An Introduction to the Controllability of Partial Differential Equations

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Introduction

These notes are a written abridged version of a course that both authors have delivered in the last five years in a number of schools and doctoral programs. Our main goal is to introduce some of the main results and tools of the modern theory of controllability of Partial Differential Equations (PDE). The notes are by no means complete. We focus the most elementary material by making a particular choice of the problems under consideration.

Roughly speaking, the *controllability problem* may be formulated as follows. Consider an evolution system (either described in terms of Partial or Ordinary Differential Equations (PDE/ODE)). We are allowed to act on the trajectories of the system by means of a suitable control (the right hand side of the system, the boundary conditions, etc.). Then, given a time interval $t \in (0, T)$, and initial and final states we have to find a control such that the solution matches both the initial state at time $t = 0$ and the final one at time $t = T$.

This is a classical problem in Control Theory and there is a large literature on the topic. We refer for instance to the book by Lee and Marcus [44] for an introduction in the context of finite-dimensional systems. We also refer to the survey paper by Russell [55] and to the book of Lions [45] for an introduction to the controllability of PDE, also referred to as Distributed Parameter Systems.

Research in this area has been very intensive in the last two decades and it would be impossible to report on the main progresses that have been made within these notes. For this reason we have chosen to collect some of the most relevant introductory material at the prize of not reaching the best results that

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are known today. The interested reader may learn more on this topic from the references above and those on the bibliography at the end of the article.

When dealing with controllability problems, to begin with, one has to distinguish between finite-dimensional systems modelled by ODE and infinite-dimensional distributed systems described by means of PDE. This modelling issue may be important in practice since finite-dimensional and infinite-dimensional systems may have quite different properties from a control theoretical point of view ([74]).

Most of these notes deal with problems related to PDE. However, we start by an introductory chapter in which we present some of the basic problems and tools of control theory for finite-dimensional systems. The theory has evolved tremendously in the last decades to deal with nonlinearity and uncertainty but here we present the simplest results concerning the controllability of linear finite-dimensional systems and focus on developing tools that will later be useful to deal with PDE. As we shall see, in the finite-dimensional context *a system is controllable if and only if the algebraic Kalman rank condition is satisfied.* According to it, when a system is controllable for some time it is controllable for all time. But this is not longer true in the context of PDE. In particular, in the frame of the wave equation, a model in which propagation occurs with finite velocity, in order for controllability properties to be true the control time needs to be large enough so that the effect of the control may reach everywhere. In this first chapter we shall develop a variational approach to the control problem.

As we shall see, whenever a system is controllable, the control can be built by minimizing a suitable quadratic functional defined on the class of solutions of the adjoint system. Suitable variants of this functional allow building different types of controls: those of minimal L^2-norm turn out to be smooth while those of minimal L^∞-norm are of bang-bang form. The main difficulty when minimizing these functionals is to show that they are coercive. This turns out to be equivalent to the so called *observability property* of the adjoint equation, a property which is equivalent to the original control property of the state equation.

In Chapters 2 and 3 we introduce the problems of interior and boundary control of the linear constant coefficient wave equation. We describe the various variants, namely, approximate, exact and null controllability, and its mutual relations. Once again, the problem of exact controllability turns out to be equivalent to the observability of the adjoint system while approximate controllability is equivalent to a weaker uniqueness or unique continuation property. In Chapter 4 we analyze the $1-d$ case by means of Fourier series expansions and the classical Ingham's inequality which is a very useful tool to solve control problems for $1-d$ wave-like and beam equations.

In Chapters 5 and 6 we discuss respectively the problems of interior and boundary control of the heat equation. We show that, as a consequence of

Holmgren Uniqueness Theorem, the adjoint heat equation posesses the property of unique continuation in an arbitrarily small time. Accordingly the multi-dimensional heat equation is approximately controllable in an arbitrarily small time and with controls supported in any open subset of the domain where the equation holds. We also show that, in one space dimension, using Fourier series expansions, the null control problem, can be reduced to a problem of moments involving a sequence of real exponentials. We then build a biorthogonal family allowing to show that the system is null controllable in any time by means of a control acting on one extreme of the space interval where the heat equation holds.

As we said above these notes are not complete. The interested reader may learn more on this topic through the survey articles [70] and [72]. For the connections between controllability and the theory of homogenization we refer to [12]. We refer to [74] for a discussion of numerical apprximation issues in controllability of PDE.

1 Controllability and stabilization of finite dimensional systems

This chapter is devoted to study some basic controllability and stabilization properties of finite dimensional systems.

The first two sections deal with the linear case. In Section 1 it is shown that the exact controllability property may be characterized by means of the *Kalman's algebraic rank condition*. In Section 2 a skew-adjoint system is considered. In the absence of control, the system is conservative and generates a group of isometries. It is shown that the system may be guaranteed to be uniformly exponentially stable if a well chosen feedback dissipative term is added to it. This is a particular case of the well known equivalence property between controllability and stabilizability of finite-dimensional systems ([65]).

1.1 Controllability of finite dimensional linear systems

Let $n, m \in \mathbb{N}^*$ and $T > 0$. We consider the following finite dimensional system:

$$\begin{cases} x'(t) = Ax(t) + Bu(t), \quad t \in (0,T), \\ x(0) = x^0. \end{cases} \tag{1}$$

In (1), A is a real $n \times n$ matrix, B is a real $n \times m$ matrix and x^0 a vector in $\mathbb{R}^n$. The function $x : [0,T] \longrightarrow \mathbb{R}^n$ represents the *state* and $u : [0,T] \longrightarrow \mathbb{R}^m$ the *control*. Both are vector functions of n and m components respectively depending exclusively on time t. Obviously, in practice $m \leq n$. The most desirable goal is, of course, controlling the system by means of a minimum number m of controls.

Given an initial datum $x^0 \in \mathbb{R}^n$ and a vector function $u \in L^2(0,T;\mathbb{R}^m)$, system (1) has a unique solution $x \in H^1(0,T;\mathbb{R}^n)$ characterized by the variation of constants formula:

$$x(t) = e^{At}x^0 + \int_0^t e^{A(t-s)}Bu(s)ds, \quad \forall t \in [0,T]. \tag{2}$$

Definition 1.1 *System (1) is* **exactly controllable** *in time $T > 0$ if given any initial and final one $x^0, x^1 \in \mathbb{R}^n$ there exists $u \in L^2(0,T,\mathbb{R}^m)$ such that the solution of (1) satisfies $x(T) = x^1$.*

According to this definition the aim of the control process consists in driving the solution x of (1) from the initial state x^0 to the final one x^1 in time T by acting on the system through the control u.

Remark that m is the number of controls entering in the system, while n stands for the number of components of the state to be controlled. As we mentioned before, in applications it is desirable to make the number of controls m to be as small as possible. But this, of course, may affect the control properties of the system. As we shall see later on, some systems with a large number of components n can be controlled with one control only (i. e. $m = 1$). But in order for this to be true, the control mechanism, i.e. the matrix (column vector when $m = 1$) B, needs to be chosen in a strategic way depending on the matrix A. Kalman's rank condition, that will be given in section 1.3, provides a simple characterization of controllability allowing to make an appropriate choice of the control matrix B.

Let us illustrate this with two examples. In the first one controllability does not hold because one of the components of the system is insensitive to the control. In the second one both components will be controlled by means of a scalar control.

Example 1. Consider the case

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{3}$$

Then the system

$$x' = Ax + Bu$$

can be written as

$$\begin{cases} x_1' = x_1 + u \\ x_2' = x_2, \end{cases}$$

or equivalently,

$$\begin{cases} x_1' = x_1 + u \\ x_2 = x_2^0 e^t, \end{cases}$$

where $x^0 = (x_1^0, x_2^0)$ are the initial data.

This system is not controllable since the control u does not act on the second component x_2 of the state which is completely determined by the initial data x_2^0. Hence, the system is not controllable. Nevertheless one can control the first component x_1 of the state. Consequently, the system is partially controllable. $\square$

Example 2. Not all systems with two components and a scalar control ($n = 2, m = 1$) behave so badly as in the previous example. This may be seen by analyzing the controlled harmonic oscillator

$$x'' + x = u, \tag{4}$$

which may be written as a system in the following way

$$\begin{cases} x' = y \\ y' = u - x. \end{cases}$$

The matrices A and B are now respectively

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Once again, we have at our disposal only one control u for both components x and y of the system. But, unlike in Example 1, now the control acts in the second equation where both components are present. Therefore, we cannot conclude immediately that the system is not controllable. In fact it is controllable. Indeed, given some arbitrary initial and final data, (x^0, y^0) and (x^1, y^1) respectively, it is easy to construct a regular function $z = z(t)$ such that

$$\begin{cases} z(0) = x^0, & z(T) = x^1, \\ z'(0) = y^0, & z'(T) = y^1. \end{cases} \tag{5}$$

In fact, there are infinitely many ways of constructing such functions. One can, for instance, choose a cubic polynomial function z. We can then define $u = z'' + z$ as being the control since the solution x of equation (4) with this control and initial data (x^0, y^0) coincides with z, i.e. $x = z$, and therefore satisfies the control requirements (5).

This construction provides an example of system with two components ($n = 2$) which is controllable with one control only ($m = 1$). Moreover, this example shows that the control u is not unique. In fact there exist infinitely many controls and different controlled trajectories fulfilling the control requirements. In practice, choosing the control which is optimal (in some sense to be made precise) is an important issue that we shall also discuss. $\square$

If we define the set of reachable states

$$R(T, x^0) = \{x(T) \in \mathbb{R}^n : x \text{ solution of (1) with } u \in (L^2(0,T))^m\}, \tag{6}$$

the exact controllability property is equivalent to the fact that $R(T, x^0) = \mathbb{R}^n$ for any $x^0 \in \mathbb{R}^n$.

Remark 1.1 *In the definition of exact controllability any initial datum* x^0 *is required to be driven to any final datum* x^1. *Nevertheless, in the view of the linearity of the system, without any loss of generality, we may suppose that* $x^1 = 0$. *Indeed, if* $x^1 \neq 0$ *we may solve*

$$\begin{cases} y' = Ay, & t \in (0,T) \\ y(T) = x^1 \end{cases} \tag{7}$$

backward in time and define the new state $z = x - y$ *which verifies*

$$\begin{cases} z' = Az + Bu \\ z(0) = x^0 - y(0). \end{cases} \tag{8}$$

Remark that $x(T) = x^1$ *if and only if* $z(T) = 0$. *Hence, driving the solution* x *of (1) from* x^0 *to* x^1 *is equivalent to leading the solution* z *of (8) from the initial data* $z^0 = x^0 - y(0)$ *to zero.* $\square$

The previous remark motivates the following definition:

Definition 1.2 *System (1) is said to be* **null-controllable** *in time* $T > 0$ *if given any initial data* $x^0 \in \mathbb{R}^n$ *there exists* $u \in L^2(0,T,\mathbb{R}^m)$ *such that* $x(T) = 0$.

Null-controllability holds if and only if $0 \in R(x^0, T)$ for any $x^0 \in \mathbb{R}^n$.

On the other hand, Remark 1.1 shows that *exact controllability and null controllability are equivalent properties in the case of finite dimensional linear systems.* But this is not necessarily the case for nonlinear systems, or, for strongly time irreversible infinite dimensional systems, for strongly time irreversible ones. For instance, the heat equation is a well known example of null-controllable system that is not exactly controllable.

1.2 Observability property

The exact controllability property is closely related to an inequality for the corresponding adjoint homogeneous system. This is the so called *observation or observability inequality.* In this section we introduce this notion and show its relation with the exact controllability property.

Let A^* be the adjoint matrix of A, i.e. the matrix with the property that $\langle Ax, y\rangle = \langle x, A^*y\rangle$ for all $x, y \in \mathbb{R}^n$. Consider the following homogeneous *adjoint system* of (1):

$$\begin{cases} -\varphi' = A^*\varphi, & t \in (0,T) \\ \varphi(T) = \varphi_T. \end{cases} \tag{9}$$

Remark that, for each $\varphi_T \in \mathbb{R}$, (9) may be solved backwards in time and it has a unique solution $\varphi \in C^\omega([0,T],\mathbb{R}^n)$ (the space of analytic functions defined in $[0,T]$ and with values in $\mathbb{R}^n$).

First of all we deduce an equivalent condition for the exact controllability property.

Lemma 1.1 *An initial datum $x^0 \in \mathbb{R}^n$ of (1) is driven to zero in time T by using a control $u \in L^2(0,T)$ if and only if*

$$\int_0^T \langle u, B^*\varphi\rangle dt + \langle x^0, \varphi(0)\rangle = 0 \tag{10}$$

for any $\varphi_T \in \mathbb{R}^n$, φ being the corresponding solution of (9).

Proof: Let φ_T be arbitrary in $\mathbb{R}^n$ and φ the corresponding solution of (9). By multiplying (1) by φ and (9) by x we deduce that

$$\langle x', \varphi\rangle = \langle Ax, \varphi\rangle + \langle Bu, \varphi\rangle; \quad -\langle x, \varphi'\rangle = \langle A^*\varphi, x\rangle.$$

Hence,

$$\frac{d}{dt}\langle x, \varphi\rangle = \langle Bu, \varphi\rangle$$

which, after integration in time, gives that

$$\langle x(T), \varphi_T\rangle - \langle x^0, \varphi(0)\rangle = \int_0^T \langle Bu, \varphi\rangle dt = \int_0^T \langle u, B^*\varphi\rangle dt. \tag{11}$$

We obtain that $x(T) = 0$ if and only if (10) is verified for any $\varphi_T \in \mathbb{R}^n$. □

It is easy to see that (10) is in fact an optimality condition for the critical points of the quadratic functional $J : \mathbb{R}^n \to \mathbb{R}^n$,

$$J(\varphi_T) = \frac{1}{2}\int_0^T \mid B^*\varphi \mid^2 dt + \langle x^0, \varphi(0)\rangle$$

where φ is the solution of the adjoint system (9) with initial data φ_T at time $t = T$.

More precisely, we have the following result:

Lemma 1.2 *Suppose that J has a minimizer $\widehat{\varphi}_T \in \mathbb{R}^n$ and let $\widehat{\varphi}$ be the solution of the adjoint system (9) with initial data $\widehat{\varphi}_T$. Then*

$$u = B^*\widehat{\varphi} \tag{12}$$

is a control of system (1) with initial data x^0.

Proof: If $\widehat{\varphi}_T$ is a point where J achieves its minimum value, then

$$\lim_{h\to 0} \frac{J\left(\widehat{\varphi}_T + h\varphi_T\right) - J\left(\widehat{\varphi}_T\right)}{h} = 0, \quad \forall \varphi_T \in \mathbb{R}^n.$$

This is equivalent to

$$\int_0^T \langle B^*\widehat{\varphi}, B^*\varphi\rangle dt + \langle x^0, \varphi(0)\rangle = 0, \quad \forall \varphi_T \in \mathbb{R}^n,$$

which, in view of Lemma 1.1, implies that $u = B^*\widehat{\varphi}$ is a control for (1). □

Remark 1.2 *Lemma 1.2 gives a variational method to obtain the control as a minimum of the functional J. This is not the unique possible functional allowing to build the control. By modifying it conveniently, other types of controls (for instance bang-bang ones) can be obtained. We shall show this in section 1.4. Remark that the controls we found are of the form $B^*\varphi$, φ being a solution of the homogeneous adjoint problem (9). Therefore, they are analytic functions of time.* □

The following notion will play a fundamental role in solving the control problems.

Definition 1.3 *System (9) is said to be* **observable** *in time $T > 0$ if there exists $c > 0$ such that*

$$\int_0^T \mid B^*\varphi \mid^2 dt \geq c \mid \varphi(0) \mid^2, \tag{13}$$

for all $\varphi_T \in \mathbb{R}^n$, φ being the corresponding solution of (9).

In the sequel (13) will be called the **observation or observability inequality**. It guarantees that the solution of the adjoint problem at $t = 0$ is uniquely determined by the observed quantity $B^*\varphi(t)$ for $0 < t < T$. In other words, the information contained in this term completely characterizes the solution of (9).

Remark 1.3 *The observation inequality (13) is equivalent to the following one: there exists $c > 0$ such that*

$$\int_0^T \mid B^*\varphi \mid^2 dt \geq c \mid \varphi_T \mid^2, \tag{14}$$

for all $\varphi_T \in \mathbb{R}^n$, φ being the solution of (9).

Indeed, the equivalence follows from the fact that the map which associates to every $\varphi_T \in \mathbb{R}^n$ the vector $\varphi(0) \in \mathbb{R}^n$, is a bounded linear transformation in $\mathbb{R}^n$ with bounded inverse. We shall use the forms (13) or (14) of the observation inequality depending of the needs of each particular problem we shall deal with. □

The following remark is very important in the context of finite dimensional spaces.

Proposition 1.1 *Inequality (13) is equivalent to the following unique continuation principle:*

$$B^*\varphi(t) = 0,\ \forall t \in [0,T] \Rightarrow \varphi_T = 0. \tag{15}$$

Proof: One of the implications follows immediately from (14). For the other one, let us define the semi-norm in $\mathbb{R}^n$

$$|\varphi_T|_* = \left[\int_0^T |\ B^*\varphi\ |^2\ dt\right]^{1/2}.$$

Clearly, $|\cdot|_*$ is a norm in $\mathbb{R}^n$ if and only if (15) holds.

Since all the norms in $\mathbb{R}^n$ are equivalent, it follows that (15) is equivalent to (14). The proof ends by taking into account the previous Remark 2.3. □

Remark 1.4 *Let us remark that (13) and (15) will no longer be equivalent properties in infinite dimensional spaces. They will give rise to different notions of controllability (exact and approximate, respectively). This issue will be further developed in the following section.* □

The importance of the observation inequality relies on the fact that it implies exact controllability of (1). In this way the controllability property is reduced to the study of an inequality for the homogeneous system (9) which, at least conceptually, is a simpler problem. Let us analyze now the relation between the controllability and observability properties.

Theorem 1.1 *System (1) is exactly controllable in time T if and only if (9) is observable in time T.*

Proof: Let us prove first that observability implies controllability. According to Lemma 1.2, the exact controllability property in time T holds if for any $x^0 \in \mathbb{R}^n$, J has a minimum. Remark that J is continuous. Consequently, the existence of a minimum is ensured if J is coercive too, i.e.

$$\lim_{|\varphi_T|\to\infty} J(\varphi_T) = \infty. \tag{16}$$

The coercivity property (16) is a consequence of the observation property in time T. Indeed, from (13) we obtain that

$$J(\varphi_T) \geq \frac{c}{2} \mid \varphi_T \mid^2 - |\langle x^0, \varphi(0) \rangle|.$$

The right hand side tends to infinity when $|\varphi_T| \to \infty$ and J satisfies (16).

Reciprocally, suppose that system (1) is exactly controllable in time T. If (9) is not observable in time T, there exists a sequence $(\varphi_T^k)_{k\geq 1} \subset \mathbb{R}^n$ such that $|\varphi_T^k| = 1$ for all $k \geq 1$ and

$$\lim_{k\to\infty} \int_0^T |B^*\varphi^k|^2 dt = 0. \tag{17}$$

It follows that there exists a subsequence of $(\varphi_T^k)_{k\geq 1}$, denoted in the same way, which converges to $\varphi_T \in \mathbb{R}^n$ and $|\varphi_T| = 1$. Moreover, if φ is the solution of (9) with initial data φ_T, from (17) it follows that

$$\int_0^T |B^*\varphi|^2 dt = 0. \tag{18}$$

Since (1) is controllable, Lemma 1.1 gives that, for any initial data $x^0 \in \mathbb{R}^n$, there exists $u \in L^2(0,T)$ such that

$$\int_0^T \langle u, B^*\varphi_k \rangle dt = -\langle x^0, \varphi_k(0) \rangle, \quad \forall k \geq 1. \tag{19}$$

By passing to the limit in (19) and by taking into account (18), we obtain that $< x^0, \varphi(0) >= 0$. Since x^0 is arbitrary in $\mathbb{R}^n$, it follows that $\varphi(0) = 0$ and, consequently, $\varphi_T = 0$. This is in contradiction with the fact that $|\varphi_T| = 1$.

The proof of the theorem is now complete. □

Remark 1.5 *The usefulness of Theorem 1.1 consists on the fact that it reduces the proof of the exact controllability to the study of the observation inequality.* □

1.3 Kalman's controllability condition

The following classical result is due to R. E. Kalman and gives a complete answer to the problem of exact controllability of finite dimensional linear systems. It shows, in particular, that the time of control is irrelevant.

Theorem 1.2 *([44]) System* (1) *is exactly controllable in some time* T *if and only if*

$$\text{rank}\,[B,\, AB, \cdots, A^{n-1}B] = n. \tag{20}$$

Consequently, if system (1) is controllable in some time $T > 0$ *it is controllable in any time.*

Remark 1.6 *From now on we shall simply say that* (A, B) *is controllable if (20) holds. The matrix* $[B, AB, \cdots, A^{n-1}B]$ *will be called the controllability matrix.* □

Examples: In Example 1 from section 1.1 we had

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{21}$$

Therefore

$$[B,\, AB] = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \tag{22}$$

which has rank 1. From Theorem 1.2 it follows that the system under consideration is not controllable. Nevertheless, in Example 2,

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{23}$$

and consequently

$$[B,\, AB] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{24}$$

which has rank 2 and the system is controllable as we have already observed. □

Proof of Theorem 1.2: " $\Rightarrow$" Suppose that $\text{rank}([B,\, AB, \cdots, A^{n-1}B]) < n$.

Then the rows of the controllability matrix $[B, AB, \cdots, A^{n-1}B]$ are linearly dependent and there exists a vector $v \in \mathbb{R}^n$, $v \neq 0$ such that

$$v^*[B,\, AB, \cdots, A^{n-1}B] = 0,$$

where the coefficients of the linear combination are the components of the vector v. Since $v^*[B,\, AB, \cdots, A^{n-1}B] = [v^*B,\, v^*AB, \cdots, v^*A^{n-1}B]$, $v^*B = v^*AB = \cdots = v^*\,A^{n-1}B = 0$. ¿From Cayley-Hamilton Theorem we deduce that there exist constants $c_1, \cdots, c_n$ such that, $A^n = c_1A^{n-1} + \cdots + c_nI$ and therefore $v^*A^nB = 0$, too. In fact, it follows that $v^*A^kB = 0$ for all $k \in \mathbb{N}$ and consequently $v^*e^{At}B = 0$ for all t as well. But, from the variation of constants formula, the solution x of (1) satisfies

$$x(t) = e^{At}x^0 + \int_0^t e^{A(t-s)}Bu(s)ds. \tag{25}$$

Therefore

$$\langle v, x(T)\rangle = \langle v, e^{AT}x^0\rangle + \int_0^T \langle v, e^{A(T-s)}Bu(s)\rangle ds = \langle v, e^{AT}x^0\rangle,$$

where $\langle\, ,\, \rangle$ denotes the canonical inner product in $\mathbb{R}^n$. Hence, $\langle v, x(T)\rangle = \langle v, e^{AT}x^0\rangle$. This shows that the projection of the solution x at time T on the vector v is independent of the value of the control u. Hence, the system is not controllable. □

Remark 1.7 *The conservation property for the quantity $\langle v, x\rangle$ we have just proved holds for any vector v for which $v[B,\ AB, \cdots, A^{n-1}B] = 0$. Thus, if the rank of the matrix $[B,\ AB, \cdots, A^{n-1}B]$ is $n-k$, the reachable set that $x(T)$ runs is an affine subspace of $\mathbb{R}^n$ of dimension $n-k$.* □

" $\Leftarrow$" Suppose now that $\text{rank}([B,\ AB, \cdots, A^{n-1}B]) = n$. According to Theorem 1.1 it is sufficient to show that system (9) is observable. By Proposition 1.1, (13) holds if and only if (15) is verified. Hence, the Theorem is proved if (15) holds. ¿From $B^*\varphi = 0$ and $\varphi(t) = e^{A^*(T-t)}\varphi_T$, it follows that $B^*e^{A^*(T-t)}\varphi_T \equiv 0$ for all $0 \le t \le T$. By computing the derivatives of this function in $t = T$ we obtain that

$$B^*[A^*]^k\varphi_T = 0 \quad \forall k \ge 0.$$

But since $\text{rank}(\big[B,\ AB, \cdots, A^{n-1}B\big]) = n$ we deduce that

$$\text{rank}(\big[B^*,\ B^*A^*, \cdots, B^*(A^*)^{n-1}\big]) = n$$

and therefore $\varphi_T = 0$. Hence, (15) is verified and the proof of Theorem 1.2 is now complete. □

Remark 1.8 *The set of controllable pairs (A, B) is open and dense. Indeed,*

- *If (A, B) is controllable there exists $\varepsilon > 0$ sufficiently small such that any (A^0, B^0) with $\mid A^0 - A \mid < \varepsilon$, $\mid B^0 - B \mid < \varepsilon$ is also controllable. This is a consequence of the fact that the determinant of a matrix depends continuously of its entries.*
- *On the other hand, if (A, B) is not controllable, for any $\varepsilon > 0$, there exists (A^0, B^0) with $\mid A - A^0 \mid < \varepsilon$ and $\mid B - B^0 \mid < \varepsilon$ such that (A^0, B^0) is controllable. This is a consequence of the fact that the determinant of a $n \times n$ matrix depends analytically of its entries and cannot vanish in a ball of $\mathbb{R}^n$.* □

The following inequality shows that the norm of the control is proportional to the distance between $e^{AT}x^0$ (the state freely attained by the system in the absence of control, i. e. with $u = 0$) and the objective x^1.

Proposition 1.2 *Suppose that the pair (A, B) is controllable in time $T > 0$ and let u be the control obtained by minimizing the functional J. There exists a constant $C > 0$, depending on T, such that the following inequality holds*

$$\| u \|_{L^2(0,T)} \leq C|e^{AT}x^0 - x^1| \tag{26}$$

for any initial data x^0 and final objective x^1.

Proof: Let us first prove (26) for the particular case $x^1 = 0$.

Let u be the control for (1) obtained by minimizing the functional J. From (10) it follows that

$$||u||^2_{L^2(0,T)} = \int_0^T |B^*\widehat{\varphi}|^2 dt = - < x^0, \widehat{\varphi}(0) > .$$

If w is the solution of

$$\begin{cases} w'(t) = Aw(t), & t \in (0, T), \\ w(0) = x^0 \end{cases} \tag{27}$$

then $w(t) = e^{At}x^0$ and

$$\frac{d}{dt} < w, \varphi >= 0$$

for all $\varphi_T \in \mathbb{R}^n$, φ being the corresponding solution of (9).

In particular, by taking $\varphi_T = \widehat{\varphi}_T$, the minimizer of J, it follows that

$$< x^0, \widehat{\varphi}(0) >=< w(0), \widehat{\varphi}(0) >=< w(T), \widehat{\varphi}_T >=< e^{AT}x^0, \widehat{\varphi}_T > .$$

We obtain that

$$||u||^2_{L^2(0,T)} = - < x^0, \widehat{\varphi}(0) >= - < e^{AT}x^0, \widehat{\varphi}_T >\leq |e^{AT}x^0|\, |\widehat{\varphi}_T|.$$

On the other hand, we have that

$$|\widehat{\varphi}_T| \leq c||B^*\widehat{\varphi}||_{L^2(0,T)} = c||u||_{L^2(0,T)}.$$

Thus, the control u verifies

$$||u||_{L^2(0,T)} \leq c|e^{AT}x^0|. \tag{28}$$

If $x^1 \neq 0$, Remark 1.1 implies that a control u driving the solution from x^0 to x^1 coincides with the one leading the solution from $x^0 - y(0)$ to zero, where y verifies (7). By using (28) we obtain that

$$||u||_{L^2(0,T)} \leq c|e^{TA}(x^0 - y(0))| = c|e^{TA}x^0 - x^1|$$

and (26) is proved. □

Remark 1.9 *Linear scalar equations of any order provide examples of systems of arbitrarily large dimension that are controllable with only one control. Indeed, the system of order* k

$$x^{(k)} + a_1 x^{(k-1)} + \ldots + a_{k-1} x = u$$

is controllable. This can be easily obtained by observing that given k *initial data and* k *final ones one can always find a trajectory* z *(in fact an infinite number of them) joining them in any time interval. This argument was already used in Example 2 for the case* $k = 2$.

It is an interesting exercise to write down the matrices A *and* B *in this case and to check that the rank condition in Theorem 1.2 is fulfilled.* □

1.4 Bang-bang controls

Let us consider the particular case

$$B \in \mathcal{M}_{n\times 1}, \tag{29}$$

i. e. $m = 1$, in which only one control $u : [0, T] \to \mathbb{R}$ is available. In order to build bang-bang controls, it is convenient to consider the quadratic functional:

$$J_{bb}(\varphi^0) = \frac{1}{2}\left[\int_0^T \mid B^*\varphi \mid dt\right]^2 + \langle x^0, \varphi(0)\rangle \tag{30}$$

where φ is the solution of the adjoint system (9) with initial data φ_T.

Note that $B^* \in \mathcal{M}_{1\times n}$ and therefore $B^*\varphi(t) : [0, T] \to \mathbb{R}$ is a scalar function. It is also interesting to note that J_{bb} differs from J in the quadratic term. Indeed, in J we took the $L^2(0, T)$-norm of $B^*\varphi$ while here we consider the $L^1(0, T)$-norm.

The same argument used in the proof of Theorem 1.2 shows that J_{bb} is also continuous and coercive. It follows that J_{bb} attains a minimum in some point $\widehat{\varphi}_T \in \mathbb{R}^n$.

On the other hand, it is easy to see that

$$\begin{aligned}\lim_{h\to 0}\frac{1}{h}&\left[\left(\int_0^T \mid f + hg \mid dt\right)^2 - \left(\int_0^T \mid f \mid\right)^2\right] = \\ &= 2\int_0^T \mid f \mid dt \int_0^T \operatorname{sgn}(f(t))g(t)dt\end{aligned} \tag{31}$$

if the Lebesgue measure of the set $\{t \in (0, T) : f(t) = 0\}$ vanishes.

The sign function "sgn" is defined as a multi-valued function in the following way

$$\operatorname{sgn}(s) = \begin{cases} 1 & \text{when } s > 0 \\ -1 & \text{when } s < 0 \\ [-1,1] & \text{when } s = 0 \end{cases}$$

Remark that in the previous limit there is no ambiguity in the definition of $\operatorname{sgn}(f(t))$ since the set of points $t \in [0,T]$ where $f = 0$ is assumed to be of zero Lebesgue measure and does not affect the value of the integral.

Identity (31) may be applied to the quadratic term of the functional J_{bb} since, taking into account that φ is the solution of the adjoint system (9), it is an analytic function and therefore, $B^*\varphi$ changes sign finitely many times in the interval $[0,T]$ except when $\widehat{\varphi}_T = 0$. In view of this, the Euler-Lagrange equation associated with the critical points of the functional J_{bb} is as follows:

$$\int_0^T \mid B^*\widehat{\varphi} \mid dt \int_0^T \operatorname{sgn}(B^*\widehat{\varphi}) B^*\psi(t) dt + \langle x^0, \varphi(0)\rangle = 0$$

for all $\varphi_T \in \mathbb{R}$, where φ is the solution of the adjoint system (9) with initial data φ_T.

Consequently, the control we are looking for is $u = \displaystyle\int_0^T \mid B^*\widehat{\varphi} \mid dt \operatorname{sgn}(B^*\widehat{\varphi})$ where $\widehat{\varphi}$ is the solution of (9) with initial data $\widehat{\varphi}_T$.

Note that the control u is of bang-bang form. Indeed, u takes only two values $\pm \int_0^T \mid B^*\widehat{\varphi} \mid dt$. The control switches from one value to the other finitely many times when the function $B^*\widehat{\varphi}$ changes sign.

Remark 1.10 *Other types of controls can be obtained by considering functionals of the form*

$$J_p(\varphi^0) = \frac{1}{2}\left(\int_0^T \mid B^*\varphi \mid^p dt\right)^{2/p} + \langle x^0, \varphi^0\rangle$$

with $1 < p < \infty$. The corresponding controls are

$$u = \left(\int_0^T \mid B^*\widehat{\varphi} \mid^p dt\right)^{(2-p)/p} |B^*\widehat{\varphi}|^{p-2} B^*\widehat{\varphi}$$

where $\widehat{\varphi}$ is the solution of (9) with initial datum $\widehat{\varphi}_T$, the minimizer of J_p.

It can be shown that, as expected, the controls obtained by minimizing this functionals give, in the limit when $p \to 1$, a bang-bang control. □

The following property gives an important characterization of the controls we have studied.

Proposition 1.3 *The control $u_2 = B^*\widehat{\varphi}$ obtained by minimizing the functional J has minimal $L^2(0,T)$ norm among all possible controls. Analogously, the control $u_\infty = \int_0^T \mid B^*\widehat{\varphi} \mid dt\,\mathrm{sgn}(B^*\widehat{\varphi})$ obtained by minimizing the functional J_{bb} has minimal $L^\infty(0,T)$ norm among all possible controls.*

Proof: Let u be an arbitrary control for (1). Then (10) is verified both by u and u_2 for any φ_T. By taking $\varphi_T = \widehat{\varphi}_T$ (the minimizer of J) in (10) we obtain that

$$\int_0^T < u, B^*\widehat{\varphi} > dt = - < x^0, \widehat{\varphi}(0) >,$$

$$||u_2||^2_{L^2(0,T)} = \int_0^T < u_2, B^*\widehat{\varphi} > dt = - < x^0, \widehat{\varphi}(0) > .$$

Hence,

$$||u_2||^2_{L^2(0,T)} = \int_0^T < u, B^*\widehat{\varphi} > dt \leq ||u||_{L^2(0,T)}||B^*\widehat{\varphi}|| = ||u||_{L^2(0,T)}||u_2||_{L^2(0,T)}$$

and the first part of the proof is complete.

For the second part a similar argument may be used. Indeed, let again u be an arbitrary control for (1). Then (10) is verified by u and u_∞ for any φ_T. By taking $\varphi_T = \widehat{\varphi}_T$ (the minimizer of J_{bb}) in (10) we obtain that

$$\int_0^T B^*\widehat{\varphi} u dt = - < x^0, \widehat{\varphi}(0) >,$$

$$||u_\infty||^2_{L^\infty(0,T)} = \left(\int_0^T |B^*\widehat{\varphi}| dt\right)^2 = \int_0^T B^*\widehat{\varphi} u_\infty dt = - < x^0, \widehat{\varphi}(0) > .$$

Hence,

$$||u_\infty||^2_{L^\infty(0,T)} = \int_0^T B^*\widehat{\varphi}\, u dt \leq$$

$$\leq ||u||_{L^\infty(0,T)} \int_0^T |B^*\widehat{\varphi}| dt = ||u||_{L^\infty(0,T)} ||u_\infty||_{L^\infty(0,T)}$$

and the proof finishes. □

1.5 Stabilization of finite dimensional linear systems

In this section we assume that A is a skew-adjoint matrix, i. e. $A^* = -A$. In this case, $< Ax, x >= 0$.

Consider the system

$$\begin{cases} x' = Ax + Bu \\ x(0) = x^0. \end{cases} \tag{32}$$

Remark 1.11 *The harmonic oscillator, $mx'' + kx = 0$, provides the simplest example of system with such properties. It will be studied with some detail at the end of the section.* □

When $u \equiv 0$, the energy of the solution of (32) is conserved. Indeed, by multiplying (32) by x, if $u \equiv 0$, one obtains

$$\frac{d}{dt}|x(t)|^2 = 0. \tag{33}$$

Hence,

$$|x(t)| = |x^0|, \quad \forall t \geq 0. \tag{34}$$

The problem of *stabilization* can be formulated in the following way. Suppose that the pair (A, B) is controllable. We then look for a matrix L such that the solution of system (32) with the *feedback* control

$$u(t) = Lx(t) \tag{35}$$

has **a uniform exponential decay**, i.e. there exist $c > 0$ and $\omega > 0$ such that

$$|x(t)| \leq ce^{-\omega t}|x^0| \tag{36}$$

for any solution.

Note that, according to the law (35), the control u is obtained in real time from the state x.

In other words, we are looking for matrices L such that the solution of the system

$$x' = (A + BL)x = Dx \tag{37}$$

has an uniform exponential decay rate.

Remark that we cannot expect more than (36). Indeed, the solutions of (37) may not satisfy $x(T) = 0$ in finite time T. Indeed, if it were the case, from the uniqueness of solutions of (37) with final state 0 in $t = T$, it would follow that $x^0 \equiv 0$. On the other hand, whatever L is, the matrix D has N eigenvalues λ_j with corresponding eigenvectors $e_j \in \mathbb{R}^n$. The solution $x(t) = e^{\lambda_j t}e_j$ of (37) shows that the decay of solutions can not be faster than exponential.

Theorem 1.3 *If A is skew-adjoint and the pair (A, B) is controllable then $L = -B^*$ stabilizes the system, i.e. the solution of*

$$\begin{cases} x' = Ax - BB^*x \\ x(0) = x^0 \end{cases} \tag{38}$$

has an uniform exponential decay (36).

Proof: With $L = -B^*$ we obtain that

$$\frac{1}{2}\frac{d}{dt}|x(t)|^2 = -< BB^*x(t), x(t) >= - \mid B^*x(t) \mid^2 \leq 0.$$

Hence, the norm of the solution decreases in time.

Moreover,

$$|x(T)|^2 - |x(0)|^2 = -2\int_0^T \mid B^*x \mid^2 dt. \tag{39}$$

To prove the uniform exponential decay it is sufficient to show that there exist $T > 0$ and $c > 0$ such that

$$|x(0)|^2 \leq c\int_0^T \mid B^*x \mid^2 dt \tag{40}$$

for any solution x of (38). Indeed, from (39) and (40) we would obtain that

$$|x(T)|^2 - |x(0)|^2 \leq -\frac{2}{c}|x(0)|^2 \tag{41}$$

and consequently

$$|x(T)|^2 \leq \gamma|x(0)|^2 \tag{42}$$

with

$$\gamma = 1 - \frac{2}{c} < 1. \tag{43}$$

Hence,

$$|x(kT)|^2 \leq \gamma^k|x^0|^2 = e^{(\ln\gamma)k}|x^0|^2 \quad \forall k \in \mathbb{N}. \tag{44}$$

Now, given any $t > 0$ we write it in the form $t = kT + \delta$, with $\delta \in [0, T)$ and $k \in \mathbb{N}$ and we obtain that

$$|x(t)|^2 \leq |x(kT)|^2 \leq e^{-|\ln(\gamma)|k}|x^0|^2 =$$
$$= e^{-|\ln(\gamma)|\left(\frac{t}{T}\right)}e^{|\ln(\gamma)|\frac{\delta}{T}}|x^0|^2 \leq \frac{1}{\gamma}e^{-\frac{|\ln(\gamma)|}{T}t}|x^0|^2.$$

We have obtained the desired decay result (36) with

$$c = \frac{1}{\gamma}, \, \omega = \frac{\mid \ln(\gamma) \mid}{T}. \tag{45}$$

To prove (40) we decompose the solution x of (38) as $x = \varphi + y$ with φ and y solutions of the following systems:

$$\begin{cases} \varphi' = A\varphi \\ \varphi(0) = x^0, \end{cases} \tag{46}$$

and

$$\begin{cases} y' = Ay - BB^*x \\ y(0) = 0. \end{cases} \tag{47}$$

Remark that, since A is skew-adjoint, (46) is exactly the adjoint system (9) except for the fact that the initial data are taken at $t = 0$.

As we have seen in the proof of Theorem 1.2, the pair (A, B) being controllable, the following observability inequality holds for system (46):

$$|x^0|^2 \leq C \int_0^T \mid B^*\varphi \mid^2 dt. \tag{48}$$

Since $\varphi = x - y$ we deduce that

$$|x^0|^2 \leq 2C \left[\int_0^T \mid B^*x \mid^2 dt + \int_0^T \mid B^*y \mid^2 dt \right].$$

On the other hand, it is easy to show that the solution y of (47) satisfies:

$$\frac{1}{2}\frac{d}{dt} \mid y \mid^2 = -\langle B^*x,\, B^*y\rangle \leq |B^*x|\,|B^*|\,|y| \leq \frac{1}{2}\left(|y|^2 + |B^*|^2|B^*x|^2\right).$$

From Gronwall's inequality we deduce that

$$\mid y(t) \mid^2 \leq |B^*|^2 \int_0^t e^{t-s} \mid B^*x \mid^2 ds \leq |B^*|^2 e^T \int_0^T \mid B^*x \mid^2 dt \tag{49}$$

and consequently

$$\int_0^T \mid B^*y \mid^2 dt \leq |B|^2 \int_0^T \mid y \mid^2 dt \leq T|B|^4 e^T \int_0^T \mid B^*x \mid^2 dt.$$

Finally, we obtain that

$$\mid x^0 \mid^2 \leq 2C \int_0^T \mid B^*x \mid^2 dt + C|B^*|^4 e^T T \int_0^T \mid B^*x \mid^2 dt \leq C' \int_0^T \mid B^*x \mid^2 dt$$

and the proof of Theorem 1.3 is complete. □

Example: Consider the damped harmonic oscillator:

$$mx'' + Rx + kx' = 0, \tag{50}$$

where m, k and R are positive constants.

Note that (50) may be written in the equivalent form

$$mx'' + Rx = -kx'$$

which indicates that an applied force, proportional to the velocity of the point-mass and of opposite sign, is acting on the oscillator.

It is easy to see that the solutions of this equation have an exponential decay properties. Indeed, it is sufficient to remark that the two characteristic roots have negative real part. Indeed,

$$mr^2 + R + kr = 0 \Leftrightarrow r_\pm = \frac{-k \pm \sqrt{k^2 - 4mR}}{2m}$$

and therefore

$$\operatorname{Re} r_\pm = \begin{cases} -\frac{k}{2m} & \text{if } k^2 \le 4mR \\ -\frac{k}{2m} \pm \sqrt{\frac{k^2}{4m} - \frac{R}{2m}} & \text{if } k^2 \ge 4mR. \end{cases}$$

Let us prove the exponential decay of the solutions of (50) by using Theorem 1.3. Firstly, we write (50) in the form (38). Setting

$$X = \begin{pmatrix} x \\ \sqrt{\frac{m}{R}}x' \end{pmatrix},$$

the conservative equation $mx'' + kx = 0$ corresponds to the system:

$$X' = AX, \quad \text{with } A = \begin{pmatrix} 0 & \sqrt{\frac{R}{m}} \\ -\sqrt{\frac{R}{m}} & 0 \end{pmatrix}.$$

Note that A is a skew-adjoint matrix. On the other hand, if we choose

$$B = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{k} \end{pmatrix}$$

we obtain that

$$BB^* = \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix}$$

and the system

$$X' = AX - BB^*X \tag{51}$$

is equivalent to (50).

Now, it is easy to see that the pair (A, B) is controllable since the rank of $[B, AB]$ is 2.

It follows that the solutions of (50) have the property of exponential decay as the explicit computation of the spectrum indicates. □

If (A, B) is controllable, we have proved the uniform stability property of the system (32), under the hypothesis that A is skew-adjoint. However, this property holds even if A is an arbitrary matrix. More precisely, we have

Theorem 1.4 *If (A, B) is controllable then it is also stabilizable. Moreover, it is possible to prescribe any complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ as the eigenvalues of the closed loop matrix $A + BL$ by an appropriate choice of the feedback matrix L so that the decay rate may be made arbitrarily fast.*

In the statement of the Theorem we use the classical term *closed loop* system to refer to the system in which the control is given in feedback form.

The proof of Theorem 1.4 is obtained by reducing system (32) to the so called *control canonical form* (see [44] and [55]).

2 Interior controllability of the wave equation

In this chapter the problem of interior controllability of the wave equation is studied. The control is assumed to act on a subset of the domain where the solutions are defined. The problem of boundary controllability, which is also important in applications and has attracted a lot of attention, will be considered in the following chapter. In the later case the control acts on the boundary of the domain where the solutions are defined.

2.1 Introduction

Let Ω be a bounded open set of $\mathbb{R}^N$ with boundary of class C^2 and ω be an open nonempty subset of Ω. Given $T > 0$ consider the following non-homogeneous wave equation:

$$\begin{cases} u'' - \Delta u = f 1_\omega & \text{in } (0,T) \times \Omega \\ u = 0 & \text{on } (0,T) \times \partial\Omega \\ u(0,\,\cdot\,) = u^0,\ u'(0,\,\cdot\,) = u^1 & \text{in } \Omega. \end{cases} \tag{52}$$

By $'$ we denote the time derivative.

In (52) $u = u(t,x)$ is the state and $f = f(t,x)$ is the interior control function with support localized in ω. We aim at changing the dynamics of the system by acting on the subset ω of the domain Ω.

It is well known that the wave equation models many physical phenomena such as small vibrations of elastic bodies and the propagation of sound. For instance (52) provides a good approximation for the small amplitude vibrations of an elastic string or a flexible membrane occupying the region Ω at rest. The control f represents then a localized force acting on the vibrating structure.

The importance of the wave equation relies not only in the fact that it models a large class of vibrating phenomena but also because it is the most relevant hyperbolic partial differential equation. As we shall see latter on, the main properties of hyperbolic equations such as time-reversibility and the lack of regularizing effects, have some very important consequences in control problems too.

Therefore it is interesting to study the controllability of the wave equation as one of the fundamental models of continuum mechanics and, at the same time, as one of the most representative equations in the theory of partial differential equations.

2.2 Existence and uniqueness of solutions

The following theorem is a consequence of classical results of existence and uniqueness of solutions of nonhomogeneous evolution equations. All the details may be found, for instance, in [14].

Theorem 2.1 *For any* $f \in L^2((0,T) \times \omega)$ *and* $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$ *equation (52) has a unique weak solution*

$$(u, u') \in C([0,T], H_0^1(\Omega) \times L^2(\Omega))$$

given by the variation of constants formula

$$(u, u')(t) = S(t)(u^0, u^1) + \int_0^t S(t-s)(0, f(s)1_\omega)ds \tag{53}$$

where $(S(t))_{t\in\mathbb{R}}$ *is the group of isometries generated by the wave operator in* $H_0^1(\Omega) \times L^2(\Omega)$.

Moreover, if $f \in W^{1,1}((0,T); L^2(\omega))$ *and* $(u^0, u^1) \in [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)$ *equation (52) has a strong solution*

$$(u, u') \in C^1([0,T], H_0^1(\Omega) \times L^2(\Omega)) \cap C([0,T], [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega))$$

and u *verifies the wave equation (52) in* $L^2(\Omega)$ *for all* $t \geq 0$.

Remark 2.1 *The wave equation is reversible in time. Hence, we may solve it for* $t \in (0,T)$ *by considering initial data* (u^0, u^1) *in* $t = 0$ *or final data* (u_T^0, u_T^1) *in* $t = T$. *In the former case the solution is given by (53) and in the later one by*

$$(u, u')(t) = S(T-t)(u_T^0, u_T^1) + \int_{T-t}^T S(s-T+t)(0, f(s)1_\omega)ds. \tag{54}$$

□

2.3 Controllability problems

Let $T > 0$ and define, for any initial data $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$, the set of reachable states

$$R(T; (u^0, u^1)) = \{(u(T), u_t(T)) : u \text{ solution of (52) with } f \in L^2((0,T) \times \omega)\}.$$

Remark that, for any $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$, $R(T; (u^0, u^1))$ is an affine subspace of $H_0^1(\Omega) \times L^2(\Omega)$.

There are different notions of controllability that need to be distinguished.

Definition 2.1 *System (52) is* **approximately controllable in time** T *if, for every initial data* $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$*, the set of reachable states* $R(T; (u^0, u^1))$ *is dense in* $H_0^1(\Omega) \times L^2(\Omega)$*.*

Definition 2.2 *System (52) is* **exactly controllable in time** T *if, for every initial data* $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$*, the set of reachable states* $R(T; (u^0, u^1))$ *coincides with* $H_0^1(\Omega) \times L^2(\Omega)$*.*

Definition 2.3 *System (52) is* **null controllable in time** T *if, for every initial data* $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$*, the set of reachable states* $R(T; (u^0, u^1))$ *contains the element* $(0, 0)$*.*

Since the only dense and convex subset of $\mathbb{R}^n$ is $\mathbb{R}^n$, it follows that the approximate and exact controllability notions are equivalent in the finite-dimensional case. Nevertheless, for infinite dimensional systems as the wave equation, these two notions do not coincide.

Remark 2.2 *In the notions of approximate and exact controllability it is sufficient to consider the case* $(u^0, u^1) \equiv 0$ *since* $R(T; (u^0, u^1)) = R(T; (0, 0)) + S(T)(u^0, u^1)$*.* $\square$

In the view of the time-reversibility of the system we have:

Proposition 2.1 *System (52) is exactly controllable if and only if it is null controllable.*

Proof: Evidently, exact controllability implies null controllability.

Let us suppose now that $(0, 0) \in R(T; (u^0, u^1))$ for any $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$. Then any initial data in $H_0^1(\Omega) \times L^2(\Omega)$ can be driven to $(0, 0)$ in time T. From the reversibility of the wave equation we deduce that any state in $H_0^1(\Omega) \times L^2(\Omega)$ can be reached in time T by starting from $(0, 0)$. This means that $R(T, (0, 0)) = H_0^1(\Omega) \times L^2(\Omega)$ and the exact controllability property holds from Remark 2.2. $\square$

The previous Proposition guarantees that (52) is exactly controllable if and only if, for any $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$ there exists $f \in L^2((0, T) \times \omega)$ such that the corresponding solution (u, u') of (52) satisfies

$$u(T, \cdot\,) = u_t(T, \cdot\,) = 0. \tag{55}$$

This is the most common form in which the exact controllability property for the wave equation is formulated.

Remark 2.3 *The following facts indicate how the main distinguishing properties of wave equation affect its controllability properties:*

- *Since the wave equation is time-reversible and does not have any regularizing effect, one may not exclude the exact controllability to hold. Nevertheless, as we have said before, there are situations in which the exact controllability property is not verified but the approximate controllability holds. This depends on the geometric properties of Ω and ω.*

- *The wave equation is a prototype of equation with finite speed of propagation. Therefore, one cannot expect the previous controllability properties to hold unless the control time T is sufficiently large.* $\square$

2.4 Variational approach and observability

Let us first deduce a necessary and sufficient condition for the exact controllability property of (52) to hold. By $\langle \cdot , \cdot \rangle_{1,-1}$ we denote the duality product between $H_0^1(\Omega)$ and its dual, $H^{-1}(\Omega)$.

For $(\varphi_T^0, \varphi_T^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, consider the following backward homogeneous equation

$$\begin{cases} \varphi'' - \Delta\varphi = 0 & \text{in } (0,T)\times\Omega \\ \varphi = 0 & \text{on } (0,T)\times\partial\Omega \\ \varphi(T,\cdot) = \varphi_T^0,\ \varphi'(T,\cdot) = \varphi_T^1 & \text{in } \Omega. \end{cases} \tag{56}$$

Let $(\varphi, \varphi') \in C([0,T], L^2(\Omega)\times H^{-1}(\Omega))$ be the unique weak solution of (56).

Lemma 2.1 *The control $f \in L^2((0,T)\times\omega)$ drives the initial data $(u^0,u^1) \in H_0^1(\Omega)\times L^2(\Omega)$ of system (52) to zero in time T if and only if*

$$\int_0^T \int_\omega \varphi f dxdt = \left\langle \varphi'(0), u^0 \right\rangle_{1,-1} - \int_\Omega \varphi(0) u^1 dx, \tag{57}$$

for all $(\varphi_T^0, \varphi_T^1) \in L^2(\Omega)\times H^{-1}(\Omega)$ where φ is the corresponding solution of (56).

Proof: Let us first suppose that $\left(u^0,u^1\right), \left(\varphi_T^0,\varphi_T^1\right) \in \mathcal{D}(\Omega)\times\mathcal{D}(\Omega)$, $f \in \mathcal{D}((0,T)\times\omega)$ and let u and φ be the (regular) solutions of (52) and (56) respectively.

We recall that $\mathcal{D}(M)$ denotes the set of $C^\infty(M)$ functions with compact support in M.

By multiplying the equation of u by φ and by integrating by parts one obtains

$$\int_0^T \int_\omega \varphi f dxdt = \int_0^T \int_\Omega \varphi\left(u'' - \Delta u\right) dxdt =$$

$$= \int_{\Omega} (\varphi u' - \varphi' u)\, dx|_0^T + \int_0^T \int_{\Omega} u\, (\varphi'' - \Delta\varphi)\, dxdt =$$
$$= \int_{\Omega} [\varphi(T)u'(T) - \varphi'(T)u(T)]\, dx - \int_{\Omega} [\varphi(0)u'(0) - \varphi'(0)u(0)]\, dx.$$

Hence,

$$\int_0^T \int_{\omega} \varphi f dxdt = \int_{\Omega} \left[\varphi_T^0 u'(T) - \varphi_T^1 u(T)\right] dx - \int_{\Omega} \left[\varphi(0)u^1 - \varphi'(0)u^0\right] dx. \quad (58)$$

From a density argument we deduce, by passing to the limit in (58), that for any $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and $(\varphi_T^0, \varphi_T^1) \in L^2(\Omega) \times H^{-1}(\Omega)$,

$$\begin{aligned} &\int_0^T \int_{\omega} \varphi f dxdt = \\ &= -\left\langle \varphi_T^1, u(T)\right\rangle_{1,-1} + \int_{\Omega} \varphi_T^0 u'(T) dx + \left\langle \varphi'(0), u^0\right\rangle_{1,-1} - \int_{\Omega} \varphi(0) u^1 dx. \end{aligned} \quad (59)$$

Now, from (59), it follows immediately that (57) holds if and only if (u^0, u^1) is controllable to zero and f is the corresponding control. This completes the proof. □

Let us define the duality product between $L^2(\Omega) \times H^{-1}(\Omega)$ and $H_0^1(\Omega) \times L^2(\Omega)$ by

$$\left\langle \left(\varphi^0, \varphi^1\right), \left(u^0, u^1\right)\right\rangle = \left\langle \varphi^1, u^0\right\rangle_{1,-1} - \int_{\Omega} \varphi^0 u^1 dx$$

for all $\left(\varphi^0, \varphi^1\right) \in L^2(\Omega) \times H^{-1}(\Omega)$ and $\left(u^0, u^1\right) \in H_0^1(\Omega) \times L^2(\Omega)$.

Remark that the map $\left(\varphi^0, \varphi^1\right) \to \left\langle \left(\varphi^0, \varphi^1\right), \left(u^0, u^1\right)\right\rangle$ is linear and continuous and its norm is equal to $||(u^0, u^1)||_{H_0^1 \times L^2}$.

For $(\varphi^0, \varphi^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, consider the following homogeneous equation

$$\begin{cases} \varphi'' - \Delta\varphi = 0 & \text{in } (0, T) \times \Omega \\ \varphi = 0 & \text{on } (0, T) \times \partial\Omega \\ \varphi(0, \cdot) = \varphi^0,\ \varphi'(0, \cdot) = \varphi^1 & \text{in } \Omega. \end{cases} \quad (60)$$

If $(\varphi, \varphi') \in C([0, T], L^2(\Omega) \times H^{-1}(\Omega))$ is the unique weak solution of (60), then

$$||\varphi||^2_{L^\infty(0,T;L^2(\Omega))} + ||\varphi'||^2_{L^\infty(0,T;H^{-1}(\Omega))} \leq ||(\varphi^0, \varphi^1)||^2_{L^2(\Omega)\times H^{-1}(\Omega)}. \quad (61)$$

Since the wave equation with homogeneous Dirichlet boundary conditions generates a group of isometries in $L^2(\Omega) \times H^{-1}(\Omega)$, Lemma 2.1 may be reformulated in the following way:

Lemma 2.2 *The initial data $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$ may be driven to zero in time T if and only if there exists $f \in L^2((0,T) \times \omega)$ such that*

$$\int_0^T \int_\omega \varphi f dx dt = \left\langle \left(\varphi^0, \varphi^1\right), \left(u^0, u^1\right) \right\rangle, \tag{62}$$

for all $(\varphi^0, \varphi^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ where φ is the corresponding solution of (60).

Relation (62) may be seen as an optimality condition for the critical points of the functional $\mathcal{J} : L^2(\Omega) \times H^{-1}(\Omega) \to \mathbb{R}$,

$$\mathcal{J}(\varphi^0, \varphi^1) = \frac{1}{2} \int_0^T \int_\omega |\varphi|^2 dx dt + \left\langle \left(\varphi^0, \varphi^1\right), \left(u^0, u^1\right) \right\rangle, \tag{63}$$

where φ is the solution of (60) with initial data $(\varphi^0, \varphi^1) \in L^2(\Omega) \times H^{-1}(\Omega)$.

We have:

Theorem 2.2 *Let $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and suppose that $(\widehat{\varphi}^0, \widehat{\varphi}^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ is a minimizer of $\mathcal{J}$. If $\widehat{\varphi}$ is the corresponding solution of (60) with initial data $(\widehat{\varphi}^0, \widehat{\varphi}^1)$ then*

$$f = \widehat{\varphi}_{|\omega} \tag{64}$$

is a control which leads (u^0, u^1) to zero in time T.

Proof: Since $\mathcal{J}$ achieves its minimum at $(\widehat{\varphi}^0, \widehat{\varphi}^1)$, the following relation holds

$$0 = \lim_{h \to 0} \frac{1}{h} \left(\mathcal{J}((\widehat{\varphi}^0, \widehat{\varphi}^1) + h(\varphi^0, \varphi^1)) - \mathcal{J}(\widehat{\varphi}^0, \widehat{\varphi}^1) \right) =$$
$$= \int_0^T \int_\omega \widehat{\varphi} \varphi dx dt + \int_\Omega u^1 \varphi^0 dx - < \varphi^1, u^0 >_{1,-1}$$

for any $(\varphi^0, \varphi^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ where φ is the solution of (60).

Lemma 2.2 shows that $f = \widehat{\varphi}_{|\omega}$ is a control which leads the initial data (u^0, u^1) to zero in time T. $\square$

Let us now give sufficient conditions ensuring the existence of a minimizer for $\mathcal{J}$.

Definition 2.4 *Equation (60) is* **observable in time** T *if there exists a positive constant $C_1 > 0$ such that the following inequality is verified*

$$C_1 \parallel \left(\varphi^0, \varphi^1\right) \parallel^2_{L^2(\Omega) \times H^{-1}(\Omega)} \leq \int_0^T \int_\omega |\varphi|^2 dx dt, \tag{65}$$

for any $\left(\varphi^0, \varphi^1\right) \in L^2(\Omega) \times H^{-1}(\Omega)$ where φ is the solution of (60) with initial data (φ^0, φ^1).

Inequality (65) is called **observation or observability inequality**. It shows that the quantity $\int_0^T \int_\omega \mid \varphi \mid^2$ (the observed one) which depends only on the restriction of φ to the subset ω of Ω, uniquely determines the solution on (60).

Remark 2.4 *The continuous dependence (61) of solutions of (60) with respect to its initial data guarantees that there exists a constant $C_2 > 0$ such that*

$$\int_0^T \int_\omega \mid \varphi \mid^2 dxdt \leq C_2 \parallel \left(\varphi^0, \varphi^1\right) \parallel^2_{L^2(\Omega)\times H^{-1}(\Omega)} \tag{66}$$

for all $(\varphi^0, \varphi^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ and φ solution of (60). $\square$

Let us show that (65) is a sufficient condition for the exact controllability property to hold. First of all let us recall the following fundamental result in the Calculus of Variations which is a consequence of the so called Direct Method of the Calculus of Variations.

Theorem 2.3 *Let H be a reflexive Banach space, K a closed convex subset of H and $\varphi : K \to \mathbb{R}$ a function with the following properties:*

1. *φ is convex*
2. *φ is lower semi-continuous*
3. *If K is unbounded then φ is coercive, i. e.*

$$\lim_{||x||\to\infty} \varphi(x) = \infty. \tag{67}$$

Then φ attains its minimum in K, i. e. there exists $x_0 \in K$ such that

$$\varphi(x_0) = \min_{x\in K} \varphi(x). \tag{68}$$

For a proof of Theorem 2.3 see [10].
We have:

Theorem 2.4 *Let $(u^0, u^1) \in H^1_0(\Omega) \times L^2(\Omega)$ and suppose that (60) is observable in time T. Then the functional $\mathcal{J}$ defined by (63) has an unique minimizer $(\widehat{\varphi}^0, \widehat{\varphi}^1) \in L^2(\Omega) \times H^{-1}(\Omega)$.*

Proof: It is easy to see that $\mathcal{J}$ is continuous and convex. Therefore, according to Theorem 2.3, the existence of a minimum is ensured if we prove that J is also coercive i.e.

$$\lim_{||(\varphi^0,\varphi^1)||_{L^2\times H^{-1}}\to\infty} \mathcal{J}(\varphi^0, \varphi^1) = \infty. \tag{69}$$

The coercivity of functional $\mathcal{J}$ follows immediately from (65). Indeed,

$$\mathcal{J}(\varphi^0,\varphi^1) \geq \frac{1}{2}\left(\int_0^T\int_\omega |\varphi|^2 - ||(u^0,u^1)||_{H_0^1(\Omega)\times L^2(\Omega)}||(\varphi^0,\varphi^1)||_{L^2(\Omega)\times H^{-1}(\Omega)}\right)$$

$$\geq \frac{C_1}{2}||(\varphi^0,\varphi^1)||^2_{L^2(\Omega)\times H^{-1}(\Omega)} - \frac{1}{2}||(u^0,u^1)||_{H_0^1(\Omega)\times L^2(\Omega)}||(\varphi^0,\varphi^1)||_{L^2(\Omega)\times H^{-1}(\Omega)}.$$

It follows from Theorem 2.3 that $\mathcal{J}$ has a minimizer $(\widehat{\varphi}^0,\widehat{\varphi}^1) \in L^2(\Omega)\times H^{-1}(\Omega)$.

To prove the uniqueness of the minimizer it is sufficient to show that $\mathcal{J}$ is strictly convex. Indeed, let $(\varphi^0,\varphi^1),\ (\psi^0,\psi^1) \in L^2(\Omega)\times H^{-1}(\Omega)$ and $\lambda \in (0,1)$. We have that

$$\mathcal{J}(\lambda(\varphi^0,\varphi^1) + (1-\lambda)(\psi^0,\psi^1)) =$$

$$= \lambda\mathcal{J}(\varphi^0,\varphi^1) + (1-\lambda)\mathcal{J}(\psi^0,\psi^1) - \frac{\lambda(1-\lambda)}{2}\int_0^T\int_\omega |\varphi-\psi|^2 dxdt.$$

From (65) it follows that

$$\int_0^T\int_\omega |\varphi-\psi|^2 dxdt \geq C_1||(\varphi^0,\varphi^1)-(\psi^0,\psi^1)||_{L^2(\Omega)\times H^{-1}(\Omega)}.$$

Consequently, for any $(\varphi^0,\varphi^1) \neq (\psi^0,\psi^1)$,

$$\mathcal{J}(\lambda(\varphi^0,\varphi^1) + (1-\lambda)(\psi^0,\psi^1)) < \lambda\mathcal{J}(\varphi^0,\varphi^1) + (1-\lambda)\mathcal{J}(\psi^0,\psi^1)$$

and $\mathcal{J}$ is strictly convex. $\square$

Theorems 2.2 and 2.4 guarantee that, under hypothesis (65), system (52) is exactly controllable. Moreover, a control may be obtained as in (64) from the solution of the homogeneous equation (60) with the initial data minimizing the functional $\mathcal{J}$. Hence, the controllability problem is reduced to a minimization problem that may be solved by the Direct Method of the Calculus of Variations. This is very useful both from a theoretical and a numerical point of view.

The following proposition shows that the control obtained by this variational method is of minimal $L^2((0,T)\times\omega)$-norm.

Proposition 2.2 *Let $f = \widehat{\varphi}_{|\omega}$ be the control given by minimizing the functional $\mathcal{J}$. If $g \in L^2((0,T)\times\omega)$ is any other control driving to zero the initial data (u^0,u^1) then*

$$||f||_{L^2((0,T)\times\omega)} \leq ||g||_{L^2((0,T)\times\omega)}. \tag{70}$$

Proof: Let $(\widehat{\varphi}^0, \widehat{\varphi}^1)$ the minimizer for the functional $\mathcal{J}$. Consider now relation (62) for the control $f = \widehat{\varphi}_{|\omega}$. By taking $(\widehat{\varphi}^0, \widehat{\varphi}^1)$ as test function we obtain that

$$||f||^2_{L^2((0,T)\times\omega)} = \int_0^T \int_\omega |\widehat{\varphi}|^2 dxdt = \left\langle \widehat{\varphi}^1, u^0 \right\rangle_{1,-1} - \int_\Omega \widehat{\varphi}^0 u^1 dx.$$

On the other hand, relation (62) for the control g and test function $(\widehat{\varphi}^0, \widehat{\varphi}^1)$ gives

$$\int_0^T \int_\omega g\widehat{\varphi} dxdt = \left\langle \widehat{\varphi}^1, u^0 \right\rangle_{1,-1} - \int_\Omega \widehat{\varphi}^0 u^1 dx.$$

We obtain that

$$||f||^2_{L^2((0,T)\times\omega)} = \left\langle \widehat{\varphi}^1, u^0 \right\rangle_{1,-1} - \int_\Omega \widehat{\varphi}^0 u^1 dx = \int_0^T \int_\omega g\widehat{\varphi} dxdt \leq$$

$$\leq ||g||_{L^2((0,T)\times\omega)} ||\widehat{\varphi}||_{L^2((0,T)\times\omega)} = ||g||_{L^2((0,T)\times\omega)} ||f||_{L^2((0,T)\times\omega)}$$

and (70) is proved. $\square$

2.5 Approximate controllability

Up to this point we have discussed only the exact controllability property of (52) which turns out to be equivalent to the observability property (65). Let us now address the approximate controllability one.

Let $\varepsilon > 0$ and $(u^0, u^1), (z^0, z^1) \in H_0^1(\Omega) \times L^2(\Omega)$. We are looking for a control function $f \in L^2((0,T)\times\omega)$ such that the corresponding solution (u, u') of (52) satisfies

$$||(u(T), u'(T)) - (z^0, z^1)||_{H_0^1(\Omega)\times L^2(\Omega)} \leq \varepsilon. \tag{71}$$

Recall that (52) is approximately controllable if, for any $\varepsilon > 0$ and (u^0, u^1), $(z^0, z^1) \in H_0^1(\Omega) \times L^2(\Omega)$, there exists $f \in L^2((0,T)\times\omega)$ such that (71) is verified.

By Remark 2.2, it is sufficient to study the case $(u^0, u^1) = (0, 0)$. From now on we assume that $(u^0, u^1) = (0, 0)$.

The variational approach considered in the previous sections may be also very useful for the study of the approximate controllability property. To see this, define the functional $\mathcal{J}_\varepsilon : L^2(\Omega) \times H^{-1}(\Omega) \to \mathbb{R}$,

$$\begin{gathered}\mathcal{J}_\varepsilon(\varphi^0, \varphi^1) = \\ = \frac{1}{2}\int_0^T \int_\omega |\varphi|^2 dxdt + \left\langle \left(\varphi^0, \varphi^1\right), \left(z^0, z^1\right)\right\rangle + \varepsilon||(\varphi^0, \varphi^1)||_{L^2\times H^{-1}},\end{gathered} \tag{72}$$

where φ is the solution of (60) with initial data $(\varphi^0, \varphi^1) \in L^2(\Omega) \times H^{-1}(\Omega)$.

As in the exact controllability case, the existence of a minimum of the functional $\mathcal{J}_\varepsilon$ implies the existence of an approximate control.

Theorem 2.5 *Let $\varepsilon > 0$ and $(z^0, z^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and suppose that $(\widehat{\varphi}^0, \widehat{\varphi}^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ is a minimizer of $\mathcal{J}_\varepsilon$. If $\widehat{\varphi}$ is the corresponding solution of (60) with initial data $(\widehat{\varphi}^0, \widehat{\varphi}^1)$ then*

$$f = \widehat{\varphi}_{|\omega} \tag{73}$$

is an approximate control which leads the solution of (52) from the zero initial data $(u^0, u^1) = (0,0)$ to the state $(u(T), u'(T))$ such that (71) is verified.

Proof: Let $(\widehat{\varphi}^0, \widehat{\varphi}^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ be a minimizer of $\mathcal{J}_\varepsilon$. It follows that, for any $h > 0$ and $(\varphi^0, \varphi^1) \in L^2(\Omega) \times H^{-1}(\Omega)$,

$$0 \leq \frac{1}{h}\left(\mathcal{J}_\varepsilon((\widehat{\varphi}^0, \widehat{\varphi}^1) + h(\varphi^0, \varphi^1)) - \mathcal{J}_\varepsilon(\widehat{\varphi}^0, \widehat{\varphi}^1)\right) \leq$$

$$\leq \int_0^T \int_\omega \widehat{\varphi}\varphi dxdt + \frac{h}{2}\int_0^T \int_\omega |\varphi|^2 dxdt + \left\langle \left(\varphi^0, \varphi^1\right), \left(z^0, z^1\right)\right\rangle + \varepsilon||(\varphi^0, \varphi^1)||_{L^2 \times H^{-1}}$$

being φ the solution of (60). By making $h \to 0$ we obtain that

$$-\varepsilon||(\varphi^0, \varphi^1)||_{L^2 \times H^{-1}} \leq \int_0^T \int_\omega \widehat{\varphi}\varphi dxdt + \left\langle \left(\varphi^0, \varphi^1\right), \left(z^0, z^1\right)\right\rangle.$$

A similar argument (with $h < 0$) leads to

$$\int_0^T \int_\omega \widehat{\varphi}\varphi dxdt + \left\langle \left(\varphi^0, \varphi^1\right), \left(z^0, z^1\right)\right\rangle \leq \varepsilon||(\varphi^0, \varphi^1)||_{L^2 \times H^{-1}}.$$

Hence, for any $(\varphi^0, \varphi^1) \in L^2(\Omega) \times H^{-1}(\Omega)$,

$$\left| \int_0^T \int_\omega \widehat{\varphi}\varphi dxdt + \left\langle \left(\varphi^0, \varphi^1\right), \left(z^0, z^1\right)\right\rangle \right| \leq \varepsilon||(\varphi^0, \varphi^1)||_{L^2 \times H^{-1}}. \tag{74}$$

Now, from (59) and (74) we obtain that

$$\left|\left\langle \left(\varphi^0, \varphi^1\right), \left[(z^0, z^1) - (u(T), u'(T))\right]\right\rangle\right| \leq \varepsilon||(\varphi^0, \varphi^1)||_{L^2 \times H^{-1}},$$

for any $(\varphi^0, \varphi^1) \in L^2(\Omega) \times H^{-1}(\Omega)$.

Consequently, (71) is verified and the proof finishes. □

As we have seen in the previous section, the exact controllability property of (52) is equivalent to the observation property (65) of system (60). An unique continuation principle of the solutions of (60), which is a weaker version of the observability inequality (65), will play a similar role for the approximate controllability property and it will give a sufficient condition for the existence of a minimizer of $\mathcal{J}_\varepsilon$. More precisely, we have

Theorem 2.6 *The following properties are equivalent:*

1. *Equation (52) is approximately controllable.*
2. *The following unique continuation principle holds for the solutions of (60)*

$$\varphi_{|_{(0,T)\times\omega}} = 0 \Rightarrow (\varphi^0, \varphi^1) = (0,0). \tag{75}$$

Proof: Let us first suppose that (52) is approximately controllable and let φ be a solution of (60) with initial data $(\varphi^0, \varphi^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ such that $\varphi_{|_{(0,T)\times\omega}} = 0$.

For any $\varepsilon > 0$ and $(z^0, z^1) \in H_0^1(\Omega) \times L^2(\Omega)$ there exists an approximate control function $f \in L^2((0,T)\times\omega)$ such that (71) is verified.

From (59) we deduce that $\left\langle (u(T), u'(T)), (\varphi^0, \varphi^1) \right\rangle = 0$. From the controllability property and the last relation we deduce that

$$\left|\left\langle (z^0, z^1), (\varphi^0, \varphi^1) \right\rangle\right| = \left|\left\langle [(z^0, z^1) - (u(T), u'(T))], (\varphi^0, \varphi^1) \right\rangle\right| \le \varepsilon ||(\varphi^0, \varphi^1)||.$$

Since the last inequality is verified by any $(z^0, z^1) \in H_0^1(\Omega) \times L^2(\Omega)$ it follows that $(\varphi^0, \varphi^1) = (0,0)$.

Hence the unique continuation principle (75) holds.

Reciprocally, suppose now that the unique continuation principle (75) is verified and let us show that (52) is approximately controllable.

In order to do that we use Theorem 2.5. Let $\varepsilon > 0$ and $(z^0, z^1) \in H_0^1(\Omega) \times L^2(\Omega)$ be given and consider the functional $\mathcal{J}_\varepsilon$. Theorem 2.5 ensures the approximate controllability property of (52) under the assumption that $\mathcal{J}_\varepsilon$ has a minimum. Let us show that this is true in our case.

The functional $\mathcal{J}_\varepsilon$ is convex and continuous in $L^2(\Omega) \times H^{-1}(\Omega)$. Thus, the existence of a minimum is ensured if $\mathcal{J}_\varepsilon$ is coercive, i. e.

$$\mathcal{J}_\varepsilon((\varphi^0, \varphi^1)) \to \infty \text{ when } ||(\varphi^0, \varphi^1)||_{L^2\times H^{-1}} \to \infty. \tag{76}$$

In fact we shall prove that

$$\liminf_{||(\varphi^0,\varphi^1)||_{L^2\times H^{-1}}\to\infty} \mathcal{J}_\varepsilon(\varphi^0, \varphi^1)/||(\varphi^0, \varphi^1)||_{L^2\times H^{-1}} \ge \varepsilon. \tag{77}$$

Evidently, (77) implies (76) and the proof of the theorem is complete.

In order to prove (77) let $\left((\varphi_j^0, \varphi_j^1)\right)_{j\ge 1} \subset L^2(\Omega) \times H^{-1}(\Omega)$ be a sequence of initial data for the adjoint system such that $\| (\varphi_j^0, \varphi_j^1) \|_{L^2\times H^{-1}} \to \infty$. We normalize them

$$(\widetilde{\varphi}_j^0, \widetilde{\varphi}_j^1) = (\varphi_j^0, \varphi_j^1)/ \| (\varphi_j^0, \varphi_j^1) \|_{L^2\times H^{-1}},$$

so that $\| (\widetilde{\varphi}_j^0, \widetilde{\varphi}_j^1) \|_{L^2\times H^{-1}} = 1$.

On the other hand, let $(\widetilde{\varphi}_j, \widetilde{\varphi}'_j)$ be the solution of (60) with initial data $(\widetilde{\varphi}^0_j, \widetilde{\varphi}^1_j)$. Then

$$\frac{\mathcal{J}_\varepsilon((\varphi^0_j, \varphi^1_j))}{\| (\varphi^0_j, \varphi^1_j) \|} = \frac{1}{2} \| (\varphi^0_j, \varphi^1_j) \| \int_0^T \int_\omega | \widetilde{\varphi}_j |^2 \, dxdt + \left\langle (z^0, z^1), (\widetilde{\varphi}^0, \widetilde{\varphi}^1) \right\rangle + \varepsilon.$$

The following two cases may occur:

1) $\liminf_{j\to\infty} \int_0^T \int_\omega | \widetilde{\varphi}_j |^2 > 0$. In this case we obtain immediately that

$$\frac{\mathcal{J}_\varepsilon((\varphi^0_j, \varphi^1_j))}{\| (\varphi^0_j, \varphi^1_j) \|} \to \infty.$$

2) $\liminf_{j\to\infty} \int_0^T \int_\omega | \widetilde{\varphi}_j |^2 = 0$. In this case since $\left(\widetilde{\varphi}^0_j, \widetilde{\varphi}^1_j\right)_{j\geq 1}$ is bounded in $L^2 \times H^{-1}$, by extracting a subsequence we can guarantee that $(\widetilde{\varphi}^0_j, \widetilde{\varphi}^1_j)_{j\geq 1}$ converges weakly to (ψ^0, ψ^1) in $L^2(\Omega) \times H^{-1}(\Omega)$.

Moreover, if (ψ, ψ') is the solution of (60) with the initial data (ψ^0, ψ^1) at $t = T$, then $(\widetilde{\varphi}_j, \widetilde{\varphi}'_j)_{j\geq 1}$ converges weakly to (ψ, ψ') in $L^2(0, T; L^2(\Omega) \times H^{-1}(\Omega)) \cap H^1(0, T; H^{-1}(\Omega) \times [H^2 \cap H^1_0(\Omega)]')$.

By lower semi-continuity,

$$\int_0^T \int_\omega \psi^2 dxdt \leq \liminf_{j\to\infty} \int_0^T \int_\omega | \widetilde{\varphi}_j |^2 \, dxdt = 0$$

and therefore $\psi = 0$ en $\omega \times (0, T)$.

From the unique continuation principle we obtain that $(\psi^0, \psi^1) = (0, 0)$ and consequently,

$$(\widetilde{\varphi}^0_j, \widetilde{\varphi}^1_j) \rightharpoonup (0, 0) \text{ weakly in } L^2(\Omega) \times H^{-1}(\Omega).$$

Hence

$$\liminf_{j\to\infty} \frac{\mathcal{J}_\varepsilon((\varphi^0_j, \varphi^1_j))}{\| (\varphi^0_j, \varphi^1_j) \|_{L^2\times H^{-1}}} \geq \liminf_{j\to\infty} [\varepsilon + \left\langle (z^0, z^1), (\widetilde{\varphi}^0, \widetilde{\varphi}^1) \right\rangle] = \varepsilon,$$

and (77) follows. □

When approximate controllability holds, then the following (apparently stronger) statement also holds:

Theorem 2.7 *Let E be a finite-dimensional subspace of $H_0^1(\Omega)\times L^2(\Omega)$ and let us denote by π_E the corresponding orthogonal projection. Then, if approximate controllability holds, for any $\left(u^0,u^1\right)$, $\left(z^0,z^1\right)\in H_0^1(\Omega)\times L^2(\Omega)$ and $\varepsilon>0$ there exists $f\in L^2((0,T)\times\omega)$ such that the solution of (52) satisfies*

$$\left\|\left(u(T)-z^0,u_t(T)-z^1\right)\right\|_{H_0^1(\Omega)\times L^2(\Omega)}\leq\varepsilon;\ \pi_E\left(u(T),u_t(T)\right)=\pi_E\left(z^0,z^1\right).$$

This property will be referred to as the **finite-approximate controllability property.** Its proof may be found in [71].

2.6 Comments

In this section we have presented some facts related with the exact and approximate controllability properties. The variational methods we have used allow to reduce these properties to an observation inequality and a unique continuation principle for the homogeneous adjoint equation respectively. The latter will be studied for some particular cases in Chapter 4 by using nonharmonic Fourier analysis.

3 Boundary controllability of the wave equation

This chapter is devoted to study the boundary controllability problem for the wave equation. The control is assumed to act on a subset of the boundary of the domain where the solutions are defined.

3.1 Introduction

Let Ω be a bounded open set of $\mathbb{R}^N$ with boundary Γ of class C^2 and Γ_0 be an open nonempty subset of Γ. Given $T>0$ consider the following nonhomogeneous wave equation:

$$\left\{\begin{array}{ll} u''-\Delta u=0 & \text{in } (0,T)\times\Omega\\ u=f1_{\Gamma_0}(x) & \text{on } (0,T)\times\Gamma\\ u(0,\,\cdot\,)=u^0,\ u'(0,\,\cdot\,)=u^1 & \text{in } \Omega.\end{array}\right. \tag{78}$$

In (78) $u=u(t,x)$ is the state and $f=f(t,x)$ is a control function which acts on Γ_0. We aim at changing the dynamics of the system by acting on Γ_0.

3.2 Existence and uniqueness of solutions

The following theorem is a consequence of the classical results of existence and uniqueness of solutions of nonhomogeneous evolution equations. Full details may be found in [46] and [68].

Theorem 3.1 *For any* $f \in L^2((0,T)\times\Gamma_0)$ *and* $(u^0,u^1) \in L^2(\Omega)\times H^{-1}(\Omega)$ *equation (78) has a unique weak solution defined by transposition*

$$(u,u') \in C([0,T], L^2(\Omega)\times H^{-1}(\Omega)).$$

Moreover, the map $\{u^0,u^1,f\} \to \{u,u'\}$ *is linear and there exists* $C = C(T) > 0$ *such that*

$$\begin{aligned} &||(u,u')||_{L^\infty(0,T;L^2(\Omega)\times H^{-1}(\Omega))} \leq \\ &\leq C\left(||(u^0,u^1)||_{L^2(\Omega)\times H^{-1}(\Omega)} + ||f||_{L^2((0,T)\times\Gamma_0)}\right). \end{aligned} \tag{79}$$

Remark 3.1 *The wave equation is reversible in time. Hence, we may solve (78) for* $t \in (0,T)$ *by considering final data at* $t = T$ *instead of initial data at* $t = 0$. $\square$

3.3 Controllability problems

Let $T > 0$ and define, for any initial data $(u^0,u^1) \in L^2(\Omega)\times H^{-1}(\Omega)$, the set of reachable states

$$R(T;(u^0,u^1)) = \{(u(T),u'(T)) : u \text{ solution of (78) with } f \in L^2((0,T)\times\Gamma_0)\}.$$

Remark that, for any $(u^0,u^1) \in L^2(\Omega)\times H^{-1}(\Omega)$, $R(T;(u^0,u^1))$ is a convex subset of $L^2(\Omega)\times H^{-1}(\Omega)$.

As in the previous chapter, several controllability problems may be addressed.

Definition 3.1 *System (78) is* **approximately controllable in time** T *if, for every initial data* $(u^0,u^1) \in L^2(\Omega)\times H^{-1}(\Omega)$*, the set of reachable states* $R(T;(u^0,u^1))$ *is dense in* $L^2(\Omega)\times H^{-1}(\Omega)$.

Definition 3.2 *System (78) is* **exactly controllable in time** T *if, for every initial data* $(u^0,u^1) \in L^2(\Omega)\times H^{-1}(\Omega)$*, the set of reachable states* $R(T;(u^0,u^1))$ *coincides with* $L^2(\Omega)\times H^{-1}(\Omega)$.

Definition 3.3 *System (78) is* **null controllable in time** T *if, for every initial data* $(u^0,u^1) \in L^2(\Omega)\times H^{-1}(\Omega)$*, the set of reachable states* $R(T;(u^0,u^1))$ *contains the element* $(0,0)$.

Remark 3.2 *In the definitions of approximate and exact controllability it is sufficient to consider the case* $(u^0,u^1) \equiv 0$ *since*

$$R(T;(u^0,u^1)) = R(T;(0,0)) + S(T)(u^0,u^1),$$

where $(S(t))_{t\in\mathbb{R}}$ *is the group of isometries generated by the wave equation in* $L^2(\Omega)\times H^{-1}(\Omega)$ *with homogeneous Dirichlet boundary conditions.* $\square$

Moreover, in view of the reversibility of the system we have

Proposition 3.1 *System (78) is exactly controllable if and only if it is null controllable.*

Proof: Evidently, exact controllability implies null controllability.

Let us suppose now that $(0,0) \in R(T;(u^0,u^1))$ for any $(u^0,u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$. It follows that any initial data in $L^2(\Omega) \times H^{-1}(\Omega)$ can be driven to $(0,0)$ in time T. From the reversibility of the wave equation we deduce that any state in $L^2(\Omega) \times H^{-1}(\Omega)$ can be reached in time T by starting from $(0,0)$. This means that $R(T,(0,0)) = L^2(\Omega) \times H^{-1}(\Omega)$ and the exact controllability property holds as a consequence of Remark 3.2. □

The previous Proposition guarantees that (78) is exactly controllable if and only if, for any $(u^0,u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ there exists $f \in L^2((0,T) \times \Gamma_0)$ such that the corresponding solution (u,u') of (78) satisfies

$$u(T,\,\cdot\,) = u'(T,\,\cdot\,) = 0. \tag{80}$$

Remark 3.3 *The following facts indicate the close connections between the controllability properties and some of the main features of hyperbolic equations:*

- *Since the wave equation is time-reversible and does not have any regularizing effect, the exact controllability property is very likely to hold. Nevertheless, as we have said before, the exact controllability property fails and the approximate controllability one holds in some situations. This is very closely related to the geometric properties of the subset Γ_0 of the boundary Γ where the control is applied.*
- *The wave equation is the prototype of partial differential equation with finite speed of propagation. Therefore, one cannot expect the previous controllability properties to hold unless the control time T is sufficiently large.* □

3.4 Variational approach

Let us first deduce a necessary and sufficient condition for the exact controllability of (78). As in the previous chapter, by $\langle\,\cdot\,,\,\cdot\,\rangle_{1,-1}$ we shall denote the duality product between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$.

For any $(\varphi_T^0,\varphi_T^1) \in H_0^1(\Omega) \times L^2(\Omega)$ let (φ,φ') be the solution of the following backward wave equation

$$\begin{cases} \varphi'' - \Delta\varphi = 0 & \text{in } (0,T)\times\Omega \\ \varphi\,|_{\partial\Omega} = 0 & \text{on } (0,T)\times\partial\Omega \\ \varphi(T,\,\cdot\,) = \varphi_T^0,\ \varphi'(T,\,\cdot\,) = \varphi_T^1. & \text{in } \Omega. \end{cases} \tag{81}$$

Lemma 3.1 *The initial data* $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ *is controllable to zero if and only if there exists* $f \in L^2((0,T) \times \Gamma_0)$ *such that*

$$\int_0^T \int_{\Gamma_0} \frac{\partial \varphi}{\partial n} f d\sigma dt + \int_\Omega u^0 \varphi'(0) dx - \left\langle u^1, \varphi(0) \right\rangle_{1,-1} = 0 \tag{82}$$

for all $(\varphi_T^0, \varphi_T^1) \in H_0^1(\Omega) \times L^2(\Omega)$ *and where* (φ, φ') *is the solution of the backward wave equation (81)*

Proof: Let us first suppose that $\left(u^0, u^1\right), \left(\varphi_T^0, \varphi_T^1\right) \in \mathcal{D}(\Omega) \times \mathcal{D}(\Omega)$, $f \in \mathcal{D}((0,T) \times \Gamma_0)$ and let u and φ be the (regular) solutions of (78) and (81) respectively.

Multiplying the equation of u by φ and integrating by parts one obtains

$$\begin{aligned} 0 = \int_0^T \int_\Omega \varphi \left(u'' - \Delta u\right) dx dt = \int_\Omega \left(\varphi u' - \varphi' u\right) dx\Big|_0^T + \\ + \int_0^T \int_\Gamma \left(-\frac{\partial u}{\partial n} \varphi + \frac{\partial \varphi}{\partial n} u \right) d\sigma dt = \int_0^T \int_{\Gamma_0} \frac{\partial \varphi}{\partial n} u d\sigma dt + \\ + \int_\Omega \left[\varphi(T) u'(T) - \varphi'(T) u(T)\right] dx - \int_\Omega \left[\varphi(0) u'(0) - \varphi'(0) u(0)\right] dx \end{aligned}$$

Hence,

$$\int_0^T \int_{\Gamma_0} \frac{\partial \varphi}{\partial n} u d\sigma dt + \int_\Omega \left[\varphi_T^0 u'(T) - \varphi_T^1 u(T)\right] dx - \int_\Omega \left[\varphi(0) u^1 - \varphi'(0) u^0\right] dx = 0.$$

By a density argument we deduce that for any $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ and $(\varphi_T^0, \varphi_T^1) \in H_0^1(\Omega) \times L^2(\Omega)$,

$$\begin{aligned} \int_0^T \int_{\Gamma_0} \frac{\partial \varphi}{\partial n} u d\sigma dt = \\ = \int_\Omega u(T) \varphi_T^1 dx + \left\langle u'(T), \varphi_T^0 \right\rangle_{1,-1} + \int_\Omega u^0 \varphi'(0) dx - \left\langle u^1, \varphi(0) \right\rangle_{1,-1}. \end{aligned} \tag{83}$$

Now, from (83), it follows immediately that (82) holds if and only if (u^0, u^1) is controllable to zero. The proof finishes. □

As in the previous chapter we introduce the duality product

$$\left\langle \left(\varphi^0, \varphi^1\right), \left(u^0, u^1\right) \right\rangle = \int_\Omega u^0 \varphi^1 dx - \left\langle u^1, \varphi^1 \right\rangle_{1,-1}$$

for all $\left(\varphi^0, \varphi^1\right) \in H_0^1(\Omega) \times L^2(\Omega)$ and $\left(u^0, u^1\right) \in L^2(\Omega) \times H^{-1}(\Omega)$.

For any $(\varphi^0, \varphi^1) \in H_0^1(\Omega) \times L^2(\Omega)$ let (φ, φ') be the finite energy solution of the following wave equation

$$\begin{cases} \varphi'' - \Delta\varphi = 0 & \text{in } (0,T) \times \Omega \\ \varphi \mid_{\partial\Omega} = 0 & \text{in } (0,T) \times \partial\Omega \\ \varphi(0,\,\cdot\,) = \varphi^0,\ \varphi'(0,\,\cdot\,) = \varphi^1. & \text{in } \Omega. \end{cases} \tag{84}$$

Since the wave equation generates a group of isometries in $H_0^1(\Omega) \times L^2(\Omega)$, Lemma 3.1 may be reformulated in the following way:

Lemma 3.2 *The initial data $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ is controllable to zero if and only if there exists $f \in L^2((0,T) \times \Gamma_0)$ such that*

$$\int_0^T \int_{\Gamma_0} \frac{\partial\varphi}{\partial n} f d\sigma dt + \left\langle \left(\varphi^0, \varphi^1\right), \left(u^0, u^1\right) \right\rangle = 0, \tag{85}$$

for all $(\varphi^0, \varphi^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and where φ is the solution of (84).

Once again, (85) may be seen as an optimality condition for the critical points of the functional $\mathcal{J} : H_0^1(\Omega) \times L^2(\Omega) \longrightarrow \mathbb{R}$, defined by

$$\mathcal{J}(\varphi^0, \varphi^1) = \frac{1}{2} \int_0^T \int_{\Gamma_0} \left| \frac{\partial\varphi}{\partial n} \right|^2 d\sigma dt + \left\langle \left(\varphi^0, \varphi^1\right), \left(u^0, u^1\right) \right\rangle, \tag{86}$$

where φ is the solution of (84) with initial data $(\varphi^0, \varphi^1) \in H_0^1(\Omega) \times L^2(\Omega)$.

We have

Theorem 3.2 *Let $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ and suppose that $(\widehat{\varphi}^0, \widehat{\varphi}^1) \in H_0^1(\Omega) \times L^2(\Omega)$ is a minimizer of $\mathcal{J}$. If $\widehat{\varphi}$ is the corresponding solution of (84) with initial data $(\widehat{\varphi}^0, \widehat{\varphi}^1)$ then $f = \frac{\partial\widehat{\varphi}}{\partial n}_{|\Gamma_0}$ is a control which leads (u^0, u^1) to zero in time T.*

Proof: Since, by assumption, $\mathcal{J}$ achieves its minimum at $(\widehat{\varphi}^0, \widehat{\varphi}^1)$, the following relation holds

$$0 = \lim_{h \to 0} \frac{1}{h} \left(\mathcal{J}((\widehat{\varphi}^0, \widehat{\varphi}^1) + h(\varphi^0, \varphi^1)) - \mathcal{J}(\widehat{\varphi}^0, \widehat{\varphi}^1) \right) =$$

$$= \int_0^T \int_{\Gamma_0} \frac{\partial\widehat{\varphi}}{\partial n} \frac{\partial\varphi}{\partial n} d\sigma dt + \int_\Omega u^0 \varphi^1 dx - < u^1, \varphi^0 >_{1,-1}$$

for any $(\varphi^0, \varphi^1) \in H_0^1(\Omega) \times L^2(\Omega)$ where φ is the solution of (84).

From Lemma 3.2 it follows that $f = \frac{\partial\widehat{\varphi}}{\partial n}_{|\Gamma_0}$ is a control which leads the initial data (u^0, u^1) to zero in time T. $\square$

Let us now give a general condition which ensures the existence of a minimizer for $\mathcal{J}$.

Definition 3.4 *Equation (84) is* **observable in time** T *if there exists a positive constant* $C_1 > 0$ *such that the following inequality is verified*

$$C_1 \parallel \left(\varphi^0, \varphi^1\right) \parallel^2_{H_0^1(\Omega)\times L^2(\Omega)} \leq \int_0^T \int_{\Gamma_0} \left|\frac{\partial \varphi}{\partial n}\right|^2 d\sigma dt, \tag{87}$$

for any $\left(\varphi^0, \varphi^1\right) \in H_0^1(\Omega) \times L^2(\Omega)$ *where* φ *is the solution of (84) with initial data* (φ^0, φ^1).

Inequality (87) is called **observation or observability inequality**. According to it, when it holds, the quantity $\int_0^T \int_{\Gamma_0} \left|\frac{\partial \varphi}{\partial n}\right|^2 d\sigma dt$ (the observed quantity) which depends only on the trace of $\frac{\partial \varphi}{\partial n}$ on $(0,T)\times\Gamma_0$, uniquely determines the solution of (84).

Remark 3.4 *One may show that there exists a constant* $C_2 > 0$ *such that*

$$\int_0^T \int_{\Gamma_0} \left|\frac{\partial \varphi}{\partial n}\right|^2 d\sigma dt \leq C_2 \parallel \left(\varphi^0, \varphi^1\right) \parallel^2_{H_0^1(\Omega)\times L^2(\Omega)} \tag{88}$$

for all $(\varphi^0, \varphi^1) \in H_0^1(\Omega) \times L^2(\Omega)$ *and* φ *solution of (84).*

Inequality (88) may be obtained by multiplier techniques (see [41] or [45]). Remark that, (88) says that $\frac{\partial \varphi}{\partial n}_{|\Gamma_0} \in L^2((0,T)\times\Gamma_0)$ *which is a "hidden" regularity result, that may not be obtained by classical trace results.* □

Let us show that (87) is a sufficient condition for the exact controllability property to hold.

Theorem 3.3 *Suppose that (84) is observable in time* T *and let* $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$. *The functional* $\mathcal{J}$ *defined by (86) has an unique minimizer* $(\widehat{\varphi}^0, \widehat{\varphi}^1) \in H_0^1(\Omega) \times L^2(\Omega)$.

Proof: It is easy to see that $\mathcal{J}$ is continuous and convex. The existence of a minimum is ensured if we prove that J is also coercive i.e.

$$\lim_{||(\varphi^0,\varphi^1)||_{H_0^1\times L^2}\to\infty} \mathcal{J}(\varphi^0, \varphi^1) = \infty. \tag{89}$$

The coercivity of the functional $\mathcal{J}$ follows immediately from (87). Indeed,

$$\mathcal{J}(\varphi^0, \varphi^1) \geq$$

$$\geq \frac{1}{2}\left(\int_0^T \int_{\Gamma_0} \left|\frac{\partial \varphi}{\partial n}\right|^2 - ||(u^0, u^1)||_{H_0^1(\Omega)\times L^2(\Omega)} ||(\varphi^0, \varphi^1)||_{L^2(\Omega)\times H^{-1}(\Omega)}\right) \geq$$

$$\geq \frac{C_1}{2}||(\varphi^0,\varphi^1)||^2_{L^2(\Omega)\times H^{-1}(\Omega)} - \frac{1}{2}||(u^0,u^1)||_{H_0^1(\Omega)\times L^2(\Omega)}||(\varphi^0,\varphi^1)||_{L^2(\Omega)\times H^{-1}(\Omega)}.$$

It follows from Theorem 2.3 that $\mathcal{J}$ has a minimizer $(\widehat{\varphi}^0,\widehat{\varphi}^1) \in H_0^1(\Omega) \times L^2(\Omega)$.

As in the proof of Theorem 2.4, it may be shown that $\mathcal{J}$ is strictly convex and therefore it achieves its minimum at a unique point. □

Theorems 3.2 and 3.3 guarantee that, under the hypothesis (87), system (78) is exactly controllable. Moreover, a control may be obtained from the solution of the homogeneous system (81) with the initial data minimizing the functional $\mathcal{J}$. Hence, the controllability is reduced to a minimization problem. This is very useful both from the theoretical and numerical point of view.

As in Proposition 2.2 the control obtained by minimizing the functional $\mathcal{J}$ has minimal L^2-norm:

Proposition 3.2 *Let $f = \frac{\partial\widehat{\varphi}}{\partial n}_{|\Gamma_0}$ be the control given by minimizing the functional $\mathcal{J}$. If $g \in L^2((0,T)\times\Gamma_0)$ is any other control driving to zero the initial data (u^0,u^1) in time T, then*

$$||f||_{L^2((0,T)\times\Gamma_0)} \leq ||g||_{L^2((0,T)\times\Gamma_0)}. \tag{90}$$

Proof: It is similar to the proof of Property 2.2. We omit the details. □

3.5 Approximate controllability

Let us now briefly present and discuss the approximate controllability property. Since many aspects are similar to the interior controllability case we only give the general ideas and let the details to the interested reader.

Let $\varepsilon > 0$ and $(u^0,u^1),(z^0,z^1) \in L^2(\Omega)\times H^{-1}(\Omega)$. We are looking for a control function $f \in L^2((0,T)\times\Gamma_0)$ such that the corresponding solution (u,u') of (78) satisfies

$$||(u(T),u'(T)) - (z^0,z^1)||_{L^2(\Omega)\times H^{-1}(\Omega)} \leq \varepsilon. \tag{91}$$

Recall that, (78) is approximately controllable if, for any $\varepsilon > 0$ and (u^0,u^1), $(z^0,z^1) \in L^2(\Omega)\times H^{-1}(\Omega)$, there exists $f \in L^2((0,T)\times\Gamma_0)$ such that (91) is verified.

By Remark 3.2, it is sufficient to study the case $(u^0,u^1) = (0,0)$. Therefore, in this section only zero initial data (u^0,u^1) will be considered.

The variational approach may be also used for the study of the approximate controllability property.

To see this, define the functional $\mathcal{J}_\varepsilon : H_0^1(\Omega)\times L^2(\Omega)\to\mathbb{R}$,

$$\mathcal{J}_\varepsilon(\varphi^0,\varphi^1)= \\ = \frac{1}{2}\int_0^T\int_{\Gamma_0}\left|\frac{\partial\varphi}{\partial n}\right|^2 dxdt + \left\langle\left(\varphi^0,\varphi^1\right),\left(z^0,z^1\right)\right\rangle + \varepsilon||(\varphi^0,\varphi^1)||_{H_0^1\times L^2}, \tag{92}$$

where φ is the solution of (81) with initial data $(\varphi^0,\varphi^1)\in H_0^1(\Omega)\times L^2(\Omega)$.

The following theorem shows how the functional $\mathcal{J}_\varepsilon$ may be used to study the approximate controllability property. In fact, as in the exact controllability case, the existence of a minimum of the functional $\mathcal{J}_\varepsilon$ implies the existence of an approximate control.

Theorem 3.4 *Let $\varepsilon>0$, $(z^0,z^1)\in L^2(\Omega)\times H^{-1}(\Omega)$. Suppose that $(\widehat{\varphi}^0,\widehat{\varphi}^1)\in H_0^1(\Omega)\times L^2(\Omega)$ is a minimizer of $\mathcal{J}_\varepsilon$. If $\widehat{\varphi}$ is the corresponding solution of (81) with initial data $(\widehat{\varphi}^0,\widehat{\varphi}^1)$ then*

$$f=\frac{\partial\widehat{\varphi}}{\partial n}\Big|_{\Gamma_0} \tag{93}$$

is an approximate control which leads the solution of (78) from the zero initial data $(u^0,u^1)=(0,0)$ to the state $(u(T),u'(T))$ such that (91) is verified.

Proof: It is similar to the proof of Theorem 3.4. □

As we have seen, the exact controllability property of (78) is related to the observation property (65) of system (81). An unique continuation property of the solutions of (81) plays a similar role in the context of approximate controllability and guarantees the existence of a minimizer of $\mathcal{J}_\varepsilon$. More precisely, we have

Theorem 3.5 *The following properties are equivalent:*

1. *Equation (78) is approximately controllable.*
2. *The following unique continuation principle holds for the solutions of (81)*

$$\frac{\partial\varphi}{\partial n}\Big|_{(0,T)\times\Gamma_0}=0\Rightarrow(\varphi^0,\varphi^1)=(0,0). \tag{94}$$

Proof: The proof of the fact that the approximate controllability property implies the unique continuation principle (94) is similar to the corresponding one in Theorem 2.6 and we omit it.

Let us prove that, if the unique continuation principle (94) is verified, (78) is approximately controllable. By Theorem 3.4 it is sufficient to prove that $\mathcal{J}_\varepsilon$

defined by (92) has a minimum. The functional $\mathcal{J}_\varepsilon$ is convex and continuous in $H_0^1(\Omega)\times L^2(\Omega)$. Thus, the existence of a minimum is ensured if $\mathcal{J}_\varepsilon$ is coercive, i. e.

$$\mathcal{J}_\varepsilon((\varphi^0,\varphi^1)) \to \infty \text{ when } ||(\varphi^0,\varphi^1)||_{H_0^1\times L^2} \to \infty. \tag{95}$$

In fact we shall prove that

$$\liminf_{||(\varphi^0,\varphi^1)||_{H_0^1\times L^2}\to\infty} \mathcal{J}_\varepsilon(\varphi^0,\varphi^1)/||(\varphi^0,\varphi^1)||_{H_0^1\times L^2} \geq \varepsilon. \tag{96}$$

Evidently, (96) implies (95) and the proof of the theorem is complete.

In order to prove (96) let $\left((\varphi_j^0,\varphi_j^1)\right)_{j\geq 1}\subset H_0^1(\Omega)\times L^2(\Omega)$ be a sequence of initial data for the adjoint system with $\|\ (\varphi_j^0,\varphi_j^1)\ \|_{H_0^1\times L^2}\to\infty$. We normalize them

$$(\widetilde{\varphi}_j^0,\widetilde{\varphi}_j^1) = (\varphi_j^0,\varphi_j^1)/\ \|\ (\varphi_j^0,\varphi_j^1)\ \|_{H_0^1\times L^2},$$

so that

$$\|\ (\widetilde{\varphi}_j^0,\widetilde{\varphi}_j^1)\ \|_{H_0^1\times L^2} = 1.$$

On the other hand, let $(\widetilde{\varphi}_j,\widetilde{\varphi}_j')$ be the solution of (81) with initial data $(\widetilde{\varphi}_j^0,\widetilde{\varphi}_j^1)$. Then

$$\frac{\mathcal{J}_\varepsilon((\varphi_j^0,\varphi_j^1))}{\|\ (\varphi_j^0,\varphi_j^1)\ \|} = \frac{1}{2}\ \|\ (\varphi_j^0,\varphi_j^1)\ \| \int_0^T\int_{\Gamma_0}\left|\frac{\partial\widetilde{\varphi}_j}{\partial n}\right|^2 d\sigma dt + \left\langle (z^0,z^1),(\widetilde{\varphi}^0,\widetilde{\varphi}^1)\right\rangle + \varepsilon.$$

The following two cases may occur:

1) $\liminf_{j\to\infty}\int_0^T\int_{\Gamma_0}\left|\frac{\partial\widetilde{\varphi}_j}{\partial n}\right|^2 > 0$. In this case we obtain immediately that

$$\frac{\mathcal{J}_\varepsilon((\varphi_j^0,\varphi_j^1))}{\|\ (\varphi_j^0,\varphi_j^1)\ \|}\to\infty.$$

2) $\liminf_{j\to\infty}\int_0^T\int_{\Gamma_0}\left|\frac{\partial\widetilde{\varphi}_j}{\partial n}\right|^2 = 0$. In this case, since $\left(\widetilde{\varphi}_j^0,\widetilde{\varphi}_j^1\right)_{j\geq1}$ is bounded in $H_0^1\times L^2$, by extracting a subsequence we can guarantee that $(\widetilde{\varphi}_j^0,\widetilde{\varphi}_j^1)_{j\geq 1}$ converges weakly to (ψ^0,ψ^1) in $H_0^1(\Omega)\times L^2(\Omega)$.

Moreover, if (ψ,ψ') is the solution of (81) with initial data (ψ^0,ψ^1) at $t=T$, then $(\widetilde{\varphi}_j,\widetilde{\varphi}_j')_{j\geq1}$ converges weakly to (ψ,ψ') in $L^2(0,T;H_0^1(\Omega)\times L^2(\Omega))\cap H^1(0,T;L^2(\Omega)\times H^{-1}(\Omega))$.

By lower semi-continuity,

$$\int_0^T\int_{\Gamma_0}\left|\frac{\partial\psi}{\partial n}\right|^2 d\sigma dt \leq \liminf_{j\to\infty}\int_0^T\int_{\Gamma_0}\left|\frac{\partial\widetilde{\varphi}_j}{\partial n}\right|^2 d\sigma dt = 0$$

and therefore $\frac{\partial\psi}{\partial n} = 0$ on $\Gamma_0 \times (0,T)$.

From the unique continuation principle we obtain that $(\psi^0, \psi^1) = (0,0)$ and consequently,

$$(\widetilde{\varphi}_j^0, \widetilde{\varphi}_j^1) \rightharpoonup (0,0) \text{ weakly in } H_0^1(\Omega) \times L^2(\Omega).$$

Hence

$$\liminf_{j\to\infty} \frac{\mathcal{J}_\varepsilon((\varphi_j^0, \varphi_j^1))}{\| (\varphi_j^0, \varphi_j^1) \|} \geq \liminf_{j\to\infty} [\varepsilon + \left\langle (z^0, z^1), (\widetilde{\varphi}^0, \widetilde{\varphi}^1) \right\rangle] = \varepsilon,$$

and (96) follows. □

As mentioned in the previous section, when approximate controllability holds, the following (apparently stronger) statement also holds (see [71]):

Theorem 3.6 *Let E be a finite-dimensional subspace of $L^2(\Omega) \times H^{-1}(\Omega)$ and let us denote by π_E the corresponding orthogonal projection. Then, if approximate controllability holds, for any $\left(u^0, u^1\right)$, $\left(z^0, z^1\right) \in L^2(\Omega) \times H^{-1}(\Omega)$ and $\varepsilon > 0$ there exists $f \in L^2((0,T) \times \Gamma_0)$ such that the solution of (78) satisfies*

$$\left\| \left(u(T) - z^0, u_t(T) - z^1\right) \right\|_{L^2(\Omega)\times H^{-1}(\Omega)} \leq \varepsilon;\ \pi_E\left(u(T), u_t(T)\right) = \pi_E\left(z^0, z^1\right).$$

3.6 Comments

In the last two sections we have presented some results concerning the exact and approximate controllability of the wave equation. The variational methods we have used allow to reduce these properties to an observation inequality and a unique continuation principle for the adjoint homogeneous equation respectively.

Let us briefly make some remarks concerning the proof of the unique continuation principles (75) and (94).

Holmgren's Uniqueness Theorem (see [33]) may be used to show that (75) and (94) hold if T is large enough. We refer to [45], chapter 1 and [13] for a discussion of this problem. Consequently, approximate controllability holds if T is large enough.

The same results hold for wave equations with analytic coefficients too. However, the problem is not completely solved in the frame of the wave equation with lower order potentials $a \in L^\infty((0,T) \times \Omega)$ of the form

$$u_{tt} - \Delta u + a(x,t)u = f 1_\omega \text{ in } (0,T) \times \Omega.$$

Once again the problem of approximate controllability of this system is equivalent to the unique continuation property of its adjoint. We refer to Alinhac [1], Tataru [62] and Robbiano-Zuilly [54] for deep results in this direction.

In the following chapter we shall prove the observability inequalities (65) and (87) in some simple one dimensional cases by using Fourier expansion of solutions. Other tools have been successfully used to prove these observability inequalities. Let us mention two of them.

1. **Multipliers techniques:** Ho in [32] proved that if one considers subsets of Γ of the form

   $$\Gamma_0 = \Gamma(x^0) = \{x \in \Gamma : (x - x^0) \cdot n(x) > 0\}$$

   for some $x^0 \in \mathbb{R}^N$ and if $T > 0$ is large enough, the boundary observability inequality (87), that is required to solve the boundary controllability problem, holds. The technique used consists of multiplying equation (84) by $q \cdot \nabla\varphi$ and integrating by parts in $(0, T) \times \Omega$. The multiplier q is an appropriate vector field defined in $\overline{\Omega}$. More precisely, $q(x) = x - x^0$ for any $x \in \overline{\Omega}$.

   Later on inequality (87) was proved in [45] for any $T > T(x^0) = 2 \parallel x - x^0 \parallel_{L^\infty(\Omega)}$. This is the optimal observability time that one may derive by means of multipliers. More recently Osses in [51] has introduced a new multiplier which is basically a rotation of the previous one and he has obtained a larger class of subsets of the boundary for which observability holds.

   Concerning the interior controllability problem, one can easily prove that (87) implies (65) when ω is a neighborhood of $\Gamma(x^0)$ in Ω, i.e. $\omega = \Omega \cap \Theta$ where Θ is a neighborhood of $\Gamma(x^0)$ in $\mathbb{R}^n$, with $T > 2 \parallel x - x^0 \parallel_{L^\infty(\Omega\setminus\omega)}$ (see in [45], vol. 1).

   An extensive presentation and several applications of multiplier techniques are given in [40] and [45].

2. **Microlocal analysis:** C. Bardos, G. Lebeau and J. Rauch [7] proved that, in the class of C^∞ domains, the observability inequality (65) holds if and only if (ω, T) satisfy the following *geometric control condition* in Ω*: Every ray of geometric optics that propagates in* Ω *and is reflected on its boundary* Γ *enters* ω *in time less than* T*.* This result was proved by means of microlocal analysis techniques. Recently the microlocal approach has been greatly simplified by N. Burq [11] by using the microlocal defect measures introduced by P. Gerard [30] in the context of the homogenization and the kinetic equations. In [11] the geometric control condition was shown to be sufficient for exact controllability for domains Ω of class C^3 and equations with C^2 coefficients.

Other methods have been developed to address the controllability problems such as moment problems, fundamental solutions, controllability via stabilization, etc. We will not present them here and we refer to the survey paper by D. L. Russell [55] for the interested reader.

4 Fourier techniques and the observability of the $1D$ wave equation

In Chapters 2 and 3 we have shown that the exact controllability problem may be reduced to the corresponding observability inequality. In this chapter we develop in detail some techniques based on Fourier analysis and more particularly on Ingham type inequalities allowing to obtain several observability results for linear 1-D wave equations. We refer to Avdonin and Ivanov [3] for a complete presentation of this approach.

4.1 Ingham's inequalities

In this section we present two inequalities which have been successfully used in the study of many 1-D control problems and, more precisely, to prove observation inequalities. They generalize the classical Parseval's equality for orthogonal sequences. Variants of these inequalities were studied in the works of Paley and Wiener at the beginning of the past century (see [53]). The main inequality was proved by Ingham (see [37]) who gave a beautiful and elementary proof (see Theorems 4.1 and 4.2 below). Since then, many generalizations have been given (see, for instance, [6], [58], [4] and [38]).

Theorem 4.1 *(Ingham [37]) Let $(\lambda_n)_{n\in\mathbb{Z}}$ be a sequence of real numbers and $\gamma > 0$ be such that*

$$\lambda_{n+1} - \lambda_n \geq \gamma > 0, \quad \forall n \in \mathbb{Z}. \tag{97}$$

For any real T with

$$T > \pi/\gamma \tag{98}$$

there exists a positive constant $C_1 = C_1(T,\gamma) > 0$ such that, for any finite sequence $(a_n)_{n\in\mathbb{Z}}$,

$$C_1 \sum_{n\in\mathbb{Z}} \mid a_n \mid^2 \leq \int_{-T}^{T} \left| \sum_{n\in\mathbb{Z}} a_n e^{i\lambda_n t} \right|^2 dt. \tag{99}$$

Proof: We first reduce the problem to the case $T = \pi$ and $\gamma > 1$. Indeed, if T and γ are such that $T\gamma > \pi$, then

$$\int_{-T}^{T} \left| \sum_n a_n e^{i\lambda_n t} \right|^2 dt \;=\; \frac{T}{\pi} \int_{-\pi}^{\pi} \left| \sum_n a_n e^{i\frac{T\lambda_n}{\pi}s} \right|^2 ds = \frac{T}{\pi} \int_{-\pi}^{\pi} \left| \sum_n a_n e^{i\mu_n s} \right|^2 ds$$

where $\mu_n = T\lambda_n/\pi$. It follows that $\mu_{n+1} - \mu_n = T\left(\lambda_{n+1} - \lambda_n\right)/\pi \geq \gamma_1 := T\gamma/\pi > 1$.

We prove now that there exists $C_1' > 0$ such that

$$C_1' \sum_{n\in\mathbb{Z}} \mid a_n \mid^2 \leq \int_{-\pi}^{\pi} \left| \sum_{n\in\mathbb{Z}} a_n e^{i\mu_n t} \right|^2 dt.$$

Define the function

$$h : \mathbb{R} \to \mathbb{R},\ h(t) = \begin{cases} \cos(t/2) & \text{if} \quad \mid t \mid \leq \pi \\ 0 & \text{if} \quad \mid t \mid > \pi \end{cases}$$

and let us compute its Fourier transform $K(\varphi)$,

$$K(\varphi) = \int_{-\pi}^{\pi} h(t)e^{it\varphi}dt = \int_{-\infty}^{\infty} h(t)e^{it\varphi}dt = \frac{4\cos\pi\varphi}{1-4\varphi^2}.$$

On the other hand, since $0 \leq h(t) \leq 1$ for any $t \in [-\pi, \pi]$, we have that

$$\int_{-\pi}^{\pi} \left| \sum_n a_n e^{i\mu_n t} \right|^2 dt \geq \int_{-\pi}^{\pi} h(t) \left| \sum_n a_n e^{i\mu_n t} \right|^2 dt = \sum_{n,m} a_n \bar{a}_m K(\mu_n - \mu_m) =$$

$$= K(0) \sum_n \mid a_n \mid^2 + \sum_{n\neq m} a_n \bar{a}_m K(\mu_n - \mu_m) \geq$$

$$\geq 4 \sum_n \mid a_n \mid^2 - \frac{1}{2} \sum_{n\neq m} \left(\mid a_n \mid^2 + \mid a_m \mid^2 \right) \mid K(\mu_n - \mu_m) \mid =$$

$$= 4 \sum_n \mid a_n \mid^2 - \sum_n \mid a_n \mid^2 \sum_{m\neq n} \mid K(\mu_n - \mu_m) \mid .$$

Remark that

$$\sum_{m\neq n} \mid K(\mu_n - \mu_m) \mid \leq \sum_{m\neq n} \frac{4}{4 \mid \mu_n - \mu_m \mid^2 - 1} \leq \sum_{m\neq n} \frac{4}{4\gamma_1^2 \mid n-m \mid^2 - 1} =$$

$$= 8 \sum_{r\geq 1} \frac{1}{4\gamma_1^2 r^2 - 1} \leq \frac{8}{\gamma_1^2} \sum_{r\geq 1} \frac{1}{4r^2 - 1} = \frac{8}{\gamma_1^2} \frac{1}{2} \sum_{r\geq 1} \left(\frac{1}{2r-1} - \frac{1}{2r+1} \right) = \frac{4}{\gamma_1^2}.$$

Hence,

$$\int_{-\pi}^{\pi} \left| \sum_n a_n e^{i\mu_n t} \right|^2 dt \geq \left(4 - \frac{4}{\gamma_1^2} \right) \sum_n \mid a_n \mid^2 .$$

If we take

$$C_1 = \frac{T}{\pi} \left(4 - \frac{4}{\gamma_1^2} \right) = \frac{4\pi}{T} \left(T^2 - \frac{\pi^2}{\gamma^2} \right)$$

the proof is concluded. □

Theorem 4.2 *Let $(\lambda_n)_{n\in\mathbb{Z}}$ be a sequence of real numbers and $\gamma > 0$ be such that*

$$\lambda_{n+1} - \lambda_n \geq \gamma > 0, \quad \forall n \in \mathbb{Z}. \tag{100}$$

For any $T > 0$ there exists a positive constant $C_2 = C_2(T, \gamma) > 0$ such that, for any finite sequence $(a_n)_{n\in\mathbb{Z}}$,

$$\int_{-T}^{T} \left| \sum_n a_n e^{i\lambda_n t} \right|^2 dt \leq C_2 \sum_n \mid a_n \mid^2 . \tag{101}$$

Proof: Let us first consider the case $T\gamma \geq \pi/2$. As in the proof of the previous theorem, we can reduce the problem to $T = \pi/2$ and $\gamma \geq 1$. Indeed,

$$\int_{-T}^{T} \left| \sum_n a_n e^{i\lambda_n t} \right|^2 dt \quad = \quad \frac{2T}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left| \sum_n a_n e^{i\mu_n s} \right|^2 ds$$

where $\mu_n = 2T\lambda_n/\pi$. It follows that $\mu_{n+1} - \mu_n = 2T\left(\lambda_{n+1} - \lambda_n\right)/\pi \geq \gamma_1 := 2T\gamma/\pi \geq 1$.

Let h be the function introduced in the proof of Theorem 4.1. Since $\sqrt{2}/2 \leq h(t) \leq 1$ for any $t \in [-\pi/2, \pi/2]$ we obtain that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left| \sum_n a_n e^{i\mu_n t} \right|^2 dt \leq 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} h(t) \left| \sum_n a_n e^{i\mu_n t} \right|^2 dt \leq$$

$$\leq 2 \int_{-\pi}^{\pi} h(t) \left| \sum_n a_n e^{i\mu_n t} \right|^2 dt = 2 \sum_{n,m} a_n \bar{a}_m K\left(\mu_n - \mu_m\right) =$$

$$= 8 \sum_n \mid a_n \mid^2 + 2 \sum_{n\neq m} a_n \bar{a}_m K\left(\mu_n - \mu_m\right) \leq$$

$$\leq 8 \sum_n \mid a_n \mid^2 + \sum_{n\neq m} \left(\mid a_n \mid^2 + \mid a_m \mid^2\right) \mid K\left(\mu_n - \mu_m\right) \mid .$$

As in the proof of Theorem 4.1 we obtain that

$$\sum_{m\neq n} \mid K(\mu_n - \mu_m) \mid \leq \frac{4}{\gamma_1^2}.$$

Hence,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left| \sum_n a_n e^{i\mu_n t} \right|^2 dt \leq 8 \sum_n \mid a_n \mid^2 + \frac{8}{\gamma_1^2} \sum_n \mid a_n \mid^2 \leq 8 \left(1 + \frac{1}{\gamma_1^2}\right) \sum_n \mid a_n \mid^2$$

and (101) follows with $C_2 = 8\left(4T^2/(\pi^2) + 1/\gamma^2\right)$.

When $T\gamma < \pi/2$ we have that

$$\int_{-T}^{T} \left|\sum a_n e^{i\lambda_n t}\right|^2 dt = \frac{1}{\gamma}\int_{-T\gamma}^{T\gamma} \left|\sum a_n e^{i\frac{\lambda_n}{\gamma}s}\right|^2 ds \le \frac{1}{\gamma}\int_{-\pi/2}^{\pi/2} \left|\sum a_n e^{i\frac{\lambda_n}{\gamma}s}\right|^2 ds.$$

Since $\lambda_{n+1}/\gamma - \lambda_n/\gamma \geq 1$ from the analysis of the previous case we obtain that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left|\sum_n a_n e^{i\frac{\lambda_n}{\gamma}s}\right|^2 ds \le 16\sum_n \mid a_n \mid^2 .$$

Hence, (101) is proved with

$$C_2 = 8\max\left\{\left(\frac{4T^2}{\pi^2} + \frac{1}{\gamma^2}\right),\ \frac{2}{\gamma}\right\}$$

and the proof concludes. □

Remark 4.1

- *Inequality (101) holds for all $T > 0$. On the contrary, inequality (99) requires the length T of the time interval to be sufficiently large. Note that, when the distance between two consecutive exponents λ_n, the gap, becomes small the value of T must increase proportionally.*
- *In the first inequality (99) T depends on the minimum γ of the distances between every two consecutive exponents (gap). However, as we shall see in the next theorem, only the asymptotic distance as $n \to \infty$ between consecutive exponents really matters to determine the minimal control time T. Note also that the constant C_1 in (99) degenerates when T goes to π/γ.*
- *In the critical case $T = \pi/\gamma$ inequality (99) may hold or not, depending on the particular family of exponential functions. For instance, if $\lambda_n = n$ for all $n \in \mathbb{Z}$, (99) is verified for $T = \pi$. This may be seen immediately by using the orthogonality property of the complex exponentials in $(-\pi, \pi)$. Nevertheless, if $\lambda_n = n - 1/4$ and $\lambda_{-n} = -\lambda_n$ for all $n > 0$, (99) fails for $T = \pi$ (see, [37] or [64]).* □

As we have said before, the length $2T$ of the time interval in (99) does not depend on the smallest distance between two consecutive exponents but on the asymptotic gap defined by

$$\liminf_{|n|\to\infty} \mid \lambda_{n+1} - \lambda_n \mid = \gamma_\infty. \tag{102}$$

An induction argument due to A. Haraux (see [31]) allows to give a result similar to Theorem 4.1 above in which condition (97) for γ is replaced by a similar one for γ_∞.

Theorem 4.3 *Let $(\lambda_n)_{n\in\mathbb{Z}}$ be an increasing sequence of real numbers such that $\lambda_{n+1}-\lambda_n \geq \gamma > 0$ for any $n\in\mathbb{Z}$ and let $\gamma_\infty > 0$ be given by (102). For any real T with*

$$T > \pi/\gamma_\infty \tag{103}$$

there exist two positive constants $C_1, C_2 > 0$ such that, for any finite sequence $(a_n)_{n\in\mathbb{Z}}$,

$$C_1 \sum_{n\in\mathbb{Z}} \mid a_n \mid^2 \leq \int_{-T}^{T} \left| \sum_{n\in\mathbb{Z}} a_n e^{i\lambda_n t} \right|^2 dt \leq C_2 \sum_{n\in\mathbb{Z}} \mid a_n \mid^2 . \tag{104}$$

Remark 4.2 *When $\gamma_\infty = \gamma$, the sequence of Theorem 4.3 satisfies $\lambda_{n+1}-\lambda_n \geq \gamma_\infty > 0$ for all $n\in\mathbb{Z}$ and we can then apply Theorems 4.1 and 4.2. However, in general, $\gamma_\infty < \gamma$ and Theorem 4.3 gives a sharper bound on the minimal time T needed for (104) to hold.*

Note that the existence of C_1 and C_2 in (104) is a consequence of Kahane's theorem (see [40]). However, if our purpose were to have an explicit estimate of C_1 or C_2 in terms of γ, γ_∞ then we would need to use the constructive argument below. It is important to note that these estimates depend strongly also on the number of eigenfrequencies λ that fail to fulfill the gap condition with the asymptotic gap γ_∞.

Proof of Theorem 4.3: The second inequality from (104) follows immediately by using Theorem 4.2. To prove the first inequality (104) we follow the induction argument due to Haraux [31].

Note that for any $\varepsilon_1 > 0$, there exists $N = N(\varepsilon_1) \in \mathbb{N}^*$ such that

$$|\lambda_{n+1} - \lambda_n| \geq \gamma_\infty - \varepsilon_1 \text{ for any } |n| > N. \tag{105}$$

We begin with the function $f_0(t) = \sum_{|n|>N} a_n e^{i\lambda_n t}$ and we add the missing exponentials one by one. From (105) we deduce that Theorems 4.1 and 4.2 may be applied to the family $\left(e^{i\lambda_n t}\right)_{|n|>N}$ for any $T > \pi/(\gamma_\infty - \varepsilon_1)$

$$C_1 \sum_{n>N} \mid a_n \mid^2 \leq \int_{-T}^{T} \mid f_0(t) \mid^2 dt \leq C_2 \sum_{n>N} \mid a_n \mid^2 . \tag{106}$$

Let now $f_1(t) = f_0 + a_N e^{i\,\lambda_N t} = \sum_{|n|>N} a_n e^{i\lambda_n t} + a_N e^{i\,\lambda_N t}$. Without loss of generality we may suppose that $\lambda_N = 0$ (since we can consider the function $f_1(t)e^{-i\lambda_N t}$ instead of $f_1(t)$).

Let $\varepsilon>0$ be such that $T'=T-\varepsilon>\pi/\gamma_\infty$. We have

$$\int_0^\varepsilon \left(f_1(t+\eta)-f_1(t)\right)d\eta=\sum_{n>N}a_n\left(\frac{e^{i\lambda_n\varepsilon}-1}{i\lambda_n}-\varepsilon\right)e^{i\lambda_n t},\quad \forall t\in[0,T'].$$

Applying now (106) to the function $h(t)=\int_0^\varepsilon \left(f_1(t+\eta)-f_1(t)\right)d\eta$ we obtain that:

$$C_1\sum_{n>N}\left|\frac{e^{i\lambda_n\varepsilon}-1}{i\lambda_n}-\varepsilon\right|^2|a_n|^2\le\int_{-T'}^{T'}\left|\int_0^\varepsilon \left(f_1(t+\eta)-f_1(t)\right)d\eta\right|^2dt. \quad (107)$$

Moreover,:

$$\left|e^{i\lambda_n\varepsilon}-1-i\lambda_n\varepsilon\right|^2=|\cos(\lambda_n\varepsilon)-1|^2+|\sin(\lambda_n\varepsilon)-\lambda_n\varepsilon|^2=$$

$$=4\sin^4\left(\frac{\lambda_n\varepsilon}{2}\right)+(\sin(\lambda_n\varepsilon)-\lambda_n\varepsilon)^2\ge\begin{cases}4\left(\frac{\lambda_n\varepsilon}{\pi}\right)^4, & \text{if } |\lambda_n|\varepsilon\le\pi\\ (\lambda_n\varepsilon)^2, & \text{if } |\lambda_n|\varepsilon>\pi.\end{cases}$$

Finally, taking into account that $|\lambda_n|\ge\gamma$, we obtain that,

$$\left|\frac{e^{i\lambda_n\varepsilon}-1}{i\lambda_n}-\varepsilon\right|^2\ge c\varepsilon^2.$$

We return now to (107) and we get that:

$$\varepsilon^2C_1\sum_{n>N}|a_n|^2\le\int_{-T'}^{T'}\left|\int_0^\varepsilon \left(f_1(t+\eta)-f_1(t)\right)d\eta\right|^2dt. \quad (108)$$

On the other hand

$$\int_{-T'}^{T'}\left|\int_0^\varepsilon \left(f_1(t+\eta)-f_1(t)\right)d\eta\right|^2dt\le\int_{-T'}^{T'}\varepsilon\int_0^\varepsilon|f_1(t+\eta)-f_1(t)|^2\,d\eta dt\le$$

$$\le 2\varepsilon\int_{-T'}^{T'}\int_0^\varepsilon\left(|f_1(t+\eta)|^2+|f_1(t)|^2\right)d\eta dt\le 2\varepsilon^2\int_{-T'}^{T}|f_1(t)|^2\,dt+$$

$$+2\varepsilon\int_0^\varepsilon\int_{-T'}^{T'}|f_1(t+\eta)|^2\,dtd\eta=2\varepsilon^2\int_{-T'}^{T}|f_1(t)|^2\,dt+2\varepsilon\int_0^\varepsilon\int_{-T'+\eta}^{T'+\eta}|f_1(s)|^2\,dsd\eta$$

$$\le 2\varepsilon^2\int_{-T}^{T}|f_1(t)|^2\,dt+2\varepsilon\int_0^\varepsilon\int_{-T}^{T}|f_1(s)|^2\,dsd\eta\le 4\varepsilon^2\int_{-T}^{T}|f_1(t)|^2\,dt.$$

From (108) it follows that

$$C_1 \sum_{n>N} |a_n|^2 \le \int_{-T}^{T} |f_1(t)|^2\, dt. \tag{109}$$

On the other hand

$$|a_N|^2 = \left| f_1(t) - \sum_{n>N} a_n e^{i\lambda_n t} \right|^2 = \frac{1}{2T} \int_{-T}^{T} \left| f_1(t) - \sum_{n>N} a_n e^{i\lambda_n t} \right|^2 dt \le$$

$$\le \frac{1}{T} \left(\int_{-T}^{T} |f_1(t)|^2\, dt + \int_{-T}^{T} \left| \sum_{n>N} a_n e^{i\lambda_n t} \right|^2 \right) dt \le$$

$$\le \frac{1}{T} \left(\int_{-T}^{T} |f_1(t)|^2\, dt + C_2^0 \sum_{n>N} |a_n|^2 \right) \le$$

$$\le \frac{1}{T} \left(1 + \frac{C_2}{C_1} \right) \int_{-T}^{T} |f_1(t)|^2\, dt.$$

From (109) we get that

$$C_1 \sum_{n\ge N} |a_n|^2 \le \int_{-T}^{T} |f_1(t)|^2\, dt.$$

Repeating this argument we may add all the terms $a_n e^{i\lambda_n t}$, $|n| \le N$ and we obtain the desired inequalities. □

4.2 Spectral analysis of the wave operator

The aim of this section is to give the Fourier expansion of solutions of the 1-D linear wave equation

$$\begin{cases} \varphi'' - \varphi_{xx} + \alpha\varphi = 0, & x \in (0,1),\ t \in (0,T) \\ \varphi(t,0) = \varphi(t,1) = 0, & t \in (0,T) \\ \varphi(0) = \varphi^0,\ \varphi'(0) = \varphi^1, & x \in (0,1) \end{cases} \tag{110}$$

where α is a real nonnegative number.

To do this let us first remark that (110) may be written as

$$\begin{cases} \varphi' = z \\ z' = \varphi_{xx} - \alpha\varphi \\ \varphi(t,0) = \varphi(t,1) = 0 \\ \varphi(0) = \varphi^0,\ z(0) = \varphi^1. \end{cases}$$

Nextly, denoting $\Phi = (\varphi, z)$, equation (110) is written in the following abstract Cauchy form:

$$\begin{cases} \Phi' + A\Phi = 0 \\ \Phi(0) = \Phi^0. \end{cases} \tag{111}$$

The differential operator A from (111) is the unbounded operator in $H = L^2(0,1) \times H^{-1}(0,1)$, $A : \mathcal{D}(A) \subset H \to H$, defined by

$$\begin{aligned} &\mathcal{D}(A) = H_0^1(0,1) \times L^2(0,1) \\ &A(\varphi, z) = (-z, -\partial_x^2\varphi + \alpha\varphi) = \begin{pmatrix} 0 & -1 \\ -\partial_x^2 + \alpha & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ z \end{pmatrix} \end{aligned} \tag{112}$$

where the Laplace operator $-\partial_x^2$ is an unbounded operator defined in $H^{-1}(0,1)$ with domain $H_0^1(0,1)$:

$$-\partial_x^2 : H_0^1(0,1) \subset H^{-1}(0,1) \to H^{-1}(0,1),$$
$$\langle -\partial_x^2\varphi, \psi\rangle = \int_0^1 \varphi_x\psi_x dx, \quad \forall\varphi,\ \psi \in H_0^1(0,1).$$

Remark 4.3 *The operator A is an isomorphism from $H_0^1(0,1) \times L^2(0,1)$ to $L^2(0,1) \times H^{-1}(0,1)$. We shall consider the space $H_0^1(0,1)$ with the inner product defined by*

$$(u, v)_{H_0^1(0,1)} = \int_0^1 (u_x)(x)v_x(x)dx + \alpha \int_0^1 u(x)v(x)dx \tag{113}$$

which is equivalent to the usual one.

Lemma 4.1 *The eigenvalues of A are $\lambda_n = \mathrm{sgn}(n)\pi i\sqrt{n^2+\alpha}$, $n \in \mathbb{Z}^*$. The corresponding eigenfunctions are given by*

$$\Phi^n = \begin{pmatrix} \frac{1}{\lambda_n} \\ -1 \end{pmatrix} \sin(n\pi x),\ n \in \mathbb{Z}^*,$$

and form an orthonormal basis in $H_0^1(0,1) \times L^2(0,1)$.

Proof: Let us first determine the eigenvalues of A. If $\lambda \in \mathbb{C}$ and $\Phi = (\varphi, z) \in H_0^1(0,1) \times L^2(0,1)$ are such that $A\Phi = \lambda\Phi$ we obtain from the definition of A that

$$\begin{cases} -z = \lambda\varphi \\ -\partial_x^2\varphi + \alpha\varphi = \lambda z. \end{cases} \tag{114}$$

It is easy to see that

$$\begin{cases} \partial_x^2\varphi - \alpha\varphi = \lambda^2\varphi \\ \varphi(0) = \varphi(1) = 0 \\ \varphi \in C^2[0,1]. \end{cases} \tag{115}$$

The solutions of (115) are given by

$$\lambda_n = \text{sgn}(n)\pi i\sqrt{n^2+\alpha}, \quad \varphi_n = c\sin(n\pi x), \quad n \in \mathbb{Z}^*$$

where c is an arbitrary complex constant.

Hence, the eigenvalues of A are $\lambda_n = \text{sgn}(n)\pi i\sqrt{n^2+\alpha}$, $n \in \mathbb{Z}^*$ and the corresponding eigenfunctions are

$$\Phi^n = \begin{pmatrix} \frac{1}{\lambda_n} \\ -1 \end{pmatrix} \sin(n\pi x),\ n \in \mathbb{Z}^*.$$

It is easy to see that

- $\| \Phi^n \|^2_{H_0^1\times L^2} = \frac{1}{(n^2+\alpha)\pi^2}\left(\int_0^1 (n\pi\cos(n\pi x))^2\,dx + \alpha\int_0^1 \sin^2(n\pi x)dx\right) + \int_0^1 (\sin(n\pi x))^2 dx = 1$
- $(\Phi^n, \Phi^m) = \frac{1}{nm\pi^2}\int_0^1 (n\pi\cos(n\pi x)m\pi\cos(m\pi x))\,dx + (\alpha+1)\int_0^1 (\sin(n\pi x)\sin(m\pi x))\,dx = \delta_{nm}.$

Hence, $(\Phi^n)_{n\in\mathbb{Z}^*}$ is an orthonormal sequence in $H_0^1(0,1)\times L^2(0,1)$.

The completeness of $(\Phi^n)_{n\in\mathbb{Z}^*}$ in $H_0^1(0,1)\times L^2(0,1)$ is a consequence of the fact that these are all the eigenfunctions of the compact skew-adjoint operator A^{-1}. It follows that $(\Phi^n)_{n\in\mathbb{Z}^*}$ is an orthonormal basis in $H_0^1(0,1)\times L^2(0,1)$. □

Remark 4.4 *Since $(\Phi^n)_{n\in\mathbb{Z}^*}$ is an orthonormal basis in $H_0^1(0,1)\times L^2(0,1)$ and A is an isomorphism from $H_0^1(0,1)\times L^2(0,1)$ to $L^2(0,1)\times H^{-1}(0,1)$ it follows immediately that $(A(\Phi^n))_{n\in\mathbb{Z}^*}$ is an orthonormal basis in $L^2(0,1)\times H^{-1}(0,1)$. Moreover $(\lambda_n\Phi^n)_{n\in\mathbb{Z}^*}$ is an orthonormal basis in $L^2(0,1)\times H^{-1}(0,1)$. We have that*

- $\Phi = \sum_{n\in\mathbb{Z}^*} a_n\,\Phi^n \in H_0^1(0,1)\times L^2(0,1)$ *if and only if* $\sum_{n\in\mathbb{Z}^*} |a_n|^2 < \infty$.
- $\Phi = \sum_{n\in\mathbb{Z}^*} a_n\,\Phi^n \in L^2(0,1)\times H^{-1}(0,1)$ *if and only if* $\sum_{n\in\mathbb{Z}^*} \frac{|a_n|^2}{|\lambda_n|^2} < \infty$. □

The Fourier expansion of the solution of (111) is given in the following Lemma.

Lemma 4.2 *The solution of (111) with the initial data*

$$W^0 = \sum_{n\in\mathbb{Z}^*} a_n\,\Phi^n \in L^2(0,1)\times H^{-1}(0,1) \tag{116}$$

is given by

$$W(t) = \sum_{n\in\mathbb{Z}^*} a_n\, e^{\lambda_n t}\Phi^n. \tag{117}$$

4.3 Observability for the interior controllability of the 1-D wave equation

Consider an interval $J\subset[0,1]$ with $\mid J\mid>0$ and a real time $T>2$. We address the following control problem discussed in 2: given $(u^0,u^1)\in H_0^1(0,1)\times L^2(0,1)$ to find $f\in L^2((0,T)\times J)$ such that the solution u of the problem

$$\begin{cases} u''-u_{xx}=f1_J, & x\in(0,1),\ t\in(0,T)\\ u(t,0)=u(t,1)=0, & t\in(0,T)\\ u(0)=u^0, u'(0)=u^1, & x\in(0,1)\end{cases} \tag{118}$$

satisfies

$$u(T,\,\cdot\,)=u'(T,\,\cdot\,)=0. \tag{119}$$

According to the developments of Chapter 2, the control problem can be solved if the following inequalities hold for any $(\varphi^0,\varphi^1)\in L^2(0,1)\times H^{-1}(0,1)$

$$C_1 \parallel (\varphi^0,\varphi^1)\parallel^2_{L^2\times H^{-1}} \leq \int_0^T\int_J \mid \varphi(t,x)\mid^2 dxdt \leq C_2 \parallel (\varphi^0,\varphi^1)\parallel^2_{L^2\times H^{-1}} \tag{120}$$

where φ is the solution of the adjoint equation (110).

In this section we prove (120) by using the Fourier expansion of the solutions of (110). Similar results can be proved for more general potentials depending on x and t by multiplier methods and sidewiese energy estimates [73] and also using Carleman inequalities [66], [67].

Remark 4.5 *In the sequel when (120) holds , for brevity, we will denote it as follows:*

$$\parallel (\varphi^0,\varphi^1)\parallel^2_{L^2(0,1)\times H^{-1}(0,1)} \asymp \int_0^T\int_J \mid\varphi(t,x)\mid^2 dxdt. \tag{121}$$

Theorem 4.4 *Let $T\geq 2$. There exist two positive constants C_1 and C_2 such that (120) holds for any $(\varphi^0,\varphi^1)\in L^2(0,1)\times H^{-1}(0,1)$ and φ solution of (110).*

Proof: Firstly, we have that

$$\| (\varphi^0, \varphi^1) \|_{L^2 \times H^{-1}}^2 = \left\| A^{-1} \left(\sum_{n \in \mathbb{Z}^*} a_n \Phi^n \right) \right\|_{H_0^1 \times L^2}^2 =$$

$$= \left\| \sum_{n \in \mathbb{Z}^*} a_n \frac{1}{in\pi} \Phi^n \right\|_{H_0^1 \times L^2}^2 = \sum_{n \in \mathbb{Z}^*} | a_n |^2 \frac{1}{n^2 \pi^2}.$$

Hence,

$$\| (\varphi^0, \varphi^1) \|_{L^2 \times H^{-1}}^2 = \sum_{n \in \mathbb{Z}^*} | a_n |^2 \frac{1}{n^2 \pi^2}. \tag{122}$$

On the other hand, since $\varphi \in C([0,T], L^2(0,1)) \subset L^2((0,T) \times (0,1))$, we obtain from Fubini's Theorem that

$$\int_0^T \int_J | w(t,x) |^2 \, dxdt = \int_J \int_0^T \left| \sum_{n \in \mathbb{Z}^*} a_n e^{in\pi t} \frac{1}{n\pi} \sin(n\pi x) \right|^2 dtdx.$$

Let first $T = 2$. From the orthogonality of the exponential functions in $L^2(0,2)$ we obtain that

$$\int_J \int_0^T \left| \sum_{n \in \mathbb{Z}^*} a_n e^{in\pi t} \frac{1}{n\pi} \sin(n\pi x) \right|^2 dtdx = \sum_{n \in \mathbb{Z}^*} \frac{|a_n|^2}{n^2\pi^2} \int_J \sin^2(n\pi x) dx.$$

If $T \geq 2$, it is immediate that

$$\int_J \int_0^T \left| \sum_{n \in \mathbb{Z}^*} \frac{a_n e^{in\pi t}}{n\pi} \sin(n\pi x) \right|^2 dtdx \geq \int_J \int_0^2 \left| \sum_{n \in \mathbb{Z}^*} \frac{a_n e^{in\pi t}}{n\pi} \sin(n\pi x) \right|^2 dtdx \geq$$

$$\geq \sum_{n \in \mathbb{Z}^*} \frac{|a_n|^2}{n^2\pi^2} \int_J \sin^2(n\pi x) dx.$$

On the other hand, by using the 2-periodicity in time of the exponentials and the fact that there exists $p > 0$ such that $2(p-1) \leq T < 2p$, it follows that

$$\int_J \int_0^T \left| \sum_{n \in \mathbb{Z}^*} \frac{a_n e^{in\pi t}}{n\pi} \sin(n\pi x) \right|^2 dtdx \leq p \int_J \int_0^2 \left| \sum_{n \in \mathbb{Z}^*} \frac{a_n e^{in\pi t}}{n\pi} \sin(n\pi x) \right|^2 dtdx$$

$$= p \sum_{n \in \mathbb{Z}^*} \frac{|a_n|^2}{n^2\pi^2} \int_J \sin^2(n\pi x) dx \leq \frac{T+2}{2} \sum_{n \in \mathbb{Z}^*} \frac{|a_n|^2}{n^2\pi^2} \int_J \sin^2(n\pi x) dx.$$

Hence, for any $T \geq 2$, we have that

$$\int_J \int_0^T \left| \sum_{n\in\mathbb{Z}^*} a_n e^{in\pi t} \frac{1}{n\pi} \sin(n\pi x) \right|^2 dxdt \asymp \sum_{n\in\mathbb{Z}^*} \frac{|a_n|^2}{n^2\pi^2} \int_J \sin^2(n\pi x)dx. \quad (123)$$

If we denote $b_n = \int_J \sin^2(n\pi x)dx$ then

$$B = \inf_{n\in\mathbb{Z}^*} b_n > 0. \quad (124)$$

Indeed,

$$b_n = \int_J \sin^2(n\pi x)dx = \frac{\mid J \mid}{2} - \int_J \frac{\cos(2n\pi x)}{2} \geq \frac{\mid J \mid}{2} - \frac{1}{2 \mid n \mid \pi}.$$

Since $1/[2 \mid n \mid \pi]$ tends to zero when n tends to infinity, there exists $n_0 > 0$ such that

$$b_n \geq \frac{\mid J \mid}{2} - \frac{1}{2 \mid n \mid \pi} > \frac{\mid J \mid}{4} > 0, \quad \forall \mid n \mid > n_0.$$

It follows that

$$B_1 = \inf_{|n|>n_0} b_n > 0 \quad (125)$$

and $B > 0$ since $b_n > 0$ for all n.

Moreover, since $b_n \leq |J|$ for any $n \in \mathbb{Z}^*$, it follows from (123) that

$$B \sum_{n\in\mathbb{Z}^*} \frac{\mid a_n \mid^2}{n^2\pi^2} \leq \int_0^T \int_J \mid \varphi(t,x) \mid^2 dxdt \leq |J| \sum_{n\in\mathbb{Z}^*} \mid a_n \mid^2 \frac{1}{n^2\pi^2}. \quad (126)$$

Finally, (120) follows immediately from (122) and (126). □

As a direct consequence of Theorem 4.4 the following controllability result holds:

Theorem 4.5 *Let $J \subset [0,1]$ with $\mid J \mid > 0$ and a real $T \geq 2$. For any $(u^0, u^1) \in H_0^1(0,1) \times L^2(0,1)$ there exists $f \in L^2((0,T) \times J)$ such that the solution u of equation (118) satisfies (119).*

Remark 4.6 *In order to obtain (123) for $T > 2$, Ingham's Theorem 4.1 could also be used. Indeed, the exponents are $\mu_n = n\pi$ and they satisfy the uniform gap condition $\gamma = \mu_{n+1} - \mu_n = \pi$, for all $n \in \mathbb{Z}^*$. It then follows from Ingham's Theorem 4.1 that, for any $T > 2\pi/\gamma = 2$, we have (123).*

Note that the result may not be ontained in the critical case $T = 2$ by using Theorems 4.1 and 4.2. The critical time $T = 2$ is reached in this case because of the orthogonality properties of the trigonometric polynomials $e^{i\pi nt}$. □

Consider now the equation

$$\begin{cases} u'' - u_{xx} + \alpha u = f 1_J, & x \in (0,1),\ t \in (0,T) \\ u(t,0) = u(t,1) = 0, & t \in (0,T) \\ u(0) = u^0, u'(0) = u^1, & x \in (0,1) \end{cases} \tag{127}$$

where α is a positive real number.

The controllability problem may be reduced once more to the proof of the following fact:

$$\int_J \int_0^T \left| \sum_{n\in\mathbb{Z}^*} a_n e^{\lambda_n t} \frac{1}{n\pi} \sin(n\pi x) \right|^2 dt dx \asymp \sum_{n\in\mathbb{Z}^*} \frac{|a_n|^2}{|\lambda_n|^2} \int_J \sin^2(n\pi x) dx \tag{128}$$

where $\lambda_n = \operatorname{sgn}(n)\pi i\sqrt{n^2+\alpha}$ are the eigenvalues of problem (127).

Remark that,

$$\begin{aligned} &\gamma = \inf\{\lambda_{n+1} - \lambda_n\} = \inf \left\{ \frac{(2n+1)\pi}{\sqrt{(n+1)^2+\alpha} + \sqrt{n^2+\alpha}} \right\} > \frac{\pi}{2\sqrt{\alpha}}, \\ &\gamma_\infty = \liminf_{n\to\infty} (\lambda_{n+1} - \lambda_n) = \pi. \end{aligned} \tag{129}$$

It follows from the generalized Ingham Theorem 4.3 that, for any $T > 2\pi/\gamma_\infty = 2$, (128) holds. Hence, the following controllability result is obtained:

Theorem 4.6 *Let $J \subset [0,1]$ with $| J |> 0$ and $T > 2$. For any $(u^0, u^1) \in H_0^1(0,1) \times L^2(0,1)$ there exists $f \in L^2((0,T) \times J)$ such that the solution u of equation (127) satisfies (119).*

Remark 4.7 *Note that if we had applied Theorem 4.1 the controllability time would have been $T > 2\pi/\gamma \geq 4\sqrt{\alpha}$. But Theorem 4.3 gives a control time T independent of α.*

Note that in this case the exponential functions $(e^{\lambda_n t})_n$ are not orthogonal in $L^2(0,T)$. Thus we can not use the same argument as in the proof on Theorem 4.4 and, accordingly, Ingham's Theorem is needed.

We have considered here the case where α is a positive constant. When α is negative the complex exponentials entering in the Fourier expansion of solutions may have eigenfrequencies λ_n which are not all purely real. In that case we can not apply directly Theorem 4.3. However, its method of proof allows also to deal with the situation where a finite number of eigenfrequencies are non real. Thus,the same result holds for all real α. $\square$

4.4 Boundary controllability of the 1-D wave equation

In this section we study the following boundary controllability problem: given $T>2$ and $(u^0,u^1)\in L^2(0,1)\times H^{-1}(0,1)$ to find a control $f\in L^2(0,T)$ such that the solution u of the problem:

$$\begin{cases} u''-u_{xx}=0 & x\in(0,1),\ t\in[0,T] \\ u(t,0)=0 & t\in[0,T] \\ u(t,1)=f(t) & t\in[0,T] \\ u(0)=u^0, u'(0)=u^1 & x\in(0,1) \end{cases} \tag{130}$$

satisfies

$$u(T,\cdot)=u'(T,\cdot)=0. \tag{131}$$

From the developments in Chapter 3 it follows that the following inequalities are a necessary and sufficient condition for the controllability of (130)

$$C_1 \parallel (\varphi^0,\varphi^1) \parallel^2_{H_0^1\times L^2} \leq \int_0^T |\varphi_x(t,1)|^2\,dt \leq C_2 \parallel (\varphi^0,\varphi^1) \parallel^2_{H_0^1\times L^2} \tag{132}$$

for any $(\varphi^0,\varphi^1)\in H_0^1(0,1)\times L^2(0,1)$ and φ solution of (110).

In order to prove (132) we use the Fourier decomposition of (110) given in the first section.

Theorem 4.7 *Let $T\geq 2$. There exist two positive constants C_1 and C_2 such that (132) holds for any $(\varphi^0,\varphi^1)\in H_0^1(0,1)\times L^2(0,1)$ and φ solution of (110).*

Proof: If $(\varphi^0,\varphi^1)=\sum_{n\in\mathbb{Z}^*} a_n\Phi_n$ we have that,

$$\parallel (\varphi^0,\varphi^1) \parallel^2_{H_0^1\times L^2} = \left\| \sum_{n\in\mathbb{Z}^*} a_n\Phi_n \right\|^2_{H_0^1\times L^2} = \sum_{n\in\mathbb{Z}^*} \mid a_n \mid^2 . \tag{133}$$

On the other hand

$$\int_0^T |\varphi_x(t,1)|^2\,dt = \int_0^T \left| \sum_{n\in\mathbb{Z}^*} (-1)^n a_n e^{in\pi t} \right|^2 dt.$$

By using the orthogonality in $L^2(0,2)$ of the exponentials $(e^{in\pi t})_n$, we get that

$$\int_0^2 \left| \sum_{n\in\mathbb{Z}^*} (-1)^n a_n e^{in\pi t} \right|^2 dt = \sum_{n\in\mathbb{Z}^*} \mid a_n \mid^2 .$$

If $T>2$, it is immediate that

$$\int_0^T \left| \sum_{n\in\mathbb{Z}^*} (-1)^n a_n e^{in\pi t} \right|^2 dt \geq \int_0^2 \left| \sum_{n\in\mathbb{Z}^*} (-1)^n a_n e^{in\pi t} \right|^2 dt = \sum_{n\in\mathbb{Z}^*} \mid a_n \mid^2 .$$

On the other hand, by using the 2-periodicity in time of the exponentials and the fact that there exists $p > 0$ such that $2(p-1) \leq T < 2p$, it follows that

$$\int_0^T \left| \sum_{n\in\mathbb{Z}^*} (-1)^n a_n e^{in\pi t} \right|^2 dt \geq p \int_0^2 \left| \sum_{n\in\mathbb{Z}^*} (-1)^n a_n e^{in\pi t} \right|^2 dt =$$

$$= p \sum_{n\in\mathbb{Z}^*} \mid a_n \mid^2 \leq \frac{T+2}{2} \sum_{n\in\mathbb{Z}^*} |a_n|^2.$$

Hence, for any $T \geq 2$, we have that

$$\int_0^T \left| \sum_{n\in\mathbb{Z}^*} (-1)^n a_n e^{in\pi t} \right|^2 dt \asymp \sum_{n\in\mathbb{Z}^*} \mid a_n \mid^2 . \tag{134}$$

Finally, from (133) and (134) we obtain that

$$\int_0^2 |\varphi_x(t,1)|^2 \, dt \asymp \| (\varphi^0, \varphi^1) \|^2_{H_0^1(0,1)\times L^2(0,1)}$$

and (132) is proved. $\square$

As a direct consequence of Theorems 4.7 the following controllability result holds:

Theorem 4.8 *Let $T \geq 2$. For any $(u^0, u^1) \in L^2(0,1) \times H^{-1}(0,1)$ there exists $f \in L^2(0,T)$ such that the solution u of equation (130) satisfies (131).*

As in the context of the interior controllability problem, one may address the following wave equation with potential

$$\begin{cases} u'' - u_{xx} + \alpha u = 0, & x \in (0,1),\ t \in (0,T) \\ u(t,0) = 0 & t \in [0,T] \\ u(t,1) = f(t) & t \in [0,T] \\ u(0) = u^0, u'(0) = u^1, & x \in (0,1) \end{cases} \tag{135}$$

where α is a positive real number.

The controllability problem is then reduced to the proof of the following inequality:

$$\int_0^T \left| \sum_{n\in\mathbb{Z}^*} (-1)^n \frac{n\pi}{\lambda_n} a_n e^{\lambda_n t} \right|^2 dt \asymp \sum_{n\in\mathbb{Z}^*} |a_n|^2 \tag{136}$$

where $\lambda_n = \text{sgn}(n)\pi i \sqrt{n^2 + \alpha}$ are the eigenvalues of problem (135).

It follows from (129) and the generalized Ingham's Theorem 4.3 that, for any $T > 2\pi/\gamma_\infty = 2$, (136) holds. Hence, the following controllability result is obtained:

Theorem 4.9 *Let $T > 2$. For any $(u^0, u^1) \in L^2(0,1) \times H^{-1}(0,1)$ there exists $f \in L^2(0,T)$ such that the solution u of equation (135) satisfies (131).*

Remark 4.8 *As we mentioned above, the classical Ingham inequality in (4.1) gives a suboptimal result in what concerns the time of control.* □

5 Interior controllability of the heat equation

In this chapter the interior controllability problem of the heat equation is studied. The control is assumed to act on a subset of the domain where the solutions are defined. The boundary controllability problem of the heat equation will be considered in the following chapter.

5.1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with boundary of class C^2 and ω a non empty open subset of Ω. Given $T > 0$ we consider the following non-homogeneous heat equation:

$$\begin{cases} u_t - \Delta u = f 1_\omega & \text{in } (0,T) \times \Omega \\ u = 0 & \text{on } (0,T) \times \partial\Omega \\ u(x,0) = u^0(x) & \text{in } \Omega. \end{cases} \tag{137}$$

In (137) $u = u(x,t)$ is the state and $f = f(x,t)$ is the control function with a support localized in ω. We aim at changing the dynamics of the system by acting on the subset ω of the domain Ω.

The heat equation is a model for many diffusion phenomena. For instance (137) provides a good description of the temperature distribution and evolution in a body occupying the region Ω. Then the control f represents a localized source of heat.

The interest on analyzing the heat equation above relies not only in the fact that it is a model for a large class of physical phenomena but also one of the most significant partial differential equation of parabolic type. As we shall see latter on, the main properties of parabolic equations such as time-irreversibility and regularizing effects have some very important consequences in control problems.

5.2 Existence and uniqueness of solutions

The following theorem is a consequence of classical results of existence and uniqueness of solutions of nonhomogeneous evolution equations. All the details may be found, for instance in [14].

Theorem 5.1 *For any $f \in L^2((0,T)\times\omega)$ and $u^0 \in L^2(\Omega)$ equation (137) has a unique weak solution $u \in C([0,T], L^2(\Omega))$ given by the variation of constants formula*

$$u(t) = S(t)u^0 + \int_0^t S(t-s)f(s)1_\omega ds \tag{138}$$

where $(S(t))_{t\in\mathbb{R}}$ is the semigroup of contractions generated by the heat operator in $L^2(\Omega)$.

Moreover, if $f \in W^{1,1}((0,T)\times L^2(\omega))$ and $u^0 \in H^2(\Omega)\cap H_0^1(\Omega)$, equation (137) has a classical solution $u \in C^1([0,T], L^2(\Omega)) \cap C([0,T], H^2(\Omega)\cap H_1^0(\Omega))$ and (137) is verified in $L^2(\Omega)$ for all $t > 0$.

Let us recall the classical energy estimate for the heat equation. Multiplying in (137) by u and integrating in Ω we obtain that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\, u\,|^2\, dx + \int_\Omega |\,\nabla u\,|^2\, dx = \int_\Omega fudx \leq \frac{1}{2}\int_\Omega |\, f\,|^2\, dx + \frac{1}{2}\int_\Omega |\, u\,|^2\, dx.$$

Hence, the scalar function $X = \int_\Omega |\, u\,|^2\, dx$ satisfies

$$X' \leq X + \int_\Omega |\, f\,|^2\, dx$$

which, by Gronwall's inequality, gives

$$X(t) \leq X(0)e^t + \int_0^t \int_\Omega |\, f\,|^2\, dxds \leq X(0)e^t + \int_0^T \int_\Omega |\, f\,|^2\, dxdt.$$

On the other hand, integrating in (137) with respect to t, it follows that

$$\frac{1}{2}\int_\Omega u^2 dx\Big|_0^T + \int_0^T \int_\Omega |\,\nabla u\,|^2\, dxdt \leq \frac{1}{2}\int_0^T \int_\Omega f^2 dxdt + \frac{1}{2}\int_0^T \int_\Omega u^2 dxdt$$

From the fact that $u \in L^\infty(0,T; L^2(\Omega))$ it follows that $u \in L^2(0,T; H_0^1(\Omega))$. Consequently, whenever $u_0 \in L^2(\Omega)$ and $f \in L^2(0,T; L^2(\omega))$ the solution u verifies

$$u \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega)).$$

5.3 Controllability problems

Let $T > 0$ and define, for any initial data $u^0 \in L^2(\Omega)$, the set of reachable states

$$R(T; u^0) = \{u(T) : u \text{ solution of (137) with } f \in L^2((0,T)\times\omega)\}. \tag{139}$$

By definition, any state in $R(T; u^0)$ is reachable in time T by starting from u^0 at time $t = 0$ with the aid of a convenient control f.

As in the case of the wave equation several notions of controllability may be defined.

Definition 5.1 *System (137) is* **approximately controllable in time** T *if, for every initial data* $u^0 \in L^2(\Omega)$*, the set of reachable states* $R(T; u^0)$ *is dense in* $L^2(\Omega)$.

Definition 5.2 *System (137) is* **exactly controllable in time** T *if, for every initial data* $u^0 \in L^2(\Omega)$*, the set of reachable states* $R(T; u^0)$ *coincides with* $L^2(\Omega)$.

Definition 5.3 *System (137) is* **null controllable in time** T *if, for every initial data* $u^0 \in L^2(\Omega)$*, the set of reachable states* $R(T; u^0)$ *contains the element* 0.

Remark 5.1 *Let us make the following remarks:*

- *One of the most important properties of the heat equation is its regularizing effect. When* $\Omega \setminus \omega \neq \emptyset$*, the solutions of (137) belong to* $C^\infty(\Omega \setminus \omega)$ *at time* $t = T$*. Hence, the restriction of the elements of* $R(T, u^0)$ *to* $\Omega \setminus \omega$ *are* C^∞ *functions. Then, the trivial case* $\omega = \Omega$ *(i. e. when the control acts on the entire domain* Ω*) being excepted, exact controllability may not hold. In this sense, the notion of exact controllability is not very relevant for the heat equation. This is due to its strong time irreversibility of the system under consideration.*

- *It is easy to see that if null controllability holds, then any initial data may be led to any final state of the form* $S(T)v^0$ *with* $v^0 \in L^2(\Omega)$*, i. e. to the range of the semigroup in time* $t = T$.

  Indeed, let $u^0, v^0 \in L^2(\Omega)$ *and remark that* $R(T; u^0 - v^0) = R(T; u^0) - S(T)v^0$*. Since* $0 \in R(T; u^0 - v^0)$*, it follows that* $S(T)v^0 \in R(T; u^0)$.

  It is known that the null controllability holds for any time $T > 0$ *and open set* ω *on which the control acts (see, for instance, [29]). The null controllability property holds in fact in a much more general setting of semilinear heat equations ([23] and [24]).*

- *Null controllability implies approximate controllability. Indeed, we have shown that, whenever null controllability holds,* $S(T)[L^2(\Omega)] \subset R(T; u^0)$ *for all* $u^0 \in L^2(\Omega)$*. Taking into account that all the eigenfunctions of the laplacian belong to* $S(T)[L^2(\Omega)]$ *we deduce that the set of reachable states is dense and, consequently, that approximate controllability holds.*

- *The problem of approximate controllability may be reduced to the case* $u^0 \equiv 0$*. Indeed, the linearity of the system we have considered implies that* $R(T, u^0) = R(T, 0) + S(T)u^0$.

- *Approximate controllability together with uniform estimates on the approximate controls as $\varepsilon \to 0$ may led to null controllability properties. More precisely, given u^1, we have that $u^1 \in R(T, u^0)$ if and only if there exists a sequence $(f_\varepsilon)_{\varepsilon>0}$ of controls such that $||u(T) - u^1||_{L^2(\Omega)} \le \varepsilon$ and $(f_\varepsilon)_{\varepsilon>0}$ is bounded in $L^2(\omega \times (0,T))$. Indeed in this case any weak limit in $L^2(\omega \times (0,T))$ of the sequence $(f_\varepsilon)_{\varepsilon>0}$ of controls gives an exact control which makes that $u(T) = u_1$.* $\square$

In this chapter we limit ourselves to study the approximate controllability problem. The main ingredients we shall develop are of variational nature. The problem will be reduced to prove unique continuation properties. Null-controllability will be addressed in the following chapter.

5.4 Approximate controllability of the heat equation

Given any $T > 0$ and any nonempty open subset ω of Ω we analyze in this section the approximate controllability problem for system (137).

Theorem 5.2 *Let ω be an open nonempty subset of Ω and $T > 0$. Then (137) is approximately controllable in time T.*

Remark 5.2 *The fact that the heat equation is approximately controllable in arbitrary time T and with control in any subset of Ω is due to the infinite velocity propagation which characterizes the heat equation.*

Nevertheless, the infinite velocity of propagation by itself does not allow to deduce quantitative estimates for the norm of the controls. Indeed, as it was proved in [50], the heat equation in an infinite domain $(0, \infty)$ of $\mathbb{R}$ is approximately controllable but, in spite of the infinite velocity of propagation, it is not null-controllable. $\square$

Remark 5.3 *There are several possible proofs for the approximate controllability property. We shall present here two of them. The first one is presented below and uses Hahn-Banach Theorem. The second one is constructive and uses a variational technique similar to the one we have used for the wave equation. We give it in the following section.* $\square$

Proof of the Theorem 5.2: As we have said before, it is sufficient to consider only the case $u^0 = 0$. Thus we assume that $u^0 = 0$.

From Hahn-Banach Theorem, $R(T, u^0)$ is dense in $L^2(\Omega)$ if the following property holds: There is no $\varphi_T \in L^2(\Omega)$, $\varphi_T \ne 0$ such that $\int_\Omega u(T)\varphi_T dx = 0$ for all u solution of (137) with $f \in L^2(\omega \times (0,T))$.

Accordingly, the proof can be reduced to showing that, if $\varphi_T \in L^2(\Omega)$ is such that $\int_\Omega u(T)\varphi_T dx = 0$, for all solution u of (137) then $\varphi_T = 0$.

To do this we consider the adjoint equation:

$$\begin{cases} \varphi_t + \Delta\varphi = 0 & \text{in } (0,T)\times\Omega \\ \varphi \mid_{\partial\Omega} = 0 & \text{on } (0,T)\times\partial\Omega \\ \varphi(T) = \varphi_T & \text{in } \Omega. \end{cases} \tag{140}$$

We multiply the equation satisfied by φ by u and then the equation of u by φ. Integrating by parts and taking into account that $u^0 \equiv 0$ the following identity is obtained

$$\int_0^T\int_\omega f\varphi dxdt = \int_{\Omega\times(0,T)}(u_t - \Delta u)\varphi dxdt = -\int_{\Omega\times(0,T)}(\varphi_t + \Delta\varphi)udxdt +$$

$$+\int_\Omega u\varphi dx\Big|_0^T + \int_0^T\int_{\partial\Omega}\left(-\frac{\partial u}{\partial n}\varphi + u\frac{\partial\varphi}{\partial n}\right)d\sigma dt = \int_\Omega u(T)\varphi_T dx.$$

Hence, $\int_\Omega u(T)\varphi_T dx = 0$ if and only if $\displaystyle\int_0^T\int_\omega f\varphi dxdt = 0$. If the later relation holds for any $f \in L^2(\omega\times(0,T))$, we deduce that $\varphi \equiv 0$ in $\omega\times(0,T)$.

Let us now recall the following result whose proof may be found in [33]:

Holmgren Uniqueness Theorem: *Let P be a differential operator with constant coefficients in $\mathbb{R}^n$. Let u be a solution of $Pu = 0$ in Q_1 where Q_1 is an open set of $\mathbb{R}^n$. Suppose that $u = 0$ in Q_2 where Q_2 is an open nonempty subset of Q_1.*

Then $u = 0$ in Q_3, where Q_3 is the open subset of Q_1 which contains Q_2 and such that any characteristic hyperplane of the operator P which intersects Q_3 also intersects Q_1.

In our particular case $P = \partial_t + \Delta_x$ is a differential operator in $\mathbb{R}^{n+1}$ and its principal part is $P_p = \Delta_x$. A hyperplane of $\mathbb{R}^{n+1}$ is characteristic if its normal vector $(\xi,\zeta) \in \mathbb{R}^{n+1}$ is a zero of P_p, i. e. of $P_p(\xi,\zeta) = |\xi|^2$. Hence, normal vectors are of the form $(0,\pm1)$ and the characteristic hyperplanes are horizontal, parallel to the hyperplane $t = 0$.

Consequently, for the adjoint heat equation under consideration (140), we can apply Holmgren's Uniqueness Theorem with $Q_1 = (0,T)\times\Omega$, $Q_2 = (0,T)\times\omega$ and $Q_3 = (0,T)\times\Omega$. Then the fact that $\varphi = 0$ in $(0,T)\times\omega$ implies $\varphi = 0$ in $(0,T)\times\Omega$. Consequently $\varphi_T \equiv 0$ and the proof is finished. □

5.5 Variational approach to approximate controllability

In this section we give a new proof of the approximate controllability result Theorem 5.2. This proof has the advantage of being constructive and it allows to compute explicitly approximate controls.

Let us fix the control time $T > 0$ and the initial datum $u^0 = 0$. Let $u^1 \in L^2(\Omega)$ be the final target and $\varepsilon > 0$ be given. Recall that we are looking for a control f such that the solution of (137) satisfies

$$||u(T) - u_1||_{L^2(\Omega)} \leq \varepsilon. \tag{141}$$

We define the following functional:

$$J_\varepsilon : L^2(\Omega) \to \mathbb{R} \tag{142}$$

$$J_\varepsilon(\varphi_T) = \frac{1}{2}\int_0^T \int_\omega \varphi^2 dxdt + \varepsilon \parallel \varphi_T \parallel_{L^2(\Omega)} - \int_\Omega u_1 \varphi_T dx \tag{143}$$

where φ is the solution of the adjoint equation (140) with initial data φ_T.

The following Lemma ensures that the minimum of J_ε gives a control for our problem.

Lemma 5.1 *If $\widehat{\varphi}_T$ is a minimum point of J_ε in $L^2(\Omega)$ and $\widehat{\varphi}$ is the solution of (140) with initial data $\widehat{\varphi}_T$, then $f = \widehat{\varphi}_{|_\omega}$ is a control for (137), i. e. (141) is satisfied.*

Proof: In the sequel we simply denote J_ε by J.

Suppose that J attains its minimum value at $\widehat{\varphi}_T \in L^2(\Omega)$. Then for any $\psi_0 \in L^2(\Omega)$ and $h \in \mathbb{R}$ we have $J(\widehat{\varphi}_T) \leq J\left(\widehat{\varphi}_T + h\psi_0\right)$. On the other hand,

$$\begin{aligned} & J\left(\widehat{\varphi}_T + h\psi_0\right) = \\ &= \frac{1}{2}\int_0^T \int_\omega |\, \widehat{\varphi} + h\psi \,|^2 \, dxdt + \varepsilon \parallel \widehat{\varphi}_T + h\psi_0 \parallel_{L^2(\Omega)} - \int_\Omega u_1(\widehat{\varphi}_T + h\psi_0)dx \\ &= \frac{1}{2}\int_0^T \int_\omega |\, \widehat{\varphi} \,|^2 \, dxdt + \frac{h^2}{2}\int_0^T \int_\omega |\, \psi \,|^2 \, dxdt + h\int_0^T \int_\omega \widehat{\varphi}\psi dxdt + \\ &\quad + \varepsilon \parallel \widehat{\varphi}_T + h\psi_0 \parallel_{L^2(\Omega)} - \int_\Omega u_1(\widehat{\varphi}_T + h\psi_0)dx. \end{aligned}$$

Thus

$$\begin{aligned} 0 \leq \quad & \varepsilon\left[\parallel \widehat{\varphi}_T + h\psi_0 \parallel_{L^2(\Omega)} - \parallel \widehat{\varphi}_T \parallel_{L^2(\Omega)}\right] + \frac{h^2}{2}\int_{(0,T)\times\omega} \psi^2 dxdt \\ & + h\left[\int_0^T \int_\omega \widehat{\varphi}\psi dxdt - \int_\Omega u_1 \psi_0 dx\right]. \end{aligned}$$

Since

$$\parallel \widehat{\varphi}_T + h\psi_0 \parallel_{L^2(\Omega)} - \parallel \widehat{\varphi}_T \parallel_{L^2(\Omega)} \leq |h| \parallel \psi_0 \parallel_{L^2(\Omega)}$$

we obtain

$$0 \leq \varepsilon \ |h| \parallel \psi_0 \parallel_{L^2(\Omega)} + \frac{h^2}{2} \int_0^T \int_\omega \psi^2 dx dt + h \int_0^T \int_\omega \widehat{\varphi} \psi dx dt - h \int_\Omega u_1 \psi_0 dx$$

for all $h \in \mathbb{R}$ and $\psi_0 \in L^2(\Omega)$.

Dividing by $h > 0$ and by passing to the limit $h \to 0$ we obtain

$$0 \leq \varepsilon \parallel \psi_0 \parallel_{L^2(\Omega)} + \int_0^T \int_\omega \widehat{\varphi} \psi dx dt - \int_\Omega u_1 \psi_0 dx. \tag{144}$$

The same calculations with $h < 0$ gives that

$$\left| \int_0^T \int_\omega \widehat{\varphi} \psi dx dt - \int_\Omega u_1 \psi_0 dx \right| \leq \varepsilon \parallel \psi_0 \parallel \quad \forall \psi_0 \in L^2(\Omega). \tag{145}$$

On the other hand, if we take the control $f = \widehat{\varphi}$ in (137), by multiplying in (137) by ψ solution of (140) and by integrating by parts we get that

$$\int_0^T \int_\omega \widehat{\varphi} \psi dx dt = \int_\Omega u(T) \psi_0 dx. \tag{146}$$

From the last two relations it follows that

$$\| \int_\Omega (u(T) - u_1) \psi_0 dx \| \leq \varepsilon ||\psi_0||_{L^2(\Omega)}, \quad \forall \psi_0 \in L^2(\Omega) \tag{147}$$

which is equivalent to

$$||u(T) - u_1||_{L^2(\Omega)} \leq \varepsilon.$$

The proof of the Lemma is now complete. □

Let us now show that J attains its minimum in $L^2(\Omega)$.

Lemma 5.2 *There exists $\widehat{\varphi}_T \in L^2(\Omega)$ such that*

$$J(\widehat{\varphi}_T) = \min_{\varphi_T \in L^2(\Omega)} J(\varphi_T). \tag{148}$$

Proof: It is easy to see that J is convex and continuous in $L^2(\Omega)$. By Theorem 2.3, the existence of a minimum is ensured if J is coercive, i. e.

$$J(\varphi_T) \to \infty \text{ when } ||\varphi_T||_{L^2(\Omega)} \to \infty. \tag{149}$$

In fact we shall prove that

$$\liminf_{||\varphi_T||_{L^2(\Omega)} \to \infty} J(\varphi_T) / ||\varphi_T||_{L^2(\Omega)} \geq \varepsilon. \tag{150}$$

Evidently, (150) implies (149) and the proof of the Lemma is complete.

In order to prove (150) let $(\varphi_{T,j}) \subset L^2(\Omega)$ be a sequence of initial data for the adjoint system with $\| \varphi_{T,j} \|_{L^2(\Omega)} \to \infty$. We normalize them

$$\widetilde{\varphi}_{T,j} = \varphi_{T,j} / \| \varphi_{T,j} \|_{L^2(\Omega)},$$

so that $\| \widetilde{\varphi}_{T,j} \|_{L^2(\Omega)} = 1$.

On the other hand, let $\widetilde{\varphi}_j$ be the solution of (140) with initial data $\widetilde{\varphi}_{T,j}$. Then

$$J(\varphi_{T,j}) / \| \varphi_{T,j} \|_{L^2(\Omega)} = \frac{1}{2} \| \varphi_{T,j} \|_{L^2(\Omega)} \int_0^T \int_\omega | \widetilde{\varphi}_j |^2 \, dxdt + \varepsilon - \int_\Omega u_1 \widetilde{\varphi}_{T,j} dx.$$

The following two cases may occur:

1) $\liminf_{j\to\infty} \int_0^T \int_\omega | \widetilde{\varphi}_j |^2 > 0$. In this case we obtain immediately that

$$J(\varphi_{T,j}) / \| \varphi_{T,j} \|_{L^2(\Omega)} \to \infty.$$

2) $\liminf_{j\to\infty} \int_0^T \int_\omega | \widetilde{\varphi}_j |^2 = 0$. In this case since $\widetilde{\varphi}_{T,j}$ is bounded in $L^2(\Omega)$, by extracting a subsequence we can guarantee that $\widetilde{\varphi}_{T,j} \rightharpoonup \psi_0$ weakly in $L^2(\Omega)$ and $\widetilde{\varphi}_j \rightharpoonup \psi$ weakly in $L^2(0,T; H_0^1(\Omega)) \cap H^1(0,T; H^{-1}(\Omega))$, where ψ is the solution of (140) with initial data ψ_0 at $t = T$. Moreover, by lower semi-continuity,

$$\int_0^T \int_\omega \psi^2 dxdt \le \liminf_{j\to\infty} \int_0^T \int_\omega | \widetilde{\varphi}_j |^2 \, dxdt = 0$$

and therefore $\psi = 0$ en $\omega \times (0,T)$.

Holmgren Uniqueness Theorem implies that $\psi \equiv 0$ in $\Omega \times (0,T)$ and consequently $\psi_0 = 0$.

Therefore, $\widetilde{\varphi}_{T,j} \rightharpoonup 0$ weakly in $L^2(\Omega)$ and consequently $\int_\Omega u_1 \widetilde{\varphi}_{T,j} dx$ tends to 0 as well.

Hence

$$\liminf_{j\to\infty} \frac{J(\varphi_{T,j})}{\| \varphi_{T,j} \|} \ge \liminf_{j\to\infty} [\varepsilon - \int_\Omega u_1 \widetilde{\varphi}_{T,j} dx] = \varepsilon,$$

and (150) follows. □

Remark 5.4 *Lemmas 5.1 and 5.2 give a second proof of Theorem 5.2. This approach does not only guarantee the existence of a control but also provides a method to obtain the control by minimizing a convex, continuous and coercive functional in* $L^2(\Omega)$.

In the proof of the coercivity, the relevance of the term $\varepsilon||\varphi_T||_{L^2(\Omega)}$ *is clear. Indeed, the coercivity of* J *depends heavily on this term. This is not only for technical reasons. The existence of a minimum of* J *with* $\varepsilon = 0$ *implies the existence of a control which makes* $u(T) = u^1$. *But this is not true unless* u^1 *is very regular in* $\Omega \setminus \omega$. *Therefore, for general* $u^1 \in L^2(\Omega)$, *the term* $\varepsilon||\varphi_T||_{L^2(\Omega)}$ *is needed.*

Note that both proofs are based on the unique continuation property which guarantees that if φ *is a solution of the adjoint system such that* $\varphi = 0$ *in* $\omega \times (0,T)$, *then* $\varphi \equiv 0$. *As we have seen, this property is a consequence of Holmgren Uniqueness Theorem.* □

The second proof, based on the minimization of J, with some changes on the definition of the functional as indicated in 1, allows proving approximate controllability by means of other controls, for instance, of bang-bang form. We address these variants in the following sections.

5.6 Finite-approximate control

Let E be a subspace of $L^2(\Omega)$ of finite dimension and Π_E be the orthogonal projection over E. As a consequence of the approximate controllability property in Theorem 5.2 the following stronger result may be proved: *given* u^0 *and* u^1 *in* $L^2(\Omega)$ *and* $\varepsilon > 0$ *there exists a control* f *such that the solution of (137) satisfies simultaneously*

$$\Pi_E(u(T)) = \Pi_E(u_1), \quad \| u(T) - u^1 \|_{L^2(\Omega)} \leq \varepsilon. \tag{151}$$

This property not only says that the distance between $u(T)$ and the target u^1 is less that ε but also that the projection of $u(T)$ and u^1 over E coincide.

This property, introduced in [71], will be called **finite-approximate controllability**. It may be proved easily by taking into account the following property of Hilbert spaces: *If* $L : E \to F$ *is linear and continuous between the Hilbert spaces* E *and* F *and the range of* L *is dense in* F, *then, for any finite set* $f_1, f_2, \ldots, f_N \in F$, *the set* $\{Le : (Le, f_j)_F = 0 \quad \forall j = 1, 2, \ldots, N\}$ *is dense in the orthogonal of* $\mathrm{Span}\{f_1, f_2, \ldots, f_N\}$.

Nevertheless, as we have said before, this result may also be proved directly, by considering a slightly modified form of the functional J used in the second proof of Theorem 5.2. We introduce

$$J_E(\varphi_T) = \frac{1}{2}\int_0^T \int_\omega \varphi^2 dxdt + \varepsilon \parallel (I - \Pi_E)\varphi_T \parallel_{L^2(\Omega)} - \int_\Omega u_1 \varphi_T dx.$$

The functional J_E is again convex and continuous in $L^2(\Omega)$. Moreover, it is coercive. The proof of the coercivity of J_E is similar to that of J. It is sufficient to note that if $\widehat{\varphi}_{T,j}$ tends weakly to zero in $L^2(\Omega)$, then $\Pi_E(\widehat{\varphi}_{T,j})$ converges (strongly) to zero in $L^2(\Omega)$.

Therefore $\| (I - \Pi_E)\widehat{\varphi}_{T,j} \|_{L^2(\Omega)} / \| \widehat{\varphi}_{T,j} \|_{L^2(\Omega)}$ tends to 1. According to this, the new functional J_E satisfies the coercivity property (150).

It is also easy to see that the minimum of J_E gives the finite-approximate control we were looking for.

5.7 Bang-bang control

In the study of finite dimensional systems we have seen that one may find "bang-bang" controls which take only two values $\pm\lambda$ for some $\lambda > 0$.

In the case of the heat equation it is also easy to construct controls of this type. In fact a convenient change in the functional J will ensure the existence of "bang-bang" controls. We consider:

$$J_{bb}(\varphi_T) = \frac{1}{2}\left(\int_0^T \int_\omega \mid \varphi \mid dxdt\right)^2 + \varepsilon \parallel \varphi_T \parallel_{L^2(\Omega)} - \int_\Omega u_1 \varphi_T dx.$$

Remark that the only change in the definition of J_{bb} is in the first term in which the norm of φ in $L^2(\omega \times (0,T))$ has been replaced by its norm in $L^1((0,T) \times \omega)$.

Once again we are dealing with a convex and continuous functional in $L^2(\Omega)$. The proof of the coercivity of J_{bb} is the same as in the case of the functional J. We obtain that:

$$\liminf_{\|\varphi_T\|_{L^2(\Omega)} \to \infty} \frac{J_{bb}(\varphi_T)}{\parallel \varphi_T \parallel_{L^2(\Omega)}} \geq \varepsilon.$$

Hence, J_{bb} attains a minimum in some $\widehat{\varphi}_T$ of $L^2(\Omega)$. It is easy to see that, if $\widehat{\varphi}$ is the corresponding solution of the adjoint system with $\widehat{\varphi}_T$ as initial data, then there exists $f \in \int_\omega \int_0^T |\widehat{\varphi}| dx \operatorname{sgn}(\widehat{\varphi})$ such that the solution of (137) with this control satisfies $||u(T) - u_1|| \leq \varepsilon$.

On the other hand, since $\widehat{\varphi}$ is a solution of the adjoint heat equation, it is real analytic in $\Omega \times (0,T)$. Hence, the set $\{t \,:\, \widehat{\varphi} = 0\}$ is of zero measure in $\Omega \times (0,T)$. Hence, we may consider

$$f = \int_\omega \int_0^T |\widehat{\varphi}| dxdt \operatorname{sgn}(\widehat{\varphi}) \tag{152}$$

which represents a bang-bang control. Remark that the sign of the control changes when the sign of $\widehat{\varphi}$ changes. Consequently, the geometry of the sets where the control has a given sign can be quite complex.

Note also that the amplitude of the bang-bang control is $\int_\omega \int_0^T |\widehat{\varphi}| dxdt$ which, evidently depends of the distance from the final target u^1 to the uncontrolled final state $S(T)u^0$ and of the control time T.

Remark 5.5 *As it was shown in [18], the bang-bang control obtained by minimizing the functional J_{bb} is the one of minimal norm in $L^\infty((0,T)\times\omega)$ among all the admissible ones. The control obtained by minimizing the functional J has the minimal norm in $L^2((0,T)\times\omega)$.* □

Remark 5.6 *The problem of finding bang-bang controls guaranteeing the finite-approximate property may also be considered. It is sufficient to take the following combination of the functionals J_E and J_{bb}:*

$$J_{bb,E}(\varphi_T) = \frac{1}{2}\left(\int_0^T \int_\omega \mid \varphi \mid dxdt\right)^2 + \varepsilon \parallel (I-\Pi_E)\varphi_T \parallel_{L^2(\Omega)} - \int_\Omega u_1 \varphi_T dx.$$

□

5.8 Comments

The null controllability problem for system (137) is equivalent to the following observability inequality for the adjoint system (140):

$$\parallel \varphi(0) \parallel^2_{L^2(\Omega)} \leq C \int_0^T \int_\omega \varphi^2 dxdt, \quad \forall \varphi^0 \in L^2(\Omega). \tag{153}$$

Once (153) is known to hold one can obtain the control with minimal L^2-norm among the admissible ones. To do that it is sufficient to minimize the functional

$$J(\varphi^0) = \frac{1}{2}\int_0^T \int_\omega \varphi^2 dxdt + \int_\Omega \varphi(0) u^0 dx \tag{154}$$

over the Hilbert space

$$H = \{\varphi^0 : \text{ the solution } \varphi \text{ of (140) satisfies } \int_0^T \int_\omega \varphi^2 dxdt < \infty\}.$$

To be more precise, H is the completion of $L^2(\Omega)$ with respect to the norm $[\int_0^T \int_\omega \varphi^2 dxdt]^{1/2}$. In fact, H is much larger than $L^2(\Omega)$. We refer to [23] for precise estimates on the nature of this space.

Observe that J is convex and continuous in H. On the other hand (153) guarantees the coercivity of J and the existence of its minimizer.

Due to the irreversibility of the system, (153) is not easy to prove. For instance, multiplier methods do not apply. Let us mention two different approaches used for the proof of (153).

1. **Results based on the observation of the wave or elliptic equations:** In [56] it was shown that if the wave equation is exactly controllable for some $T > 0$ with controls supported in ω, then the heat equation (137) is null controllable for all $T > 0$ with controls supported in ω. As a consequence of this result and in view of the controllability results for the wave equation, it follows that the heat equation (137) is null controllable for all $T > 0$ provided ω satisfies the geometric control condition. However, the geometric control condition does not seem to be natural at all in the context of the heat equation.

   Later on, Lebeau and Robbiano [42] proved that the heat equation (137) is null controllable for every open, non-empty subset ω of Ω and $T > 0$. This result shows, as expected, that the geometric control condition is unnecessary in the context of the heat equation. A simplified proof of it was given in [43] where the linear system of thermoelasticity was addressed. The main ingredient in the proof is the following observability estimate for the eigenfunctions $\{\psi_j\}$ of the Laplace operator

$$\int_\omega \left| \sum_{\lambda_j \le \mu} a_j \psi_j(x) \right|^2 dx \ge C_1 e^{-C_2\sqrt{\mu}} \sum_{\lambda_j \le \mu} | a_j |^2 \tag{155}$$

   which holds for any $\{a_j\} \in \ell^2$ and for all $\mu > 0$ and where $C_1, C_2 > 0$ are two positive constants.

   This result was implicitly used in [42] and it was proved in [43] by means of Carleman's inequalities for elliptic equations.

2. **Carleman inequalities for parabolic equations:** The null controllability of the heat equation with variable coefficients and lower order time-dependent terms has been studied by Fursikov and Imanuvilov (see for instance [16], [26], [27], [28], [34] and [35]). Their approach is based on the use of the Carleman inequalities for parabolic equations and is different to the one we have presented above. In [29], Carleman estimates are systematically applied to solve observability problem for linearized parabolic equations.

In [21] the boundary null controllability of the heat equation was proved in one space dimension using moment problems and classical results on the linear independence in $L^2(0,T)$ of families of real exponentials. We shall describe this method in the next chapter.

6 Boundary controllability of the heat equation

In this chapter the boundary null-controllability problem of the heat equation is studied. We do it by reducing the control problem to an equivalent problem of moments. The latter is solved with the aid of a biorthogonal sequence to a family of real exponential functions. This technique was used in the study of several control problems (the heat equation being one of the most relevant examples of application) in the late 60's and early 70's by R. D. Russell and H. O. Fattorini (see, for instance, [21] and [22]).

6.1 Introduction

Given $T>0$ arbitrary, $u^0\in L^2(0,1)$ and $f\in L^2(0,T)$ we consider the following non-homogeneous 1-D problem:

$$\left\{\begin{array}{ll} u_t-u_{xx}=0 & x\in(0,1),\ t\in(0,T)\\ u(t,0)=0,\quad u(t,1)=f(t) & t\in(0,T)\\ u(0,x)=u^0(x) & x\in(0,1).\end{array}\right. \tag{156}$$

In (156) $u=u(x,t)$ is the state and $f=f(t)$ is the control function which acts on the extreme $x=1$. We aim at changing the dynamics of the system by acting on the boundary of the domain $(0,1)$.

6.2 Existence and uniqueness of solutions

The following theorem is a consequence of classical results of existence and uniqueness of solutions of nonhomogeneous evolution equations. All the details may be found, for instance in [47].

Theorem 6.1 *For any $f\in L^2(0,T)$ and $u^0\in L^2(\Omega)$ equation (156) has a unique weak solution $u\in C([0,T],H^{-1}(\Omega))$.*

Moreover, the map $\{u^0,f\}\to\{u\}$ is linear and there exists $C=C(T)>0$ such that

$$||u||_{L^\infty(0,T;H^{-1}(\Omega))}\leq C\left(||u^0||_{L^2(\Omega)}+||f||_{L^2(0,T)}\right). \tag{157}$$

6.3 Controllability and problem of moments

In this section we introduce several notions of controllability.

Let $T>0$ and define, for any initial data $u^0\in L^2(\Omega)$, the set of reachable states

$$R(T;u^0)=\{u(T):u \text{ solution of (156) with } f\in L^2(0,T)\}. \tag{158}$$

An element of $R(T,u^0)$ is a state of (156) reachable in time T by starting from u^0 with the aid of a control f.

As in the previous chapter, several notions of controllability may be defined.

Definition 6.1 *System (156) is* **approximately controllable in time** T *if, for every initial data* $u^0 \in L^2(\Omega)$*, the set of reachable states* $R(T; u^0)$ *is dense in* $L^2(\Omega)$.

Definition 6.2 *System (156) is* **exactly controllable in time** T *if, for every initial data* $u^0 \in L^2(\Omega)$*, the set of reachable states* $R(T; u^0)$ *coincides with* $L^2(\Omega)$.

Definition 6.3 *System (156) is* **null controllable in time** T *if, for every initial data* $u^0 \in L^2(\Omega)$*, the set of reachable states* $R(T; u^0)$ *contains the element* 0.

Remark 6.1 *Note that the regularity of solutions stated above does not guarantee that* $u(T)$ *belongs to* $L^2(\Omega)$*. In view of this it could seem that the definitions above do not make sense. Note however that, due to the regularizing effect of the heat equation, if the control* f *vanishes in an arbitrarily small neighborhood of* $t = T$ *then* $u(T)$ *is in* C^∞ *and in particular in* $L^2(\Omega)$*. According to this, the above definitions make sense by introducing this minor restrictions on the controls under consideration.* □

Remark 6.2 *Let us make the following remarks, which are very close to those we did in the context of interior control:*

- *The linearity of the system under consideration implies that* $R(T, u^0) = R(T, 0) + S(T)u^0$ *and, consequently, without loss of generality one may assume that* $u^0 = 0$.
- *Due to the regularizing effect the solutions of (156) are in* C^∞ *far away from the boundary at time* $t = T$*. Hence, the elements of* $R(T, u^0)$ *are* C^∞ *functions in* $[0, 1)$*. Then, exact controllability may not hold.*
- *It is easy to see that if null controllability holds, then any initial data may be led to any final state of the form* $S(T)v^0$ *with* $v^0 \in L^2(\Omega)$.

  Indeed, let $u^0, v^0 \in L^2(\Omega)$ *and remark that* $R(T; u^0 - v^0) = R(T; u^0) - S(T)v^0$*. Since* $0 \in R(T; u^0 - v^0)$*, it follows that* $S(T)v^0 \in R(T; u^0)$.
- *Null controllability implies approximate controllability. Indeed we have that* $S(T)[L^2(\Omega)] \subset R(T; u^0)$ *and* $S(T)[L^2(\Omega)]$ *is dense in* $L^2(\Omega)$.
- *Note that* $u^1 \in R(T, u^0)$ *if and only if there exists a sequence* $(f_\varepsilon)_{\varepsilon>0}$ *of controls such that* $||u(T) - u^1||_{L^2(\Omega)} \leq \varepsilon$ *and* $(f_\varepsilon)_{\varepsilon>0}$ *is bounded in* $L^2(0, T)$*. Indeed, in this case, any weak limit in* $L^2(0, T)$ *of the sequence* $(f_\varepsilon)_{\varepsilon>0}$ *gives an exact control which makes that* $u(T) = u_1$. □

Remark 6.3 *As we shall see, null controllability of the heat equation holds in an arbitrarily small time. This is due to the infinity speed of propagation. It is important to underline, however, that, despite of the infinite speed of propagation, the null controllability of the heat equation does not hold in an infinite domain. We refer to [50] for a further discussion of this issue.*

The techniques we shall develop in this section do not apply in unbounded domains. Although, as shown in [50], using the similarity variables, one can find a spectral decomposition of solutions of the heat equation on the whole or half line, the spectrum is too dense and biorthogonal families do not exist. □

In this chapter the null-controllability problem will be considered. Let us first give the following characterization of the null-controllability property of (156).

Lemma 6.1 *Equation (156) is null-controllable in time $T>0$ if and only if, for any $u^0 \in L^2(0,1)$ there exists $f \in L^2(0,T)$ such that the following relation holds*

$$\int_0^T f(t)\varphi_x(t,1)dt = \int_0^1 u^0(x)\varphi(0,x)dx, \tag{159}$$

for any $\varphi_T \in L^2(0,1)$, where $\varphi(t,x)$ is the solution of the backward adjoint problem

$$\left\{\begin{array}{ll} \varphi_t + \varphi_{xx} = 0 & x \in (0,1),\ t \in (0,T) \\ \varphi(t,0) = \varphi(t,1) = 0 & t \in (0,T) \\ \varphi(T,x) = \varphi_T(x) & x \in (0,1). \end{array}\right. \tag{160}$$

Proof: Let $f \in L^2(0,T)$ be arbitrary and u the solution of (156). If $\varphi_T \in L^2(0,1)$ and φ is the solution of (160) then, by multiplying (156) by φ and by integrating by parts we obtain that

$$0 = \int_0^T \int_0^1 (u_t - u_{xx})\varphi dx dt = \int_0^1 u\varphi dx\Big|_0^T + \int_0^T (-u_x\varphi + u\varphi_x)dt\Big|_0^1 +$$

$$+ \int_0^T \int_0^1 u(-\varphi_t - \varphi_{xx})dxdt = \int_0^1 u\varphi dx\Big|_0^T + \int_0^T f(t)\varphi_x(t,1)dt.$$

Consequently

$$\int_0^T f(t)\varphi_x(t,1)dt = \int_0^1 u^0(x)\varphi(0,x)dx - \int_0^1 u(T,x)\varphi_T(x)dx. \tag{161}$$

Now, if (159) is verified, it follows that $\int_0^1 u(T,x)\varphi_T(x)dx = 0$, for all $\varphi^1 \in L^2(0,1)$ and $u(T)=0$.

Hence, the solution is controllable to zero and f is a control for (156).

Reciprocally, if f is a control for (156), we have that $u(T) = 0$. From (161) it follows that (159) holds and the proof finishes. $\square$

From the previous Lemma we deduce the following result:

Proposition 6.1 *Equation (156) is null-controllable in time* $T > 0$ *if and only if for any* $u^0 \in L^2(0,1)$, *with Fourier expansion*

$$u^0(x) = \sum_{n\geq 1} a_n \sin(\pi n x),$$

there exists a function $w \in L^2(0,T)$ *such that,*

$$\int_0^T w(t) e^{-n^2\pi^2 t} dt = (-1)^n \frac{a_n}{2n\pi} e^{-n^2\pi^2 T}, \quad n = 1, 2, ... \tag{162}$$

Remark 6.4 *Problem (162) is usually refered to as* **problem of moments.**

Proof: From the previous Lemma we know that $f \in L^2(0,T)$ is a control for (156) if and only if it satisfies (159). But, since $(\sin(n\pi x))_{n\geq 1}$ forms an orthogonal basis in $L^2(0,1)$, (159) is verified if and only if it is verified by $\varphi_n^1 = \sin(n\pi x)$, $n = 1, 2, ...$

If $\varphi_n^1 = \sin(n\pi x)$ then the corresponding solution of (160) is $\varphi(t,x) = e^{-n^2\pi^2(T-t)} \sin(n\pi x)$ and from (159) we obtain that

$$\int_0^T f(t)(-1)^n n\pi e^{-n^2\pi^2(T-t)} = \frac{a_n}{2} e^{-n^2\pi^2 T}.$$

The proof ends by taking $w(t) = f(T-t)$. $\square$

The control property has been reduced to the problem of moments (162). The latter will be solved by using biorthogonal techniques. The main ideas are due to R.D. Russell and H.O. Fattorini (see, for instance, [21] and [22]).

The eigenvalues of the heat equation are $\lambda_n = n^2\pi^2$, $n \geq 1$. Let $\Lambda = \left(e^{-\lambda_n t}\right)_{n\geq 1}$ be the family of the corresponding real exponential functions.

Definition 6.4 $(\theta_m)_{m\geq 1}$ *is a* **biorthogonal sequence** *to* Λ *in* $L^2(0,T)$ *if and only if*

$$\int_0^T e^{-\lambda_n t} \theta_m(t) dt = \delta_{nm}, \quad \forall n, m = 1, 2, ...$$

If there exists a biorthogonal sequence $(\theta_m)_{m\geq 1}$, the problem of moments (162) may be solved immediately by setting

$$w(t) = \sum_{m\geq 1} (-1)^m \frac{a_m}{2m\pi} e^{-m^2\pi^2 T} \theta_m(t). \tag{163}$$

As soon as the series converges in $L^2(0,T)$, this provides the solution to (162).

We have the following controllability result:

Theorem 6.2 *Given $T>0$, suppose that there exists a biorthogonal sequence $(\theta_m)_{m\geq 1}$ to Λ in $L^2(0,T)$ such that*

$$||\theta_m||_{L^2(0,T)} \leq Me^{\omega m}, \quad \forall m \geq 1 \tag{164}$$

where M and ω are two positive constants.

Then (156) is null-controllable in time T.

Proof: From Proposition 6.1 it follows that it is sufficient to show that for any $u^0 \in L^2(0,1)$ with Fourier expansion

$$u^0 = \sum_{n\geq 1} a_n \sin(n\pi x),$$

there exists a function $w \in L^2(0,T)$ which verifies (162).

Consider

$$w(t) = \sum_{m\geq 1} (-1)^m \frac{a_m}{2m\pi} e^{-m^2\pi^2 T}\theta_m(t). \tag{165}$$

Note that the series which defines w is convergent in $L^2(0,T)$. Indeed,

$$\sum_{m\geq 1} \left|\left|(-1)^m \frac{a_m}{2m\pi} e^{-m^2\pi^2 T}\theta_m\right|\right|_{L^2(0,T)} = \sum_{m\geq 1} \frac{|a_m|}{2m\pi} e^{-m^2\pi^2 T}||\theta_m||_{L^2(0,T)} \leq$$

$$\leq M \sum_{m\geq 1} \frac{|a_m|}{2m\pi} e^{-m^2\pi^2 T + \omega m} < \infty$$

where we have used the estimates (164) of the norm of the biorthogonal sequence (θ_m).

On the other hand, (165) implies that w satisfies (162) and the proof finishes. □

Theorem 6.2 shows that, the null-controllability problem (156) is solved if we prove the existence of a biorthogonal sequence $(\theta_m)_{m\geq 1}$ to Λ in $L^2(0,T)$ which verifies (164). The following sections are devoted to accomplish this task.

6.4 Existence of a biorthogonal sequence

The existence of a biorthogonal sequence to the family Λ is a consequence of the following Theorem (see, for instance, [57]).

Theorem 6.3 *(Münz) Let $0 < \mu_1 \leq \mu_2 \leq ... \leq \mu_n \leq ...$ be a sequence of real numbers. The family of exponential functions $(e^{-\mu_n t})_{n\geq 1}$ is complete in $L^2(0,T)$ if and only if*

$$\sum_{n\geq 1} \frac{1}{\mu_n} = \infty. \tag{166}$$

Given any $T > 0$, from Münz's Theorem we obtain that the space generated by the family Λ is a proper space of $L^2(0,T)$ since

$$\sum_{n\geq 1} \frac{1}{\lambda_n} = \sum_{n\geq 1} \frac{1}{n^2\pi^2} < \infty.$$

Let $E(\Lambda, T)$ be the space generated by Λ in $L^2(0,T)$ and $E(m, \Lambda, T)$ be the subspace generated by $\left(e^{-\lambda_n t}\right)_{\substack{n\geq 1\\ n\neq m}}$ in $L^2(0,T)$.

We also introduce the notation $p_n(t) = e^{-\lambda_n t}$.

Theorem 6.4 *Given any $T > 0$, there exists a unique sequence $(\theta_m(T, \cdot))_{m\geq 1}$, biorthogonal to the family Λ, such that*

$$(\theta_m(T, \cdot))_{m\geq 1} \subset E(\Lambda, T).$$

Moreover, this biorthogonal sequence has minimal $L^2(0,T)$-norm.

Proof: Since Λ is not complete in $L^2(0,T)$, it is also minimal. Thus, $p_m \notin E(m, \Lambda, T)$, for each $m \in I$.

Let r_m be the orthogonal projection p_m over the space $E(m, \Lambda, T)$ and define

$$\theta_m(T, \cdot) = \frac{p_m - r_m}{||p_m - r_m||^2_{L^2(0,T)}}. \tag{167}$$

From the projection properties (see [10], pp. 79-80), it follows that

1. $r_m \in E(m, \Lambda, T)$ verifies $||p_m(t) - r_m(t)||_{L^2(0,T)} = \min_{r\in E(m,\Lambda,T)} ||p_m - r||_{L^2(0,T)}$
2. $(p_m - r_m) \perp E(m, \Lambda, T)$
3. $(p_m - r_m) \perp p_n \in E(m, \Lambda, T)$, $\forall n \neq m$
4. $(p_m - r_m) \perp r_m \in E(m, \Lambda, T)$.

From the previous properties and (167) we deduce that

1. $\int_0^T \theta_m(T,t) p_n(t) dt = \delta_{m,n}$
2. $\theta_m(T, \cdot) = \frac{p_m - r_m}{||p_m - r_m||^2} \in E(\Lambda, T)$.

Thus, (167) gives a biorthogonal sequence $(\theta_m(T,\cdot))_{m\geq 1} \subset E(\Lambda,T)$ to the family Λ.

The uniqueness of the biorthogonal sequence is obtained immediately. Indeed, if $(\theta'_m)_{m\geq 1} \subset E(\Lambda,T)$ is another biorthogonal sequence to the family Λ, then

$$\left.\begin{array}{l}(\theta_m - \theta'_m) \in E(\Lambda,T)\\ p_n \perp (\theta_m - \theta'_m),\ \forall n \geq 1\end{array}\right\} \Rightarrow \theta_m - \theta'_m = 0$$

where we have taken into account that $(p_m)_{m\geq 1}$ is complete in $E(\Lambda,T)$.

To prove the minimality of the norm of $(\theta_m(T,\cdot))_{m\geq 1}$, let us consider any other biorthogonal sequence $(\theta'_m)_{m\geq 1} \subset L^2(0,T)$.

$E(\Lambda,T)$ being closed in $L^2(0,T)$, its orthogonal complement, $E(\Lambda,T)^\perp$, is well defined. Thus, for any $m \geq 1$, there exists a unique $q_m \in E(\Lambda,T)^\perp$ such that $\theta'_m = \theta_m + q_m$.

Finally,

$$||\theta'_m||^2 = ||\theta_m + q_m||^2 = ||\theta_m||^2 + ||q_m||^2 \geq ||\theta_m||^2$$

and the proof ends. □

Remark 6.5 *The previous Theorem gives a biorthogonal sequence of minimal norm. This property is important since the convergence of the series of (163) depends directly of these norms.* □

The existence of a biorthogonal sequence $(\theta_m)_{m\geq 1}$ to the family Λ being proved, the next step is to evaluate its $L^2(0,T)$-norm. This will be done in two steps. First for the case $T = \infty$ and next for $T < \infty$.

6.5 Estimate of the norm of the biorthogonal sequence: $T = \infty$

Theorem 6.5 *There exist two positive constants M and ω such that the biorthogonal of minimal norm $(\theta_m(\infty,\cdot)_{m\geq 1}$ given by Theorem 6.4 satisfies the following estimate*

$$||\theta_m(\infty,\cdot)||_{L^2(0,\infty)} \leq M\pi e^{\omega m}, \quad \forall m \geq 1. \tag{168}$$

Proof: Let us introduce the following notations: $E^n := E^n(\Lambda,\infty)$ is the subspace generated by $\Lambda^n := \left(e^{-\lambda_k t}\right)_{1\leq k\leq n}$ in $L^2(0,T)$ and $E^n_m := E^2(m,\Lambda,\infty)$ is the subspace generated by $\left(e^{-\lambda_k t}\right)_{\substack{1\leq k\leq n\\ k\neq m}}$ in $L^2(0,T)$.

Remark that E^n and E^n_m are finite dimensional spaces and

$$E(\Lambda,\infty) = \cup_{n\geq 1} E^n, \quad E(m,\Lambda,\infty) = \cup_{n\geq 1} E^n_m.$$

We have that, for each $n \geq 1$, there exists a unique biorthogonal family $(\theta_m^n)_{1\leq m\leq n} \subset E^n$, to the family of exponentials $\left(e^{-\lambda_k t}\right)_{1\leq k\leq n}$. More precisely,

$$\theta_m^n = \frac{p_m - r_m^n}{||p_m - r_m^n||^2_{L^2(0,\infty)}}, \tag{169}$$

where r_m^n is the orthogonal projection of p_m over E_m^n.

If

$$\theta_m^n = \sum_{k=1}^n c_k^m p_k \tag{170}$$

then, by multiplying (170) by p_l and by integrating in $(0,\infty)$, it follows that

$$\delta_{m,l} = \sum_{k\geq 1} c_k^m \int_0^T p_l(t) p_k(t) dt, \quad 1 \leq m, l \leq n. \tag{171}$$

Moreover, by multiplying in (170) by θ_m^n and by integrating in $(0,\infty)$, we obtain that

$$||\theta_m^n||^2_{L^2(0,\infty)} = c_m^m. \tag{172}$$

If G denotes the Gramm matrix of the family Λ, i. e. the matrix of elements

$$g_k^l = \int_0^\infty p_k(t) p_l(t) dt, \quad 1 \leq k, l \leq n$$

we deduce from (171) that c_k^m are the elements of the inverse of G. Cramer's rule implies that

$$c_m^m = \frac{|G_m|}{|G|} \tag{173}$$

where $|G|$ is the determinant of matrix G and $|G_m|$ is the determinant of the matrix G_m obtained by changing the $m-$th column of G by the $m-$th vector of the canonical basis.

It follows that

$$||\theta_m^n||_{L^2(0,\infty)} = \sqrt{\frac{|G_m|}{|G|}}. \tag{174}$$

The elements of G may be computed explicitly

$$g_k^n = \int_0^\infty p_k(t) p_n(t) = \int_0^\infty e^{-(n^2+k^2)\pi^2 t} dt = \frac{1}{n^2\pi^2 + k^2\pi^2}.$$

Remark 6.6 *A formula, similar to (174), may be obtained for any $T > 0$. Nevertheless, the determinants may be estimated only in the case $T = \infty$.* □

To compute the determinats $|G|$ and $|G_m|$ we use the following lemma (see [17]):

Lemma 6.2 *If* $C = (c_{ij})_{1\le i,j\le n}$ *is a matrix of coefficients* $c_{ij} = \frac{1}{a_i+b_j}$ *then*

$$|C| = \frac{\prod_{1\le i<j\le n}(a_i - a_j)(b_i - b_j)}{\prod_{1\le i,j\le n}(a_i + b_j)}. \tag{175}$$

It follows that

$$|G| = \frac{\prod_{1\le i<j\le n}(i^2\pi^2 - j^2\pi^2)^2}{\prod_{1\le i,j\le n}(i^2\pi^2 + j^2\pi^2)}, \quad |G_m| = \frac{\prod'_{1\le i<j\le n}(i^2\pi^2 - j^2\pi^2)^2}{\prod'_{1\le i,j\le n}(i^2\pi^2 + j^2\pi^2)}$$

where $'$ means that the index m has been skipped in the product.

Hence,

$$\frac{|G_m|}{|G|} = 2m^2\pi^2 {\prod_{k=1}^{n}}' \frac{(m^2+k^2)^2}{(m^2-k^2)^2}. \tag{176}$$

From (174) and (176) we deduce that

$$||\theta_m^n||_{L^2(0,\infty)} = \sqrt{2}m\pi {\prod_{k=1}^{n}}' \frac{m^2+k^2}{|m^2-k^2|}. \tag{177}$$

Lemma 6.3 *The norm of the biorthogonal sequence* $(\theta_m(\infty,\cdot))_{m\ge1}$ *to the family* Λ *in* $L^2(0,\infty)$ *given by Theorem 6.4, verifies*

$$||\theta_m(\infty,\cdot)||_{L^2(0,\infty)} = \sqrt{2}m\pi {\prod_{k=1}^{\infty}}' \frac{m^2+k^2}{|m^2-k^2|}. \tag{178}$$

Proof: It consists in passing to the limit in (177) as $n\to\infty$. Remark first that, for each $m\ge1$, the product

$${\prod_{k=1}^{\infty}}' \frac{m^2+k^2}{|m^2-k^2|}$$

is convergent since

$$1 \le {\prod_{k=1}^{\infty}}' \frac{m^2+k^2}{|m^2-k^2|} = \exp\left({\sum_{k=1}^{\infty}}' \ln\left(\frac{m^2+k^2}{|m^2-k^2|}\right)\right) \le$$

$$\le \exp\left({\sum_{k=1}^{\infty}}' \ln\left(1+\frac{2m^2}{|m^2-k^2|}\right)\right) \le \exp\left(2m^2{\sum_{k=1}^{\infty}}' \frac{1}{|m^2-k^2|}\right) < \infty.$$

Consequently, the limit $\lim_{n\to\infty}||\theta_m^n||_{L^2(0,\infty)} = L \geq 1$ exists. The proof ends if we prove that

$$\lim_{n\to\infty}||\theta_m^n||_{L^2(0,\infty)} = ||\theta_m||_{L^2(0,\infty)}. \tag{179}$$

Identity (169) implies that $\lim_{n\to\infty}||p_m - r_m^n||_{L^2(0,\infty)} = 1/L$ and (179) is equivalent to

$$\lim_{n\to\infty}||p_m - r_m^n||_{L^2(0,\infty)} = ||p_m - r_m||_{L^2(0,\infty)}. \tag{180}$$

Let now $\varepsilon > 0$ be arbitrary. Since $r_m \in E(m,\Lambda,\infty)$ it follows that there exist $n(\varepsilon) \in \mathbb{N}^*$ and $r_m^\varepsilon \in E_m^{n(\varepsilon)}$ with

$$||r_m - r_m^\varepsilon||_{L^2(0,\infty)} < \varepsilon.$$

For any $n \geq n(\varepsilon)$ we have that

$$||p_m - r_m|| = \min_{r\in E(m,\Lambda,\infty)}||p_m - r|| \leq ||p_m - r_m^n|| = \min_{r\in E_m^n}||p_m - r|| \leq$$

$$\leq ||p_m - r_m^\varepsilon|| \leq ||p_m - r_m|| + ||r_m - r_m^\varepsilon|| < ||p_m - r_m|| + \varepsilon.$$

Thus, (180) holds and Lemma 6.3 is proved. □

Finally, to evaluate $\theta_m(\infty,\cdot)$ we use the following estimate

Lemma 6.4 *There exist two positive constants M and ω such that for any $m \geq 1$,*

$$\prod_{k=1}^{\infty}{}' \frac{m^2+k^2}{|m^2-k^2|} \leq Me^{\omega m}. \tag{181}$$

Proof: Remark that

$$\prod_k{}' \frac{m^2+k^2}{|m^2-k^2|} = \exp\left[\sum_k{}' \ln\left(\frac{m^2+k^2}{|m^2-k^2|}\right)\right] \leq \exp\left[\sum_k{}' \ln\left(1+\frac{2m^2}{|m^2-k^2|}\right)\right].$$

Now

$$\sum_k{}' \ln\left(1+\frac{2m^2}{|m^2-k^2|}\right) \leq \int_1^m \ln\left(1+\frac{2m^2}{m^2-x^2}\right)dx +$$

$$+\int_m^{2m} \ln\left(1+\frac{2m^2}{x^2-m^2}\right)dx + \int_{2m}^{\infty} \ln\left(1+\frac{2m^2}{x^2-m^2}\right)dx =$$

$$= m\left[\int_0^1 \ln\left(1+\frac{2}{1-x^2}\right)dx + \int_1^2 \ln\left(1+\frac{2}{x^2-1}\right)dx +\right.$$

$$+\int_2^\infty \ln\left(1+\frac{2}{x^2-1}\right)dx\bigg] = m\left(I_1+I_2+I_3\right).$$

We evaluate now each one of these integrals.

$$I_1 = \int_0^1 \ln\left(1+\frac{2}{1-x^2}\right)dx = \int_0^1 \ln\left(1+\frac{2}{(1-x)(1+x)}\right)dx \leq$$

$$\int_0^1 \ln\left(1+\frac{2}{1-x}\right)dx == -\int_0^1 (1-x)' \ln\left(1+\frac{2}{1-x}\right)dx =$$

$$== -(1-x)\ln\left(1+\frac{2}{1-x}\right)\bigg|_0^1 + \int_0^1 \frac{2}{3-x}dx = c_1 < \infty,$$

$$I_2 = \int_1^2 \ln\left(1+\frac{2}{x^2-1}\right)dx \leq \int_1^2 \ln\left(1+\frac{2}{(x-1)^2}\right)dx =$$

$$= \int_1^2 (x-1)' \ln\left(1+\frac{2}{(x-1)^2}\right)dx =$$

$$= -(x-1)\ln\left(1+\frac{2}{(x-1)^2}\right)\bigg|_0^1 + \int_1^2 \frac{2}{2+(x-1)^2}dx = c_2 < \infty.$$

$$I_3 = \int_2^\infty \ln\left(1+\frac{2}{x^2-1}\right)dx \leq \int_2^\infty \ln\left(1+\frac{2}{(x-1)^2}\right)dx \leq$$

$$\leq \int_2^\infty \frac{2}{(x-1)^2}dx = c_3 < \infty.$$

The proof finishes by taking $\omega = c_1 + c_2 + c_3$. □

The proof of Theorem 6.5 ends by taking into account relation (178) and Lemma 6.4. □

6.6 Estimate of the norm of the biorthogonal sequence: $T < \infty$

We consider now $T < \infty$. To evaluate the norm of the biorthogonal sequence $(\theta_m(T,\cdot))_{m\geq 1}$ in $L^2(0,T)$ the following result is necessary. The first version of this result may be found in [57] (see also [21] and [58]).

Theorem 6.6 *Let Λ be the family of exponential functions $\left(e^{-\lambda_n t}\right)_{n\geq 1}$ and let T be arbitrary in $(0,\infty)$. The restriction operator*

$$R_T : E(\Lambda,\infty) \to E(\Lambda,T), \quad R_T(v) = v_{|[0,T]}$$

is invertible and there exists a constant $C > 0$, which only depends on T, such that

$$||R_T^{-1}|| \leq C. \tag{182}$$

Proof: Suppose that, on the contrary, for some $T > 0$ there exists a sequence of exponential polynomials

$$P_k(t) = \sum_{n=1}^{N(k)} a_{kn} e^{-\lambda_n t} \subset E(\Lambda, T)$$

such that

$$\lim_{k\to\infty} ||P_k||_{L^2(0,T)} = 0 \tag{183}$$

and

$$||P_k||_{L^2(0,\infty)} = 1, \quad \forall k \geq 1. \tag{184}$$

By using the estimates from Theorem 6.5 we obtain that

$$|a_{mn}| = \left| \int_0^\infty P_k(t)\theta_m(\infty, t)dt \right| \leq ||P_k||_{L^2(0,\infty)} ||\theta_m(\infty, \cdot)||_{L^2(0,\infty)} \leq M e^{\omega m}.$$

Thus

$$|P_k(z)| \leq \sum_{n=1}^{N(k)} |a_{kn}| \left| e^{-\lambda_n z} \right| \leq M \sum_{n=1}^{\infty} e^{\omega n - n^2\pi^2 \mathcal{R}e(z)}. \tag{185}$$

If $r > 0$ is given let $\Delta_r = \{z \in \mathbb{C} : \mathcal{R}e(z) > r\}$. For all $z \in \Delta_r$, we have that

$$|P_k(z)| \leq M \sum_{n=1}^{\infty} e^{\omega n - n^2\pi^2 r} \leq M(\omega, r). \tag{186}$$

Hence, the family $(P_k)_{k\geq 1}$ consists of uniformly bounded entire functions. From Montel's Theorem (see [15]) it follows that there exists a subsequence, denoted in the same way, which converges uniformly on compact sets of Δ_r to an analytic function P.

Choose $r < T$. From (183) it follows that $\lim_{k\to\infty} ||P_k||_{L^2(r,T)} = 0$ and therefore $P(t) = 0$ for all $t \in (r, T)$. Since P is analytic in Δ_r, P must be identically zero in Δ_r.

Hence, $(P_k)_{k\geq 1}$ converges uniformly to zero on compact sets of Δ_r.

Let us now return to (185). There exists $r_0 > 0$ such that

$$|P_k(z)| \leq M e^{-\mathcal{R}e(z)}, \quad \forall z \in \Delta_{r_0}. \tag{187}$$

Indeed, there exists $r_0 > 0$ such that

$$\omega n - n^2\pi^2 \mathcal{R}e(z) \leq -\mathcal{R}e(z) - n, \quad \forall z \in \Delta_{r_0}$$

and therefore, for any $z \in \Delta_{r_0}$,

$$|P_k(z)| \leq M \sum_{n\geq 1} e^{\omega n - n^2\pi^2 \mathcal{R}e(z)} \leq M e^{-\mathcal{R}e(z)} \sum_{n\geq 1} e^{-n} = \frac{M}{e-1} e^{-\mathcal{R}e(z)}.$$

Lebesgue's Theorem implies that

$$\lim_{k\to\infty} ||P_k||_{L^2(r,\infty)} = 0$$

and consequently

$$\lim_{k\to\infty} ||P_k||_{L^2(0,r)} = 1.$$

If we take $r < T$ the last relation contradicts (184) and the proof ends. □

We can now evaluate the norm of the biorthogonal sequence.

Theorem 6.7 *There exist two positive constants M and ω with the property that*

$$||\theta_m(T,\cdot)||_{L^2(0,T)} \leq M e^{\omega m}, \quad \forall m \geq 1 \tag{188}$$

where $(\theta_m(T,\cdot))_{m\geq 1}$ is the biorthogonal sequence to the family Λ in $L^2(0,T)$ which belongs to $E(\Lambda,T)$ and it is given in Theorem 6.4.

Proof: Let $(R_T^{-1})^* : E(\Lambda,\infty) \to E(\Lambda,T)$ be the adjoint of the bounded operator R_T^{-1}. We have that

$$\delta_{kj} = \int_0^\infty p_k(t)\theta_j(\infty,t)dt = \int_0^\infty (R_T^{-1}R_T)(p_k(t))\theta_j(\infty,t)dt =$$

$$= \int_0^T R_T(p_k(t))(R_T^{-1})^*(\theta_j(\infty,t))dt.$$

Since $(R_T^{-1})^*(\theta_j(\infty,\cdot)) \in E(\Lambda,T)$, from the uniqueness of the biorthogonal sequence in $E(\Lambda,T)$, we finally obtain that

$$(R_T^{-1})^*(\theta_j(\infty,\cdot)) = \theta_j(T,\cdot), \quad \forall j \geq 1.$$

Hence

$$||\theta_j(T,\cdot)||_{L^2(0,T)} = \left|\left|(R_T^{-1})^*(\theta_j(\infty,\cdot))\right|\right|_{L^2(0,T)} \leq ||R_T^{-1}||\,||\theta_j(\infty,\cdot)||_{L^2(0,\infty)},$$

since $||(R_T^{-1})^*|| = ||R_T^{-1}||$.

The proof finishes by taking into account the estimates from Theorem 6.5. □

Remark 6.7 *From the proof of Theorem 6.7 it follows that the constant ω does not depend of T.* □

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