 1, 2$) are $\mathcal{F}-$ multipliers, we consider the mapping $f_1 \boxplus f_2 : I_1 \cap I_2 \to A/\theta_{\mathcal{F}}$ defined by*

$$(f_1 \boxplus f_2)(x) = (f_1(x) \oplus f_2(x)) \wedge x/\theta_{\mathcal{F}}$$

for any $x \in I_1 \cap I_2$, and for any $\mathcal{F}-$ multiplier $f : I \to A/\theta_{\mathcal{F}}$ (with $I \in \mathcal{F}$) we consider the mappings

$$f^-, f^\sim : I \to A/\theta_{\mathcal{F}}$$

defined by

$$f^-(x) = x/\theta_{\mathcal{F}} \odot (f(x))^-$$

and

$$f^\sim(x) = (f(x))^\sim \odot x/\theta_{\mathcal{F}}$$

for any $x \in I$. If f_1, f_2 and f are strong - $\mathcal{F}-$ multipliers, then the multipliers $f_1 \boxplus f_2, f^-, f^\sim$ are also strong - $\mathcal{F}-$ multipliers. Clearly, if $e \in I \cap B(A)$, then

$$(f_1 \boxplus f_2)(e) = [f_1(e) \oplus f_2(e)] \wedge e/\theta_{\mathcal{F}} \in B(A/\theta_{\mathcal{F}})$$
$$f^-(e) = e/\theta_{\mathcal{F}} \odot [f(e)]^- \in B(A/\theta_{\mathcal{F}})$$
$$f^\sim(e) = [f(e)]^\sim \odot e/\theta_{\mathcal{F}} \in B(A/\theta_{\mathcal{F}}).$$

For $e \in I_1 \cap I_2 \cap B(A)$ and $x \in I_1 \cap I_2$ we have:

$$x/\theta_{\mathcal{F}} \wedge (f_1 \boxplus f_2)(e) = x/\theta_{\mathcal{F}} \wedge [(f_1(e) \oplus f_2(e)) \wedge e/\theta_{\mathcal{F}}] = (f_1(e) \oplus f_2(e)) \wedge x/\theta_{\mathcal{F}} \wedge e/\theta_{\mathcal{F}}$$
$$\overset{psmv-c_{43}}{=} (f_1(e) \oplus f_2(e)) \wedge x/\theta_{\mathcal{F}},$$

and

$$e/\theta_{\mathcal{F}} \wedge (f_1 \boxplus f_2)(x) = e/\theta_{\mathcal{F}} \wedge [(f_1(x) \oplus f_2(x)) \wedge x/\theta_{\mathcal{F}}] = e/\theta_{\mathcal{F}} \odot [(f_1(x) \oplus f_2(x)) \wedge x/\theta_{\mathcal{F}}]$$
$$\overset{psmv-c_{26}}{=} [e/\theta_{\mathcal{F}} \odot (f_1(x) \oplus f_2(x))] \wedge (e \odot x)/\theta_{\mathcal{F}} \overset{psmv-c_{43}}{=}$$
$$\overset{psmv-c_{43}}{=} [(e/\theta_{\mathcal{F}} \odot f_1(x)) \oplus (e/\theta_{\mathcal{F}} \odot f_2(x))] \wedge (e \odot x)/\theta_{\mathcal{F}}$$
$$= [x/\theta_{\mathcal{F}} \odot f_1(e) \oplus x/\theta_{\mathcal{F}} \odot f_2(e)] \wedge (e \odot x)/\theta_{\mathcal{F}} = [(f_1(e) \wedge x/\theta_{\mathcal{F}}) \oplus (f_2(e) \wedge x/\theta_{\mathcal{F}})] \wedge (e \wedge x)/\theta_{\mathcal{F}}$$
$$= [[(f_1(e) \wedge x/\theta_{\mathcal{F}}) \oplus (f_2(e) \wedge x/\theta_{\mathcal{F}})] \wedge x/\theta_{\mathcal{F}}] \wedge e/\theta_{\mathcal{F}}$$
$$\overset{psmv-c_{40}}{=} ((f_1(e) \oplus f_2(e)) \wedge x/\theta_{\mathcal{F}}) \wedge e/\theta_{\mathcal{F}} = ((f_1(e) \oplus f_2(e)) \wedge e/\theta_{\mathcal{F}}) \wedge x/\theta_{\mathcal{F}}$$
$$\overset{psmv-c_{43}}{=} (f_1(e) \oplus f_2(e)) \wedge x/\theta_{\mathcal{F}}$$

hence

$$x/\theta_{\mathcal{F}} \wedge (f_1 \boxplus f_2)(e) = e/\theta_{\mathcal{F}} \wedge (f_1 \boxplus f_2)(x).$$

Since $f \in M(I, A/\theta_{\mathcal{F}})$, for $e \in I \cap B(A)$ and $x \in I$ we have:

$$x/\theta_{\mathcal{F}} \wedge f(e) = e/\theta_{\mathcal{F}} \wedge f(x) \Rightarrow (x/\theta_{\mathcal{F}})^- \vee (f(e))^- = (e/\theta_{\mathcal{F}})^- \vee (f(x))^-$$
$$\Rightarrow (x/\theta_{\mathcal{F}})^- \oplus (f(e))^- = (e/\theta_{\mathcal{F}})^- \oplus (f(x))^-$$
$$\Rightarrow e/\theta_{\mathcal{F}} \odot x/\theta_{\mathcal{F}} \odot [(x/\theta_{\mathcal{F}})^- \oplus (f(e))^-] = x/\theta_{\mathcal{F}} \odot e/\theta_{\mathcal{F}} \odot [(e/\theta_{\mathcal{F}})^- \oplus (f(x))^-] \Rightarrow$$
$$\Rightarrow e/\theta_{\mathcal{F}} \odot [x/\theta_{\mathcal{F}} \wedge (f(e))^-] = x/\theta_{\mathcal{F}} \odot [e/\theta_{\mathcal{F}} \wedge (f(x))^-]$$
$$\Rightarrow e/\theta_{\mathcal{F}} \odot x/\theta_{\mathcal{F}} \odot (f(e))^- = x/\theta_{\mathcal{F}} \odot e/\theta_{\mathcal{F}} \odot (f(x))^-$$
$$\Rightarrow x/\theta_{\mathcal{F}} \odot [e/\theta_{\mathcal{F}} \odot (f(e))^-] = e/\theta_{\mathcal{F}} \odot [x/\theta_{\mathcal{F}} \odot (f(x))^-]$$
$$\Rightarrow x/\theta_{\mathcal{F}} \wedge [e/\theta_{\mathcal{F}} \odot (f(e))^-] = e/\theta_{\mathcal{F}} \wedge [x/\theta_{\mathcal{F}} \odot (f(x))^-] \Rightarrow x/\theta_{\mathcal{F}} \wedge f^-(e) = e/\theta_{\mathcal{F}} \wedge f^-(x),$$

and

$$x/\theta_{\mathcal{F}} \wedge f(e) = e/\theta_{\mathcal{F}} \wedge f(x) \Rightarrow (x/\theta_{\mathcal{F}})^{\sim} \vee (f(e))^{\sim} = (e/\theta_{\mathcal{F}})^{\sim} \vee (f(x))^{\sim}$$
$$\Rightarrow (f(e))^{\sim} \oplus (x/\theta_{\mathcal{F}})^{\sim} = (f(x))^{\sim} \oplus (e/\theta_{\mathcal{F}})^{\sim}$$
$$\Rightarrow [(f(e))^{\sim} \oplus (x/\theta_{\mathcal{F}})^{\sim}] \odot x/\theta_{\mathcal{F}} \odot e/\theta_{\mathcal{F}} = [(f(x))^{\sim} \oplus (e/\theta_{\mathcal{F}})^{\sim}] \odot e/\theta_{\mathcal{F}} \odot x/\theta_{\mathcal{F}} \Rightarrow$$
$$\Rightarrow [(f(e))^{\sim} \wedge x/\theta_{\mathcal{F}}] \odot e/\theta_{\mathcal{F}} = [(f(x))^{\sim} \wedge e/\theta_{\mathcal{F}}] \odot x/\theta_{\mathcal{F}}$$
$$\Rightarrow (f(e))^{\sim} \odot x/\theta_{\mathcal{F}} \odot e/\theta_{\mathcal{F}} = (f(x))^{\sim} \odot e/\theta_{\mathcal{F}} \odot x/\theta_{\mathcal{F}}$$
$$\Rightarrow [(f(e))^{\sim} \odot e/\theta_{\mathcal{F}}] \odot x/\theta_{\mathcal{F}} = [(f(x))^{\sim} \odot x/\theta_{\mathcal{F}}] \odot e/\theta_{\mathcal{F}}$$
$$\Rightarrow [(f(e))^{\sim} \odot e/\theta_{\mathcal{F}}] \wedge x/\theta_{\mathcal{F}} = [(f(x))^{\sim} \odot x/\theta_{\mathcal{F}}] \wedge e/\theta_{\mathcal{F}}$$
$$\Rightarrow x/\theta_{\mathcal{F}} \wedge f^{\sim}(e) = e/\theta_{\mathcal{F}} \wedge f^{\sim}(x).$$

REMARK 7.17. *Analogous as in the case of $\mathcal{F}-$ multipliers if we work with strong -$\mathcal{F}-$ multipliers we obtain a pseudo $MV-$ subalgebra of $A_{\mathcal{F}}$ denoted by $s-A_{\mathcal{F}}$ which will be called the strong -localization pseudo $MV-$ algebra of A with respect to the topology $\mathcal{F}$.*

REMARK 7.18. *If $\mathcal{F} = \mathcal{I}(A) \cap \mathcal{R}(A)$ is the topology of regular ordered ideals (see Example 7.2), then $\theta_{\mathcal{F}}$ is the identity congruence of A and*

$$s - A_{\mathcal{F}} = \varinjlim_{I \in \mathcal{F}} M(I, A),$$

where $M(I, A)$ is the set of multipliers of A having the domain I (in the sense of Definitions 7.3).

In these situations we obtain:

PROPOSITION 7.27. *In the case $\mathcal{F} = \mathcal{I}(A) \cap \mathcal{R}(A)$, $s - A_{\mathcal{F}}$ is exactly a maximal pseudo $MV-$ algebra $Q(A)$ of quotients of A.*

2.3. Application 3: Pseudo MV algebra of frations relative to a $\wedge-$closed system. Let A be a pseudo MV -algebra. We denote by $\mathcal{S}(A)$ the set of all $\wedge-$closed system of A (see Definition 6.1). Clearly $\{1\}, A \in \mathcal{S}(A)$.

For $S \in \mathcal{S}(A)$, on the pseudo $MV-$ algebra A we consider the relation θ_S defined by

$$(x, y) \in \theta_S \text{ iff there exists } e \in S \cap B(A) \text{ such that } x \wedge e = y \wedge e.$$

LEMMA 7.28. *θ_S is a congruence on A.*

Proof. The reflexivity (since $1 \in S \cap B(A)$) and the symmetry of θ_S are immediately. To prove the transitivity of θ_S, let $(x, y), (y, z) \in \theta_S$. Thus there exists $e, f \in S \cap B(A)$ such that $x \wedge e = y \wedge e$ and $y \wedge f = z \wedge f$. If denote $g = e \wedge f \in S \cap B(A)$, then $g \wedge x = (e \wedge f) \wedge x = (e \wedge x) \wedge f = (y \wedge e) \wedge f = (y \wedge f) \wedge e = (z \wedge f) \wedge e = z \wedge (f \wedge e) = z \wedge g$, hence $(x, z) \in \theta_S$.

To prove the compatibility of θ_S with the operations $\oplus, ^-$ and $^{\sim}$, let $x, y, z, t \in A$ such that $(x, y) \in \theta_S$ and $(z, t) \in \theta_S$. Thus there exists $e, f \in S \cap B(A)$ such that $x \wedge e = y \wedge e$ and $z \wedge f = t \wedge f$; we denote $g = e \wedge f \in S \cap B(A)$.

By $psmv - c_{43}$ we obtain:

$$(x \oplus z) \wedge g = (g \wedge x) \oplus (g \wedge z) = (e \wedge f \wedge x) \oplus (e \wedge f \wedge z) =$$
$$= (y \wedge e \wedge f) \oplus (e \wedge t \wedge f) = (g \wedge y) \oplus (g \wedge t) = (y \oplus t) \wedge g,$$

hence $(x \oplus z, y \oplus t) \in \theta_S$

From $x \wedge e = y \wedge e$ we deduce

$$x \odot e = y \odot e \Rightarrow (x \odot e)^- = (y \odot e)^- \Rightarrow e^- \oplus x^- = e^- \oplus y^-,$$

so $e \odot (e^- \oplus x^-) = e \odot (e^- \oplus y^-)$, hence $x^- \wedge e = y^- \wedge e$, that is $(x^-, y^-) \in \theta_S$.

From $x \wedge e = y \wedge e$ we deduce $e^\sim \oplus x^\sim = e^\sim \oplus y^\sim$. Since $e \in B(A)$ it follows that $x^\sim \oplus e^\sim = y^\sim \oplus e^\sim$. So, $(x^\sim \oplus e^\sim) \odot e = (y^\sim \oplus e^\sim) \odot e$, hence $x^\sim \wedge e = y^\sim \wedge e$, that is $(x^\sim, y^\sim) \in \theta_S$.■

For x we denote by x/S the equivalence class of x relative to θ_S and by

$$A[S] = A/\theta_S.$$

By $p_S : A \to A[S]$ we denote the canonical map defined by $p_S(x) = x/S$, for every $x \in A$. Clearly, in $A[S]$, $\mathbf{0} = 0/S$, $\mathbf{1} = 1/S$ and for every $x, y \in A$,

$$x/S \oplus y/S = (x \oplus y)/S,$$

$$(x/S)^- = x^-/S,$$

$$(x/S)^\sim = x^\sim/S,$$

So, p_S is an onto morphism of pseudo $MV-$ algebras.

Remark 7.19. *Since for every $s \in S \cap B(A)$, $s \wedge s = s \wedge 1$ we deduce that $s/S = 1/S = \mathbf{1}$, hence $p_S(S \cap B(A)) = \{\mathbf{1}\}$.*

Proposition 7.29. *If $a \in A$, then $a/S \in B(A[S])$ iff there exists $e \in S \cap B(A)$ such that $e \wedge a \in B(A)$. So, if $e \in B(A)$, then $e/S \in B(A[S])$.*

Proof. For $a \in A$, we have $a/S \in B(A[S]) \Leftrightarrow a/S \oplus a/S = a/S \Leftrightarrow (a \oplus a)/S = a/S \Leftrightarrow$ there exists $e \in S \cap B(A)$ such that $(a \oplus a) \wedge e = a \wedge e \overset{psmv-c_{43}}{\Leftrightarrow} (a \wedge e) \oplus (a \wedge e) = a \wedge e \Leftrightarrow a \wedge e \in B(A)$.

If $e \in B(A)$, since $1 \in S \cap B(A)$ and $1 \wedge e = e \in B(A)$ we deduce that $e/S \in B(A[S])$. ■

Theorem 7.30. *If A' is a pseudo $MV-$ algebra and $f : A \to A'$ is a morphism of pseudo $MV-$ algebras such that $f(S \cap B(A)) = \{\mathbf{1}\}$, then there exists a unique morphism of pseudo $MV-$ algebras $f' : A[S] \to A'$ such that the diagram*

$$\begin{array}{ccc} A & \xrightarrow{p_S} & A[S] \\ & \searrow^{f} \quad \swarrow_{f'} & \\ & A' & \end{array}$$

is commutative (i.e. $f' \circ p_S = f$).

Proof. If $x, y \in A$ and $p_S(x) = p_S(y)$, then $(x, y) \in \theta_S$, hence there exists $e \in S \cap B(A)$ such that $x \wedge e = y \wedge e$. Since f is morphism of pseudo $MV-$ algebras, we obtain that $f(x \wedge e) = f(y \wedge e) \Leftrightarrow f(x) \wedge f(e) = f(y) \wedge f(e) \Leftrightarrow f(x) \wedge \mathbf{1} = f(y) \wedge \mathbf{1} \Leftrightarrow f(x) = f(y)$.

From this observation we deduce that the map $f' : A[S] \to A'$ defined for $x \in A$ by $f'(x/S) = f(x)$ is correctly defined. Clearly, f' is a morphism of pseudo $MV-$ algebras. The unicity of f' follows from the fact that p_S is a onto map.■

Remark 7.20. *Theorem 7.30 allows us to call $A[S]$ the pseudo $MV-$ algebra of fractions relative to the $\wedge-$closed system S .*

REMARK 7.21. *If pseudo $MV-$ algebra $(A,\oplus,\odot,^-,^\sim,0,1)$ is an $MV-$ algebra (i.e. $x\oplus y=y\oplus x$, for all $x,y\in A$), then $x/S\oplus y/S=(x\oplus y)/S=(y\oplus x)/S=y/S\oplus x/S$, for all $x,y\in A$. So, in this case, pseudo $MV-$ algebra $(A[S],\oplus,\odot,^-,^\sim,\mathbf{0},\mathbf{1})$ is an $MV-$ algebra. In Chapter 2 us to call $A[S]$ the $MV-$ algebra of fractions relative to the $\wedge-$closed system S.*

EXAMPLE 7.7. *If $S=\{1\}$ or S is such that $1\in S$ and $S\cap(B(A)\backslash\{1\})=\varnothing$, then for $x,y\in A,(x,y)\in\theta_S\Longleftrightarrow x\wedge 1=y\wedge 1\Longleftrightarrow x=y$, hence in this case $A[S]=A$.*

EXAMPLE 7.8. *If S is an $\wedge-$closed system such that $0\in S$ (for example $S=A$ or $S=B(A)$), then for every $x,y\in A$, $(x,y)\in\theta_S$ (since $x\wedge 0=y\wedge 0$ and $0\in S\cap B(A)$), hence in this case $A[S]=\mathbf{0}$.*

If $\mathcal{F}_S$ is the topology associated with an $\wedge-$closed system $S\subseteq A$ (see Example 7.3), then:

PROPOSITION 7.31. *The pseudo $MV-$ algebra $s-A_{\mathcal{F}_S}$ is isomorphic with $B(A[S])$.*

Proof. For $x,y\in A$ we have $(x,y)\in\theta_{\mathcal{F}_S}\Leftrightarrow$ there exists $I\in\mathcal{F}_S$ (hence $I\cap S\cap B(A)\neq\oslash$) such that $x\wedge e=y\wedge e$ for any $e\in I\cap B(A)$. Since $\theta_{\mathcal{F}_S}=\theta_S$ we have $A[S]=A/\theta_S$; therefore an strong $\mathcal{F}_S-$multiplier can be considered in this case as a mapping $f:I\to A[S]$ $(I\in\mathcal{F}_S)$ having the properties $f(e\odot x)=e/S\odot f(x)$, $f(x)\leq x/S$, for every $x\in I$, and if $e\in I\cap B(A)$, then $f(e)\in B(A[S])$ and for every $e\in I\cap B(A)$ and $x\in I$,

$$(e/S)\wedge f(x)=(x/S)\wedge f(e)$$

(x/S denotes the congruence class of x relative to θ_S).

We can define the mapping injective and surjective (see the proof of Proposition 6.34):

$$\alpha:s-A_{\mathcal{F}_S}=\varinjlim_{I\in\mathcal{F}_S}M(I,A[S])\to B(A[S])$$

by putting

$$\alpha(\widehat{(I,f)})=f(s)\in B(A[S])$$

where $s\in I\cap S\cap B(A)$.

This mapping is a morphism of pseudo $MV-$ algebras.

Indeed, $\alpha(\mathbf{0})=\alpha(\widehat{(A,\mathbf{0})})=\mathbf{0}(e)=0/S=\mathbf{0}$ for every $e\in S\cap B(A)$. If $\widehat{(I,f)}\in s-A_{\mathcal{F}_S}$, we have

$$\alpha(\widehat{(I,f)^-})=\alpha(\widehat{(I,f^-)})=f^-(e)=(e/S)\odot[f(e)]^-=$$
$$=1\odot(f(e))^-=(f(e))^-=(\alpha(\widehat{(I,f)}))^-$$

and

$$\alpha(\widehat{(I,f)^\sim})=\alpha(\widehat{(I,f^\sim)})=f^\sim(e)=[f(e)]^\sim\odot(e/S)=(f(e))^\sim\odot 1$$
$$=(f(e))^\sim=(\alpha(\widehat{(I,f)}))^\sim$$

(with $e\in I\cap S\cap B(A)$). Also, for every $\widehat{(I_i,f_i)}\in s-A_{\mathcal{F}_S},i=1,2$ we have:

$$\alpha[\widehat{(I_1,f_1)}+\widehat{(I_2,f_2)}]=\alpha[(I_1\cap I_2,f_1\boxplus f_2).]=$$
$$=(f_1\boxplus f_2)(e)=(f_1(e)\oplus f_2(e))\wedge(e/S)=f_1(e)\oplus f_2(e)=$$
$$=\alpha[\widehat{(I_1,f_1)}]\oplus\alpha[\widehat{(I_2,f_2)}]$$

(with $e\in I_1\cap I_2\cap S\cap B(A)$).

So, α is an isomorphism of pseudo $MV-$ algebras.■

REMARK 7.22. *In the proof of Proposition 7.31 the axiom $m - psMV_4$ is not necessarily.*

2.4. Application 4: Localization of *lu*-groups. Pseudo $MV-$ algebras can be studied within the context of lattice-ordered groups with strong units (lu-groups). This viewpoint is made possible by the fundamental result of Dvurečenskij [**58**].

We shall often write (G, u) to indicate that G is an lu-group (with strong unit u). If (G, u) is an lu-group then the unit interval of G is

$$[0, u]_G = \{g \in G : 0 \leq g \leq u\}.$$

It has a canonical pseudo $MV-$ algebra structure given by the Example 4.2. Dvurečenskij's result says that for any pseudo $MV-$ algebra A there is an lu-group (G_A, u) such that A and $[0, u]_{G_A}$ are isomorphic. The categorical equivalence means that the entire theory of lu-groups applies to pseudo $MV-$ algebras. The main work involved has the flavor of translation. We take on the task of translating the theory of localization pseudo $MV-$ algebras into the language of localization lu-groups .

If (G, u) and (H, v) are lu-groups, then an *lu-groups morphism* is an l-groups morphism $f : G \to H$ such that $f(u) = v$.

We denote by $\mathcal{PMV}$ the category of pseudo $MV-$ algebras and by $\mathcal{LUG}$ the category of lu-groups. The definition of the Dvurečenskij functor

$$\Gamma : \mathcal{LUG} \to \mathcal{PMV}$$

is strainghtforward (see [**58**]):

$$\Gamma(G, u) := [0, u]_G,$$
$$\Gamma(h) := h_{|[0,u]},$$

if $h : (G, u) \to (H, v)$ is an lu-groups morphism.

EXAMPLE 7.9. [**58**] *Let $G = (Z \times Z \times Z, +, (0, 0, 0), \leq)$ be the Scrimger 2-group: the group operation $+$ is defined by*

$$(x_1, y_1, n_1) + (x_2, y_2, n_2) := \begin{cases} (y_1 + x_2, y_2 + x_1, n_1 + n_2), & \text{if } n_2 \text{ is odd} \\ (x_1 + x_2, y_1 + y_2, n_1 + n_2), & \text{if } n_2 \text{ is even,} \end{cases}$$

the order relation is $(x_1, y_1, n_1) \leq (x_2, y_2, n_2)$ iff $(n_1 < n_2)$ or $(n_1 = n_2, x_1 \leq x_2, y_1 \leq y_2)$. We remark that G is a non-abelian l-group which is not linearly ordered and that $u = (1, 1, 1)$ is a strong unit of G. The corresponding interval pseudo MV -algebra has the form

$$M = \Gamma(G, u) = (Z_+ \times Z_+ \times \{0\}) \cup (Z_{\leq 1} \times Z_{\leq 1} \times \{1\}),$$

where $Z_{\leq 1} := \{x \in Z : x \leq 1\}$. The pseudo MV - algebra operations are defined as follows:

$$(x, y, 0)^- = (1 - x, 1 - y, 1), (x, y, 0)^\sim = (1 - y, 1 - x, 1),$$
$$(x, y, 1)^- = (1 - y, 1 - x, 0), (x, y, 1)^\sim = (1 - x, 1 - y, 0),$$
$$(x_1, y_1, 0) \oplus (x_2, y_2, 0) = (x_1 + x_2, y_1 + y_2, 0),$$
$$(x_1, y_1, 0) \oplus (x_2, y_2, 1) = ((y_1 + x_2) \wedge 1, (x_1 + y_2) \wedge 1, 1),$$
$$(x_1, y_1, 1) \oplus (x_2, y_2, 0) = ((x_1 + x_2) \wedge 1, (y_1 + y_2) \wedge 1, 1),$$
$$(x_1, y_1, 1) \oplus (x_2, y_2, 1) = (1, 1, 1),$$

One can see [**58**] *for more details on G and M.*

For the definition of the functor

$$\Xi : \mathcal{PMV} \to \mathcal{LUG}$$

(the inverse of the functor Γ which together with Γ determine a categorical equivalence) see [**58**], (where for the pseudo $MV-$ algebra A, $\Xi(A)$ is denoted by (G_A, u_A)). With the notations of [**58**] we have:

THEOREM 7.32. *([**58**])For every pseudo $MV-$ algebra A there exists an lu-group G_A with strong unit u_A and an isomorphism of pseudo $MV-$ algebras $\varphi_A : A \to \Gamma(G_A, u_A) = [0, u_A]$.*

In the sequel G will designate an *lu*-group with strong unit u and A will designate $[0, u]_G$.

As in abelian case (see Definition 2.15) we define:

DEFINITION 7.8. For any integer k, let $\pi_k : G \to A$ be defined by

$$\pi_k(g) = ((g - ku) \wedge u) \vee 0.$$

From Remark 2.22 we deduce:

PROPOSITION 7.33. *The maps π_k have the following properties for all $f, g \in G$:*

$(psmv - c_{45})$ $\pi_{0|A} = 1_A$;
$(psmv - c_{46})$ $\pi_k(g) \geq \pi_{k+1}(g)$, *for all* $k \in Z$;
$(psmv - c_{47})$ $\pi_k(f \vee g) = \pi_k(f) \vee \pi_k(g)$ *and* $\pi_k(f \wedge g) = \pi_k(f) \wedge \pi_k(g)$, *for all* $k \in Z$, *(hence π_k is an increasing map for all $k \in Z$).*

As for abelian lu-groups, we have for non-commutative case (*lu*- groups) the next analogous definitions and results:

PROPOSITION 7.34. (*i*) *If $H \in \mathcal{I}(G)$, then $\overline{H} = H \cap A \in \mathcal{I}(A)$,*
(*ii*) *If $I \in \mathcal{I}(A)$, then*

$$H_I = \{g \in G : \pi_k(g) \in I \text{ for all } k \geq 0\}$$

is the order ideal of G generated by I in G (that is $H_I \in \mathcal{I}(G)$ and $H_I =< I >_G$). Moreover, $\overline{H_I} = H_I \cap A = I$,
(*iii*) *For every $K \in \mathcal{I}(G), K = H_I$ where $I = K \cap A \in \mathcal{I}(A)$,*
(*iii*) *There is a bijective correspondence between $\mathcal{I}(G)$ and $\mathcal{I}(A)$.*

The proof of Proposition 7.34 is analogous to the proof of Proposition 6.35 for commutative case.

Let (G, u) be an *lu*-group. A nonempty set $\mathcal{F}$ of elements $I \in \mathcal{I}(G)$ will be called a *topology* on G (or a *Gabriel filter* on $\mathcal{I}(G)$) if verify the properties from Definition 6.14.

For an *lu*-group (G, u) we define the *boolean center* $B(G)$ of G by

$$B(G, u) = B(A)$$

(where $A = \Gamma(G, u)$). Hence

$$B(G, u) = \{x \in [0, u] : (x + x) \wedge u = x\}.$$

Clearly, $0, u \in B(G, u)$ and we deduce that $B(G, u) \approx B(G_A, u_A) = B(\Xi(A))$.

We recall ([**68**]) that for every pseudo $MV-$ algebra A, $B(A)$ is a subalgebra of A.

REMARK 7.23. *If A, B are pseudo $MV-$ algebras, $\varphi : A \to B$ an isomorphism of pseudo $MV-$ algebras and $\mathcal{F}$ a topology on A, then $\varphi(\mathcal{F}) = \{\varphi(I) : I \in \mathcal{F}\}$ is a topology on B and $A_{\mathcal{F}} \approx B_{\varphi(\mathcal{F})}$.*

EXAMPLE 7.10. *If $H \in \mathcal{I}(G)$, then the set*

$$\mathcal{F}(H) = \{H' \in \mathcal{I}(G) : H \subseteq H'\}$$

is a topology on G.

EXAMPLE 7.11. *A non-empty set $H \subseteq G$ will be called regular if for every $x, y \in G$ such that $e \wedge x = e \wedge y$ for every $e \in H \cap B(G)$, we have $x = y$. If we denote $\mathcal{R}(G) = \{H \subseteq G : H$ is a regular subset of $G\}$, then $\mathcal{I}(G) \cap \mathcal{R}(G)$ is a topology on G .*

EXAMPLE 7.12. *A subset $S \subseteq G$ is called $\wedge-$ closed if $u \in S$ and if $x, y \in S$ implies $x \wedge y \in S$. For any $\wedge-$ closed subset S of G we set $\mathcal{F}_S = \{H \in \mathcal{I}(G) : H \cap S \cap B(G) \neq \oslash\}$. Then $\mathcal{F}_S$ is a topology on G .*

PROPOSITION 7.35. *Let (G, u) an lu-group and $A = \Gamma(G, u) = [0, u]_G$.*

(i) If $\mathcal{F}$ is a topology on G, then $\mathcal{F}_{\mathcal{A}} = \{H \cap A : H \in \mathcal{F}\}$ is a topology on A,

(ii) If $\mathcal{F}$ is a topology on A, then $\mathcal{F}_{\mathcal{G}} = \{H_I : I \in \mathcal{F}\}$ is a topology on G (where H_I is defined by Proposition 7.34, (ii)); if denote $\mathcal{F}_{\mathcal{G}} \cap A = \{H \cap A : H \in \mathcal{F}_{\mathcal{G}}\}$, then $\mathcal{F}_{\mathcal{G}} \cap A = \mathcal{F}_{\mathcal{A}}$,

(iii) There is a bijective correspondence betwen the topologies on G and the topologies on A.

Proof. See the proof of Proposition 6.36.

In the sequel (G, u) is an *lu*-group, $A = \Gamma(G, u) = [0, u]_G$ and $\mathcal{F}$ is a topology on G.

Now we are in the situation to define the notion of lu-group of localization of G with respect to the topology $\mathcal{F}$.

By Proposition 7.35, (i), $\mathcal{F}_{\mathcal{A}} = \{H \cap A : H \in \mathcal{F}\}$ is a topology on A. We can construct the pseudo $MV-$ algebra of localization of A with respect to the topology $\mathcal{F}_{\mathcal{A}}$, denoted by $A_{\mathcal{F}_{\mathcal{A}}}$.

DEFINITION 7.9. We denote the *lu*-group $\Xi(A_{\mathcal{F}_{\mathcal{A}}})$ by $G_{\mathcal{F}}$ and will be called the *localization lu-group of G with respect to the topology $\mathcal{F}$.*

Let now A be a pseudo $MV-$ algebra and $\mathcal{F}$ a topology on A. We consider $\Xi(A) = (G_A, u_A)$ and the isomorphism of pseudo $MV-$ algebras $\varphi_A : A \to B = [0, u_A] = \Gamma(G_A, u_A)$. By Remark 7.23, $\varphi_A(\mathcal{F}) = \{\varphi_A(I) : I \in \mathcal{F}\}$ is a topology on B and $A_{\mathcal{F}} \approx B_{\varphi_A(\mathcal{F})}$. Then $\Xi(A_{\mathcal{F}}) \approx \Xi(B_{\varphi_A(\mathcal{F})}) = \Xi(A)_{\varphi_A(\mathcal{F})}$ (see Definition 7.9).

So, we obtain:

THEOREM 7.36. *Let A be a pseudo $MV-$ algebra and $\mathcal{F}$ a topology on A. Then*

$$\Xi(A)_{\varphi_A(\mathcal{F})} \approx \Xi(A_{\mathcal{F}}).$$

If A is a pseudo $MV-$ algebra and $S \subseteq A$ is a $\wedge-$ closed system, then we define the notion of *pseudo $MV-$ algebra of fraction relative to S* (denoted by $A[S]$) and the *maximal pseudo $MV-$ algebra of quotients of A* (denoted by $Q(A)$).

We shall now define the analogous notions for lu-groups using the functor Ξ.

We continue the running assumption that (G, u) is an *lu*-group with unit interval $A = [0, u]$.

If $S \subseteq G$ is an $\wedge-$ closed system in G, then $\overline{S} = S \cap A$ is an $\wedge-$ closed system in A. So, we can consider the pseudo $MV-$ algebra of fractions relative to $\overline{S}$ (denoted by $A[\overline{S}]$).

DEFINITION 7.10. We denote the *lu*-group $\Xi(A[\overline{S}])$ by $G[S]$ and will be called the *lu-group of fraction* of G relative to the $\wedge-$ closed system S. Also, we denote the *lu*-group $\Xi(Q(A))$ by $Q(G)$ and will be called a *maximal lu-group of quotients* of G.

As in the case of pseudo $MV-$ algebras in the following we describe for an *lu*-group (G, u) the localization *lu*-group $G_{\mathcal{F}}$ in some special instances.

We recall that for the next two examples we work with strong-$\mathcal{F}-$ multipliers (see Definitions 7.7).

1. If $\mathcal{F} = \mathcal{I}(G) \cap \mathcal{R}(G)$, then $\mathcal{F}_{\mathcal{A}} = \mathcal{I}(A) \cap \mathcal{R}(A)$ where we recall that A is the pseudo $MV-$ algebra $[0, u]$ and $\mathcal{F}_{\mathcal{A}} = \mathcal{F} \cap A = \{H \cap A : H \in \mathcal{F}\}$. Then $s - A_{\mathcal{F}_{\mathcal{A}}} = Q(A)$, so $G_{\mathcal{F}} = \Xi(s - A_{\mathcal{F}_{\mathcal{A}}}) = \Xi(Q(A)) = Q(G)$ (that is $G_{\mathcal{F}}$ is the maximal *lu*-group of quotients of G).

2. If $S \subseteq G$ is an $\wedge-$ closed system of and $\mathcal{F}_{\mathcal{S}}$ is the topology $\mathcal{F}_{\mathcal{S}} = \{H \in \mathcal{I}(G) : H \cap S \cap B(G) \neq \varnothing\}$, then $\overline{S} = S \cap A$ is an $\wedge-$ closed system of A and $\mathcal{F}_{\overline{\mathcal{S}}} = \{I \in \mathcal{I}(A) : I \cap S \cap B(A) \neq \varnothing\}$ (since $B(G, u) = B(A)$).

Thus by Proposition 7.31, $s - A_{\mathcal{F}_{\overline{\mathcal{S}}}} \approx B(A[\overline{S}])$, hence

$$G_{\mathcal{F}_{\mathcal{S}}} = \Xi(s - A_{\mathcal{F}_{\overline{\mathcal{S}}}}) \approx \Xi(B(A[\overline{S}])) \approx \Xi(B(\Xi(A[\overline{S}]))) = B(G[S]).$$

CHAPTER 8

Localization of pseudo BL-algebras

The aim of this chapter is to define the localization (strong localization) pseudo BL-algebra of a pseudo BL -algebra A with respect to a topology $\mathcal{F}$ on A and to prove that the maximal pseudo BL - algebra of quotients and the pseudo BL- algebra of fractions relative to an $\wedge-$ closed system are strong pseudo BL -algebras of localization (see Proposition 8.39 and Proposition 8.40).

The concepts of pseudo BL -algebra of localization was defined in [**39**].

If the pseudo BL- algebra A is a pseudo MV- algebra or a BL- algebra then we obtain the results from Chapter 7 and 6; so, the results of this chapter are generalizations of the results for MV, pseudo MV and BL- algebras.

1. Pseudo-BL algebra of fractions relative to an $\wedge-$ closed system

As in the case of BL we denote by $S(A)$ the set of all $\wedge-$closed system of A, (see Definition 6.1).

For $S \in S(A)$, on the pseudo - BL algebra A we consider the relation θ_S defined by

$$(x, y) \in \theta_S \text{ iff there exists } e \in S \cap B(A) \text{ such that } x \wedge e = y \wedge e.$$

LEMMA 8.1. *θ_S is a congruence on A.*

Proof. The reflexivity, symmetry and transitivity of θ_S are immediately. To prove the compatibility of θ_S with the operations $\wedge, \vee, \odot$ see the proof of Lemma 6.1.

To prove the compatibility of θ_S with the operations $\rightarrow$ and $\rightsquigarrow$, let $x, y, z, t \in A$ such that $(x, y) \in \theta_S$ and $(z, t) \in \theta_S$. Thus there exists $e, f \in S \cap B(A)$ such that $x \wedge e = y \wedge e$ and $z \wedge f = t \wedge f$; we denote $g = e \wedge f \in S \cap B(A)$.

By $psbl - c_{61}$ we obtain :

$$(x \rightarrow z) \wedge g = (x \rightarrow z) \odot g = [(x \odot g) \rightarrow (z \odot g)] \odot g$$
$$= [(y \odot g) \rightarrow (t \odot g)] \odot g = (y \rightarrow t) \odot g = (y \rightarrow t) \wedge g,$$

hence $(x \rightarrow z, y \rightarrow t) \in \theta_S$ and

$$(x \rightsquigarrow z) \wedge g = g \odot (x \rightsquigarrow z) = g \odot [(g \odot x) \rightsquigarrow (g \odot z)]$$
$$= g \odot [(g \odot y) \rightsquigarrow (g \odot t)] = g \odot (y \rightsquigarrow t) = (y \rightsquigarrow t) \wedge g,$$

hence $(x \rightsquigarrow z, y \rightsquigarrow t) \in \theta_S$.■

For x we denote by x/S the equivalence class of x relative to θ_S and by

$$A[S] = A/\theta_S.$$

By $p_S : A \rightarrow A[S]$ we denote the canonical map defined by $p_S(x) = x/S$, for every $x \in A$. Clearly, in $A[S]$, $\mathbf{0} = 0/S$, $\mathbf{1} = 1/S$ and for every $x, y \in A$,

$$x/S \wedge y/S = (x \wedge y)/S, x/S \vee y/S = (x \vee y)/S, x/S \odot y/S = (x \odot y)/S,$$

$$x/S \to y/S = (x \to y)/S, x/S \rightsquigarrow y/S = (x \rightsquigarrow y)/S.$$

So, p_S is an onto morphism of pseudo-BL algebras.

REMARK 8.1. *Since for every* $s \in S \cap B(A)$, $s \wedge s = s \wedge 1$ *we deduce that* $s/S = 1/S = \mathbf{1}$, *hence* $p_S(S \cap B(A)) = \{\mathbf{1}\}$.

PROPOSITION 8.2. *If* $a \in A$, *then* $a/S \in B(A[S])$ *iff there exists* $e \in S \cap B(A)$ *such that* $e \wedge a \in B(A)$. *So, if* $e \in B(A)$, *then* $e/S \in B(A[S])$.

Proof. For $a \in A$, we have $a/S \in B(A[S]) \Leftrightarrow a/S \odot a/S = a/S$ and $((a/S)^-)^\sim = ((a/S)^\sim)^- = a/S$.

From $a/S \odot a/S = a/S$ we deduce that $(a \odot a)/S = a/S \Leftrightarrow$ there exists $g \in S \cap B(A)$ such that $(a \odot a) \wedge g = a \wedge g \Leftrightarrow (a \odot a) \odot g = a \wedge g \Leftrightarrow (a \odot g) \odot (a \odot g) = a \wedge g \Leftrightarrow (a \wedge g) \odot (a \wedge g) = a \wedge g$.

From $((a/S)^-)^\sim = ((a/S)^\sim)^- = a/S$ we deduce that exists $f, h \in S \cap B(A)$ such that $(a^-)^\sim \wedge f = a \wedge f$ and $(a^\sim)^- \wedge h = a \wedge h$. If denote $e = g \wedge f \wedge h \in S \cap B(A)$, then

$$(a \wedge e) \odot (a \wedge e) = (a \wedge g \wedge f \wedge h) \odot (a \wedge g \wedge f \wedge h) =$$
$$(a \odot g) \odot f \odot h \odot (a \odot g) \odot f \odot h = a \odot g \odot f \odot h = a \wedge g \wedge f \wedge h = a \wedge e$$

and

$$((a \wedge e)^-)^\sim \overset{psbl-c_{50}}{=} (a^- \vee e^-)^\sim \overset{psbl-c_{49}}{=} (a^-)^\sim \wedge (e^-)^\sim = (a^-)^\sim \wedge e$$
$$= (a^-)^\sim \wedge g \wedge f \wedge h = [(a^-)^\sim \wedge f] \wedge g \wedge h = (a \wedge f) \wedge g \wedge h = a \wedge e$$

and

$$((a \wedge e)^\sim)^- \overset{psbl-c_{49}}{=} (a^\sim \vee e^\sim)^- \overset{psbl-c_{50}}{=} (a^\sim)^- \wedge (e^\sim)^- = (a^\sim)^- \wedge e$$
$$= (a^\sim)^- \wedge g \wedge f \wedge h = [(a^\sim)^- \wedge h] \wedge g \wedge f = (a \wedge h) \wedge g \wedge f = a \wedge e,$$

so,

$$((a \wedge e)^\sim)^- = ((a \wedge e)^-)^\sim = a \wedge e,$$

hence $a \wedge e \in B(A)$.

If $e \in B(A)$, since $1 \in S \cap B(A)$ and $1 \wedge e = e \in B(A)$ we deduce that $e/S \in B(A[S])$. ■

As in the case of $BL-$ algebras we have the following result :

THEOREM 8.3. *If* A' *is a pseudo-BL algebra and* $f : A \to A'$ *is an morphism of pseudo-BL algebras such that* $f(S \cap B(A)) = \{\mathbf{1}\}$, *then there exists an unique morphism of pseudo-BL algebras* $f' : A[S] \to A'$ *such that the diagram*

$$\begin{array}{ccc} A & \xrightarrow{p_S} & A[S] \\ & \searrow^{f} \quad \swarrow_{f'} & \\ & A' & \end{array}$$

is commutative (i.e. $f' \circ p_S = f$*).*

REMARK 8.2. *Theorem 8.3 allows us to call* $A[S]$ *the* pseudo-BL algebra of fractions relative to the $\wedge-$closed system S.

REMARK 8.3. *If pseudo* $BL-$ *algebra* A *is a* $BL-$ *algebra (i.e.* $x \to y = x \rightsquigarrow y$, *for all* $x, y \in A$, *see Remark 5.1), then* $(x/S) \to (y/S) = (x \to y)/S = (x \rightsquigarrow y)/S = (x/S) \rightsquigarrow (y/S)$, *so* $A[S]$ *is a* $BL-$ *algebra, called the* BL-algebra of fractions relative to the $\wedge-$closed system S *(see Chapter 6).*

REMARK 8.4. *If pseudo $BL-$ algebra A is a pseudo $MV-$ algebra (i.e. $(x^-)^\sim = x = (x^\sim)^-$, for all $x \in A$, see Corollary 6.31), then $[(x/S)^-]^\sim = (x^-)^\sim/S = x/S = (x^\sim)^-/S = [(x/S)^\sim]^-$, so $A[S]$ is a pseudo $MV-$ algebra, called the* pseudo MV-algebra of fractions relative to the $\wedge-$closed system S *(see Chapter 7).*

EXAMPLE 8.1. *If A is a pseudo $BL-$ algebra and $S = \{1\}$ or S is such that $1 \in S$ and $S \cap (B(A)\backslash\{1\}) = \varnothing$, then for $x, y \in A, (x, y) \in \theta_S \Longleftrightarrow x \wedge 1 = y \wedge 1 \Longleftrightarrow x = y$, hence in this case $A[S] = A$.*

EXAMPLE 8.2. *If A is a pseudo $BL-$ algebra and S is an $\wedge-$closed system such that $0 \in S$ (for example $S = A$ or $S = B(A)$), then for every $x, y \in A$, $(x, y) \in \theta_S$ (since $x \wedge 0 = y \wedge 0$ and $0 \in S \cap B(A)$), hence in this case $A[S] = \mathbf{0}$.*

2. Pseudo-BL algebra of fractions and maximal pseudo BL-algeba of quotients

2.1. Strong multipliers on a pseudo-BL algebra. We denote by $\mathcal{I}(A)$ the set of all *order ideals* of A (see Definition 6.2) and by $I_d(A)$ the set of all *ideals* of the lattice $L(A)$.

DEFINITION 8.1. *By* partial strong multiplier *of A we mean a map $f : I \to A$, where $I \in \mathcal{I}(A)$, which verifies the next conditions:*

$(sm - psBL_1)$ $f(e \odot x) = e \odot f(x)$, *for every $e \in B(A)$ and $x \in I$;*
$(sm - psBL_2)$ $f(x) \leq x$, *for every $x \in I$;*
$(sm - psBL_3)$ *If $e \in I \cap B(A)$, then $f(e) \in B(A)$;*
$(sm - psBL_4)$ $x \wedge f(e) = e \wedge f(x)$, *for every $e \in I \cap B(A)$ and $x \in I$.*

REMARK 8.5. *If A is a $BL-$ algebra or a pseudo $MV-$ algebra the definition of strong multiplier on A is the same as Definitions 6.3 for the case of $BL-$ algebras and Definition 7.3 for pseudo $MV-$ algebras.*

Clearly, $f(0) = 0$.

As in the case of $BL-$ algebras, by $dom(f) \in \mathcal{I}(A)$ we denote the domain of f; if $dom(f) = A$, we called f *total*.

To simplify the language, we will use *strong multiplier* instead *partial strong multiplier* using *total* to indicate that the domain of a certain multiplier is A.

EXAMPLE 8.3. *The map $\mathbf{0} : A \to A$ defined by $\mathbf{0}(x) = 0$, for every $x \in A$ is a total strong multiplier on A;*

EXAMPLE 8.4. *The map $\mathbf{1} : A \to A$ defined by $\mathbf{1}(x) = x$, for every $x \in A$ is also a total strong multiplier on A;*

EXAMPLE 8.5. *For $a \in B(A)$ and $I \in \mathcal{I}(A)$, the map $f_a : I \to A$ defined by $f_a(x) = a \wedge x$, for every $x \in I$ is a strong multiplier on A (called principal).*

REMARK 8.6. *The condition $sm-psBL_4$ is not a consequence of $sm-psBL_1, sm-psBL_2$ and $sm - psBL_3$. As example, $f : I \to A, f(x) = x \wedge x^-$ for every $x \in I$, verify $sm - psBL_1, sm - psBL_2$ and $sm - psBL_3$.*

REMARK 8.7. *In general, if consider $a \in A$, then $f_a : I \to A$ verifies only $sm - psBL_1, sm - psBL_2$ and $sm - psBL_4$ but does not verify $sm - psBL_3$.*

If $dom(f_a) = A$, we denote f_a by $\overline{f_a}$; clearly, $\overline{f_0} = \mathbf{0}$.

For $I \in \mathcal{I}(A)$, we denote

$$M(I, A) = \{f : I \to A : f \text{ is a strong multiplier on } A\}$$

and

$$M(A) = \bigcup_{I \in \mathcal{I}(A)} M(I, A).$$

DEFINITION 8.2. *If $I_1, I_2 \in \mathcal{I}(A)$ and $f_i \in M(I_i, A), i = 1, 2$, we define $f_1 \wedge f_2$, $f_1 \vee f_2$, $f_1 \boxdot f_2$, $f_1 \to f_2, f_1 \rightsquigarrow f_2 : I_1 \cap I_2 \to A$ by*

$$(f_1 \wedge f_2)(x) = f_1(x) \wedge f_2(x),$$
$$(f_1 \vee f_2)(x) = f_1(x) \vee f_2(x),$$
$$(f_1 \boxdot f_2)(x) = [x \to f_1(x)] \odot f_2(x) \overset{psbl-c_{64}}{=} f_1(x) \odot [x \rightsquigarrow f_2(x)],$$
$$(f_1 \to f_2)(x) = [f_1(x) \to f_2(x)] \odot x,$$
$$(f_1 \rightsquigarrow f_2)(x) = x \odot [f_1(x) \rightsquigarrow f_2(x)],$$

for every $x \in I_1 \cap I_2$.

LEMMA 8.4. $f_1 \wedge f_2 \in M(I_1 \cap I_2, A)$.

Proof. See the proof of Lemma 6.4. ■

LEMMA 8.5. $f_1 \vee f_2 \in M(I_1 \cap I_2, A)$.

Proof. See the proof of Lemma 6.5. ■

LEMMA 8.6. $f_1 \boxdot f_2 \in M(I_1 \cap I_2, A)$.

Proof. If $x \in I_1 \cap I_2$ and $e \in B(A)$, then

$$(f_1 \boxdot f_2)(e \odot x) = [(e \odot x) \to f_1(e \odot x)] \odot f_2(e \odot x) = [(e \odot x) \to (e \odot f_1(x))] \odot [e \odot f_2(x)] =$$
$$= [((e \odot x) \to (e \odot f_1(x))) \odot e] \odot f_2(x) \overset{psbl-c_{74}}{=} [(x \to f_1(x)) \odot e] \odot f_2(x) =$$
$$= [(x \to f_1(x)) \odot f_2(x)] \odot e = (f_1 \boxdot f_2)(x) \odot e.$$

Clearly, $(f_1 \boxdot f_2)(x) = [x \to f_1(x)] \odot f_2(x) \leq f_2(x) \leq x$, for every $x \in I_1 \cap I_2$ and if $e \in I_1 \cap I_2 \cap B(A)$, then by Proposition 5.13, we have

$$(f_1 \boxdot f_2)(e) = [e \to f_1(e)] \odot f_2(e) = (e^- \vee f_1(e)) \odot f_2(e) \in B(A).$$

For $e \in I_1 \cap I_2 \cap B(A)$ and $x \in I_1 \cap I_2$, we have:

$$x \wedge (f_1 \boxdot f_2)(e) = x \wedge [(e \to f_1(e)) \odot f_2(e)] =$$
$$= [(e \to f_1(e)) \odot f_2(e)] \odot x = [(e \to f_1(e)) \odot x] \odot f_2(e) =$$
$$\overset{psbl-c_{75}}{=} [((e \odot x) \to (f_1(e) \odot x)) \odot x] \odot f_2(e) = [(e \odot x) \to (f_1(e) \odot x)] \odot [x \odot f_2(e)] =$$
$$[(e \odot x) \to (e \odot f_1(x))] \odot [e \odot f_2(x)] = [((e \odot x) \to (e \odot f_1(x))) \odot e] \odot f_2(x) =$$
$$\overset{psbl-c_{74}}{=} [(x \to f_1(x)) \odot e] \odot f_2(x) = [(x \to f_1(x)) \odot f_2(x)] \odot e =$$
$$= [(f_1 \boxdot f_2)(x)] \odot e = e \wedge (f_1 \boxdot f_2)(x),$$

hence, we have $x \wedge (f_1 \boxdot f_2)(e) = e \wedge (f_1 \boxdot f_2)(x)$, that is, $f_1 \boxdot f_2 \in M(I_1 \cap I_2, A)$. ■

LEMMA 8.7. $f_1 \to f_2 \in M(I_1 \cap I_2, A)$.

Proof. If $x \in I_1 \cap I_2$ and $e \in B(A)$, then

$$(f_1 \to f_2)(e \odot x) = [f_1(e \odot x) \to f_2(e \odot x)] \odot (e \odot x) = [(e \odot f_1(x)) \to (e \odot f_2(x))] \odot (e \odot x) =$$

$$= [((e \odot f_1(x)) \to (e \odot f_2(x))) \odot e] \odot x \overset{psbl-c_{74}}{=} [(f_1(x) \to f_2(x)) \odot e] \odot x =$$

$$= [(f_1(x) \to f_2(x)) \odot x] \odot e = [(f_1 \to f_2)(x)] \odot e.$$

Clearly, $(f_1 \to f_2)(x) = [f_1(x) \to f_2(x)] \odot x \leq x$, for every $x \in I_1 \cap I_2$ and if $e \in I_1 \cap I_2 \cap B(A)$, then by Proposition 5.13 we have

$$(f_1 \to f_2)(e) = [f_1(e) \to f_2(e)] \odot e = [(f_1(e))^- \vee f_2(e)] \odot e \in B(A).$$

For $e \in I_1 \cap I_2 \cap B(A)$ and $x \in I_1 \cap I_2$, we have:

$$e \wedge (f_1 \to f_2)(x) = [(f_1(x) \to f_2(x)) \odot x] \wedge e =$$

$$= [(f_1(x) \to f_2(x)) \odot x] \odot e = [(f_1(x) \to f_2(x)) \odot e] \odot x =$$

$$\overset{psbl-c_{74}}{=} [((f_1(x) \odot e) \to (f_2(x) \odot e)) \odot e] \odot x = [((x \odot f_1(e)) \to (x \odot f_2(e))) \odot e] \odot x =$$

$$[((x \odot f_1(e)) \to (x \odot f_2(e))) \odot x] \odot e \overset{psbl-c_{75}}{=} [(f_1(e) \to f_2(e)) \odot x] \odot e =$$

$$= [(f_1(e) \to f_2(e)) \odot e] \odot x = [(f_1 \to f_2)(e)] \odot x = x \wedge (f_1 \to f_2)(e)$$

hence, we have $x \wedge (f_1 \to f_2)(e) = e \wedge (f_1 \to f_2)(x)$, that is, $f_1 \to f_2 \in M(I_1 \cap I_2, A)$. ■

LEMMA 8.8. *$f_1 \rightsquigarrow f_2 \in M(I_1 \cap I_2, A)$.*

Proof. If $x \in I_1 \cap I_2$ and $e \in B(A)$, then

$$(f_1 \rightsquigarrow f_2)(e \odot x) = (e \odot x) \odot [f_1(e \odot x) \rightsquigarrow f_2(e \odot x)] = (e \odot x) \odot [(e \odot f_1(x)) \rightsquigarrow (e \odot f_2(x))] =$$

$$= x \odot [e \odot ((e \odot f_1(x)) \rightsquigarrow (e \odot f_2(x)))] \overset{psbl-c_{74}}{=} x \odot [e \odot (f_1(x) \rightsquigarrow f_2(x))] =$$

$$= e \odot [x \odot (f_1(x) \rightsquigarrow f_2(x))] = e \odot (f_1 \rightsquigarrow f_2)(x).$$

Clearly, $(f_1 \rightsquigarrow f_2)(x) = x \odot [f_1(x) \rightsquigarrow f_2(x)] \leq x$, for every $x \in I_1 \cap I_2$ and if $e \in I_1 \cap I_2 \cap B(A)$, then, by Proposition 5.13, we have

$$(f_1 \rightsquigarrow f_2)(e) = e \odot [f_1(e) \rightsquigarrow f_2(e)] = e \odot [(f_1(e))^\sim \vee f_2(e)] \in B(A).$$

For $e \in I_1 \cap I_2 \cap B(A)$ and $x \in I_1 \cap I_2$, we have:

$$e \wedge (f_1 \rightsquigarrow f_2)(x) = e \wedge [x \odot (f_1(x) \rightsquigarrow f_2(x))] =$$

$$= (e \odot x) \odot [f_1(x) \rightsquigarrow f_2(x)] = x \odot [e \odot (f_1(x) \rightsquigarrow f_2(x))] =$$

$$\overset{psbl-c_{74}}{=} x \odot [e \odot ((e \odot f_1(x)) \rightsquigarrow (e \odot f_2(x)))] = x \odot [e \odot ((x \odot f_1(e)) \rightsquigarrow (x \odot f_2(e)))] =$$

$$= e \odot [x \odot ((x \odot f_1(e)) \rightsquigarrow (x \odot f_2(e)))] \overset{psbl-c_{75}}{=} e \odot [x \odot (f_1(e) \rightsquigarrow f_2(e))] =$$

$$= x \odot [e \odot (f_1(e) \rightsquigarrow f_2(e))] = x \odot (f_1 \rightsquigarrow f_2)(e) = x \wedge (f_1 \rightsquigarrow f_2)(e)$$

hence, we have $x \wedge (f_1 \rightsquigarrow f_2)(e) = e \wedge (f_1 \rightsquigarrow f_2)(x)$, that is, $f_1 \rightsquigarrow f_2 \in M(I_1 \cap I_2, A)$. ■

PROPOSITION 8.9. *$(M(A), \wedge, \vee, \boxdot, \to, \rightsquigarrow, \mathbf{0}, \mathbf{1})$ is a pseudo - BL algebra.*

Proof. We now verify the axioms of pseudo BL-algebras.

$(psBL_1)$. It is obvious that $(M(A), \wedge, \vee, \mathbf{0}, \mathbf{1})$ is a bounded lattice.

$(psBL_2)$. Let $f_i \in M(I_i, A)$ where $I_i \in I(A)$, $i = 1, 2, 3$. Then, it is clear that $f_1 \boxdot f_2 \in M(A)$ (see Lemma 8.6).

Thus, for $x \in I_1 \cap I_2 \cap I_3$, we have

$$[(f_1 \boxdot f_2) \boxdot f_3](x) = ((f_1 \boxdot f_2)(x)) \odot (x \rightsquigarrow f_3(x)) =$$
$$= [(x \to f_1(x)) \odot f_2(x)] \odot (x \rightsquigarrow f_3(x)) = (x \to f_1(x)) \odot [f_2(x) \odot (x \rightsquigarrow f_3(x))] =$$
$$= (x \to f_1(x)) \odot [(f_2 \boxdot f_3)(x)] = [f_1 \boxdot (f_2 \boxdot f_3)](x),$$

that is, the operation $\boxdot$ is associative.

Let $f \in M(I, A)$ with $I \in I(A)$. If $x \in I$, then

$$(f \boxdot \mathbf{1})(x) = f(x) \odot (x \rightsquigarrow \mathbf{1}(x)) = f(x) \odot (x \rightsquigarrow x) = f(x) \odot 1 = f(x),$$
$$(\mathbf{1} \boxdot f)(x) = \mathbf{1}(x) \odot (x \rightsquigarrow f(x)) = x \odot (x \rightsquigarrow f(x)) = x \wedge f(x) = f(x),$$

hence, $f \boxdot \mathbf{1} = \mathbf{1} \boxdot f = f$, that is, $(M(A), \boxdot, \mathbf{1})$ is a monoid.

$(psBL_3)$. Let $f_i \in M(I_i, A)$, where $I_i \in I(A)$, $i = 1, 2, 3$.

Since $f_1 \leq f_2 \to f_3$ for $x \in I_1 \cap I_2 \cap I_3$, we have

$$f_1(x) \leq (f_2 \to f_3)(x) \Leftrightarrow f_1(x) \leq [f_2(x) \to f_3(x)] \odot x.$$

So, by $psbl - c_3$,we derive that

$$f_1(x) \odot [x \rightsquigarrow f_2(x)] \leq [f_2(x) \to f_3(x)] \odot x \odot [x \rightsquigarrow f_2(x)] \Leftrightarrow$$
$$f_1(x) \odot [x \rightsquigarrow f_2(x)] \leq (f_2(x) \to f_3(x)) \odot (x \wedge f_2(x)) \Leftrightarrow$$
$$f_1(x) \odot [x \rightsquigarrow f_2(x)] \leq (f_2(x) \to f_3(x)) \odot f_2(x) \Leftrightarrow$$
$$f_1(x) \odot [x \rightsquigarrow f_2(x)] \leq f_2(x) \wedge f_3(x) \leq f_3(x) \Leftrightarrow$$
$$(f_1 \boxdot f_2)(x) \leq f_3(x),$$

for every $x \in I_1 \cap I_2 \cap I_3$, that is, $f_1 \boxdot f_2 \leq f_3$.

Conversely, if $(f_1 \boxdot f_2)(x) \leq f_3(x)$, then we have $[x \to f_1(x)] \odot f_2(x) \leq f_3(x)$,for every $x \in I_1 \cap I_2 \cap I_3$.

Obviously,

$$[x \to f_1(x)] \leq f_2(x) \to f_3(x) \overset{psbl-c_3}{\Rightarrow} (x \to f_1(x)) \odot x \leq (f_2(x) \to f_3(x)) \odot x$$
$$\Rightarrow x \wedge f_1(x) \leq (f_2(x) \to f_3(x)) \odot x \Rightarrow f_1(x) \leq (f_2 \to f_3)(x).$$

Hence, $f_1 \leq f_2 \to f_3$ iff $f_1 \boxdot f_2 \leq f_3$, for all $f_1, f_2, f_3 \in M(A)$.

Since $f_2 \leq f_1 \rightsquigarrow f_3$ for $x \in I_1 \cap I_2 \cap I_3$, we have

$$f_2(x) \leq (f_1 \rightsquigarrow f_3)(x) \Leftrightarrow f_2(x) \leq x \odot [f_1(x) \rightsquigarrow f_3(x)].$$

So, by $psbl - c_3$, we have

$$[x \to f_1(x)] \odot f_2(x) \leq [x \to f_1(x)] \odot x \odot [f_1(x) \rightsquigarrow f_3(x)] \Leftrightarrow$$
$$(f_1 \boxdot f_2)(x) \leq (x \wedge f_1(x)) \odot (f_1(x) \rightsquigarrow f_3(x)) \Leftrightarrow (f_1 \odot f_2)(x) \leq f_1(x) \odot (f_1(x) \rightsquigarrow f_3(x)) \Leftrightarrow$$
$$(f_1 \boxdot f_2)(x) \leq f_1(x) \wedge f_3(x) \leq f_3(x) \Leftrightarrow (f_1 \boxdot f_2)(x) \leq f_3(x),$$

for every $x \in I_1 \cap I_2 \cap I_3$, that is, $f_1 \boxdot f_2 \leq f_3$.

Conversely if $(f_1 \boxdot f_2)(x) \leq f_3(x)$, then we have $f_1(x) \odot [x \rightsquigarrow f_2(x)] \leq f_3(x)$,for every $x \in I_1 \cap I_2 \cap I_3$.

It is obvious that

$$(x \rightsquigarrow f_2(x)) \leq f_1(x) \rightsquigarrow f_3(x) \overset{psbl-c_3}{\Rightarrow} x \odot (x \rightsquigarrow f_2(x)) \leq x \odot (f_1(x) \rightsquigarrow f_3(x))$$

$$\Rightarrow x \wedge f_2(x) \leq x \odot (f_1(x) \rightsquigarrow f_3(x)) \Rightarrow f_2(x) \leq (f_1 \rightsquigarrow f_3)(x).$$

Hence, $f_2 \leq f_1 \rightsquigarrow f_3$ iff $f_1 \boxdot f_2 \leq f_3$ for all $f_1, f_2, f_3 \in M(A)$.

$(psBL_4)$. Let $f_i \in M(I_i, A)$ where $I_i \in I(A)$, $i = 1, 2$.

Then, for $x \in I_1 \cap I_2$, we have

$$[(f_1 \rightarrow f_2) \boxdot f_1](x) = [(f_1 \rightarrow f_2)(x)] \odot [x \rightsquigarrow f_1(x)]$$

$$= ([f_1(x) \rightarrow f_2(x)] \odot x) \odot [x \rightsquigarrow f_1(x)] = [f_1(x) \rightarrow f_2(x)] \odot (x \odot [x \rightsquigarrow f_1(x)]) =$$

$$= [f_1(x) \rightarrow f_2(x)] \odot [x \wedge f_1(x)] = [f_1(x) \rightarrow f_2(x)] \odot f_1(x) = f_1(x) \wedge f_2(x) = (f_1 \wedge f_2)(x),$$

and

$$[f_1 \boxdot (f_1 \rightsquigarrow f_2)](x) = [x \rightarrow f_1(x)] \odot [(f_1 \rightsquigarrow f_2)(x)]$$

$$= [x \rightarrow f_1(x)] \odot (x \odot [f_1(x) \rightsquigarrow f_2(x)]) = ([x \rightarrow f_1(x)] \odot x) \odot [f_1(x) \rightsquigarrow f_2(x)] =$$

$$= [x \wedge f_1(x)] \odot [f_1(x) \rightsquigarrow f_2(x)] = f_1(x) \odot [f_1(x) \rightsquigarrow f_2(x)] = f_1(x) \wedge f_2(x) = (f_1 \wedge f_2)(x).$$

So, $f_1 \wedge f_2 = (f_1 \rightarrow f_2) \boxdot f_1 = f_1 \boxdot (f_1 \rightsquigarrow f_2)$.

$(psBL_5)$. We have

$$[(f_1 \rightarrow f_2) \vee (f_2 \rightarrow f_1)](x) = [(f_1 \rightarrow f_2)(x)] \vee [(f_2 \rightarrow f_1)(x)] =$$

$$= [(f_1(x) \rightarrow f_2(x)) \odot x] \vee [(f_2(x) \rightarrow f_1(x)) \odot x] =$$

$$\overset{psbl-c_{36}}{=} [(f_1(x) \rightarrow f_2(x)) \vee (f_2(x) \rightarrow f_1(x))] \odot x = 1 \odot x = x = \mathbf{1}(x),$$

and

$$[(f_1 \rightsquigarrow f_2) \vee (f_2 \rightsquigarrow f_1)](x) = [(f_1 \rightsquigarrow f_2)(x)] \vee [(f_2 \rightsquigarrow f_1)(x)] =$$

$$= [x \odot (f_1(x) \rightsquigarrow f_2(x))] \vee [x \odot (f_2(x) \rightsquigarrow f_1(x))] =$$

$$\overset{psbl-c_{36}}{=} x \odot [(f_1(x) \rightsquigarrow f_2(x)) \vee (f_2(x) \rightsquigarrow f_1(x))] = x \odot 1 = x = \mathbf{1}(x),$$

hence $(f_1 \rightarrow f_2) \vee (f_2 \rightarrow f_1) = (f_1 \rightsquigarrow f_2) \vee (f_2 \rightsquigarrow f_1) = \mathbf{1}$. ■

REMARK 8.8. *To prove that $(M(A), \wedge, \vee, \boxdot, \rightarrow, \rightsquigarrow, \mathbf{0}, \mathbf{1})$ is a pseudo BL-algebra it is sufficient to ask for multipliers only the axioms $sm - psBL_1$ and $sm - psBL_2$.*

PROPOSITION 8.10. *If pseudo $BL-$ algebra $(A, \vee, \wedge, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ is a pseudo $MV-$ algebra $(A, \odot, \oplus, ^-, ^\sim, 0, 1)$ (i.e. $(x^-)^\sim = (x^\sim)^- = x$, for all $x \in A$), then pseudo $BL-$ algebra $(M(A), \wedge, \vee, \boxdot, \rightarrow, \rightsquigarrow, \mathbf{0}, \mathbf{1})$ is a pseudo $MV-$ algebra $(M(A), \boxdot, \boxplus, ^-, ^\sim, \mathbf{0}, \mathbf{1})$.*

Proof. To prove that pseudo $BL-$ algebra $M(A)$ is a pseudo $MV-$ algebra let $f \in M(I, A)$ with $I \in \mathcal{I}(A)$.

Then

$$(f^-)^\sim(x) = x \odot [(f(x))^- \odot x]^\sim \overset{psbl-c_{48}}{=} x \odot [x \rightsquigarrow ((f(x))^-)^\sim]$$

$$= x \odot (x \rightsquigarrow f(x)) = x \wedge f(x) = f(x)$$

and

$$(f^\sim)^-(x) = [x \odot (f(x))^\sim]^- \odot x \overset{psbl-c_{48}}{=} [x \rightarrow ((f(x))^\sim)^-] \odot x$$

$$= (x \rightarrow f(x)) \odot x = x \wedge f(x) = f(x)$$

(since $f(x) \in A$ which is a pseudo $MV-$ algebra), for all $x \in I$.

So, $(f^-)^\sim = (f^\sim)^- = f$, for all $f \in M(A)$ and pseudo $BL-$ algebra $M(A)$ is a pseudo MV -algebra (see Proposition 7.15). ■

PROPOSITION 8.11. *If pseudo BL -algebra $(A, \vee, \wedge, \odot, \to, \rightsquigarrow, 0, 1)$ is a BL -algebra (i.e $x \odot y = y \odot x$ for all $x, y \in A$ and in particular $x \to y = x \rightsquigarrow y$ for all $x, y \in A$), then pseudo BL -algebra $(M(A), \wedge, \vee, \boxdot, \to, \rightsquigarrow, \mathbf{0}, \mathbf{1})$ is a BL -algebra $(M(A), \wedge, \vee, \boxdot, \to, \mathbf{0}, \mathbf{1})$. Indeed if $I_1, I_2 \in \mathcal{I}(A)$ and $f_i \in M(I_i, A)$, $i = 1, 2$ we have*

$$(f_1 \to f_2)(x) = [f_1(x) \to f_2(x)] \odot x = x \odot [f_1(x) \rightsquigarrow f_2(x)] = (f_1 \rightsquigarrow f_2)(x),$$

for all $x \in I_1 \cap I_2$, then $f_1 \to f_2 = f_1 \rightsquigarrow f_2$, so pseudo BL -algebra $(M(A), \wedge, \vee, \boxdot, \to , \rightsquigarrow, \mathbf{0}, \mathbf{1})$ is commutative (see Remark 5.1), so is a BL -algebra (see Proposition 6.8).

REMARK 8.9. *For every $I \in \mathcal{I}(A)$ the algebra of multipliers $M_{ps\mathcal{BL}}(I, A)$ for a pseudo $BL-$ algebra is in fact a generalization of the algebra of multipliers $M_{ps\mathcal{MV}}(I, A)$ for pseudo $MV-$ algebras, defined in Chapter 7 and algebra of multipliers $M_{\mathcal{BL}}(I, A)$ for $BL-$ algebras, defined in Chapter 6.*

LEMMA 8.12. *The map $v_A : B(A) \to M(A)$ defined by $v_A(a) = \overline{f_a}$ for every $a \in B(A)$ is a monomorphism of pseudo BL-algebras.*

Proof. Clearly, $v_A(0) = \overline{f_0} = \mathbf{0}$. Let $a, b \in B(A)$ and $x \in A$. We have:

$$(v_A(a) \boxdot v_A(b))(x) = v_A(a)(x) \odot (x \rightsquigarrow v_A(b)(x)) = (a \wedge x) \odot (x \rightsquigarrow (b \wedge x))$$

$$= (a \odot x) \odot (x \rightsquigarrow (b \wedge x)) = a \odot [x \odot (x \rightsquigarrow (b \wedge x))] = a \odot [x \wedge (b \wedge x)]$$

$$= a \wedge [x \wedge (b \wedge x)] = a \wedge (b \wedge x) = (a \wedge b) \wedge x = (v_A(a \wedge b))(x) = (v_A(a \odot b))(x),$$

hence $v_A(a \odot b) = v_A(a) \boxdot v_A(b)$.

Also, we have

$$(v_A(a) \to v_A(b))(x) = [v_A(a)(x) \to v_A(b)(x)] \odot x = [(a \wedge x) \to (b \wedge x)] \odot x$$

$$= [(x \odot a) \to (x \odot b)] \odot x \overset{psbl-c_{75}}{=} (a \to b) \odot x = x \wedge (a \to b)$$

(since $a \to b \in B(A)$)

$$= v_A(a \to b)(x),$$

and

$$(v_A(a) \rightsquigarrow v_A(b))(x) = x \odot [v_A(a)(x) \rightsquigarrow v_A(b)(x)] = x \odot [(a \wedge x) \rightsquigarrow (b \wedge x)]$$

$$= x \odot [(x \odot a) \rightsquigarrow (x \odot b)] \overset{psbl-c_{75}}{=} x \odot (a \rightsquigarrow b) = x \wedge (a \rightsquigarrow b)$$

(since $a \rightsquigarrow b \in B(A)$)

$$= v_A(a \rightsquigarrow b)(x).$$

Consequently, we have

$$v_A(a) \to v_A(b) = v_A(a \to b),\ v_A(a) \rightsquigarrow v_A(b) = v_A(a \rightsquigarrow b).$$

This proves that v_A is a morphism of pseudo BL-algebras.

The injectivity of v_A is obviously. ■

As in the case of BL-algebras, we denote by $\mathcal{R}(A) = \{I \subseteq A : I$ is a regular subset of $A\}$, see Definition 6.5.

REMARK 8.10. *The condition $I \in \mathcal{R}(A)$ is equivalent with the condition: for every $x, y \in A$, if $f_{x|I \cap B(A)} = f_{y|I \cap B(A)}$, then $x = y$.*

LEMMA 8.13. *If $I_1, I_2 \in \mathcal{I}(A) \cap \mathcal{R}(A)$, then $I_1 \cap I_2 \in \mathcal{I}(A) \cap \mathcal{R}(A)$.*

REMARK 8.11. *By Lemma 8.13, we deduce that*

$$M_r(A) = \{f \in M(A) : dom(f) \in \mathcal{I}(A) \cap \mathcal{R}(A)\}$$

is a pseudo BL - subalgebra of $M(A)$.

PROPOSITION 8.14. *$M_r(A)$ is a Boolean subalgebra of $M(A)$.*

Proof. Let $f : I \to A$ be a strong multiplier on A with $I \in \mathcal{I}(A) \cap \mathcal{R}(A)$. Then for all $x \in I$,

$$e \wedge [f \boxdot f](x) = e \wedge [(x \to f(x)) \odot f(x)] = e \odot [x \to f(x)] \odot f(x) =$$
$$\overset{psbl-c_{74}}{=} [(x \odot e) \to (f(x) \odot e)] \odot e \odot f(x) = [(e \odot x) \to (f(x) \odot e)] \odot e \odot f(x) =$$
$$= [(e \odot x) \to (f(e) \odot x)] \odot x \odot f(e) =$$
$$\overset{psbl-c_{75}}{=} [e \to f(e)] \odot x \odot f(e) = [e \to f(e)] \odot e \odot f(x) = [e \wedge f(e)] \odot f(x) =$$
$$= e \odot f(e) \odot f(x) = [e \odot f(x)] \odot f(e) = x \odot f(e) \odot f(e) = x \odot f(e) =$$
$$= e \odot f(x) = e \wedge f(x),$$

hence $(x \to f(x)) \odot f(x) = f(x)$, (since $I \in \mathcal{R}(A)$), hence $f \boxdot f = f$.

We have

$$e \wedge (f^-)^\sim(x) = e \wedge x \odot [x \rightsquigarrow ((f(x))^-)^\sim] = e \odot x \odot [x \rightsquigarrow ((f(x))^-)^\sim] =$$
$$= x \odot e \odot [(e \odot x) \rightsquigarrow (e \odot ((f(x))^-)^\sim)] =$$
$$= x \odot e \odot [(e \odot x) \rightsquigarrow (e \odot [(f(x))^- \rightsquigarrow 0])] =$$
$$= x \odot e \odot [(e \odot x) \rightsquigarrow (e \odot [(e \odot (f(x))^-) \rightsquigarrow 0])] =$$
$$= x \odot e \odot [(e \odot x) \rightsquigarrow (e \odot [(e \odot (f(x))^-)]^\sim)] =$$
$$= x \odot e \odot [(e \odot x) \rightsquigarrow (e \odot [e \odot [f(x) \to 0]]^\sim)] =$$
$$= x \odot e \odot [(e \odot x) \rightsquigarrow (e \odot e \odot ([e \odot f(x)]^-)^\sim)] =$$
$$= x \odot e \odot [(e \odot x) \rightsquigarrow (e \odot ([x \odot f(e)]^-)^\sim)] =$$
$$= x \odot e \odot [x \rightsquigarrow ([x \odot f(e)]^-)^\sim] = x \odot e \odot [x \rightsquigarrow ([x \wedge f(e)]^-)^\sim] =$$
$$= x \odot e \odot [x \rightsquigarrow [x^- \vee (f(e))^-]^\sim] = x \odot e \odot [x \rightsquigarrow [(x^-)^\sim \wedge f(e)]] =$$
$$\overset{psbl-c_{36}}{=} x \odot e \odot ([x \rightsquigarrow (x^-)^\sim] \wedge [x \rightsquigarrow f(e)]) =$$
$$\overset{psbl-c_{42}}{=} x \odot e \odot (1 \wedge [x \rightsquigarrow f(e)]) = x \odot e \odot [x \rightsquigarrow f(e)] =$$
$$= e \odot x \odot [x \rightsquigarrow f(e)] = e \odot [x \wedge f(e)] = e \wedge x \wedge f(e) = x \wedge f(e) = e \wedge f(x),$$

and

$$e \wedge (f^\sim)^-(x) = e \wedge [x \to ((f(x))^\sim)^-] \odot x = [x \to ((f(x))^\sim)^-] \odot e \odot x =$$
$$= [(x \odot e) \rightsquigarrow (((f(x))^\sim)^- \odot e)] \odot e \odot x =$$
$$= [(x \odot e) \rightsquigarrow (([e \odot f(x)]^\sim)^- \odot e)] \odot e \odot x =$$
$$= [(x \odot e) \rightsquigarrow (([x \odot f(e)]^\sim)^- \odot e)] \odot e \odot x =$$
$$= [x \to ([x \odot f(e)]^\sim)^-] \odot x \odot e = [x \to [(x^\sim)^- \wedge f(e)]] \odot x \odot e =$$
$$= ([x \to (x^\sim)^-] \wedge [x \to f(e)]) \odot x \odot e =$$
$$= (1 \wedge [x \to f(e)]) \odot x \odot e = [x \to f(e)] \odot x \odot e =$$
$$= [x \wedge f(e)] \odot e = e \wedge f(x) \wedge e = e \wedge f(x).$$

So, $f \boxdot f = f$ and $f = (f^-)^\sim = (f^\sim)^-$, that is, $M_r(A)$ is a Boolean algebra. ■

REMARK 8.12. *The axioms $sm-BL_3, sm-BL_4$ are necessary in the proof of Proposition 8.14.*

We recall that for two strong multipliers f_1, f_2 on A, we say that f_2 extends f_1 if $dom(f_1) \subseteq dom(f_2)$ and $f_{2|dom(f_1)} = f_1$ and we write $f_1 \leq f_2$ if f_2 extends f_1. A strong multiplier f is called maximal if f can not be extended to a strictly larger domain.

As in the case of $BL-$ algebras we have the following results:

LEMMA 8.15. *If $f_1, f_2 \in M(A)$, $f \in M_r(A)$ and $f \leq f_1, f \leq f_2$, then f_1 and f_2 coincide on the $dom(f_1) \cap dom(f_2)$.*

LEMMA 8.16. *Every strong multiplier $f \in M_r(A)$ can be extended to a maximal strong multiplier.*

LEMMA 8.17. *Each principal strong multiplier f_a with $a \in B(A)$ and $dom(f_a) \in \mathcal{I}(A) \cap \mathcal{R}(A)$ can be uniquely extended to a total multiplier $\overline{f_a}$ and each non-principal strong multiplier can be extended to a maximal non-principal one.*

On the Boolean algebra $M_r(A)$ we consider the relation ρ_A defined by

$(f_1, f_2) \in \rho_A$ iff f_1 and f_2 coincide on the intersection of their domains.

LEMMA 8.18. *ρ_A is a congruence on Boolean algebra $M_r(A)$.*

Proof. See the proof of Lemma 6.14. ■

DEFINITION 8.3. *For $f \in M_r(A)$ with $I = dom(f) \in \mathcal{I}(A) \cap \mathcal{R}(A)$, we denote by $[f, I]$ the congruence class of f modulo ρ_A and $Q(A) = M_r(A)/\rho_A$.*

COROLLARY 8.19. *By Proposition 6.12 and Lemma 8.18 we deduce that $Q(A)$ is a Boolean algebra.*

LEMMA 8.20. *Let the map $\overline{v_A} : B(A) \to Q(A)$ defined by $\overline{v_A}(a) = [\overline{f_a}, A]$ for every $a \in B(A)$. Then*

(i) $\overline{v_A}$ is an injective morphism of Boolean algebras,
(ii) $\overline{v_A}(B(A)) \in \mathcal{R}(A'')$.

Proof. See the proof of Lemma 6.15. ■

REMARK 8.13. *Since for every $a \in B(A)$, $\overline{f_a}$ is the unique maximal strong multiplier on $[\overline{f_a}, A]$ we can identify $[\overline{f_a}, A]$ with $\overline{f_a}$. So, since $\overline{v_A}$ is injective map, the elements of $B(A)$ can be identified with the elements of the set $\{\ \overline{f_a} : a \in B(A)\}$.*

LEMMA 8.21. *In view of the identifications made above, if $[f, dom(f)] \in Q(A)$ (with $f \in M_r(A)$ and $I = dom(f) \in \mathcal{I}(A) \cap \mathcal{R}(A)$), then*

$$I \cap B(A) \subseteq \{a \in B(A) : \overline{f_a} \wedge [f, dom(f)] \in B(A)\}.$$

REMARK 8.14. *The axiom $sm-BL_4$ is necessary in the proof of Lemma* **??**.

2.2. Maximal pseudo BL-algebra of quotients.

DEFINITION 8.4. A pseudo BL -algebra F is called *pseudo BL-algebra of fractions of A* if:

($psBLfr_1$) $B(A)$ is a pseudo BL-subalgebra of F;
($psBLfr_2$) For every $a', b', c' \in F, a' \neq b'$, there exists $e \in B(A)$ such that $e \wedge a' \neq e \wedge b'$ and $e \wedge c' \in B(A)$.

So, pseudo - BL algebra $B(A)$ is a pseudo BL - algebra of fractions of itself (since $1 \in B(A)$).

As a notational convenience, we write $A \preceq F$ to indicate that F is a pseudo BL-algebra of fractions of A.

DEFINITION 8.5. *A pseudo $BL-$ algebra A_M* is a *maximal pseudo BL-algebra of quotients of A* if $A \preceq A_M$ and for every pseudo BL-algebra F with $A \preceq F$ there exists a monomorphism of pseudo BL-algebras $i : F \to A_M$.

REMARK 8.15. *If $A \preceq F$, then F is a Boolean algebra. Indeed, if $a' \in F$ such that $a' \neq a' \odot a'$ or $((a')^-)^\sim \neq a'$ or $((a')^\sim)^- \neq a'$ then there exists $e, f, g \in B(A)$ such that $e \wedge a', f \wedge a', g \wedge a' \in B(A)$ and*

$$e \wedge a' \neq e \wedge (a' \odot a') = (e \wedge a') \odot (e \wedge a') \text{ or}$$

$$f \wedge a' \neq f \wedge ((a')^-)^\sim = ((f \wedge a')^-)^\sim \text{ or}$$

$$g \wedge a' \neq g \wedge ((a')^\sim)^- = ((g \wedge a')^\sim)^-,$$

a contradiction !.

As in the case of $BL-$ algebras we have:

LEMMA 8.22. *Let $A \preceq F$; then for every $a', b' \in F, a' \neq b'$, and any finite sequence $c'_1, ..., c'_n \in F$, there exists $e \in B(A)$ such that $e \wedge a' \neq e \wedge b'$ and $e \wedge c'_i \in B(A)$ for $i = 1, 2, ..., n$ $(n \geq 2)$.*

LEMMA 8.23. *Let $A \preceq F$ and $a' \in F$. Then*

$$I_{a'} = \{e \in B(A) : e \wedge a' \in B(A)\} \in \mathcal{I}(B(A)) \cap \mathcal{R}(A).$$

THEOREM 8.24. *For every pseudo $BL-$ algebra A, $Q(A)$ is the maximal pseudo BL-algebra of quotients of A.*

Proof. See the proof of Theorem 6.19. ■

REMARK 8.16. *If pseudo $BL-$ algebra A is a $BL-$ algebra, then $Q(A)$ is the maximal $BL-$ algebra of quotients of A; if pseudo $BL-$ algebra A is a pseudo $MV-$ algebra, then $Q(A)$ is the maximal pseudo $MV-$ algebra of quotients of A.*

REMARK 8.17. *If A is a Boolean algebra, then $B(A) = A$. By Remark 8.15, $Q(A)$ is a Boolean algebra and the axioms $sm - psBL_1, sm - psBL_2, sm - psBL_3$ and $sm - psBL_4$ are equivalent with $sm - psBL_1$, hence $Q(A)$ is in this case just the classical Dedekind-MacNeille completion of A (see* [**122**]*, p.687). In contrast to the general situation, the Dedekind-MacNeille completion of a Boolean algebra is again distributive and, in fact, is a Boolean algebra (*[**2**]*, p.239).*

PROPOSITION 8.25. *Let A be a pseudo BL - algebra. Then the following statements are equivalent:*

(i) Every maximal strong multiplier on A has domain A;
(ii) For every strong multiplier $f \in M(I, A)$ there is $a \in B$ such that $f = f_a$ (that is $f(x) = a \wedge x$ for every $x \in I$);
(iii) $Q(A) \approx B(A)$.

DEFINITION 8.6. If A verify one of condition of Proposition 8.25, we call A *rationnaly complete.*

REMARK 8.18. 1. *If A is a pseudo - BL algebra with $B(A)=\{0,1\}=L_2$ and $A\preceq F$ then $F=\{0,1\}$, hence $Q(A)=A''\approx L_2$. Indeed, if $a,b,c\in F$ with $a\neq b$, then by $psBLfr_2$ there exists $e\in B(A)$ such $e\wedge a\neq e\wedge b$ (hence $e\neq 0$) and $e\wedge c\in B(A)$. Clearly, $e=1$, hence $c\in B(A)$, that is $F=B(A)$. As examples of pseudo BL-algebras with this property we have local pseudo BL-algebras and pseudo - BL chains.*

2. *More general, if A is a pseudo $BL-$ algebra, $B(A)$ is a finite and $A\preceq F$, then $F=B(A)$, hence in this case $Q(A)=A''\approx B(A)$. Indeed, since $A\preceq F$ we have $B(A)\subseteq B(F)\subseteq F$. If consider $a\in F$, then there exists $e\in B(A)$ such that $e\wedge x\in B(A)$ (for example $e=0$). $B(A)$ being finite, there exists a largest element $e_a\in B(A)$ such $e_a\wedge a\in B(A)$. Suppose $e_a\vee a\neq e_a$, then there would exists $e\in B(A)$ such that $e\wedge(e_a\vee a)\neq e\wedge e_a$ and $e\wedge a\in B(A)$. But $e\wedge a\in B(A)$ implies $e\leq e_a$ and thus we obtain $e=e\wedge(e_a\vee a)\neq e\wedge e_a=e$, a contradiction. Hence $e_a\vee a=e_a$, so $a\leq e_a$, consequently $a=a\wedge e_a\in B(A)$, that is, $F\subseteq B(A)$. Then $F=B(A)$, hence $Q(A)\approx B(A)$.*

3. Localization of pseudo BL-algebras

3.1. Topologies on a pseudo BL-algebra. We recall that, as in the case of $BL-$ algebras, a non-empty set $\mathcal{F}$ of elements of $I\in\mathcal{I}(A)$ will be called a *topology* on a pseudo $BL-$ algebra A if verifies the conditions of Definition 6.10.

EXAMPLE 8.6. *If $I\in\mathcal{I}(A)$, then the set $\mathcal{F}(I)=\{I'\in\mathcal{I}(A):I\subseteq I'\}$ is a topology on A.*

EXAMPLE 8.7. *If we denote $\mathcal{R}(A)=\{I\subseteq A:I$ is a regular subset of $A\}$, then $\mathcal{I}(A)\cap\mathcal{R}(A)$ is a topology on A.*

EXAMPLE 8.8. *If we denote by $D(A)$ the set of all dense subsets of A, then $\mathcal{R}(A)\subseteq D(A)$ and $\mathcal{F}=\mathcal{I}(A)\cap D(A)$ is a topology on A.*

EXAMPLE 8.9. *For any $\wedge-$ closed subset S of A, we set $\mathcal{F}_S=\{I\in\mathcal{I}(A):I\cap S\cap B(A)\neq\oslash\}$ is a topology on A.*

3.2. $\mathcal{F}$-multipliers and localization pseudo BL-algebras. Let $\mathcal{F}$ a topology on A. As in the case of $BL-$ algebras, the relation $\theta_{\mathcal{F}}$ of A defined in the following way:

$$(x,y)\in\theta_{\mathcal{F}}\Leftrightarrow\text{there exists } I\in\mathcal{F}\text{ such that } e\wedge x=e\wedge y\text{ for any } e\in I\cap B(A),$$

is a congruence on A.

We denote by $x/\theta_{\mathcal{F}}$ the congruence class of an element $x\in A$ and by $p_{\mathcal{F}}:A\to A/\theta_{\mathcal{F}}$the canonical morphism of pseudo BL-algebras.

PROPOSITION 8.26. *For $a\in A$, $a/\theta_{\mathcal{F}}\in B(A/\theta_{\mathcal{F}})$ iff there exists $I\in\mathcal{F}$ such that $a\wedge e\in B(A)$ for every $e\in I\cap B(A)$. So, if $a\in B(A)$, then $a/\theta_{\mathcal{F}}\in B(A/\theta_{\mathcal{F}})$.*

Proof. For $a\in A$, we have $a/\theta_{\mathcal{F}}\in B(A/\theta_{\mathcal{F}})\Leftrightarrow a/\theta_{\mathcal{F}}\odot a/\theta_{\mathcal{F}}=a/\theta_{\mathcal{F}}$ and $[(a/\theta_{\mathcal{F}})^{\sim}]^{-}=[(a/\theta_{\mathcal{F}})^{-}]^{\sim}=a/\theta_{\mathcal{F}}\Leftrightarrow(a\odot a)/\theta_{\mathcal{F}}=a/\theta_{\mathcal{F}}$ and $(a^{\sim})^{-}/\theta_{\mathcal{F}}=(a^{-})^{\sim}/\theta_{\mathcal{F}}=a/\theta_{\mathcal{F}}\Leftrightarrow$ there exists $J,K,G\in\mathcal{F}$ such that $(a\odot a)\wedge j=a\wedge j$, for every $j\in J\cap B(A)$, $(a^{\sim})^{-}\wedge k=a\wedge k$, for every $k\in K\cap B(A)$ and $(a^{-})^{\sim}\wedge g=a\wedge g$, for every $g\in G\cap B(A)$.

From $psbl-c_{73}$, we deduce that $(a\wedge j)\odot(a\wedge j)=(a\odot a)\wedge j=a\wedge j$, for every $j\in J\cap B(A)$.

If denote $I=J\cap K\cap G$, then $I\in\mathcal{F}$ and for every $e\in I\cap B(A)$,

$$(a\wedge e)\odot(a\wedge e)=a\wedge e,$$

$$[(a\wedge e)^{\sim}]^{-}\overset{psbl-c_{49}}{=}(a^{\sim}\vee e^{\sim})^{-}\overset{psbl-c_{50}}{=}(a^{\sim})^{-}\wedge(e^{\sim})^{-}=(a^{\sim})^{-}\wedge e=a\wedge e$$

and

$$[(a\wedge e)^{-}]^{\sim}\overset{psbl-c_{50}}{=}(a^{-}\vee e^{-})^{\sim}\overset{psbl-c_{49}}{=}(a^{-})^{\sim}\wedge(e^{-})^{\sim}=(a^{-})^{\sim}\wedge e=a\wedge e,$$

so, $a\wedge e\in B(A)$ for every $e\in I\cap B(A)$.

So, if $a\in B(A)$, then for every $I\in\mathcal{F}$, $a\wedge e\in B(A)$ for every $e\in I\cap B(A)$, hence $a/\theta_{\mathcal{F}}\in B(A/\theta_{\mathcal{F}})$.■

Corollary 8.27. *If $\mathcal{F}=\mathcal{I}(A)\cap\mathcal{R}(A)$, then for $a\in A$, $a\in B(A)$ iff $a/\theta_{\mathcal{F}}\in B(A/\theta_{\mathcal{F}})$.*

Definition 8.7. *Let $\mathcal{F}$ be a topology on A. A* partial $\mathcal{F}-$multiplier *is a mapping $f:I\to A/\theta_{\mathcal{F}}$ where $I\in\mathcal{F}$ and for every $x\in I$ and $e\in B(A)$ the following axioms are fulfilled:*

$(m-psBL_1)$ $f(e\odot x)=e/\theta_{\mathcal{F}}\wedge f(x)=e/\theta_{\mathcal{F}}\odot f(x)$;
$(m-psBL_2)$ $f(x)\leq x/\theta_{\mathcal{F}}$.

By $dom(f)\in\mathcal{F}$ we denote the domain of f; if $dom(f)=A$, we called f *total*.

To simplify language, we will use *$\mathcal{F}-$multiplier* instead *partial $\mathcal{F}-$multiplier,* using *total* to indicate that the domain of a certain $\mathcal{F}-$multiplier is A.

The maps $\mathbf{0},\mathbf{1}:A\to A/\theta_{\mathcal{F}}$ defined by $\mathbf{0}(x)=0/\theta_{\mathcal{F}}$ and $\mathbf{1}(x)=x/\theta_{\mathcal{F}}$ for every $x\in A$ are multipliers in the sense of Definition 8.7.

Also for $a\in B(A)$, $f_a:A\to A/\theta_{\mathcal{F}}$ defined by $f_a(x)=a/\theta_{\mathcal{F}}\wedge x/\theta_{\mathcal{F}}$ for every $x\in A$, is an $\mathcal{F}-$ multiplier. If $dom(f_a)=A$, we denote f_a by $\overline{f_a}$; clearly, $\overline{f_0}=\mathbf{0}$.

We shall denote by $M(I,A/\theta_{\mathcal{F}})$ the set of all $\mathcal{F}-$ multipliers having the domain $I\in\mathcal{F}$ and

$$M(A/\theta_{\mathcal{F}})=\underset{I\in\mathcal{F}}{\cup}M(I,A/\theta_{\mathcal{F}}).$$

If $I_1,I_2\in\mathcal{F}$, $I_1\subseteq I_2$ we have a canonical mapping

$$\varphi_{I_1,I_2}:M(I_2,A/\theta_{\mathcal{F}})\to M(I_1,A/\theta_{\mathcal{F}})$$

defined by

$$\varphi_{I_1,I_2}(f)=f_{|I_1}\text{ for }f\in M(I_2,A/\theta_{\mathcal{F}}).$$

Let us consider the directed system of sets

$$\left\langle\{M(I,A/\theta_{\mathcal{F}})\}_{I\in\mathcal{F}},\{\varphi_{I_1,I_2}\}_{I_1,I_2\in\mathcal{F},I_1\subseteq I_2}\right\rangle$$

and denote by $A_{\mathcal{F}}$ the inductive limit (in the category of sets):

$$A_{\mathcal{F}}=\varinjlim_{I\in\mathcal{F}}M(I,A/\theta_{\mathcal{F}}).$$

For any $\mathcal{F}-$ multiplier $f:I\to A/\theta_{\mathcal{F}}$ we shall denote by $\widehat{(I,f)}$ the equivalence class of f in $A_{\mathcal{F}}$.

REMARK 8.19. *If $f_i : I_i \to A/\theta_{\mathcal{F}}$, $i = 1, 2$, are $\mathcal{F}-$ multipliers, then $\widehat{(I_1, f_1)} = \widehat{(I_2, f_2)}$ (in $A_{\mathcal{F}}$) iff there exists $I \in \mathcal{F}$, $I \subseteq I_1 \cap I_2$ such that $f_{1|I} = f_{2|I}$.*

Let $f_i : I_i \to A/\theta_{\mathcal{F}}$, (with $I_i \in \mathcal{F}$, $i = 1, 2$), $\mathcal{F}-$multipliers. Let us consider the mappings:

$$f_1 \wedge f_2 : I_1 \cap I_2 \to A/\theta_{\mathcal{F}}$$
$$f_1 \vee f_2 : I_1 \cap I_2 \to A/\theta_{\mathcal{F}}$$
$$f_1 \boxdot f_2 : I_1 \cap I_2 \to A/\theta_{\mathcal{F}}$$
$$f_1 \to f_2 : I_1 \cap I_2 \to A/\theta_{\mathcal{F}}$$
$$f_1 \rightsquigarrow f_2 : I_1 \cap I_2 \to A/\theta_{\mathcal{F}}$$

defined by

$$(f_1 \wedge f_2)(x) = f_1(x) \wedge f_2(x),$$
$$(f_1 \vee f_2)(x) = f_1(x) \vee f_2(x)$$
$$(f_1 \boxdot f_2)(x) = [x/\theta_{\mathcal{F}} \to f_1(x)] \odot f_2(x) \overset{psbl-c_{77}}{=} f_1(x) \odot [x/\theta_{\mathcal{F}} \rightsquigarrow f_2(x)],$$
$$(f_1 \to f_2)(x) = [f_1(x) \to f_2(x)] \odot x/\theta_{\mathcal{F}},$$
$$(f_1 \rightsquigarrow f_2)(x) = x/\theta_{\mathcal{F}} \odot [f_1(x) \rightsquigarrow f_2(x)],$$

for any $x \in I_1 \cap I_2$, and let

$$\widehat{(I_1, f_1)} \curlywedge \widehat{(I_2, f_2)} = (I_1 \cap \widehat{I_2, f_1} \wedge f_2),$$
$$\widehat{(I_1, f_1)} \curlyvee \widehat{(I_2, f_2)} = (I_1 \cap \widehat{I_2, f_1} \vee f_2),$$
$$\widehat{(I_1, f_1)} \cdot \widehat{(I_2, f_2)} = (I_1 \cap \widehat{I_2, f_1} \boxdot f_2),$$
$$\widehat{(I_1, f_1)} \longmapsto \widehat{(I_2, f_2)} = (I_1 \cap \widehat{I_2, f_1} \to f_2).$$

$$\widehat{(I_1, f_1)} \leftrightsquigarrow \widehat{(I_2, f_2)} = (I_1 \cap \widehat{I_2, f_1} \rightsquigarrow f_2).$$

Clearly the definitions of the operations $\curlywedge, \curlyvee, \cdot, \longmapsto$ and $\leftrightsquigarrow$ on $A_{\mathcal{F}}$ are correct.

LEMMA 8.28. $f_1 \wedge f_2 \in M(I_1 \cap I_2, A/\theta_{\mathcal{F}})$.

Proof. See the proof of Lemma 6.24. ■

LEMMA 8.29. $f_1 \vee f_2 \in M(I_1 \cap I_2, A/\theta_{\mathcal{F}})$.

Proof. See the proof of Lemma 6.25. ■

LEMMA 8.30. $f_1 \boxdot f_2 \in M(I_1 \cap I_2, A/\theta_{\mathcal{F}})$.

Proof. If $x \in I_1 \cap I_2$ and $e \in B(A)$, then

$$(f_1 \boxdot f_2)(e \odot x) = [(e \odot x)/\theta_{\mathcal{F}} \to f_1(e \odot x)] \odot f_2(e \odot x)$$
$$= [(e \odot x)/\theta_{\mathcal{F}} \to (e/\theta_{\mathcal{F}} \odot f_1(x))] \odot [e/\theta_{\mathcal{F}} \odot f_2(x)] =$$
$$= [((e/\theta_{\mathcal{F}} \odot x/\theta_{\mathcal{F}}) \to (e/\theta_{\mathcal{F}} \odot f_1(x))) \odot e/\theta_{\mathcal{F}}] \odot f_2(x) \overset{psbl-c_{74}}{=} [(x/\theta_{\mathcal{F}} \to f_1(x)) \odot e/\theta_{\mathcal{F}}] \odot f_2(x) =$$
$$= [(x/\theta_{\mathcal{F}} \to f_1(x)) \odot f_2(x)] \odot e/\theta_{\mathcal{F}} = (f_1 \boxdot f_2)(x) \odot e/\theta_{\mathcal{F}}.$$

Clearly, $(f_1 \boxdot f_2)(x) = [x/\theta_{\mathcal{F}} \to f_1(x)] \odot f_2(x) \leq f_2(x) \leq x/\theta_{\mathcal{F}}$, for every $x \in I_1 \cap I_2$, that is $f_1 \odot f_2 \in M(I_1 \cap I_2, A/\theta_{\mathcal{F}})$. ■

LEMMA 8.31. . $f_1 \to f_2 \in M(I_1 \cap I_2, A/\theta_{\mathcal{F}})$.

Proof. If $x \in I_1 \cap I_2$ and $e \in B(A)$, then

$$(f_1 \to f_2)(e \odot x) = [f_1(e \odot x) \to f_2(e \odot x)] \odot (e \odot x)/\theta_{\mathcal{F}} =$$
$$= [(e/\theta_{\mathcal{F}} \odot f_1(x)) \to (e/\theta_{\mathcal{F}} \odot f_2(x))] \odot (e \odot x)/\theta_{\mathcal{F}} =$$
$$= [((e/\theta_{\mathcal{F}} \odot f_1(x)) \to (e/\theta_{\mathcal{F}} \odot f_2(x))) \odot e/\theta_{\mathcal{F}}] \odot x/\theta_{\mathcal{F}} \overset{psbl-c_{74}}{=} [(f_1(x) \to f_2(x)) \odot e/\theta_{\mathcal{F}}] \odot x/\theta_{\mathcal{F}} =$$
$$= [(f_1(x) \to f_2(x)) \odot x/\theta_{\mathcal{F}}] \odot e/\theta_{\mathcal{F}} = [(f_1 \to f_2)(x)] \odot e/\theta_{\mathcal{F}}.$$

Clearly, $(f_1 \to f_2)(x) = [f_1(x) \to f_2(x)] \odot x/\theta_{\mathcal{F}} \leq x/\theta_{\mathcal{F}}$, for every $x \in I_1 \cap I_2$, that is $f_1 \to f_2 \in M(I_1 \cap I_2, A/\theta_{\mathcal{F}})$. ■

LEMMA 8.32. . $f_1 \rightsquigarrow f_2 \in M(I_1 \cap I_2, A/\theta_{\mathcal{F}})$.

Proof. If $x \in I_1 \cap I_2$ and $e \in B(A)$, then

$$(f_1 \rightsquigarrow f_2)(e \odot x) = (e \odot x)/\theta_{\mathcal{F}} \odot [f_1(e \odot x) \rightsquigarrow f_2(e \odot x)]$$
$$= (e \odot x)/\theta_{\mathcal{F}} \odot [(e/\theta_{\mathcal{F}} \odot f_1(x)) \rightsquigarrow (e/\theta_{\mathcal{F}} \odot f_2(x))] =$$
$$= x/\theta_{\mathcal{F}} \odot [e/\theta_{\mathcal{F}} \odot ((e/\theta_{\mathcal{F}} \odot f_1(x)) \rightsquigarrow (e/\theta_{\mathcal{F}} \odot f_2(x)))]$$
$$\overset{psbl-c_{74}}{=} x/\theta_{\mathcal{F}} \odot [e/\theta_{\mathcal{F}} \odot (f_1(x) \rightsquigarrow f_2(x))] =$$
$$= e/\theta_{\mathcal{F}} \odot [x/\theta_{\mathcal{F}} \odot (f_1(x) \rightsquigarrow f_2(x))] = e/\theta_{\mathcal{F}} \odot (f_1 \rightsquigarrow f_2)(x).$$

Clearly, $(f_1 \rightsquigarrow f_2)(x) = x/\theta_{\mathcal{F}} \odot [f_1(x) \rightsquigarrow f_2(x)] \leq x/\theta_{\mathcal{F}}$, for every $x \in I_1 \cap I_2$, that is $f_1 \rightsquigarrow f_2 \in M(I_1 \cap I_2, A/\theta_{\mathcal{F}})$. ■

PROPOSITION 8.33. $(A_{\mathcal{F}}, \curlywedge, \curlyvee, \cdot, \longmapsto, \leftrightsquigarrow, \mathbf{0} = \widehat{(A, \mathbf{0})}, \mathbf{1} = \widehat{(A, \mathbf{1})})$ *is a pseudo - BL algebra.*

Proof. We will verify the axioms of pseudo - BL algebras.

$(psBL_1)$. Obviously $(A_{\mathcal{F}}, \curlywedge, \curlyvee, \mathbf{0} = \widehat{(A, \mathbf{0})}, \mathbf{1} = \widehat{(A, \mathbf{1})})$ is a bounded lattice.

$(psBL_2)$. Let $f_i \in M(I_i, A/\theta_{\mathcal{F}})$ where $I_i \in \mathcal{F}$, $i = 1, 2, 3$.

Clearly, $f_1 \boxdot f_2 \in M(A/\theta_{\mathcal{F}})$ (see Lemma 8.30) and

$$\widehat{(I_1, f_1)} \cdot \widehat{(I_2, f_2)} = (I_1 \cap \widehat{I_2, f_1 \boxdot f_2}) \in A_{\mathcal{F}}.$$

Thus, for $x \in I_1 \cap I_2 \cap I_3$ we have

$$[(f_1 \boxdot f_2) \boxdot f_3](x) = ((f_1 \boxdot f_2)(x)) \odot (x/\theta_{\mathcal{F}} \rightsquigarrow f_3(x)) =$$
$$= [(x/\theta_{\mathcal{F}} \to f_1(x)) \odot f_2(x)] \odot (x/\theta_{\mathcal{F}} \rightsquigarrow f_3(x)) =$$
$$= (x/\theta_{\mathcal{F}} \to f_1(x)) \odot [f_2(x) \odot (x/\theta_{\mathcal{F}} \rightsquigarrow f_3(x))] =$$
$$= (x/\theta_{\mathcal{F}} \to f_1(x)) \odot [(f_2 \boxdot f_3)(x)] = [f_1 \boxdot (f_2 \boxdot f_3)](x),$$

so

$$\widehat{(I_1, f_1)} \cdot [\widehat{(I_2, f_2)} \cdot \widehat{(I_3, f_3)}] = [\widehat{(I_1, f_1)} \cdot \widehat{(I_2, f_2)}] \cdot \widehat{(I_3, f_3)},$$

that is the operation $\cdot$ is associative on $A_{\mathcal{F}}$.

Let $f \in M(I, A/\theta_{\mathcal{F}})$ with $I \in \mathcal{F}$. If $x \in I$, then

$$(f \boxdot \mathbf{1})(x) = f(x) \odot (x/\theta_{\mathcal{F}} \rightsquigarrow \mathbf{1}(x)) = f(x) \odot (x/\theta_{\mathcal{F}} \rightsquigarrow x/\theta_{\mathcal{F}}) = f(x) \odot 1/\theta_{\mathcal{F}} = f(x),$$

and

$$(\mathbf{1} \boxdot f)(x) = \mathbf{1}(x) \odot (x/\theta_{\mathcal{F}} \rightsquigarrow f(x)) = x/\theta_{\mathcal{F}} \odot (x/\theta_{\mathcal{F}} \rightsquigarrow f(x)) = x/\theta_{\mathcal{F}} \wedge f(x) = f(x),$$

hence

$$f \boxdot \mathbf{1} = \mathbf{1} \boxdot f = f,$$

that is

$$\widehat{(I,f)} \cdot \widehat{(A,\mathbf{1})} = \widehat{(A,\mathbf{1})} \cdot \widehat{(I,f)} = \widehat{(I,f)},$$

and $(A_{\mathcal{F}}, \cdot, \mathbf{1} = \widehat{(A,\mathbf{1})})$ is a monoid.

$(psBL_3)$. Let $f_i \in M(I_i, A/\theta_{\mathcal{F}})$ where $I_i \in \mathcal{F}$, $i = 1, 2, 3$.

Since $f_1 \leq f_2 \to f_3$ for $x \in I_1 \cap I_2 \cap I_3$ we have

$$f_1(x) \leq (f_2 \to f_3)(x) \Leftrightarrow f_1(x) \leq [f_2(x) \to f_3(x)] \odot x/\theta_{\mathcal{F}}.$$

So, by $psbl - c_3$

$$f_1(x) \odot [x/\theta_{\mathcal{F}} \rightsquigarrow f_2(x)] \leq [f_2(x) \to f_3(x)] \odot x/\theta_{\mathcal{F}} \odot [x/\theta_{\mathcal{F}} \rightsquigarrow f_2(x)] \Leftrightarrow$$
$$f_1(x) \odot [x/\theta_{\mathcal{F}} \rightsquigarrow f_2(x)] \leq (f_2(x) \to f_3(x)) \odot (x/\theta_{\mathcal{F}} \wedge f_2(x)) \Leftrightarrow$$
$$f_1(x) \odot [x/\theta_{\mathcal{F}} \rightsquigarrow f_2(x)] \leq (f_2(x) \to f_3(x)) \odot f_2(x) \Leftrightarrow$$
$$f_1(x) \odot [x/\theta_{\mathcal{F}} \rightsquigarrow f_2(x)] \leq f_2(x) \wedge f_3(x) \leq f_3(x) \Leftrightarrow$$
$$(f_1 \boxdot f_2)(x) \leq f_3(x),$$

for every $x \in I_1 \cap I_2 \cap I_3$, that is

$$f_1 \boxdot f_2 \leq f_3.$$

Conversely if $(f_1 \boxdot f_2)(x) \leq f_3(x)$ we have

$$[x/\theta_{\mathcal{F}} \to f_1(x)] \odot f_2(x) \leq f_3(x),$$

for every $x \in I_1 \cap I_2 \cap I_3$.

Obviously,

$$[x/\theta_{\mathcal{F}} \to f_1(x)] \leq f_2(x) \to f_3(x)$$
$$\overset{psbl-c_3}{\Leftrightarrow} (x/\theta_{\mathcal{F}} \to f_1(x)) \odot x/\theta_{\mathcal{F}} \leq (f_2(x) \to f_3(x)) \odot x/\theta_{\mathcal{F}}$$
$$\Leftrightarrow x/\theta_{\mathcal{F}} \wedge f_1(x) \leq (f_2(x) \to f_3(x)) \odot x/\theta_{\mathcal{F}}$$
$$\Leftrightarrow f_1(x) \leq (f_2 \to f_3)(x).$$

So $f_1 \leq f_2 \to f_3$ iff $f_1 \boxdot f_2 \leq f_3$ for all $f_1, f_2, f_3 \in M(A/\theta_{\mathcal{F}})$.

Since $f_2 \leq f_1 \rightsquigarrow f_3$ for $x \in I_1 \cap I_2 \cap I_3$ we have

$$f_2(x) \leq (f_1 \rightsquigarrow f_3)(x) \Leftrightarrow f_2(x) \leq x/\theta_{\mathcal{F}} \odot [f_1(x) \rightsquigarrow f_3(x)].$$

So, by $psbl - c_3$

$$[x/\theta_{\mathcal{F}} \to f_1(x)] \odot f_2(x) \leq [x/\theta_{\mathcal{F}} \to f_1(x)] \odot x/\theta_{\mathcal{F}} \odot [f_1(x) \rightsquigarrow f_3(x)] \Leftrightarrow$$
$$(f_1 \boxdot f_2)(x) \leq (x/\theta_{\mathcal{F}} \wedge f_1(x)) \odot (f_1(x) \rightsquigarrow f_3(x)) \Leftrightarrow$$
$$(f_1 \boxdot f_2)(x) \leq f_1(x) \odot (f_1(x) \rightsquigarrow f_3(x)) \Leftrightarrow$$
$$(f_1 \boxdot f_2)(x) \leq f_1(x) \wedge f_3(x) \leq f_3(x) \Leftrightarrow$$
$$(f_1 \boxdot f_2)(x) \leq f_3(x),$$

for every $x \in I_1 \cap I_2 \cap I_3$, that is

$$f_1 \boxdot f_2 \leq f_3.$$

Conversely if $(f_1 \boxdot f_2)(x) \leq f_3(x)$ we have

$$f_1(x) \odot [x/\theta_{\mathcal{F}} \rightsquigarrow f_2(x)] \leq f_3(x),$$

for every $x \in I_1 \cap I_2 \cap I_3$.

Obviously,

$$(x/\theta_{\mathcal{F}} \rightsquigarrow f_2(x)) \leq f_1(x) \rightsquigarrow f_3(x)$$

$$\overset{psbl-c_3}{\Leftrightarrow} x/\theta_{\mathcal{F}} \odot (x/\theta_{\mathcal{F}} \rightsquigarrow f_2(x)) \leq x/\theta_{\mathcal{F}} \odot (f_1(x) \rightsquigarrow f_3(x))$$
$$\Leftrightarrow x/\theta_{\mathcal{F}} \wedge f_2(x) \leq x/\theta_{\mathcal{F}} \odot (f_1(x) \rightsquigarrow f_3(x))$$
$$\Leftrightarrow f_2(x) \leq (f_1 \rightsquigarrow f_3)(x).$$

So, $f_2 \leq f_1 \rightsquigarrow f_3$ iff $f_1 \boxdot f_2 \leq f_3$ for all $f_1, f_2, f_3 \in M(A/\theta_{\mathcal{F}})$. So,

$$\widehat{(I_1, f_1)} \leq \widehat{(I_2, f_2)} \longmapsto \widehat{(I_3, f_3)} \text{ iff } \widehat{(I_1, f_1)} \cdot \widehat{(I_2, f_2)} \leq \widehat{(I_3, f_3)}$$

and

$$\widehat{(I_2, f_2)} \leq \widehat{(I_1, f_1)} \leftrightsquigarrow \widehat{(I_3, f_3)} \text{ iff } \widehat{(I_1, f_1)} \cdot \widehat{(I_2, f_2)} \leq \widehat{(I_3, f_3)}.$$

$(psBL_4)$. Let $f_i \in M(I_i, A/\theta_{\mathcal{F}})$ where $I_i \in \mathcal{F}$, $i = 1, 2$.

Thus, for $x \in I_1 \cap I_2$ we have

$$[(f_1 \to f_2) \boxdot f_1](x) = [(f_1 \to f_2)(x)] \odot [x/\theta_{\mathcal{F}} \rightsquigarrow f_1(x)]$$
$$= ([f_1(x) \to f_2(x)] \odot x/\theta_{\mathcal{F}}) \odot [x/\theta_{\mathcal{F}} \rightsquigarrow f_1(x)] =$$
$$= [f_1(x) \to f_2(x)] \odot (x/\theta_{\mathcal{F}} \odot [x/\theta_{\mathcal{F}} \rightsquigarrow f_1(x)]) =$$
$$= [f_1(x) \to f_2(x)] \odot [x/\theta_{\mathcal{F}} \wedge f_1(x)] = [f_1(x) \to f_2(x)] \odot f_1(x) =$$
$$= f_1(x) \wedge f_2(x) = (f_1 \wedge f_2)(x),$$

and

$$[f_1 \boxdot (f_1 \rightsquigarrow f_2)](x) = [x/\theta_{\mathcal{F}} \to f_1(x)] \odot [(f_1 \rightsquigarrow f_2)(x)]$$
$$= [x/\theta_{\mathcal{F}} \to f_1(x)] \odot (x/\theta_{\mathcal{F}} \odot [f_1(x) \rightsquigarrow f_2(x)]) =$$
$$= ([x/\theta_{\mathcal{F}} \to f_1(x)] \odot x/\theta_{\mathcal{F}}) \odot [f_1(x) \rightsquigarrow f_2(x)] =$$
$$= [x/\theta_{\mathcal{F}} \wedge f_1(x)] \odot [f_1(x) \rightsquigarrow f_2(x)] = f_1(x) \odot [f_1(x) \rightsquigarrow f_2(x)] =$$
$$= f_1(x) \wedge f_2(x) = (f_1 \wedge f_2)(x).$$

So,

$$f_1 \wedge f_2 = (f_1 \to f_2) \boxdot f_1 = f_1 \boxdot (f_1 \rightsquigarrow f_2)$$

and

$$\widehat{(I_1, f_1)} \curlywedge \widehat{(I_2, f_2)} = [\widehat{(I_1, f_1)} \longmapsto \widehat{(I_2, f_2)}] \cdot \widehat{(I_1, f_1)} = \widehat{(I_1, f_1)} \cdot [\widehat{(I_1, f_1)} \leftrightsquigarrow \widehat{(I_2, f_2)}].$$

$(psBL_5)$. We have

$$[(f_1 \to f_2) \vee (f_2 \to f_1)](x) = [(f_1 \to f_2)(x)] \vee [(f_2 \to f_1)(x)] =$$
$$= [(f_1(x) \to f_2(x)) \odot x/\theta_{\mathcal{F}}] \vee [(f_2(x) \to f_1(x)) \odot x/\theta_{\mathcal{F}}] =$$
$$\overset{psbl-c_{36}}{=} [(f_1(x) \to f_2(x)) \vee (f_2(x) \to f_1(x))] \odot x/\theta_{\mathcal{F}} = 1/\theta_{\mathcal{F}} \odot x/\theta_{\mathcal{F}} = x/\theta_{\mathcal{F}} = \mathbf{1}(x),$$

and

$$[(f_1 \rightsquigarrow f_2) \vee (f_2 \rightsquigarrow f_1)](x) = [(f_1 \rightsquigarrow f_2)(x)] \vee [(f_2 \rightsquigarrow f_1)(x)] =$$
$$= [x/\theta_{\mathcal{F}} \odot (f_1(x) \rightsquigarrow f_2(x))] \vee [x/\theta_{\mathcal{F}} \odot (f_2(x) \rightsquigarrow f_1(x))] =$$
$$\overset{psbl-c_{36}}{=} x/\theta_{\mathcal{F}} \odot [(f_1(x) \rightsquigarrow f_2(x)) \vee (f_2(x) \rightsquigarrow f_1(x))] = x/\theta_{\mathcal{F}} \odot 1/\theta_{\mathcal{F}} = x/\theta_{\mathcal{F}} = \mathbf{1}(x),$$

hence

$$(f_1 \to f_2) \vee (f_2 \to f_1) = (f_1 \rightsquigarrow f_2) \vee (f_2 \rightsquigarrow f_1) = \mathbf{1}$$

and

$$[\widehat{(I_1, f_1)} \longmapsto \widehat{(I_2, f_2)}] \curlyvee [\widehat{(I_2, f_2)} \longmapsto \widehat{(I_1, f_1)}] =$$
$$= [\widehat{(I_1, f_1)} \leftrightsquigarrow \widehat{(I_2, f_2)}] \curlyvee [\widehat{(I_2, f_2)} \leftrightsquigarrow \widehat{(I_1, f_1)}] = \widehat{(A, \mathbf{1})}.\blacksquare$$

REMARK 8.20. *$(M(A/\theta_{\mathcal{F}}), \wedge, \vee, \boxdot, \to, \rightsquigarrow, \mathbf{0}, \mathbf{1})$ is a pseudo - BL algebra.*

DEFINITION 8.8. The pseudo - BL algebra $A_{\mathcal{F}}$ will be called the *localization pseudo - BL algebra of A with respect to the topology $\mathcal{F}$.*

PROPOSITION 8.34. *If pseudo $BL-$ algebra $(A, \vee, \wedge, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ is a pseudo $MV-$ algebra $(A, \odot, \oplus, ^-, ^\sim, 0, 1)$ (i.e. $(x^-)^\sim = (x^\sim)^- = x$, for all $x \in A$), then pseudo $BL-$ algebra $(M(A/\theta_{\mathcal{F}}), \wedge, \vee, \boxdot, \rightarrow, \rightsquigarrow, \mathbf{0}, \mathbf{1})$ is a pseudo $MV-$ algebra*

$$(M(A/\theta_{\mathcal{F}}), \boxdot, \boxplus, ^-, ^\sim, \mathbf{0}, \mathbf{1}),$$

where for $f_i : I_i \rightarrow A/\theta_{\mathcal{F}}$, (with $I_i \in \mathcal{F}$, $i = 1, 2$), $\mathcal{F}-$multipliers we have the mapping

$$f_1 \boxplus f_2 : I_1 \cap I_2 \rightarrow A/\theta_{\mathcal{F}},$$

$$(f_1 \boxplus f_2)(x) = (f_1(x) \oplus f_2(x)) \wedge x/\theta_{\mathcal{F}}$$

for any $x \in I_1 \cap I_2$, and for any $\mathcal{F}-$multiplier $f : I \rightarrow A/\theta_{\mathcal{F}}$ (with $I \in \mathcal{F}$) we have the mappings

$$f^- = f \rightarrow \mathbf{0} : I \rightarrow A/\theta_{\mathcal{F}},$$

$$f^-(x) = (f \rightarrow \mathbf{0})(x) = [f(x) \rightarrow \mathbf{0}(x)] \odot x/\theta_{\mathcal{F}} = [f(x)]^- \odot x/\theta_{\mathcal{F}}$$

for any $x \in I$, and

$$f^\sim = f \rightsquigarrow \mathbf{0} : I \rightarrow A/\theta_{\mathcal{F}},$$

$$f^\sim(x) = (f \rightsquigarrow \mathbf{0})(x) = x/\theta_{\mathcal{F}} \odot [f(x) \rightsquigarrow \mathbf{0}(x)] = x/\theta_{\mathcal{F}} \odot [f(x)]^\sim$$

for any $x \in I$.

Proof. To prove that pseudo $BL-$ algebra $M(A/\theta_{\mathcal{F}})$ is a pseudo $MV-$ algebra let $f \in M(I, A/\theta_{\mathcal{F}})$, where $I \in \mathcal{F}$.

Then

$$(f^-)^\sim(x) = x/\theta_{\mathcal{F}} \odot [(f(x))^- \odot x/\theta_{\mathcal{F}}]^\sim \overset{psbl-c_{48}}{=} x/\theta_{\mathcal{F}} \odot [x/\theta_{\mathcal{F}} \rightsquigarrow ((f(x))^-)^\sim]$$

$$= x/\theta_{\mathcal{F}} \odot (x/\theta_{\mathcal{F}} \rightsquigarrow f(x)) = x/\theta_{\mathcal{F}} \wedge f(x) = f(x)$$

and

$$(f^\sim)^-(x) = [x/\theta_{\mathcal{F}} \odot (f(x))^\sim]^- \odot x/\theta_{\mathcal{F}} \overset{psbl-c_{48}}{=} [x/\theta_{\mathcal{F}} \rightarrow ((f(x))^\sim)^-] \odot x/\theta_{\mathcal{F}}$$

$$= (x/\theta_{\mathcal{F}} \rightarrow f(x)) \odot x/\theta_{\mathcal{F}} = x/\theta_{\mathcal{F}} \wedge f(x) = f(x)$$

(since A is a pseudo $MV-$ algebra then $A/\theta_{\mathcal{F}}$ is a pseudo $MV-$ algebra and $f(x) \in A/\theta_{\mathcal{F}}$, for all $x \in I$).

So, $(f^-)^\sim = (f^\sim)^- = f$, for all $f \in M(A/\theta_{\mathcal{F}})$ and pseudo $BL-$ algebra $M(A/\theta_{\mathcal{F}})$ is a pseudo MV -algebra.

We have $f_1 \boxplus f_2 = (f_2^- \boxdot f_1^-)^\sim$.

Clearly,

$$(f_1 \boxplus f_2)(x) = x/\theta_{\mathcal{F}} \odot [f_2^-(x) \odot (x/\theta_{\mathcal{F}} \rightsquigarrow f_1^-(x))]^\sim$$

$$= x/\theta_{\mathcal{F}} \odot [(f_2(x))^- \odot x/\theta_{\mathcal{F}} \odot (x/\theta_{\mathcal{F}} \rightsquigarrow (f_1(x))^- \odot x/\theta_{\mathcal{F}})]^\sim =$$

$$= x/\theta_{\mathcal{F}} \odot [(f_2(x))^- \odot (x/\theta_{\mathcal{F}} \wedge [(f_1(x))^- \odot x/\theta_{\mathcal{F}}])]^\sim = x/\theta_{\mathcal{F}} \odot [(f_2(x))^- \odot (f_1(x))^- \odot x/\theta_{\mathcal{F}}]^\sim$$

$$\overset{psbl-c_{48}}{=} x/\theta_{\mathcal{F}} \odot [x/\theta_{\mathcal{F}} \rightsquigarrow ((f_2(x))^- \odot (f_1(x))^-)^\sim]$$

$$= x/\theta_{\mathcal{F}} \wedge [(f_2(x))^- \odot (f_1(x))^-]^\sim = x/\theta_{\mathcal{F}} \wedge (f_1(x) \oplus f_2(x)),$$

for all $x \in I_1 \cap I_2$.■

COROLLARY 8.35. *If pseudo $BL-$ algebra A is a pseudo $MV-$ algebra then pseudo $BL-$ algebra $(A_{\mathcal{F}}, \curlywedge, \curlyvee, \cdot, \longmapsto, \leftrightsquigarrow, \mathbf{0} = \widehat{(A, \mathbf{0})}, \mathbf{1} = \widehat{(A, \mathbf{1})})$ is a pseudo $MV-$ algebra $(A_{\mathcal{F}}, \cdot, +, ^{-}, ^{\sim}, \mathbf{0} = \widehat{(A, \mathbf{0})}, \mathbf{1} = \widehat{(A, \mathbf{1})})$, where*

$$\widehat{(I_1, f_1)} \cdot \widehat{(I_2, f_2)} = (I_1 \cap \widehat{I_2, f_1 \boxdot f_2}),$$

$$\widehat{(I_1, f_1)} + \widehat{(I_2, f_2)} = (I_1 \cap \widehat{I_2, f_1 \boxplus f_2}),$$

and

$$\widehat{(I, f)}^{-} = \widehat{(I, f^{-})},$$
$$\widehat{(I, f)}^{\sim} = \widehat{(I, f^{\sim})}.$$

In this case we obtain the results from Corollary 7.10.

PROPOSITION 8.36. *If pseudo BL -algebra $(A, \vee, \wedge, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ is a BL -algebra (i.e $x \odot y = y \odot x$ for all $x, y \in A$ and in particular $x \rightarrow y = x \rightsquigarrow y$ for all $x, y \in A$), then pseudo BL -algebra $(M(A/\theta_{\mathcal{F}}), \wedge, \vee, \boxdot, \rightarrow, \rightsquigarrow, \mathbf{0}, \mathbf{1})$ is a BL -algebra $(M(A/\theta_{\mathcal{F}}), \wedge, \vee, \boxdot, \rightarrow, \mathbf{0}, \mathbf{1})$. Indeed if $I_1, I_2 \in \mathcal{F}$ and $f_i \in M(I_i, A/\theta_{\mathcal{F}})$, $i = 1, 2$ we have*

$$(f_1 \rightarrow f_2)(x) = [f_1(x) \rightarrow f_2(x)] \odot x/\theta_{\mathcal{F}} = x/\theta_{\mathcal{F}} \odot [f_1(x) \rightsquigarrow f_2(x)] = (f_1 \rightsquigarrow f_2)(x),$$

for all $x \in I_1 \cap I_2$, then $f_1 \rightarrow f_2 = f_1 \rightsquigarrow f_2$, so pseudo BL -algebra $(M(A/\theta_{\mathcal{F}}), \wedge, \vee, \boxdot, \rightarrow, \rightsquigarrow, \mathbf{0}, \mathbf{1})$ is commutative (see Remark 5.1), so is a BL -algebra (see Proposition 6.28).

COROLLARY 8.37. *If pseudo $BL-$ algebra A is a $BL-$ algebra then pseudo $BL-$ algebra $(A_{\mathcal{F}}, \curlywedge, \curlyvee, \cdot, \longmapsto, \leftrightsquigarrow, \mathbf{0} = \widehat{(A, \mathbf{0})}, \mathbf{1} = \widehat{(A, \mathbf{1})})$ is a $BL-$ algebra $(A_{\mathcal{F}}, \curlywedge, \curlyvee, \cdot, \longmapsto, \mathbf{0} = \widehat{(A, \mathbf{0})}, \mathbf{1} = \widehat{(A, \mathbf{1})})$, where*

$$\widehat{(I_1, f_1)} \curlywedge \widehat{(I_2, f_2)} = (I_1 \cap \widehat{I_2, f_1 \wedge f_2}),$$
$$\widehat{(I_1, f_1)} \curlyvee \widehat{(I_2, f_2)} = (I_1 \cap \widehat{I_2, f_1 \vee f_2}),$$
$$\widehat{(I_1, f_1)} \cdot \widehat{(I_2, f_2)} = (I_1 \cap \widehat{I_2, f_1 \boxdot f_2}),$$
$$\widehat{(I_1, f_1)} \longmapsto \widehat{(I_2, f_2)} = (I_1 \cap \widehat{I_2, f_1 \rightarrow f_2}).$$

In this case we obtain the results from Corollary 6.29.

LEMMA 8.38. *Let the map $v_{\mathcal{F}} : B(A) \rightarrow A_{\mathcal{F}}$ defined by $v_{\mathcal{F}}(a) = \widehat{(A, \overline{f_a})}$ for every $a \in B(A)$. Then:*

(i) $v_{\mathcal{F}}$ is a morphism of pseudo - BL algebras;
(ii) For $a \in B(A)$, $\widehat{(A, \overline{f_a})} \in B(A_{\mathcal{F}})$;
(iii) $v_{\mathcal{F}}(B(A)) \in \mathcal{R}(A_{\mathcal{F}})$.

Proof. (i). We have $v_{\mathcal{F}}(0) = \widehat{(A, \overline{f_0})} = \widehat{(A, \mathbf{0})} = \mathbf{0}$.
For $a, b \in B(A)$ and $x \in A$ we have

$$(a \wedge x) \odot (x \rightsquigarrow (b \wedge x)) = (a \odot x) \odot (x \rightsquigarrow (b \wedge x)) =$$
$$= a \odot [x \odot (x \rightsquigarrow (b \wedge x))] = a \odot [x \wedge (b \wedge x)]$$
$$= a \wedge [x \wedge (b \wedge x)] = a \wedge (b \wedge x) = (a \wedge b) \wedge x = (a \odot b) \wedge x$$

and

$$[(a\wedge x)\to(b\wedge x)]\odot x=[(x\odot a)\to(x\odot b)]\odot x \overset{psbl-c_{75}}{=} (a\to b)\odot x=x\wedge(a\to b),$$

and

$$x\odot[(a\wedge x)\rightsquigarrow(b\wedge x)]=x\odot[(x\odot a)\rightsquigarrow(x\odot b)] \overset{psbl-c_{75}}{=} x\odot(a\rightsquigarrow b)=x\wedge(a\rightsquigarrow b),$$

hence

$$v_{\mathcal{F}}(a)\cdot v_{\mathcal{F}}(b)=\widehat{(A,\overline{f_a})}\cdot(A,\overline{f_b})=(A,\widehat{\overline{f_a}\boxdot\overline{f_b}})=\widehat{(A,\overline{f_{a\odot b}})}=v_{\mathcal{F}}(a\odot b),$$

$$v_{\mathcal{F}}(a)\longmapsto v_{\mathcal{F}}(b)=\widehat{(A,\overline{f_a})}\longmapsto(A,\overline{f_b})=(A,\widehat{\overline{f_a}\to\overline{f_b}})=\widehat{(A,\overline{f_{a\to b}})}=v_{\mathcal{F}}(a\to b),$$

and

$$v_{\mathcal{F}}(a)\leftrightsquigarrow v_{\mathcal{F}}(b)=\widehat{(A,\overline{f_a})}\leftrightsquigarrow(A,\overline{f_b})=(A,\widehat{\overline{f_a}\rightsquigarrow\overline{f_b}})=\widehat{(A,\overline{f_a\quad b})}=v_{\mathcal{F}}(a\rightsquigarrow b),$$

hence $v_{\mathcal{F}}$ is a morphism of pseudo - BL algebras.

(ii). For $a\in B(A)$ we have $a\odot a=a$ and $(a^{\sim})^{-}=(a^{-})^{\sim}=a$, hence

$$(a\wedge x)\odot[x\rightsquigarrow(a\wedge x)]=(a\odot x)\odot[x\rightsquigarrow(a\wedge x)]$$

$$=a\odot[x\odot(x\rightsquigarrow(a\wedge x))]=a\odot[x\wedge(a\wedge x)]=a\odot(a\wedge x)=a\wedge(a\wedge x)=(a\wedge x),$$

and

$$[x\odot(a\wedge x)^{\sim}]^{-}\odot x \overset{psbl-c_{49}}{=} [x\odot(a^{\sim}\vee x^{\sim})]^{-}\odot x \overset{psbl-c_{36}}{=} [(x\odot a^{\sim})\vee(x\odot x^{\sim})]^{-}\odot x \overset{psbl-c_{38}}{=}$$

$$\overset{psbl-c_{38}}{=} [(x\odot a^{\sim})\vee 0]^{-}\odot x=[(x\odot a^{\sim})]^{-}\odot x=$$

(since $a\in B(A)$)

$$=(x\to a)\odot x=x\wedge a,$$

and

$$x\odot[(a\wedge x)^{-}\odot x]^{\sim} \overset{psbl-c_{50}}{=} x\odot[(a^{-}\vee x^{-})\odot x]^{\sim} \overset{psbl-c_{36}}{=} x\odot[(a^{-}\odot x)\vee(x^{-}\odot x)]^{\sim} \overset{psbl-c_{38}}{=}$$

$$\overset{psbl-c_{38}}{=} x\odot[(a^{-}\odot x)\vee 0]^{\sim}=x\odot[(a^{-}\odot x)]^{\sim}=$$

(since $a\in B(A)$)

$$=x\odot(x\rightsquigarrow a)=x\wedge a,$$

for every $x\in A$.

Since $A\in\mathcal{F}$ we deduce that

$$(a/\theta_{\mathcal{F}}\wedge x/\theta_{\mathcal{F}})\odot[x/\theta_{\mathcal{F}}\rightsquigarrow(a/\theta_{\mathcal{F}}\wedge x/\theta_{\mathcal{F}})]=(a/\theta_{\mathcal{F}}\wedge x/\theta_{\mathcal{F}})$$

and

$$[x/\theta_{\mathcal{F}}\odot(a/\theta_{\mathcal{F}}\wedge x/\theta_{\mathcal{F}})^{\sim}]^{-}\odot x/\theta_{\mathcal{F}}=a/\theta_{\mathcal{F}}\wedge x/\theta_{\mathcal{F}},$$
$$x/\theta_{\mathcal{F}}\odot[(a/\theta_{\mathcal{F}}\wedge x/\theta_{\mathcal{F}})^{-}\odot x/\theta_{\mathcal{F}}]^{\sim}=a/\theta_{\mathcal{F}}\wedge x/\theta_{\mathcal{F}},$$

hence

$$\overline{f_a}\boxdot\overline{f_a}=\overline{f_a}$$

and

$$[(\overline{f_a})^{\sim}]^{-}=[(\overline{f_a})^{-}]^{\sim}=\overline{f_a},$$

that is,

$$\widehat{(A,\overline{f_a})}\in B(A_{\mathcal{F}}).$$

(iii). See the proof of Lemma 6.32, (iii). ■

3.3. Strong $\mathcal{F}$-multipliers and strong localization pseudo BL-algebras. As in the case of BL -algebras, to obtain the maximal pseudo BL -algebra of quotients $Q(A)$ as a localization relative to a topology $\mathcal{F}$ we will develope another theory of $\mathcal{F}-$ multipliers.

DEFINITION 8.9. Let $\mathcal{F}$ be a topology on A. A *strong - $\mathcal{F}-$ multiplier* is a mapping $f : I \to A/\theta_{\mathcal{F}}$ *(*where $I \in \mathcal{F}$) which verifies the axioms $m - psBL_1, m - psBL_2$ (see Definition 8.7) and

$(m - psBL_3)$ If $e \in I \cap B(A)$, then $f(e) \in B(A/\theta_{\mathcal{F}})$;

$(m - psBL_4)$ $(x/\theta_{\mathcal{F}}) \wedge f(e) = (e/\theta_{\mathcal{F}}) \wedge f(x)$, *for every* $e \in I \cap B(A)$ *and* $x \in I$.

If $\mathcal{F} = \{A\}$, then $\theta_{\mathcal{F}}$ is the identity congruence of A so an strong $\mathcal{F}-$ multiplier is a strong total multiplier.

REMARK 8.21. *If $(A, \wedge, \vee, \odot, \to, \rightsquigarrow, 0, 1)$ is a pseudo $BL-$ algebra, the maps $\mathbf{0}, \mathbf{1} : A \to A/\theta_{\mathcal{F}}$ defined by $\mathbf{0}(x) = 0/\theta_{\mathcal{F}}$ and $\mathbf{1}(x) = x/\theta_{\mathcal{F}}$ for every $x \in A$ are strong - $\mathcal{F}-$ multipliers. We recall that if $f_i : I_i \to A/\theta_{\mathcal{F}}$, (with $I_i \in \mathcal{F}$, $i = 1, 2$) are $\mathcal{F}-$multipliers we consider the mappings $f_1 \wedge f_2, f_1 \vee f_2, f_1 \boxdot f_2, f_1 \to f_2, f_1 \rightsquigarrow f_2 : I_1 \cap I_2 \to A/\theta_{\mathcal{F}}$ defined by*

$$(f_1 \wedge f_2)(x) = f_1(x) \wedge f_2(x),$$
$$(f_1 \vee f_2)(x) = f_1(x) \vee f_2(x),$$
$$(f_1 \boxdot f_2)(x) = [x/\theta_{\mathcal{F}} \to f_1(x)] \odot f_2(x) \overset{psbl-c_{77}}{=} f_1(x) \odot [x/\theta_{\mathcal{F}} \rightsquigarrow f_2(x)],$$
$$(f_1 \to f_2)(x) = [f_1(x) \to f_2(x)] \odot x/\theta_{\mathcal{F}},$$
$$(f_1 \rightsquigarrow f_2)(x) = x/\theta_{\mathcal{F}} \odot [f_1(x) \rightsquigarrow f_2(x)],$$

for any $x \in I_1 \cap I_2$. If f_1, f_2 are strong - $\mathcal{F}-$ multipliers, then the multipliers $f_1 \wedge f_2, f_1 \vee f_2, f_1 \boxdot f_2, f_1 \to f_2, f_1 \rightsquigarrow f_2$ are also strong - $\mathcal{F}-$ multipliers. Indeed, if $e \in I_1 \cap I_2 \cap B(A)$, then

$$(f_1 \wedge f_2)(e) = f_1(e) \wedge f_2(e) \in B(A/\theta_{\mathcal{F}}),$$
$$(f_1 \vee f_2)(e) = f_1(e) \vee f_2(e) \in B(A/\theta_{\mathcal{F}}).$$

By Proposition 5.13 we have

$$(f_1 \boxdot f_2)(e) = [e/\theta_{\mathcal{F}} \to f_1(e)] \odot f_2(e) = [(e^-)/\theta_{\mathcal{F}} \vee f_1(e)] \odot f_2(e) \in B(A/\theta_{\mathcal{F}}),$$
$$(f_1 \to f_2)(e) = [f_1(e) \to f_2(e)] \odot e/\theta_{\mathcal{F}} = [(f_1(e))^- \vee f_2(e)] \odot e/\theta_{\mathcal{F}} \in B(A/\theta_{\mathcal{F}}),$$

and

$$(f_1 \rightsquigarrow f_2)(e) = e/\theta_{\mathcal{F}} \odot [f_1(e) \rightsquigarrow f_2(e)] = e/\theta_{\mathcal{F}} \odot [(f_1(e))^\sim \vee f_2(e)] \in B(A/\theta_{\mathcal{F}}).$$

For $e \in I_1 \cap I_2 \cap B(A)$ and $x \in I_1 \cap I_2$ we have:

$$x/\theta_{\mathcal{F}} \wedge (f_1 \wedge f_2)(e) = x/\theta_{\mathcal{F}} \wedge f_1(e) \wedge f_2(e) =$$
$$= [x/\theta_{\mathcal{F}} \wedge f_1(e)] \wedge [x/\theta_{\mathcal{F}} \wedge f_2(e)] =$$
$$= [e/\theta_{\mathcal{F}} \wedge f_1(x)] \wedge [e/\theta_{\mathcal{F}} \wedge f_2(x)] = e/\theta_{\mathcal{F}} \wedge (f_1 \wedge f_2)(x)$$

and

$$x/\theta_{\mathcal{F}} \wedge (f_1 \vee f_2)(e) = x/\theta_{\mathcal{F}} \wedge [f_1(e) \vee f_2(e)] =$$
$$= [x/\theta_{\mathcal{F}} \wedge f_1(e)] \vee [x/\theta_{\mathcal{F}} \wedge f_2(e)] =$$
$$= [e/\theta_{\mathcal{F}} \wedge f_1(x)] \vee [e/\theta_{\mathcal{F}} \wedge f_2(x)] =$$
$$= e/\theta_{\mathcal{F}} \wedge [f_1(x) \vee f_2(x)] = e/\theta_{\mathcal{F}} \wedge (f_1 \vee f_2)(x)$$

and

$$x/\theta_{\mathcal{F}} \wedge (f_1 \boxdot f_2)(e) = x/\theta_{\mathcal{F}} \wedge [(e/\theta_{\mathcal{F}} \to f_1(e)) \odot f_2(e)]$$
$$= [(e/\theta_{\mathcal{F}} \to f_1(e)) \odot f_2(e)] \odot x/\theta_{\mathcal{F}} = [(e/\theta_{\mathcal{F}} \to f_1(e)) \odot x/\theta_{\mathcal{F}}] \odot f_2(e)$$
$$\overset{psbl-c_{75}}{=} [((e \odot x)/\theta_{\mathcal{F}} \to (f_1(e) \odot x/\theta_{\mathcal{F}})) \odot x/\theta_{\mathcal{F}}] \odot f_2(e)$$
$$= [(e \odot x)/\theta_{\mathcal{F}} \to (f_1(e) \odot x/\theta_{\mathcal{F}})] \odot [x/\theta_{\mathcal{F}} \odot f_2(e)]$$
$$= [(e \odot x)/\theta_{\mathcal{F}} \to (e/\theta_{\mathcal{F}} \odot f_1(x))] \odot [e/\theta_{\mathcal{F}} \odot f_2(x)]$$
$$= [((e/\theta_{\mathcal{F}} \odot x/\theta_{\mathcal{F}}) \to (e/\theta_{\mathcal{F}} \odot f_1(x))) \odot e/\theta_{\mathcal{F}}] \odot f_2(x)$$
$$\overset{psbl-c_{74}}{=} [(x/\theta_{\mathcal{F}} \to f_1(x)) \odot e/\theta_{\mathcal{F}}] \odot f_2(x) = [(x/\theta_{\mathcal{F}} \to f_1(x)) \odot f_2(x)] \odot e/\theta_{\mathcal{F}}$$
$$= [(f_1 \boxdot f_2)(x)] \odot e/\theta_{\mathcal{F}} = e/\theta_{\mathcal{F}} \wedge (f_1 \boxdot f_2)(x),$$

hence

$$x/\theta_{\mathcal{F}} \wedge (f_1 \boxdot f_2)(e) = e/\theta_{\mathcal{F}} \wedge (f_1 \boxdot f_2)(x).$$

Also

$$e/\theta_{\mathcal{F}} \wedge (f_1 \to f_2)(x) = [(f_1(x) \to f_2(x)) \odot x/\theta_{\mathcal{F}}] \wedge e/\theta_{\mathcal{F}}$$
$$= [(f_1(x) \to f_2(x)) \odot x/\theta_{\mathcal{F}}] \odot e/\theta_{\mathcal{F}} = [(f_1(x) \to f_2(x)) \odot e/\theta_{\mathcal{F}}] \odot x/\theta_{\mathcal{F}}$$
$$\overset{psbl-c_{74}}{=} [((f_1(x) \odot e/\theta_{\mathcal{F}}) \to (f_2(x) \odot e/\theta_{\mathcal{F}})) \odot e/\theta_{\mathcal{F}}] \odot x/\theta_{\mathcal{F}}$$
$$= [((x/\theta_{\mathcal{F}} \odot f_1(e)) \to (x/\theta_{\mathcal{F}} \odot f_2(e))) \odot e/\theta_{\mathcal{F}}] \odot x/\theta_{\mathcal{F}} =$$
$$= [((x/\theta_{\mathcal{F}} \odot f_1(e)) \to (x/\theta_{\mathcal{F}} \odot f_2(e))) \odot x/\theta_{\mathcal{F}}] \odot e/\theta_{\mathcal{F}} \overset{psbl-c_{75}}{=} [(f_1(e) \to f_2(e)) \odot x/\theta_{\mathcal{F}}] \odot e/\theta_{\mathcal{F}} =$$
$$= [(f_1(e) \to f_2(e)) \odot e/\theta_{\mathcal{F}}] \odot x/\theta_{\mathcal{F}} = [(f_1 \to f_2)(e)] \odot x/\theta_{\mathcal{F}} = x/\theta_{\mathcal{F}} \wedge (f_1 \to f_2)(e),$$

hence

$$x/\theta_{\mathcal{F}} \wedge (f_1 \to f_2)(e) = e/\theta_{\mathcal{F}} \wedge (f_1 \to f_2)(x).$$

Also

$$e/\theta_{\mathcal{F}} \wedge (f_1 \rightsquigarrow f_2)(x) = e/\theta_{\mathcal{F}} \wedge [x/\theta_{\mathcal{F}} \odot (f_1(x) \rightsquigarrow f_2(x))]$$
$$= (e \odot x)/\theta_{\mathcal{F}} \odot [f_1(x) \rightsquigarrow f_2(x)] = x/\theta_{\mathcal{F}} \odot [e/\theta_{\mathcal{F}} \odot (f_1(x) \rightsquigarrow f_2(x))]$$
$$\overset{psbl-c_{74}}{=} x/\theta_{\mathcal{F}} \odot [e/\theta_{\mathcal{F}} \odot ((e/\theta_{\mathcal{F}} \odot f_1(x)) \rightsquigarrow (e/\theta_{\mathcal{F}} \odot f_2(x)))]$$
$$= x/\theta_{\mathcal{F}} \odot [e/\theta_{\mathcal{F}} \odot ((x/\theta_{\mathcal{F}} \odot f_1(e)) \rightsquigarrow (x/\theta_{\mathcal{F}} \odot f_2(e)))] =$$
$$= e/\theta_{\mathcal{F}} \odot [x/\theta_{\mathcal{F}} \odot ((x/\theta_{\mathcal{F}} \odot f_1(e)) \rightsquigarrow (x/\theta_{\mathcal{F}} \odot f_2(e)))] \overset{psbl-c_{75}}{=} e/\theta_{\mathcal{F}} \odot [x/\theta_{\mathcal{F}} \odot (f_1(e) \rightsquigarrow f_2(e))] =$$
$$= x/\theta_{\mathcal{F}} \odot [e/\theta_{\mathcal{F}} \odot (f_1(e) \rightsquigarrow f_2(e))] = x/\theta_{\mathcal{F}} \odot (f_1 \rightsquigarrow f_2)(e) = x/\theta_{\mathcal{F}} \wedge (f_1 \rightsquigarrow f_2)(e),$$

hence

$$x/\theta_{\mathcal{F}} \wedge (f_1 \rightsquigarrow f_2)(e) = e/\theta_{\mathcal{F}} \wedge (f_1 \rightsquigarrow f_2)(x).$$

REMARK 8.22. *Analogous as in the case of $\mathcal{F}-$ multipliers if we work with strong-$\mathcal{F}-$ multipliers we obtain a pseudo $BL-$ subalgebra of $A_{\mathcal{F}}$ denoted by $s-A_{\mathcal{F}}$ which will be called the* strong-localization pseudo $BL-$ algebra of A with respect to the topology $\mathcal{F}$.

3.4. Applications. In the following we describe the localization (strong localization) pseudo - BL algebra $A_{\mathcal{F}}$ ($s-A_{\mathcal{F}}$) in some special instances.

1. If $I \in \mathcal{I}(A)$ and $\mathcal{F}$ is the topology $\mathcal{F}(I) = \{I' \in \mathcal{I}(A) : I \subseteq I'\}$, then $A_{\mathcal{F}}$ is isomorphic with $M(I, A/\theta_{\mathcal{F}})$ and $v_{\mathcal{F}} : B(A) \to A_{\mathcal{F}}$ is defined by $v_{\mathcal{F}}(a) = \overline{f_{a|I}}$ for every $a \in B(A)$.

If I is a regular subset of A, then $\theta_{\mathcal{F}}$ is the identity, hence $A_{\mathcal{F}}$ is isomorphic with $M(I, A)$.

2. If $\mathcal{F} = \mathcal{I}(A) \cap \mathcal{R}(A)$ is the topology of regular ideals, then $\theta_{\mathcal{F}}$ is the identity congruence of A and we obtain the Definition 8.9 for strong multipliers of A, so

$$s - A_{\mathcal{F}} = \varinjlim_{I \in \mathcal{F}} M(I, A),$$

where $M(I, A)$ is the set of multipliers of A having the domain I in the sense of Definition 8.9.

In this situation we obtain:

PROPOSITION 8.39. *In the case $\mathcal{F} = \mathcal{I}(A) \cap \mathcal{R}(A)$, $s - A_{\mathcal{F}}$ is exactly the maximal pseudo BL-algebra $Q(A)$ of quotients of A, which is a Boolean algebra.*

REMARK 8.23. *If pseudo $BL-$ algebra A is a pseudo $MV-$ algebra, $s - A_{\mathcal{F}}$ is exactly the maximal pseudo MV-algebra $Q(A)$ of quotients of A introduced in Definition 7.6.*

REMARK 8.24. *If pseudo $BL-$ algebra A is a $BL-$ algebra, $s - A_{\mathcal{F}}$ is exactly the maximal BL-algebra $Q(A)$ of quotients of A introduced in Definition 6.8.*

3. Denoting by $\mathcal{D}$ the topology of dense ordered ideals of A (that is $\mathcal{D} = \mathcal{I}(A) \cap D(A)$ - see Example 10 from Subsection 5.1), then (since $\mathcal{R}(A) \subseteq D(A)$) there exists a morphism of pseudo BL -algebras $\alpha : Q(A) \to s - A_{\mathcal{D}}$ such that the diagrame

$$\begin{array}{ccc} B(A) & \xrightarrow{\overline{v_A}} & Q(A) \\ \searrow & & \swarrow \\ {\scriptstyle v_D} & & {\scriptstyle \alpha} \\ & s - A_{\mathcal{D}} & \end{array}$$

is commutative (i.e. $\alpha \circ \overline{v_A} = v_{\mathcal{D}}$). Indeed, if $[f, I] \in Q(A)$ (with $I \in \mathcal{I}(A) \cap \mathcal{R}(A)$ and $f : I \to A$ is a strong multiplier in the sense of Definition 8.9) we denote by $f_{\mathcal{D}}$ the strong - $\mathcal{D}-$multiplier $f_{\mathcal{D}} : I \to A/\theta_{\mathcal{D}}$ defined by $f_{\mathcal{D}}(x) = f(x)/\theta_{\mathcal{D}}$ for every $x \in I$. Thus, α is defined by $\alpha([f, I]) = [f_{\mathcal{D}}, I]$.

4. Let $S \subseteq A$ an $\wedge-$closed system of A.

PROPOSITION 8.40. *If $\mathcal{F}_S$ is the topology associated with an $\wedge-$closed system $S \subseteq A$ (see Example 11 from Subsection 5.1), then the pseudo BL-algebra $s - A_{\mathcal{F}_S}$ is isomorphic with $B(A[S])$.*

Proof. See the proof of Proposition 6.34. ■

Bibliography

[1] F. W. Anderson, *Lattice-ordered rings of quotients*, Canadian Journal of Mathematics, **17** (1965), 434-448.

[2] R. Balbes, P. Dwinger, *Distributive lattices*, Columbia, Missouri: University of Missouri Press. XIII (1974).

[3] R. N. Ball, G. Georgescu and I. Leustean, *Cauchy completions of MV-algebras*, Algebra Universs. 47 (2002),367-407.

[4] B. Banaschewski, *Maximal rings of quotients of semi-simple commutative rings,* Arch. Math., vol **XVI**, (1965), 414-420.

[5] L. P. Belluce, *Semisimple algebras of infinite valued logic and bold fuzzy set theory,* Can. J. Math. **38**, No. 6 (1986), 1356-1379.

[6] L. P. Belluce, *Spectral spaces and non-commutative rings,* Comm. Algebra **19** (1991), 1855-1865.

[7] L. P. Belluce, A. Di Nola, A. Lettieri, *Subalgebras, direct products and associated lattices of MV-algebras,* Glasgow Math. J. **34** (1992), 301-307.

[8] L. P. Belluce, A. Di Nola, A. Lettieri, *Local MV-algebras,* Rend. Circolo Mat. Palermo, Serie II **42** (1993), 347-361.

[9] L. P. Belluce, A. Di Nola, S. Sessa, *The prime spectrum of an MV-algebra,* Math. Log. Q. **40**, No.3 (1994), 331-346.

[10] A. Bigard, K. Keimel, S. Wolfenstein, *Groupes et Anneaux Réticulés,* Lecture Notes in Mathematics 608, Berlin-Heidelberg-New York: Springer-Verlag (1977).

[11] G. Birkfoff, *Lattice theory (3rd ed.)*, Colloquim Publications 25, Amer. Math. Soc, 1967.

[12] T. S. Blyth, M. F. Janovitz, *Residuation Theory*, Pergamon Press, 1972.

[13] W. J. Blok, D. Pigozzi, *Algebraizable Logics*, Memoirs of the American Mathematical Society, No. 396, Amer. Math. Soc, Providence, 1989.

[14] K. Blount, C. Tsinakis, *The structure of residuated lattices*, Internat. J. Algebra Comput., 13 (2003), no. 4, 437-461.

[15] V. Boicescu, A. Filipoiu, G. Georgescu, S. Rudeanu, *Łukasiewicz-Moisil Algebras,* Annals of Discrete Mathematics **49**, Amsterdam etc.: North-Holland (1991).

[16] B. Bosbach, *Concerning bricks,* Acta Mathematica Academiae Scientiarum Hungaricae 38 (1981), 89-104.

[17] H. W. Buff, *Decidable and undecidable MV-algebras,* Algebra Universalis, 21 (1985), 234-249.

[18] S. Buris, H. P. Sankappanavar: *A Course in Universal Algebra*, Graduate Texts in Mathematics, No. 78, Springer, 1981.

[19] D. Buşneag, *Hilbert algebra of fractions relative to an $\veebar$-closed system*, Analele Universităţii din Craiova, Seria Matematica - Fizica - Chimie, vol. XVI, (1988), 34-38.

[20] D. Buşneag, *Hilbert algebra of fractions and maximal Hilbert algebras of quotients,* Kobe Journal of Mathematics, **5** (1988), 161-172.

[21] D. Buşneag, *F-multipliers and the localization of Hilbert algebras,* Zeitschr. f. math. Logik und Grundlagen d. Math. Bd. **36** (1990), 331-338.

[22] D. Buşneag, *Categories of algebraic logic*, Ed. Academiei Române, Bucureşti, (2006).

[23] D. Buşneag, **D. Piciu**, *On the lattice of ideals of an MV-algebra*, Scientiae Mathematicae Japonicae, **56**, No.2 (2002), 367-372.

[24] D. Buşneag, **D. Piciu**, *Meet-irreducible ideals in an MV-algebra*, Analele Universităţii din Craiova, Seria Matematica-Informatica, vol. XXVIII, (2001), 110-119.

[25] D. Buşneag, **D. Piciu**, *MV-algebra of fractions relative to an $\wedge-$closed system*, Annals of the University of Craiova, Mathematics and Computer Science series, Volume **30**, No.2, (2003), 48-53.

[26] D. Buşneag, **D. Piciu**, *MV-algebra of fractions and maximal MV-algebra of quotients,* Journal of Multiple-Valued Logic and Soft Computing, Volume **10**, No.4, (2004), 363-383.

[27] D. Buşneag, **D. Piciu**, *Boolean MV-algebra of fractions,* Mathematical Reports, Vol. **7** (57), Nr. 4 (2005), 265-280.

[28] D. Buşneag, **D. Piciu**, *Localization of* MV*-algebras and lu-groups*, Algebra Universalis, **50**, (2003), 359-380.

[29] D. Buşneag, **D. Piciu**, *Localization of pseudo-*MV *algebras and l_u-groups with strong unit*, to appear in International Review of Fuzzy Mathematics.

[30] D. Buşneag, **D. Piciu**, *Injective objects in the BL-algebras category (Part I)*, Analele Universităţii din Craiova, Seria Matematica -Informatica, Vol. XXIX, (2002), 32-39.

[31] D. Buşneag, **D. Piciu**, *On the lattice of deductive systems of a BL-algebra*, Central European Journal of Mathematics, CEJM **1** (2), (2003), 221-238.

[32] D. Buşneag, **D. Piciu**, *BL-algebra of fractions relative to an $\wedge$-closed system*, Analele Ştiinţifice ale Universităţii Ovidius Constanţa, Seria Matematica, vol. XI, Fascicula 1, 2003, 39-48.

[33] D. Buşneag, **D. Piciu**, *BL-algebra of fractions and maximal BL-algebra of quotients,* Soft Computing (Springer - Verlag), No.9, (2005), 544-555.

[34] D. Buşneag, **D. Piciu**, *Boolean BL-algebra of fractions,* Annals of the University of Craiova, Mathematics and Computer Science series, Volume **31**, (2004), 1-19.

[35] D. Buşneag, **D. Piciu**, *Localization of BL-algebras,* Soochow Journal of Mathematics, Volume **32**, No.1, (2006), 127-159.

[36] D. Buşneag, **D. Piciu**, *On the lattice of filters of a pseudo BL-algebra*, Journal of Multiple-Valued Logic and Soft Computing, Volume **12**, No. **3-4**, (2006), 217-248.

[37] D. Buşneag, **D. Piciu**, *Pseudo BL-algebra of fractions relative to an $\wedge-$ closed system*, Analele Ştiinţifice ale Universităţii A.I. Cuza, Iaşi, Tomul L, s. I a, Matematică, (2004), f.2, 459-472.

[38] D. Buşneag, **D. Piciu**, *Pseudo BL-algebra of fractions and maximal pseudo BL-algebra of quotients,* Southeast Assian Bulletin of Mathematics (Springer - Verlag Hong - Kong Ltd), Volume **31**, (2007), 1-27.

[39] D. Buşneag, **D. Piciu**, *Localization of pseudo BL-algebras,* Revue Roumaine de Mathématique Pures and Appliqées, Tome L, No. **5-6**, (2005), 495-513.

[40] D. Buşneag, **D. Piciu**, *Residuated lattices of fractions relative to an $\wedge$-closed system*, Bull. Math. Soc. Math. Roumanie, Tome 49, (97), No. **1**, (2006), 13-24.

[41] V. Cavaccini, C. Cella, G. Georgescu, *Pure ideals of MV-algebras,* Math. Japonica, **45**, (1997), 1-8.

[42] C. C. Chang, *Algebraic analysis of many-valued logic*, Trans. Amer. Math. Soc., **88** (1958), 467-490.

[43] C. C. Chang, *A new proof of the completeness of the Łukasiewicz axioms*, Trans. Amer. Math. Soc., **93** (1959), 74-80.

[44] R. Cignoli, F. Esteva, L. Godo, A. Torrens, *Basic Fuzzy Logic in the logic of continuous t-norms and their residua,* Soft Computing **4** (2000), 106-112.

[45] R. Cignoli, I.M.L. D'Ottaviano, D. Mundici, *Algebraic Foundations of many-valued Reasoning*, Trends in Logic-Studia Logica Library 7, Dordrecht: Kluwer Academic Publishers (2000).

[46] R. Cignoli, A. Torrens, *An algebraic analysis of product logic,* Mult.-Valued Logic **5** (2000), 45-65.

[47] W. H. Cornish, *The multiplier extension of a distributive lattice*, Journal of Algebra, **32** (1974), 339-355.

[48] W. H. Cornish, *A multiplier approach to implicative BCK-algebras*, Mathematics Seminar Notes, Kobe University, **8**, No.1 (1980), 157-169.

[49] C. Dan, *F-multipliers and the localization of Heyting algebras*, Analele Universităţii din Craiova, Seria Matematica-Informatica, vol. XXIV, (1997), 98-109.

[50] M. R. Darnel, *Theory of Lattice Ordered Groups*, Pure and Applied Mathematics 187, Marcel Dekker, Inc., New York, 1955.

[51] A. Diego, *Sur les algebrès de Hilbert*, Ed. Hermann, Collection de Logique Mathématique, Serie A, XXI, Paris (1966).

[52] R. P. Dilworth, *Non-commutative residuated lattices*, Transactions of the American Mathematical Society 46 (1939), 426-444.

[53] A. Di Nola, G. Georgescu, A. Iorgulescu, *Pseudo-BL algebras: Part I,* Multiple Valued Logic **8**, No. 5-6 (2002), 673-714.

[54] A. Di Nola, G. Georgescu, A. Iorgulescu, *Pseudo-BL algebras: Part II,* Multiple Valued Logic **8**, No. 5-6 (2002), 717-750.

[55] A. Di Nola, G. Georgescu, L. Leuştean, *Boolean products of BL-algebras,* J. Math. Anal. Appl. **251**, No. 1 (2000), 106-131.

[56] A. Di Nola, A. Lettieri, *Perfect MV-algebras are categorically equivalent to abelian l-groups,* Studia Logica **53** (1994), 417-432.

[57] A. Di Nola, S. Sessa, F. Esteva, L. Godo, P. Garcia, *The variety generated by perfect BL-algebras: an algebraic approach in a fuzzy logic setting,* Ann Math. Artif. Intell. **35**, No.1-4 (2002), 197-214.

[58] A. Dvurečenskij, *Pseudo MV-algebras are intervals in l-groups*, Journal of Australian Mathematical Society 72 (2002), 427-445.

[59] A. Dvurečenskij, S. Pulmannová, *New trends in Quantum Structures,* Kluwer (2000).

[60] A. Filipoiu, *About Baer extensions of MV-algebras*, Math. Japonica **40**, No.2 (1994), 235-241.

[61] A. Filipoiu, G. Georgescu, A. Lettieri, *Maximal MV-algebras*, Mathware Soft Comput. **4**, No.1 (1997), 53-62.

[62] J. M. Font, A. J. Rodriguez, A. Torrens, Wajsberg algebras, Stochastica 8 (1984), 5-31.

[63] H. Freytes: *Injectives in residuated algebras*, Algebra universalis, 51 (2004), 373-393.

[64] G. Georgescu, *F-multipliers and localization of distributive lattices,* Algebra Universalis, **21** (1985), 181-197.

[65] G. Georgescu, *F-multipliers and localization of distributive lattices II,* Zeitschr. f. math. Logik und Grundlagen d. Math. Bd. **37** (1991), 293-300.

[66] G. Georgescu, A. Iorgulescu, *Pseudo-MV Algebras: a Noncommutative Extension of MV Algebras,* The Proceedings of the Fourth International Symposium on Economic Informatics, Bucharest, Romania, May 1999, 961-968.

[67] G. Georgescu, A. Iorgulescu, *Pseudo-BL Algebras: A noncommutative extension of BL Algebras,* Abstracts of The Fifth International Conference FSTA, 2000, Slovakia, February 2000, 90-92.

[68] G. Georgescu, A. Iorgulescu, *Pseudo MV - algebras*, Multi. Val. Logic, **6** (2001), 95-135.

[69] G. Georgescu, M. Ploščica, *Values and minimal spectrum of an algebraic lattice*, Math. Slovaca, **52**, No.3 (2002), 247-253.

[70] G. Georgescu, L. Leuştean, *Some classes of pseudo-BL algebras*, J. Aust. Math. Soc. **73**, No.1 (2002), 127-153.

[71] G. Georgescu, L. Leuştean, *Semilocal and maximal BL algebras*, submitted (2003).

[72] S. Gottwald, Gottwald, Siegfried, *A treatise on many valued logics, Studies in Logic and Computation,* 9, Baldock: Research Studies Press (2000).

[73] G. Grätzer, *Lattice Theory. First Concepts and Distributive Lattices*, A Series of Books in Mathematics, San Francisco: W. H. Freeman and Company (1972).

[74] P. Hájek, *Basic fuzzy logic and BL-algebras*, Soft Computing **2** (1998), 124-128.

[75] P. Hájek, *Metamathematics of Fuzzy Logic*, Trends in Logic-Studia Logica Library 4, Dordrecht: Kluwer Academic Publishers (1998).

[76] P. Hájek: *Fuzzy logics with non-commutative conjunction*, Journal of Logic and Computation, (to appear).

[77] P. Hájek: *Observation of non-commutative logics*, Soft Computing, (to appear).

[78] C. S. Hoo, *Semilocal MV-algebras*, Math. Japonica **40**, No.3 (1994), 451-453.

[79] U. Hőhle, *Commutative residuated monoids*, in: U. Hőhle, P. Klement (eds), Non-classical Logics and Their Aplications to Fuzzy Subsets, Kluwer Academic Publishers, 1995.

[80] P. M. Idziak, *Lattice operations in BCK-algebras*, Mathematica Japonica, 29(1984), 839-846.

[81] I. D. Ion, N. Radu, *Algebra (in romanian)*, Ed. Didactică şi Pedagogică, Bucureşti, 1991.

[82] A. Iorgulescu: *Classes of BCK algebras-Part I,* Preprint Series of The Institute of Mathematics of the Romanian Academy, preprint nr.1 (2004), 1-33.

[83] A. Iorgulescu: *Classes of BCK algebras-Part II,* Preprint Series of The Institute of Mathematics of the Romanian Academy, preprint nr.2 (2004), 1-27.

[84] A. Iorgulescu: *Classes of BCK algebras-Part III,* Preprint Series of The Institute of Mathematics of the Romanian Academy, preprint nr.3 (2004), 1-37.

[85] A. Iorgulescu: *Classes of BCK algebras-Part IV,* Preprint Series of The Institute of Mathematics of the Romanian Academy, preprint nr.4 (2004), 1-25.

[86] A. Iorgulescu: *Classes of BCK algebras-Part V,* Preprint Series of The Institute of Mathematics of the Romanian Academy, preprint nr.5 (2004), 1-22.

[87] A. Iorgulescu: *Iséki algebras. Connections with BL-algebras,* to appear in Soft Computing.

[88] K. Iséki, S. Tanaka, *An introduction to the theory of BCK-algebras,* Mathematica Japonica, 23 (1978), 1-26.

[89] E. P. Klement, R. Mesiar, E. Pap, *Triangular norms*, Trends in Logic-Studia Logica Library 8, Dordrecht/Boston/London: Kluwer Academic Publishers (2000).

[90] Y. Komori, *Super-Łukasiewicz propositional logics,* Nagoya Mathematical Journal, 84 (1981), 119-133.

[91] T. Kowalski: *The bottom of the lattice of BCK-varieties,* Reports on mathematical Logic, 29(1995), 87-93.

[92] T. Kowalski, H. Ono: *Residuated lattices: an algebraic glimpse at logic without contraction*, 2001.

[93] W. Krull: *Axiomatische Begründung der allgemeinen Ideal theorie*, Sitzungsberichte der physikalisch medizinischen Societäd der Erlangen 56 (1924), 47-63.

[94] J. Kühr, *Pseudo-BL algebras and DRL-monoids,* Math. Bohem., **128** (2003), 199-208.

[95] J. Kühr: *Prime ideals and polars in DRL-monoids and pseudo-BL algebras,* Math. Slovaca, **53** (2003), 233-246.

[96] J. Lambek, *Lectures on Rings and Modules*, Blaisdell Publishing Company (1966).

[97] L. Leuştean, *Some algebraic properties of non-commutative fuzzy structures*, Studies in Informatics and Control **9**, No.4 (2000), 365-370.

[98] L. Leuştean, *The prime and maximal spectra and the reticulation of BL algebras*, to appear in Central European Journal of Mathematics.

[99] L. Leuştean, *Representations of many-valued algebras, Ph. D. Thesis,* University of Bucharest, 2004.

[100] I. Leuştean: *Local Pseudo-MV algebras*, Soft Computing, 5 (2001), 386-395.

[101] I. Leuştean, *Contributions to the study of MV-algebras: MV-modules, Ph. D. Thesis,* University of Bucharest, 2004.

[102] V. Marra, D. Mundici, *Łukasiewicz logic and Chang's MV-algebras in action,* Trends in Logic, vol. 20, Special volume for the 50 yers of Studia Logica, V. F. Hendricks and J. Malinowski (eds.), Kluwer (2003), 129-176.

[103] J. Martinez, *The maximal ring of quotient f- ring*, Algebra Universalis, **33** (1995), 355-369.

[104] J. Martinez, W. W.Mc Govern, *Rings of quotients of f- ring*, Communications in algebra, **27(7)** (1999), 3495-3511.

[105] D. Mundici, *Interpretation of AFC^*-algebras in Łukasiewicz sentential calculus*, J. Functional Analysis **65** (1986), 15-63.

[106] C. Năstăsescu, *Inele. Module. Categorii.*, Ed. Academiei, Bucureşti, (1976).

[107] M. Okada, K. Terui, *The finite model property for various fragments of intuitionistic linear logic*, Journal of Symbolic Logic, 64 (1999), 790-802.

[108] H. Ono, Y. Komori, *Logics without the contraction rule*, Journal of Symbolic Logic, 50 (1985), 169-201.

[109] G. Panti, *Multi - valued Logics. Handbook Chapter, In: Quantified Representation of Uncertainly and Imprecision,* vol. 1, (Smets, P. Ed.), Kluwer (1998), 25-74.

[110] E. Pap, Editor, *Handbook of Measure Theory, I, II,* North-Holland Amsterdam (2002).

[111] J. Pavelka, *On fuzzy logic II. Enriched residuated lattices and semantics of propositional calculi*, Zeitschrift für mathematische Logik und Grundlagen der Mathematik 25 (1979), 119-134.

[112] N. Popescu, *Abelian categories*, Ed. Academiei, Bucureşti, (1971).

[113] N. Popescu, *Abelian categories with applications to rings and modules*, Academic Press, New York (1973).

[114] N. Popescu, L. Popescu, *Theory of categories,* Editura Academiei, Bucureşti si Ed Sijthoff & Noordhoff International Publishers, (1979).

[115] **D. Piciu**, *Pseudo- MV algebra of fractions relative to an $\wedge-$closed system*, Annals of University of Craiova, Math. Comp. Sci. series, Volume **30**, No. 2 (2003), 140-145.

[116] **D. Piciu**, *Pseudo-MV algebra of fractions and maximal pseudo-MV algebra of quotients,* Central European Journal of Mathematics, CEJM **2** (2)(2004), 199-217.

[117] **D. Piciu**, *Localizations of MV and BL-algebras, Ph. D. Thesis,* University of Bucharest, 2004.

[118] H. Rasiowa, *An algebraic Approach to Non-Classical Logics,* PWN and North-Holland Publishing Company (1974).

[119] B. Riečam, T. Neubrunn, *Integral, measure, and ordering,* Kluwer (1997).

[120] S. Rudeanu, *Axiomele laticilor şi ale algebrelor booleene (Axioms of Lattices and Boolean Algebras)*, Ed. Academiei, Bucureşti (1963).

[121] J. Schmid, *Multipliers on distributive lattices and rings of quotients*, Houston Journal of Mathematics **6**, No.3 (1980), 401-425.

[122] J. Schmid, *Distributive lattices and rings of quotients*, Coll. Math. Societatis Janos Bolyai, 33, Szeged, Hungary, (1980).

[123] H. Simmons, *Reticulated rings*, Journal of Algebra **66** (1980), 169-192.

[124] B. Strenström, *Platnes and localization over monoids*, Math. Nachrichten **48** (1971), 315-334.

[125] B. Strenström, *Ring of Quotients*, Springer-Verlag Berlin Heidelberg (1975).

[126] B. Strenström, *Rings and modules of quotients*, Lecture Notes in Math. 237, Springer-Verlag (1971).

[127] A. Tarski, J. Łukasiewicz, *Investigations into the Sentential calculus in Logic, Semantics, Metamathematics*, Oxford University Press, (1956), 38-59.

[128] T. Traczyk, *On the variety of bounded commutative BCK-algebras,* Matematica Japonica 24 (1979), 238-292.

[129] E. Turunen, *Mathematics Behind Fuzzy Logic*, Advances in Soft Computing, Heidelberg: Physica-Verlag (1999).

[130] E. Turunen, *BL-algebras of basic fuzzy logic*, Mathware Soft Computing **6**, No. 1 (1999), 49-61.

[131] E. Turunen, *Boolean deductive systems of BL-algebras*, Arch. Math. Logic **40**, No.6 (2001), 467-473.

[132] E. Turunen, S. Sessa, *Local BL-algebras*, Multiple Valued Logic **6**, No.1-2 (2001), 229-249.

[133] Y. Utumi, *On quotient rings*, Osaka Math. Journal, 8 (1956).

[134] H. Yutami, *On a system of axioms of commutative BCK-algebras,* Mathematics Seminar Notes of the Kobe University, 5 (1977), 255-256.

[135] M. Ward: *Residuated distributive lattices*, Duke Mathematical Journal 6 (1940), 641-651.

[136] M. Ward, R. P. Dilworth: *Residuated lattices*, Transactions of the American Mathematical Society 45 (1939), 335-354.