

Numerical approximation of the boundary control for the wave equation with mixed finite elements in a square

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This paper studies the numerical approximation of the boundary control for the wave equation in a square domain. It is known that the discrete and semi-discrete models obtained by discretizing the wave equation with the usual finite-difference or finite-element methods do not provide convergent sequences of approximations to the boundary control of the continuous wave equation as the mesh size goes to zero. Here, we introduce and analyse a new semi-discrete model based on the space discretization of the wave equation using a mixed finite-element method with two different basis functions for the position and velocity. The main theoretical result is a uniform observability inequality which allows us to construct a sequence of approximations converging to the minimal L^2 -norm control of the continuous wave equation. We also introduce a fully discrete system, obtained from our semi-discrete scheme, for which we conjecture that it provides a convergent sequence of discrete approximations as both h and Δt , the time discretization parameter, go to zero. We illustrate this fact with several numerical experiments.

Keywords: exact controllability; observability; numerical approximation of controls; wave equation.

1. Introduction

Let us consider $\Omega = (0, 1) \times (0, 1) \subset \mathbb{R}^2$ with boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ divided as follows:

$$\begin{cases} \Gamma_0 = \{(x, 0): 0 \leq x \leq 1\} \cup \{(0, y): 0 \leq y \leq 1\}, \\ \Gamma_1 = \{(x, 1): 0 < x < 1\} \cup \{(1, y): 0 < y \leq 1\}. \end{cases} \quad (1.1)$$

We are concerned with the following exact boundary controllability property for the wave equation in Ω : given T sufficiently large and $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, there exists a control $(v(t, y),$

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$z(t, x) \in [L^2((0, T) \times (0, 1))]^2$ such that the solution of the equation

$$\begin{cases} u'' - \Delta u = 0 & \text{for } (x, y) \in \Omega, \quad t > 0, \\ u(t, x, y) = 0 & \text{for } (x, y) \in \Gamma_0, \quad t > 0, \\ u(t, 1, y) = v(t, y) & \text{for } y \in (0, 1), \quad t > 0, \\ u(t, x, 1) = z(t, x) & \text{for } x \in (0, 1), \quad t > 0, \\ u(0, x, y) = u^0(x, y) & \text{for } (x, y) \in \Omega, \\ u'(0, x, y) = u^1(x, y) & \text{for } (x, y) \in \Omega \end{cases} \quad (1.2)$$

satisfies

$$u(T, \cdot) = u'(T, \cdot) = 0. \quad (1.3)$$

By $'$ we denote the time derivative.

The Hilbert uniqueness method (HUM) introduced by J.-L. Lions provides a control (v, z) with minimal L^2 -norm (see Lions, 1988). This control is unique and it will be referred in the sequel as the HUM control. We briefly describe this method at the beginning of Section 2 below.

In the last years, many works have dealt with the numerical approximations of the control problem (1.2)–(1.3). For instance, in Glowinski (1991), Glowinski *et al.* (1989, 1990) and Glowinski & Lions (1996), numerical algorithms based on the finite-difference and finite-element approximations of (1.2) were described. However, these algorithms do not converge when the discretization parameters go to zero.

Let us briefly explain this fact. When we are dealing with the exact controllability problem, a uniform time $T > 0$ for the control of ‘all solutions’ is required. This time T depends on the size of the domain and the velocity of propagation of waves. In general, any semi-discrete dynamics generates spurious high-frequency oscillations that do not exist at the continuous level. Moreover, a numerical dispersion phenomenon appears and the velocity of propagation of some of these high-frequency numerical waves may possibly converge to zero when the mesh size h does. Consequently, the controllability property for the semi-discrete system will not be uniform for a fixed time T . This is the case when the semi-discrete model is obtained by discretizing the wave equation with the usual finite-difference or finite-element method (see Infante & Zuazua (1999) for a detailed analysis of the 1D case and Zuazua (1999) for the 2D case, in the context of the dual observability problem).

From the numerical point of view, several techniques have been proposed as possible cures of the low velocity of propagation of the high-frequency spurious oscillations (see, for instance, Glowinski, 1991; Glowinski *et al.*, 1989, 1990; Glowinski & Lions, 1996). To our knowledge, no proof of convergence has been given for any of these methods, as h tends to 0, so far.

In this paper, we construct, for any T sufficiently large but independent of h , a convergent sequence of semi-discrete approximations of the HUM control (v, z) of (1.2). The main idea is to introduce a new space discretization scheme for the wave equation (1.2), based on a ‘mixed finite-element method’, in which different base functions for the position u and the velocity u' are considered. More precisely, while the usual linear finite elements are used for the former, discontinuous elements approximate the latter. This new scheme still has spurious high-frequency oscillations but the numerical dispersion makes them to have larger velocity of propagation. Consequently, the velocity of propagation of all waves is bounded from below by a uniform positive constant.

The semi-discrete approximations $(v_h, z_h)_{h>0}$ of the HUM control (v, z) of (1.2) are obtained by minimizing a functional \mathcal{J}_h depending on the associated space-discretized adjoint system (see (5.1)). The main result of the paper is Theorem 4.2 which gives a uniform (in h) observability inequality for

this homogeneous semi-discrete adjoint system. This is equivalent to the uniform coercivity of \mathcal{J}_h . Theorem 4.2 permits to show that if a weakly convergent sequence of approximations of the continuous initial data (u^0, u^1) is considered, the sequence of approximations $(v_h, z_h)_{h>0}$ converges weakly to (v, z) (Theorems 6.3 and 6.4).

To our knowledge, the scheme described in this paper was used by the first time in the context of the wave equation in Banks *et al.* (1991) in order to obtain a uniform decay rate of the energy associated to the semi-discrete wave equation by a boundary dissipation. This scheme is different from the mixed element method applied in Glowinski *et al.* (1989) where two different basis functions are considered for u and ∇u .

In this paper, we concentrate on the simplest 2D domain consisting of a unit square. The mixed finite-element method may be applied to general domains but our proofs of the uniform observability and convergence strongly depend on the particular geometry of the square and cannot be generalized.

We also introduce a fully discrete approximation of the wave equation, based on the semi-discrete scheme, for which the velocity of propagation of all numerical waves does not vanish as both h and Δt , the time discretization parameter, tend to zero. Based on this fact, we conjecture that this fully discrete scheme also provides convergent approximations of the control. At the end, we include two numerical experiments that illustrate this fact.

The rest of the paper is organized in the following way: Section 2 briefly recalls some controllability results for the wave equation (1.2) and introduces the HUM. In Section 3, the semi-discrete model under consideration is deduced. In Section 4, the main properties of this system are discussed and, in particular, the fundamental uniform observability inequality (Theorem 4.2). Its technical proof is given in Appendix A at the end of the paper. In Section 5, an approximation sequence is constructed and in Section 6, its convergence to the HUM control of the continuous equation (1.2) is proved. Section 7 is devoted to present the fully discrete scheme and the numerical results.

2. The continuous problem: results and notations

In this section, we recall some of the controllability properties of the wave equation (1.2) and we briefly describe the HUM. Also, we introduce some notations that will be used in the article. The following classical result may be found, for instance, in Lions (1988).

THEOREM 2.1 Given any $T > 2\sqrt{2}$ and $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, there exists a control function $(v, z) \in [L^2((0, T) \times (0, 1))]^2$ such that the solution (u, u') of (1.2) verifies (1.3).

In general, there are infinitely many controls when they exist. However, the one with minimal L^2 -norm is unique and can be characterized by the minimizer of a suitable functional. Let us introduce the map $\mathcal{J}: H_0^1(\Omega) \times L^2(\Omega) \rightarrow \mathbb{R}$ defined by

$$\begin{aligned} \mathcal{J}(w^0, w^1) &= \frac{1}{2} \int_0^T \int_0^1 (w_x)^2(t, 1, y) dy dt + \frac{1}{2} \int_0^T \int_0^1 (w_y)^2(t, x, 1) dx dt \\ &\quad + \int_{\Omega} u^0(x, y) w'(0, x, y) dx dy - \langle u^1, w(0, \cdot) \rangle_{-1,1}, \end{aligned} \quad (2.1)$$

where (w, w') is the solution of the backward homogeneous equation

$$\begin{cases} w'' - \Delta w = 0 & \text{for } (x, y) \in \Omega, \quad t > 0, \\ w(t, 0, y) = w(t, x, 0) = w(t, x, 1) = w(t, 1, y) = 0 & \text{for } x, y \in [0, 1], \quad t > 0, \\ w(T, x, y) = w^0(x, y), \quad w'(T, x, y) = w^1(x, y) & \text{for } (x, y) \in \Omega. \end{cases} \quad (2.2)$$

In (2.1), $\langle \cdot, \cdot \rangle_{-1,1}$ denotes the duality product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

THEOREM 2.2 Given any $T > 2\sqrt{2}$ and $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, \mathcal{J} has a unique minimizer $(\widehat{w}^0, \widehat{w}^1) \in H_0^1(\Omega) \times L^2(\Omega)$. If $(\widehat{w}, \widehat{w}')$ is the solution of (2.2) with initial data $(\widehat{w}^0, \widehat{w}^1)$, then

$$(v(t, y), z(t, x)) = (\widehat{w}_x(t, 1, y), \widehat{w}_y(t, x, 1)) \quad (2.3)$$

is the control of (1.2) with minimal L^2 -norm.

The method we have just presented was introduced by J.-L. Lions (see Lions, 1988) and named HUM. The control (v, z) given by (2.3) is usually called the HUM control.

We recall that the main ingredient of the proof of Theorem 2.2 is the following observability inequality for (2.2): given $T > 2\sqrt{2}$, there exists a constant $C > 0$ such that the following inequality holds for any solution of (2.2):

$$\int_{\Omega} (|\nabla w|^2 + |w_t|^2) dx dy \leq C \left(\int_0^T \int_0^1 |w_x(t, 1, y)|^2 dy dt + \int_0^T \int_0^1 |w_y(t, x, 1)|^2 dx dt \right). \quad (2.4)$$

Indeed, (2.4) implies that \mathcal{J} is coercive and ensures the existence of a minimizer, as stated in Theorem 2.2.

REMARK 2.3 For the continuous wave equation (1.2), the velocity of propagation of all waves is one and the bound of the minimal controllability time, $T > 2\sqrt{2}$, is exactly the minimum time that requires a wave, starting at any $x \in \Omega$ in any direction, to arrive to the controllability zone.

REMARK 2.4 The control (v, z) from Theorem 2.2 is characterized by the following two properties:

1. (v, z) is a control for (1.2) or, equivalently,

$$\begin{aligned} & \int_0^T \int_0^1 v(t, y) w_x(t, 1, y) dy dt + \int_0^T \int_0^1 z(t, x) w_y(t, x, 1) dx dt \\ &= \langle u^1, w(0) \rangle_{-1,1} - \int_{\Omega} u^0(x, y) w'(0, x, y) dx dy \end{aligned} \quad (2.5)$$

for any $(w^0, w^1) \in H_0^1(\Omega) \times L^2(\Omega)$, being w the solution of the adjoint equation (2.2).

2. There exists $(\widehat{w}^0, \widehat{w}^1) \in H_0^1(\Omega) \times L^2(\Omega)$ such that $v(t, y) = \widehat{w}_x(t, 1, y)$ and $z(t, x) = \widehat{w}_y(t, x, 1)$, where $(\widehat{w}, \widehat{w}')$ is the solution of the adjoint system (2.2) with initial data $(\widehat{w}^0, \widehat{w}^1)$.

Much of our analysis will be based on Fourier expansion of solutions. Therefore, let us now introduce the eigenvalues of the wave equation (2.2):

$$\lambda^{nm} = \operatorname{sgn}(n) \sqrt{n^2 + m^2} \pi \quad (2.6)$$

and the corresponding eigenfunctions:

$$\Psi^{nm}(x, y) = \sqrt{2} \begin{pmatrix} i\lambda^{nm} & -1 \\ -1 & \end{pmatrix} \sin(n\pi x) \sin(m\pi y), \quad (n, m) \in \mathbb{Z}^* \times \mathbb{N}^*, \quad i = \sqrt{-1}. \quad (2.7)$$

The sequence $(\Psi^{nm})_{(n,m) \in \mathbb{Z}^* \times \mathbb{N}^*}$ forms an orthonormal basis in $H_0^1(\Omega) \times L^2(\Omega)$. Moreover,

$$\|\Psi^{nm}\|_{L^2(\Omega) \times H^{-1}(\Omega)} = \frac{1}{\lambda^{nm}}.$$

The following characterization of any control of (1.2) in terms of the Fourier coefficients of initial data is useful.

PROPOSITION 2.5 Given any $T > 2\sqrt{2}$ and $(u^0, u^1) = \sum_{(n,m) \in \mathbb{Z}^* \times \mathbb{N}^*} \alpha_{nm}^0 \Phi^{nm} \in L^2(\Omega) \times H^{-1}(\Omega)$, $(v, z) \in [L^2((0, T) \times (0, 1))]^2$ is a control for (1.2) if and only if for all $(n, m) \in \mathbb{Z}^* \times \mathbb{N}^*$,

$$\int_0^T e^{i\lambda^{nm}t} \left((-1)^n n \int_0^1 v(t, y) \sin(m\pi y) dy + (-1)^m m \int_0^1 z(t, x) \sin(n\pi x) dx \right) dt = \frac{\alpha_{nm}^0}{\sqrt{2\pi}}. \quad (2.8)$$

Proof. From the continuity of the linear form $A: H_0^1(\Omega) \times L^2(\Omega) \rightarrow \mathbb{C}$, defined by

$$\begin{aligned} A(w^0, w^1) &= \int_0^T \int_0^1 v(t, y) w_x(t, 1, y) dy dt + \int_0^T \int_0^1 z(t, x) w_y(t, x, 1) dx dt \\ &\quad - \langle u^1, w(0) \rangle_{H^{-1}, H_0^1} + \int_{\Omega} u^0(x, y) w'(0, x, y) dx dy, \end{aligned}$$

it follows that (2.5) holds for any $(w^0, w^1) \in H_0^1(\Omega) \times L^2(\Omega)$ if and only if it is verified on a basis of the space $H_0^1(\Omega) \times L^2(\Omega)$. Thus, by considering $(w^0, w^1) = \Psi^{nm}$ in (2.5), we obtain that the control (v, z) drives to zero the initial data of (1.2) if and only if (2.8) is verified. \square

3. The semi-discrete problem

In this section, we introduce a suitable semi-discretization of the homogeneous adjoint equation (2.2). By minimizing the HUM functional corresponding to this semi-discrete system, a convergent sequence of discrete approximations $(v_h, z_h)_{h>0}$ of the HUM control (v, z) of (1.2) is obtained.

We introduce $N \in \mathbb{N}^*$ and $h = 1/(N + 1)$, we consider the points $(x_i, y_j) = (ih, jh)$, $0 \leq i, j \leq N + 1$, and we denote $w_{ij} = w(x_i, y_j)$.

Let us also introduce the new variable $\zeta(t, x, y) = w'(t, x, y)$. Equation (2.2) may be written in the following variational form:

$$\left\{ \begin{array}{l} \text{Find } (w, \zeta) = (w, \zeta)(t, x, y) \text{ with } (w(t), \zeta(t)) \in H_0^1(\Omega) \times L^2(\Omega) \quad \forall t \in (0, T) \text{ and} \\ \frac{d}{dt} \int_0^1 \int_0^1 w(t, x, y) \psi(x, y) dx dy = \int_0^1 \int_0^1 \zeta(t, x, y) \psi(x, y) dx dy \quad \forall \psi \in L^2(\Omega), \\ \frac{d}{dt} \langle \zeta(t, \cdot), \varphi \rangle_{-1,1} = \int_0^1 \int_0^1 \nabla w(t, x, y) \nabla \varphi(x, y) dx dy \quad \forall \varphi \in H_0^1(\Omega), \\ w(T, x, y) = w^0(x, y), \quad \zeta(T, x, y) = w^1(x, y) \quad \forall (x, y) \in \Omega. \end{array} \right. \quad (3.1)$$

We now discretize (3.1) by using a mixed finite-element method (see, for instance, [Roberts & Thomas, 1989](#)). Let \mathbb{Q}_1 be the space of all polynomials of degree less than or equal to one with respect to each of the variables x, y and \mathbb{Q}_0 the space of constant functions. We introduce the basis functions in the following way: For each $1 \leq i, j \leq N$, let $\mathcal{Q}_{ij}^h = (x_i, x_{i+1}) \times (y_j, y_{j+1})$ be such that $\bigcup_{0 \leq i, j \leq N} \mathcal{Q}_{ij}^h = \Omega = (0, 1)^2$ and define the functions

$$\left\{ \begin{array}{l} \psi_{ij|_{\mathcal{Q}_{kl}^h}} \in \mathbb{Q}_0, \quad \psi_{ij} = \begin{cases} \frac{1}{2}, & \text{if } (x, y) \in \mathcal{Q}_{ij}^h \cup \mathcal{Q}_{i-1j}^h \cup \mathcal{Q}_{ij-1}^h \cup \mathcal{Q}_{i-1j-1}^h, \\ 0, & \text{otherwise,} \end{cases} \\ \varphi_{ij|_{\mathcal{Q}_{kl}^h}} \in \mathbb{Q}_1, \quad \varphi_{ij}(x_k, y_l) = \delta_{ij}^{kl}. \end{array} \right.$$

The variational formulation (3.1) is then reduced to find

$$w_h(t, x, y) = \sum_{i,j=1}^N w_{ij}(t) \varphi_{ij}(x, y) \quad \text{and} \quad \zeta_h(t, x, y) = \sum_{i,j=1}^N \zeta_{ij}(t) \psi_{ij}(x, y) \quad (3.2)$$

that satisfy

$$\begin{cases} \frac{d}{dt} \int_0^1 \int_0^1 w_h(t, x, y) \psi_{ij}(x, y) dx dy = \int_0^1 \int_0^1 \zeta_h(t, x, y) \psi_{ij}(x, y) dx dy & \forall 1 \leq i, j \leq N, \\ \frac{d}{dt} \langle \zeta_h(t, \cdot), \varphi_{ij} \rangle_{-1,1} = \int_0^1 \int_0^1 \nabla w_h(t, x, y) \nabla \varphi_{ij}(x, y) dx dy & \forall 1 \leq i, j \leq N, \\ w_h(T, x, y) = w_h^0(x, y), \quad \zeta_h(T, x, y) = w_h^1(x, y) & \forall (x, y) \in \Omega. \end{cases} \quad (3.3)$$

The variables ζ_{ij} may be eliminated from (3.2) and (3.3) leading to the following semi-discrete system for $w_{ij}(t)$ in $t \in (0, T)$:

$$\begin{cases} \frac{h^2}{16} (4w''_{ij} + 2w''_{i+1j} + 2w''_{i-1j} + 2w''_{ij+1} + 2w''_{ij-1} + w''_{i+1j+1} + w''_{i+1j-1} + w''_{i-1j+1} \\ \quad + w''_{i-1j-1}) + \frac{1}{3} (8w_{ij} - w_{i+1j} - w_{i-1j} - w_{ij+1} - w_{ij-1} - w_{i+1j+1} - w_{i+1j-1} \\ \quad - w_{i-1j+1} - w_{i-1j-1}) = 0, \text{ for } 1 \leq i, j \leq N, \\ w_{i0} = w_{iN+1} = 0, \quad w_{0j} = w_{N+1j} = 0 \quad \text{for } 0 \leq i \leq N+1, \\ w_{ij}(T) = w_{ij}^0, \quad w'_{ij}(T) = w_{ij}^1 \quad \text{for } 0 \leq i, j \leq N+1. \end{cases} \quad (3.4)$$

The convergence of the scheme (3.4) is given in [Kappel & Ito \(1998\)](#). We shall consider that the initial data are zero on the boundary of Ω , which in the discrete equation corresponds to

$$\begin{cases} w_{0,j}^0 = w_{0,j}^1 = 0, \quad w_{N+1,j}^0 = w_{N+1,j}^1 = 0 & \text{for } 0 \leq j \leq N+1, \\ w_{i,0}^0 = w_{i,0}^1 = 0, \quad w_{i,N+1}^0 = w_{i,N+1}^1 = 0 & \text{for } 0 \leq i \leq N+1. \end{cases} \quad (3.5)$$

The same property will be also satisfied by the corresponding solutions of (3.4).

If we denote the unknown

$$W_h(t) = (w_{11}(t), w_{21}(t), \dots, w_{N1}, \dots, w_{1N}(t), w_{2N}(t), \dots, w_{NN}(t))^T,$$

then (3.4) may be written in vectorial form as follows:

$$\begin{cases} M_h W_h''(t) + K_h W_h(t) = 0, & \text{for } t > 0, \\ W_h(T) = W_h^0, \quad W_h'(T) = W_h^1, \end{cases} \quad (3.6)$$

where $(W_h^0, W_h^1) = (w_{ij}^0, w_{ij}^1)_{1 \leq i, j \leq N} \in \mathbb{R}^{2N^2}$ are the initial data and the corresponding solution of (3.4) is given by $(W_h, W_h') = (w_{ij}, w'_{ij})_{1 \leq i, j \leq N}$.

The entries of the block-three-diagonal matrices M_h and K_h belonging to $\mathcal{M}_{N^2}(\mathbb{R})$ may be easily deduced from (3.4).

4. Properties of the semi-discrete system

In this section, we study some of the properties of the semi-discrete adjoint system (3.4) related to the controllability problem. More precisely, the aim of this section consists in giving a uniform (in h) observability inequality for (3.4). But before that, let us briefly explain why the semi-discretization introduced in this paper is likely to provide a uniform observability property rather than others like the usual finite-difference semi-discretization implemented in Glowinski *et al.* (1990).

As we have mentioned in Remark 2.3, in order to have an observability inequality for the continuous wave equation (2.2) of the type (2.4), it is necessary to consider $T > 2\sqrt{2}$. This is due to the finite velocity of propagation of waves. More precisely, a planar wave of the form $e^{i(\xi \cdot (x,y) - \omega t)}$ propagates in any spatial direction $v = (v_1, v_2) \in \mathbb{R}^2$ with group velocity $\nabla_\xi \omega \cdot v$, where $\omega = |\xi|$. Let us denote

$$\zeta = \min_{\xi \in \mathbb{R}^2} \max_{v \in \mathbb{R}^2, |v|=1} \nabla_\xi \omega \cdot v = \min_{\xi \in \mathbb{R}^2} |\nabla_\xi \omega|. \quad (4.1)$$

The observability time T and ζ are inversely proportional. In our particular case, $T > 2 \text{diam}(\Omega)/\zeta$ (see Lions, 1988) and $\zeta = 1$. Thus $T > 2\sqrt{2}$.

In a similar way, we may introduce the velocity of waves for the semi-discrete problem (see Trefethen, 1982). Let $w_{ij} = e^{i(\xi \cdot (x_i, y_j) - \omega t)}$, $\xi = (\xi_1, \xi_2) \in (-\pi/h, \pi/h)^2$, be a discrete plane wave which propagates in any spatial direction $v = (v_1, v_2) \in \mathbb{R}^2$ with group velocity $\nabla_\xi \omega \cdot v$. In the mixed finite-element method,

$$\omega = \omega_{\text{mfe}}(\xi) = \frac{2}{h} \sqrt{\tan^2\left(\frac{\xi_1 h}{2}\right) + \tan^2\left(\frac{\xi_2 h}{2}\right) + \frac{2}{3} \tan^2\left(\frac{\xi_1 h}{2}\right) \tan^2\left(\frac{\xi_2 h}{2}\right)}, \quad (4.2)$$

while for the finite-difference method,

$$\omega = \omega_{\text{fd}}(\xi) = \frac{2}{h} \sqrt{\sin^2\left(\frac{\xi_1 h}{2}\right) + \sin^2\left(\frac{\xi_2 h}{2}\right)}. \quad (4.3)$$

Note that $\zeta_{\text{mfe}} = \min_{\xi \in (-\pi/h, \pi/h)^2} |\nabla_\xi \omega_{\text{mfe}}| = 1$ and $\zeta_{\text{fd}} = \min_{\xi \in (-\pi/h, \pi/h)^2} |\nabla_\xi \omega_{\text{fd}}| = O(h)$. This is illustrated in Fig. 1. Thus, the observability time T can be uniformly bounded, in h , only for the mixed finite-element method.

In the rest of this section, we prove that indeed this property holds for system (3.4).

Since the matrices M_h and K_h are positive definite, we may define the inner product

$$\langle (f_1, f_2), (g_1, g_2) \rangle_0 = \langle K_h f_1, g_1 \rangle + \langle M_h f_2, g_2 \rangle \quad (4.4)$$

for any $(f_1, f_2), (g_1, g_2) \in \mathbb{R}^{2N^2}$, where $\langle \cdot, \cdot \rangle$ denotes the canonical inner product in \mathbb{R}^{N^2} . The corresponding norm will be denoted $\|\cdot\|_0$.

We introduce the following discrete version of the continuous energy of (2.2):

$$E_h(t) = \frac{1}{2} \|(W_h, W'_h)(t)\|_0^2. \quad (4.5)$$

The following proposition shows that as in the corresponding continuous case, the energy E_h defined by (4.5) is conserved along trajectories.

PROPOSITION 4.1 For any $h > 0$ and any solution of (3.4), the following holds:

$$E_h(t) = E_h(0) \quad \forall t > 0.$$

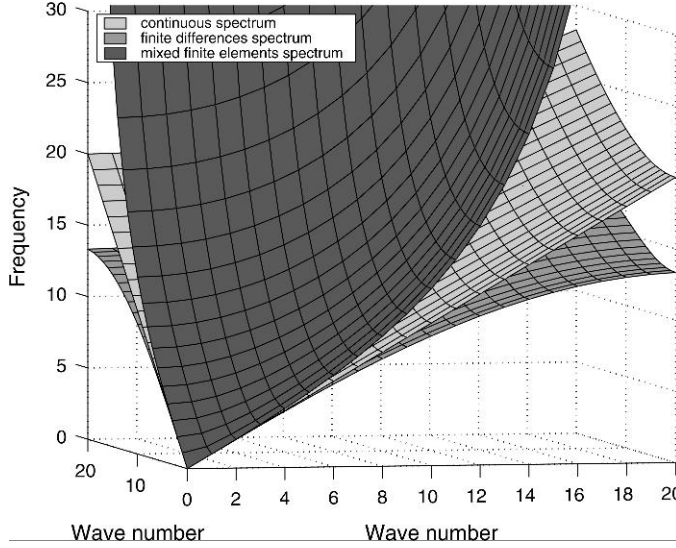


FIG. 1. $\omega(\xi)$ with $\xi \in [0, \pi/h)^2$ and $h = 1/21$ for the mixed finite-element semi-discretization (upper surface), continuous wave equation (medium surface) and the usual finite-difference semi-discretization (lower surface). We observe that the norm of the gradient $|\nabla_\xi \omega(\xi)|$ is always one in the continuous case, it is greater than one for the mixed finite-element scheme and it becomes zero for the usual finite-difference scheme as ξ approaches $(\pi/h, 0)$.

Proof. Multiplying (3.6) by W'_h , we obtain that

$$0 = \langle M_h W''_h, W'_h \rangle + \langle K_h W_h, W'_h \rangle = \frac{1}{2} [\langle M_h W'_h, W'_h \rangle + \langle K_h W_h, W_h \rangle]' = \frac{d}{dt} E_h(t)$$

and the proof finishes. \square

The following result shows that a discrete version of the observability inequality (2.4) is valid for the solutions of system (3.4).

THEOREM 4.2 Given $T > 2\sqrt{3}$, there exists a constant $C(T) > 0$ independent of the discretization step h such that the following inequality holds:

$$E_h(0) \leq C(T) \frac{h}{2} \left\{ \int_0^T \left[\frac{1}{h^2} \langle C_h W'_{N\cdot}, W'_{N\cdot} \rangle + \frac{1}{h^2} \langle C_h W'_{\cdot N}, W'_{\cdot N} \rangle \right] dt - \int_0^T \left[\frac{1}{h^2} \langle B_h W_{N\cdot}, W_{N\cdot} \rangle + \frac{1}{h^2} \langle B_h W_{\cdot N}, W_{\cdot N} \rangle \right] dt \right\}, \quad (4.6)$$

where $W_{N\cdot} = (w_{Nj})_{1 \leq j \leq N} \in \mathbb{R}^N$ and $W_{\cdot N} = (w_{iN})_{1 \leq i \leq N} \in \mathbb{R}^N$. The matrixes $B_h, C_h \in \mathcal{M}_{N^2}(\mathbb{R})$ are defined as follows: $B_h = -\frac{1}{3}T$ and $C_h = \frac{h^2}{16}(T + I_d)$ where $I_d \in \mathcal{M}_{N^2}(\mathbb{R})$ is the identity matrix and $T\mathcal{M}_{N^2}(\mathbb{R})$ is the tridiagonal matrix whose elements are all ones.

REMARK 4.3 The method used in the proof of the observability inequality (4.6) works only if $T > 2\sqrt{3}$. Probably this time is not sharp and the same is true for $T > 2\sqrt{2}$, which is the necessary and sufficient time condition for controllability in the continuous case (see Remark 2.3).

The proof of Theorem 4.2 is technical and it is given in Appendix A.

5. Construction of the discrete approximations

In this section, we explicitly construct a sequence of approximations $(v_h, z_h)_{h>0}$ of the HUM control (v, z) of (1.2). This will be done by minimizing the HUM functional of the semi-discrete adjoint system (3.4).

Suppose that $(U_h^0, U_h^1) = (u_j^0, u_j^1)_{1 \leq j \leq N} \in \mathbb{R}^{2N^2}$ is a discretization of the continuous initial data of (1.2) to be controlled. We define the functional $\mathcal{J}_h: \mathbb{R}^{2N^2} \rightarrow \mathbb{R}$,

$$\begin{aligned} \mathcal{J}_h(W_h^0, W_h^1) &= -\langle (-K_h^{-1} M_h U_h^1, U_h^0), (W_h(0), W_h'(0)) \rangle_0 \\ &\quad + \frac{1}{2h} \int_0^T [\langle C_h W_{N\cdot}, W_{N\cdot}' \rangle + \langle C_h W_{N\cdot}', W_{N\cdot}' \rangle] dt \\ &\quad + \frac{1}{2h} \int_0^T [\langle B_h W_{N\cdot}, W_{N\cdot} \rangle + \langle B_h W_{N\cdot}, W_{N\cdot}' \rangle] dt, \end{aligned} \quad (5.1)$$

where (W_h, W_h') is the solution of (3.6) with initial data $(W_h^0, W_h^1) \in \mathbb{R}^{2N^2}$, and we have noted $W_{N\cdot} = (w_{Nj})_{1 \leq j \leq N} \in \mathbb{R}^N$ and $W_{N\cdot}' = (w_{iN}')_{1 \leq i \leq N} \in \mathbb{R}^N$.

We show now that \mathcal{J}_h has a minimizer $(\widehat{W}_h^0, \widehat{W}_h^1)$.

THEOREM 5.1 For any $T > 2\sqrt{3}$, the functional \mathcal{J}_h (5.1) has a unique minimizer $(\widehat{W}_h^0, \widehat{W}_h^1)$.

Proof. Since \mathcal{J}_h is continuous, convex and defined in a finite-dimensional space, the theorem is proved if we show that \mathcal{J}_h is coercive. This is a consequence of (4.6). More precisely,

$$\begin{aligned} \mathcal{J}_h(W_h^0, W_h^1) &\geq \frac{h}{32} \int_0^T \left(\sum_{j=0}^N |w_{Nj+1}'(t) + w_{Nj}'(t)|^2 + \sum_{i=0}^N |w_{i+1N}'(t) + w_{iN}'(t)|^2 \right) dt \\ &\quad + \frac{1}{6h} \int_0^T \left(\sum_{j=0}^N |w_{Nj+1}(t) + w_{Nj}(t)|^2 + \sum_{i=0}^N |w_{i+1N}(t) + w_{iN}(t)|^2 \right) dt \\ &\quad - \frac{1}{6h} \int_0^T \left(\sum_{j=0}^N |w_{Nj}(t)|^2 + \sum_{i=0}^N |w_{iN}(t)|^2 \right) dt \\ &\quad - \|(-K_h^{-1} M_h U_h^1, U_h^0)\|_0 \| (W_h(0), W_h'(0)) \|_0 \\ &\geq C(T) \| (W_h^0, W_h^1) \|_0^2 - \|(-K_h^{-1} M_h U_h^1, U_h^0)\|_0 \| (W_h^0, W_h^1) \|_0, \end{aligned}$$

and therefore

$$\lim_{\| (W_h^0, W_h^1) \|_0 \rightarrow \infty} \mathcal{J}_h(W_h^0, W_h^1) = \infty.$$

□

REMARK 5.2 The main tool in the proof of the previous result is the observability inequality (4.6) stated in Theorem 4.2. It ensures the coercivity of \mathcal{J} and consequently the existence of a minimizer. Moreover, as we shall see in Theorem 6.1, the constant $C(T)$ appearing in (4.6) is an upper bound for the sequences of minimizers and controls.

Let $(\widehat{W}_h^0, \widehat{W}_h^1)$ be the minimizer of the functional \mathcal{J}_h given by Theorem 5.1. We define $v_h = (v_{h,j})_{1 \leq j \leq N} \in L^2(0, T; \mathbb{R}^N)$ and $z_h = (z_{h,i})_{1 \leq i \leq N} \in L^2(0, T; \mathbb{R}^N)$ by

$$v_{h,j}(t) = -\frac{\widehat{w}_{Nj}}{h}, \quad z_{h,i}(t) = -\frac{\widehat{w}_{iN}}{h} \quad \forall 1 \leq i, j \leq N, \quad (5.2)$$

where $(\widehat{W}_h, \widehat{W}_h')$ is the solution of (3.6) with initial data $(\widehat{W}_h^0, \widehat{W}_h^1)$.

Our aim is to show that the sequence $(v_h, z_h)_{h>0}$ converges to a control (v, z) of the continuous equation (1.2). Since v_h and z_h belong to $L^2(0, T; \mathbb{R}^N)$ whereas v and z are in $L^2(0, T; L^2(0, 1))$, the convergence is stated in terms of the Fourier coefficients. This is done in Section 6.1.

In the rest of this section, we introduce the eigenfunctions and the eigenvalues of the semi-discrete problem (3.6). Let $\mathcal{J}_N = \{(n, m) \in \mathbb{Z}^* \times \mathbb{N}^*: 1 \leq |n| \leq N, 1 \leq m \leq N\}$.

LEMMA 5.3 The eigenvalues λ_h^{nm} , $(n, m) \in \mathcal{J}_N$, of the semi-discrete problem (3.6) are given by

$$\lambda_h^{nm} = \operatorname{sgn}(n) \frac{2}{h} \sqrt{\tan^2\left(\frac{m\pi h}{2}\right) + \tan^2\left(\frac{n\pi h}{2}\right) + \frac{2}{3} \tan^2\left(\frac{m\pi h}{2}\right) \tan^2\left(\frac{n\pi h}{2}\right)}. \quad (5.3)$$

The corresponding eigenfunctions are

$$\Psi_h^{nm} = \frac{\sqrt{2}}{\cos\left(\frac{n\pi h}{2}\right) \cos\left(\frac{m\pi h}{2}\right)} \begin{pmatrix} (i\lambda_h^{nm})^{-1} \Phi_h^{nm} \\ -\Phi_h^{nm} \end{pmatrix} \quad \forall (n, m) \in \mathcal{J}_N, \quad (5.4)$$

where $\Phi_h^{nm} = (\phi_h^n \sin(p\pi h))_{1 \leq p \leq N} \in \mathbb{R}^{N^2}$ and $\phi_h^n = (\sin(jn\pi h))_{1 \leq j \leq N} \in \mathbb{R}^N$.

A straightforward computation shows that $(\Psi_h^{nm})_{(n,m) \in \mathcal{J}_N}$ constitutes an orthonormal basis in \mathbb{R}^{2N^2} with respect to the inner product $\langle \cdot, \cdot \rangle_0$.

For any $(f^1, f^2), (g^1, g^2) \in \mathbb{R}^{2N^2}$, we introduce the notations

$$\begin{aligned} \langle (f^1, f^2), (g^1, g^2) \rangle_{-1} &= \langle (-K_h^{-1} M_h f^2, f^1), (-K_h^{-1} M_h g^2, g^1) \rangle_0, \\ \|(f^1, f^2)\|_{-1} &= \|(-K_h^{-1} M_h f^2, f^1)\|_0. \end{aligned}$$

Remark that $\langle \cdot, \cdot \rangle_{-1}$ is an inner product and $\|\cdot\|_{-1}$ is a norm on \mathbb{R}^{2N^2} .

6. Convergence of the discrete approximations

In this section, we prove the weak convergence of the sequence $(v_h, z_h)_{h>0}$ to the HUM control of the continuous equation (1.2). Let us first show the following boundedness property of the initial data from which (v_h, z_h) were constructed.

THEOREM 6.1 Assume that $T > 2\sqrt{3}$. The sequence of minimizers of \mathcal{J}_h given by Theorem 5.1, $(\widehat{W}_h^0, \widehat{W}_h^1)_{h>0}$, verifies

$$\|(\widehat{W}_h^0, \widehat{W}_h^1)\|_0 \leq \frac{1}{C} \|(-K_h^{-1} M_h U_h^1, U_h^0)\|_0, \quad (6.1)$$

where $C = C(T)$ is the observability constant of (4.6) which is independent of h .

If the sequence of discretizations $(U_h^0, U_h^1)_{h>0}$ is uniformly bounded in the $\|\cdot\|_{-1}$ -norm, then the sequence $(\widehat{W}_h^0, \widehat{W}_h^1)_{h>0}$ is bounded in the $\|\cdot\|_0$ -norm.

Proof. From the observability inequality (4.6), we have that

$$\begin{aligned} C\|(\widehat{W}_h^0, \widehat{W}_h^1)\|_0^2 &\leq \frac{h}{2} \int_0^T [\langle C_h v'_h, v'_h \rangle + \langle C_h z'_h, z'_h \rangle] dt - \frac{h}{2} \int_0^T [\langle B_h v_h, v_h \rangle + \langle B_h z_h, z_h \rangle] dt \\ &= \mathcal{J}_h(\widehat{W}_h^0, \widehat{W}_h^1) + \langle (-K_h^{-1} M_h U_h^1, U_h^0), (\widehat{W}_h(0), \widehat{W}_h'(0)) \rangle_0. \end{aligned}$$

Now, since $\mathcal{J}_h(\widehat{W}_h^0, \widehat{W}_h^1) \leq \mathcal{J}_h(0, 0) = 0$, it follows that

$$\begin{aligned} C\|(\widehat{W}_h^0, \widehat{W}_h^1)\|_0^2 &\leq \langle (-K_h^{-1} M_h U_h^1, U_h^0), (\widehat{W}_h(0), \widehat{W}_h'(0)) \rangle_0 \\ &\leq \|(-K_h^{-1} M_h U_h^1, U_h^0)\|_0 \|(\widehat{W}_h(0), \widehat{W}_h'(0))\|_0 = \|(-K_h^{-1} M_h U_h^1, U_h^0)\|_0 \|(\widehat{W}_h^0, \widehat{W}_h^1)\|_0, \end{aligned}$$

which is equivalent to (6.1). \square

REMARK 6.2 Theorem 6.1 shows that the sequence of initial data $(\widehat{W}_h^0, \widehat{W}_h^1)_{h>0}$ which give (v_h, z_h) is uniformly bounded in h for the $\|\cdot\|_0$ -norm if the sequence of discretizations $(U_h^0, U_h^1)_{h>0}$ is bounded in the $\|\cdot\|_{-1}$ -norm. The sequences $(v_h, z_h)_{h>0}$ verify the following inequality:

$$\begin{aligned} &\frac{h}{2} \int_0^T [\langle C_h v'_h, v'_h \rangle + \langle C_h z'_h, z'_h \rangle - \langle B_h v_h, v_h \rangle - \langle B_h z_h, z_h \rangle] dt \\ &\leq \frac{1}{C} \|(-K_h^{-1} M_h U_h^1, U_h^0)\|_0^2 = \frac{1}{C} \|(U_h^0, U_h^1)\|_{-1}^2. \end{aligned} \quad (6.2)$$

6.1 Weak convergence of the approximations

Assume that the sequence of discretizations of the continuous initial data on (1.2), $(U_h^0, U_h^1)_{h>0}$, converges weakly to (u^0, u^1) in $L^2(\Omega) \times H^{-1}(\Omega)$. This should be understood in the sense of the convergence of the Fourier coefficients. More precisely, if

$$(U_h^0, U_h^1) = \sum_{(n,m) \in \mathcal{J}_N} \alpha_{nm}^h \Phi_h^{nm}, \quad (u^0, u^1) = \sum_{(n,m) \in \mathbb{Z}^* \times \mathbb{N}^*} \alpha_{nm} \Phi^{nm},$$

then the following weak convergence holds in ℓ^2 :

$$\left(\frac{\alpha_{nm}^h}{\lambda_h^{nm}} \right)_{(n,m) \in \mathcal{J}_N} \rightharpoonup \left(\frac{\alpha_{nm}}{\lambda^{nm}} \right)_{(n,m) \in \mathbb{Z}^* \times \mathbb{N}^*} \quad \text{when } h \rightarrow 0. \quad (6.3)$$

Now, assume that the minimizer $(\widehat{W}_h^0, \widehat{W}_h^1)$ has the following expansion:

$$(\widehat{W}_h^0, \widehat{W}_h^1) = \sum_{(n,m) \in \mathcal{J}_N} a_{nm}^h \Psi_h^{nm}. \quad (6.4)$$

Inequality (6.1) is equivalent to

$$\sum_{(n,m) \in \mathcal{I}_N} |a_{nm}^h|^2 = \|(\widehat{W}_h^0, \widehat{W}_h^1)\|_0^2 \leq \frac{1}{C^2} \|(-K_h^{-1} M_h U_h^1, U_h^0)\|_0^2 = \frac{1}{C^2} \sum_{(n,m) \in I_N} \left| \frac{\alpha_{nm}^h}{\lambda_h^{nm}} \right|^2.$$

Here, the right-hand side is bounded due to the weak convergence stated in (6.3). Hence, the sequence of the Fourier coefficients $(a_{nm}^h)_{(n,m) \in \mathcal{I}_N}$ is bounded in ℓ^2 and there exists a subsequence, denoted in the same way, and $(a_{nm})_{(n,m) \in \mathbb{Z}^* \times \mathbb{N}^*} \in \ell^2$ such that

$$(a_{nm}^h)_{(n,m) \in \mathcal{I}_N} \rightharpoonup (a_{nm})_{(n,m) \in \mathbb{Z}^* \times \mathbb{N}^*} \quad \text{in } \ell^2 \quad \text{when } h \rightarrow 0. \quad (6.5)$$

Let us now introduce the continuous initial data

$$(\widehat{w}^0, \widehat{w}^1) = \sum_{(n,m) \in \mathbb{Z}^* \times \mathbb{N}^*} a_{nm} \Psi^{nm} \in H_0^1(\Omega) \times L^2(\Omega) \quad (6.6)$$

and the corresponding solution $(\widehat{w}, \widehat{w}') \in C([0, T]; H_0^1(\Omega) \times L^2(\Omega))$. We have that

$$\begin{aligned} \widehat{w}_x(t, 1, y) &= \sum_{m \in \mathbb{N}^*} \left(\sum_{n \in \mathbb{Z}^*} i a_{nm} (-1)^{n+1} \frac{\sqrt{2} n \pi}{\lambda^{nm}} e^{i \lambda^{nm} t} \right) \sin(m \pi y) := v, \\ \widehat{w}_y(t, x, 1) &= \sum_{n \in \mathbb{Z}^*} \left(\sum_{m \in \mathbb{N}^*} i a_{nm} (-1)^{m+1} \frac{\sqrt{2} m \pi}{\lambda^{nm}} e^{i \lambda^{nm} t} \right) \sin(n \pi x) := w. \end{aligned}$$

If $(\widehat{W}_h, \widehat{W}_h')$ is the corresponding solution of (3.6) with initial data $(\widehat{W}_h^0, \widehat{W}_h^1)$, it follows that

$$\begin{aligned} v_h &= \sum_{1 \leq m \leq N} \left(\sum_{1 \leq |n| \leq N} i a_{nm}^h (-1)^{n+1} \frac{\sqrt{2}}{\lambda_h^{nm} \cos(\frac{n\pi h}{2}) \cos(\frac{m\pi h}{2})} \sin(n\pi h) e^{i \lambda_h^{nm} t} \right) \phi_h^m, \\ z_h &= \sum_{1 \leq |n| \leq N} \left(\sum_{1 \leq m \leq N} i a_{nm}^h (-1)^{m+1} \frac{\sqrt{2}}{\lambda_h^{nm} \cos(\frac{n\pi h}{2}) \cos(\frac{m\pi h}{2})} \sin(m\pi h) e^{i \lambda_h^{nm} t} \right) \phi_h^n. \end{aligned}$$

We denote

$$\begin{aligned} b_m^h &= \begin{cases} \sum_{1 \leq |n| \leq N} i a_{nm}^h (-1)^{n+1} \frac{\sqrt{2}}{\lambda_h^{nm} \cos(\frac{n\pi h}{2}) \cos(\frac{m\pi h}{2})} \sin(n\pi h) e^{i \lambda_h^{nm} t}, & \text{if } 1 \leq m \leq N, \\ 0, & \text{if } m > N, \end{cases} \\ b_m &= \sum_{n \in \mathbb{Z}^*} i a_{nm} (-1)^{n+1} \frac{\sqrt{2} n \pi}{\lambda^{nm}} e^{i \lambda^{nm} t}, \\ d_n^h &= \begin{cases} \sum_{1 \leq m \leq N} i a_{nm}^h (-1)^{m+1} \frac{\sqrt{2}}{\lambda_h^{nm} \cos(\frac{n\pi h}{2}) \cos(\frac{m\pi h}{2})} \sin(m\pi h) e^{i \lambda_h^{nm} t}, & \text{if } 1 \leq |n| \leq N, \\ 0, & \text{if } |n| > N, \end{cases} \\ d_n &= \sum_{m \in \mathbb{N}^*} i a_{nm} (-1)^{m+1} \frac{\sqrt{2} m \pi}{\lambda^{nm}} e^{i \lambda^{nm} t}. \end{aligned}$$

THEOREM 6.3 Assume that the sequence of discretizations $(U_h^0, U_h^1)_{h>0}$ converges weakly to (u^0, u^1) in the sense of (6.3). The following convergencies hold weakly in $L^2(0, T; \ell^2)$ when h tends to zero:

$$\begin{aligned} (b_m^h)_{m \in \mathbb{N}^*} &\rightharpoonup (b_m)_{m \in \mathbb{N}^*}, & (d_n^h)_{n \in \mathbb{Z}^*} &\rightharpoonup (d_n)_{n \in \mathbb{Z}^*}, \\ (h(b_m^h)')_{m \in \mathbb{N}^*} &\rightharpoonup 0, & (h(d_n^h)')_{n \in \mathbb{Z}^*} &\rightharpoonup 0. \end{aligned}$$

In particular, $(v_h, z_h)_{h>0}$ converges weakly to (v, z) in $[L^2((0, T) \times (0, 1))]^2$.

Proof. We show the first convergence, the other ones being similar. Let us introduce

$$\begin{aligned} \tilde{b}_m^h(t) &= \sum_{1 \leq |n| \leq N} i a_{nm}^h (-1)^{n+1} \frac{\sqrt{2} \sin(n\pi h)}{\lambda_h^{nm} \cos(\frac{n\pi h}{2}) \cos(\frac{m\pi h}{2})} \frac{1}{(\lambda_h^{nm})^2} e^{i\lambda_h^{nm} t}, \\ \tilde{b}_m(t) &= \sum_{n \in \mathbb{Z}^*} i a_{nm} (-1)^{n+1} \frac{\sqrt{2} n \pi}{\lambda^{nm}} \frac{1}{(\lambda^{nm})^2} e^{i\lambda^{nm} t}. \end{aligned}$$

The convergence $(b_m^h)_{m \in \mathbb{N}^*} \rightharpoonup (b_m)_{m \in \mathbb{N}^*}$ is proved if we show that

$$\int_0^T \sum_{m \geq 1} |\tilde{b}_m^h(t) - \tilde{b}_m(t)|^2 dt \longrightarrow 0 \quad \text{when } h \rightarrow 0. \quad (6.7)$$

In order to prove (6.7), we consider an arbitrary $\varepsilon > 0$ and show that there exists an N sufficiently large such that

$$\int_0^T \sum_{m > N} |\tilde{b}_m(t)|^2 dt \leq \frac{\varepsilon}{2} \quad (6.8)$$

and

$$\int_0^T \sum_{1 \leq m \leq N} |\tilde{b}_m^h(t) - \tilde{b}_m(t)|^2 dt \leq \frac{\varepsilon}{2}. \quad (6.9)$$

Remark that (6.8) and (6.9) imply (6.7) immediately.

To prove (6.8) note that since $(a_{nm}) \in \ell^2$, there exists an $N_1 > 0$ independent of h such that for any $N > N_1$, we have

$$\begin{aligned} \int_0^T \sum_{m > N} |\tilde{b}_m(t)|^2 dt &\leq \int_0^T \sum_{m > N} \left(\sum_{n \in \mathbb{Z}^*} \frac{1}{|\lambda^{nm}|^4} \right) \left(\sum_{n \in \mathbb{Z}^*} \left| i a_{nm} (-1)^{n+1} \frac{\sqrt{2} n \pi}{\lambda^{nm}} e^{i\lambda^{nm} t} \right|^2 \right) dt \\ &\leq 2 \left(\sum_{m > N} \sum_{n \in \mathbb{Z}^*} \frac{1}{|\lambda^{nm}|^4} \right) \int_0^T \left(\sum_{m > N} \sum_{n \in \mathbb{Z}^*} |a_{nm}|^2 dt \right) \leq cT \sum_{m > N} \sum_{n \in \mathbb{Z}^*} |a_{nm}|^2 \leq \frac{\varepsilon}{2}. \end{aligned}$$

Let us now show that for h sufficiently small (or, equivalently, for N sufficiently large), (6.9) also holds. We have that

$$\begin{aligned} & \frac{1}{2} \sum_{1 \leq m \leq N} |\tilde{b}_m^h - \tilde{b}_m|^2 \\ & \leq \sum_{1 \leq m \leq N} \left| \sum_{1 \leq |n| \leq N} (-1)^{n+1} i a_{nm}^h \left(\frac{\sqrt{2} \sin(n\pi h)}{\lambda_h^{nm} \cos(\frac{n\pi h}{2}) \cos(\frac{m\pi h}{2})} \frac{1}{(\lambda_h^{nm})^2} e^{i\lambda_h^{nm} t} - \frac{\sqrt{2}n\pi}{\lambda^{nm}} \frac{1}{(\lambda^{nm})^2} e^{i\lambda^{nm} t} \right) \right|^2 \\ & + \sum_{1 \leq m \leq N} \left| \sum_{1 \leq |n| \leq N} i(-1)^{n+1} (a_{nm}^h - a_{nm}) \frac{\sqrt{2}n\pi}{\lambda^{nm}} \frac{1}{(\lambda^{nm})^2} e^{i\lambda^{nm} t} \right|^2. \end{aligned}$$

According to the weak convergence of the sequence $(a_{nm}^h)_{nm}$ to $(a_{nm})_{nm}$ and the presence of the weights $1/(\lambda^{nm})^2$, for h sufficiently small,

$$\sum_{1 \leq m \leq N} \left| \sum_{1 \leq |n| \leq N} i(-1)^{n+1} (a_{nm}^h - a_{nm}) \frac{\sqrt{2}n\pi}{\lambda^{nm}} \frac{1}{(\lambda^{nm})^2} e^{i\lambda^{nm} t} \right|^2 \leq \frac{\varepsilon}{4}.$$

On the other hand,

$$\begin{aligned} & \sum_{1 \leq m \leq N} \left| \sum_{1 \leq |n| \leq N} (-1)^{n+1} i a_{nm}^h \left(\frac{\sqrt{2} \sin(n\pi h)}{\lambda_h^{nm} \cos(\frac{n\pi h}{2}) \cos(\frac{m\pi h}{2})} \frac{e^{i\lambda_h^{nm} t}}{(\lambda_h^{nm})^2} - \frac{\sqrt{2}n\pi}{\lambda^{nm}} \frac{e^{i\lambda^{nm} t}}{(\lambda^{nm})^2} \right) \right|^2 \\ & \leq \sum_{1 \leq m \leq N} \left(\sum_{1 \leq |n| \leq N} |a_{nm}^h|^2 \sum_{1 \leq |n| \leq N} \left| \frac{\sqrt{2} \sin(n\pi h)}{\lambda_h^{nm} \cos(\frac{n\pi h}{2}) \cos(\frac{m\pi h}{2})} \frac{e^{i\lambda_h^{nm} t}}{(\lambda_h^{nm})^2} - \frac{\sqrt{2}n\pi}{\lambda^{nm}} \frac{e^{i\lambda^{nm} t}}{(\lambda^{nm})^2} \right|^2 \right). \end{aligned}$$

Since $(a_{nm}^h)_{nm}$ is bounded in ℓ^2 , there exists a $c > 0$ such that

$$\sum_{1 \leq |n| \leq N} |a_{nm}^h|^2 \leq \sum_{1 \leq m \leq N} \sum_{1 \leq |n| \leq N} |a_{nm}^h|^2 \leq c,$$

and (6.9) follows if we prove that

$$\sum_{1 \leq m \leq N} \sum_{1 \leq |n| \leq N} \left| \frac{\sqrt{2} \sin(n\pi h)}{\lambda_h^{nm} \cos(\frac{n\pi h}{2}) \cos(\frac{m\pi h}{2})} \frac{1}{(\lambda_h^{nm})^2} e^{i\lambda_h^{nm} t} - \frac{\sqrt{2}n\pi}{\lambda^{nm}} \frac{1}{(\lambda^{nm})^2} e^{i\lambda^{nm} t} \right|^2 \leq \frac{\varepsilon}{4c}. \quad (6.10)$$

Note that

$$\max \left\{ \left| \frac{\sqrt{2} \sin(n\pi h)}{\lambda_h^{nm} \cos(\frac{n\pi h}{2}) \cos(\frac{m\pi h}{2})} \right|, \frac{\sqrt{2}n\pi}{\lambda^{nm}} \right\} \leq \sqrt{3}$$

and thus there exists an $n_\varepsilon > 0$ independent of h such that

$$\begin{aligned} & \sum_{1 \leq m \leq N} \sum_{n_\varepsilon + 1 \leq |n| \leq N} \left| \frac{\sqrt{2} \sin(n\pi h)}{\lambda_h^{nm} \cos(\frac{n\pi h}{2}) \cos(\frac{m\pi h}{2})} \frac{1}{(\lambda_h^{nm})^2} e^{i\lambda_h^{nm} t} - \frac{\sqrt{2} n \pi}{\lambda^{nm}} \frac{1}{(\lambda^{nm})^2} e^{i\lambda^{nm} t} \right|^2 \\ & + \sum_{n_\varepsilon + 1 \leq m \leq N} \sum_{1 \leq |n| \leq n_\varepsilon} \left| \frac{\sqrt{2} \sin(n\pi h)}{\lambda_h^{nm} \cos(\frac{n\pi h}{2}) \cos(\frac{m\pi h}{2})} \frac{1}{(\lambda_h^{nm})^2} e^{i\lambda_h^{nm} t} - \frac{\sqrt{2} n \pi}{\lambda^{nm}} \frac{1}{(\lambda^{nm})^2} e^{i\lambda^{nm} t} \right|^2 \\ & \leq 6 \sum_{1 \leq m \leq N} \sum_{n_\varepsilon + 1 \leq |n| \leq N} \frac{1}{(\lambda^{nm})^2} + 6 \sum_{n_\varepsilon + 1 \leq m \leq N} \sum_{1 \leq |n| \leq n_\varepsilon} \frac{1}{(\lambda^{nm})^2} \leq \frac{\varepsilon}{8c}. \end{aligned} \quad (6.11)$$

Let us now analyse the case $1 \leq m, |n| \leq n_\varepsilon$. Since $\lambda_h^{nm} \rightarrow \lambda^{nm}$ when h tends to zero, it follows that for h sufficiently small,

$$\begin{aligned} & \left| \frac{\sqrt{2} \sin(n\pi h)}{\lambda_h^{nm} \cos(\frac{n\pi h}{2}) \cos(\frac{m\pi h}{2})} \frac{1}{(\lambda_h^{nm})^2} e^{i\lambda_h^{nm} t} - \frac{\sqrt{2} n \pi}{\lambda^{nm}} \frac{1}{(\lambda^{nm})^2} e^{i\lambda^{nm} t} \right|^2 \\ & \leq \frac{\sqrt{2}}{(\lambda^{nm})^4} \left| \frac{\frac{\sin(n\pi h)}{n\pi} \lambda^{nm}}{\lambda_h^{nm} \cos(\frac{n\pi h}{2}) \cos(\frac{m\pi h}{2})} \frac{(\lambda^{nm})^2}{(\lambda_h^{nm})^2} e^{i(\lambda_h^{nm} - \lambda^{nm})t} - 1 \right|^2 \leq \frac{\varepsilon}{8cn_\varepsilon^2}. \end{aligned}$$

Consequently,

$$\sum_{1 \leq m \leq n_\varepsilon} \sum_{1 \leq |n| \leq n_\varepsilon} \left| \frac{\sqrt{2} \sin(n\pi h)}{\lambda_h^{nm} \cos(\frac{n\pi h}{2}) \cos(\frac{m\pi h}{2})} \frac{1}{(\lambda_h^{nm})^2} e^{i\lambda_h^{nm} t} - \frac{\sqrt{2} n \pi}{\lambda^{nm}} \frac{1}{(\lambda^{nm})^2} e^{i\lambda^{nm} t} \right|^2 \leq \frac{\varepsilon}{8c}. \quad (6.12)$$

From (6.11) and (6.12), (6.10) follows immediately and the proof ends. \square

6.2 Identification of the limit control

In this section, we show that the limit (v, w) of the sequence $(v_h, z_h)_{h>0}$ from Theorem 6.3 is the HUM control for the continuous equation (1.2).

THEOREM 6.4 We have that $(v, z) = (\widehat{w}_x(t, 1, y), \widehat{w}_y(t, x, 1))$ is the HUM control for (1.2), where $(\widehat{w}, \widehat{w}')$ is the solution of (2.2) with initial data $(\widehat{w}^0, \widehat{w}^1)$ given by (6.6).

Proof. By taking into account Proposition 2.5, the proof consists of verifying (2.8).

The optimality condition for the minimizer of \mathcal{J}_h provides the following characterization of v_h and z_h :

$$\begin{aligned} & \langle (-K_h^{-1} M_h U_h^1, U_h^0), (W_h(0), W_h'(0)) \rangle_0 \\ & = \frac{h^2}{16} \int_0^T \left(\sum_{j=1}^N (2v'_{h,j} + v'_{h,j+1} + v'_{h,j-1}) w'_{Nj} + \sum_{i=1}^N (2z'_{h,i} + z'_{h,i+1} + z'_{h,i-1}) w'_{iN} \right) dt \\ & + \frac{1}{3} \int_0^T \left(\sum_{j=1}^N (v_{h,j} + v_{h,j+1} + v_{h,j-1}) w_{Nj} + \sum_{i=1}^N (z_{h,i} + z_{h,i+1} + z_{h,i-1}) w_{iN} \right) dt = 0 \end{aligned} \quad (6.13)$$

for any $(W_h^0, W_h^1) \in \mathbb{R}^{2N^2}$, where (W_h, W_h') is the corresponding solution of (3.6).

Now, we evaluate (6.13) for $(W_h^0, W_h^1) = \Psi_h^{nm}$. We obtain that for any $(n, m) \in \mathcal{I}_N$,

$$\begin{aligned} & \frac{\cos\left(\frac{n\pi h}{2}\right) \cos\left(\frac{m\pi h}{2}\right)}{\sqrt{2}} \left\langle (-K_h^{-1} M_h U_h^1, U_h^0), \Psi_h^{mn} e^{i\lambda_h^{nm} T} \right\rangle_0 \\ &= \int_0^T e^{i\lambda_h^{nm}(t-T)} [(-1)^{n+1} \sin(n\pi h) \langle C_h v_h', \phi_h^m \rangle + (-1)^{m+1} \sin(m\pi h) \langle C_h z_h', \phi_h^n \rangle] dt \\ &+ \int_0^T \frac{e^{i\lambda_h^{nm}(t-T)}}{i\lambda_h^{nm}} [(-1)^{n+1} \sin(n\pi h) \langle B_h v_h, \phi_h^m \rangle + (-1)^{m+1} \sin(m\pi h) \langle B_h z_h, \phi_h^n \rangle] dt, \end{aligned}$$

which is equivalent to

$$\begin{aligned} & i \cos\left(\frac{n\pi h}{2}\right) \cos\left(\frac{m\pi h}{2}\right) \langle (-K_h^{-1} M_h U_h^1, U_h^0), \Psi_h^{mn} \rangle_0 \\ &= \frac{\sqrt{2}h^2 i}{4} \int_0^T e^{i\lambda_h^{nm} t} \left[(-1)^{n+1} \sin(n\pi h) \cos^2\left(\frac{m\pi h}{2}\right) \langle v_h', \phi_h^m \rangle \right. \\ &\quad \left. + (-1)^{m+1} \sin(m\pi h) \cos^2\left(\frac{n\pi h}{2}\right) \langle z_h', \phi_h^n \rangle \right] dt \\ &\quad - \frac{\sqrt{2}}{3\lambda_h^{nm}} \int_0^T e^{i\lambda_h^{nm} t} [(-1)^{n+1} \sin(n\pi h)(1 + 2 \cos(m\pi h)) \langle v_h, \phi_h^m \rangle \\ &\quad + (-1)^{m+1} \sin(m\pi h)(1 + 2 \cos(n\pi h)) \langle z_h, \phi_h^n \rangle] dt. \end{aligned} \tag{6.14}$$

We have that

$$\langle (-K_h^{-1} M_h U_h^1, U_h^0), \Psi_h^{mn} \rangle_0 = \frac{1}{i\lambda_h^{nm}} \alpha_{nm}^h, \quad \langle v_h, \phi_h^m \rangle = \frac{1}{2h} b_m^h(t), \quad \langle z_h, \phi_h^n \rangle = \frac{1}{2h} d_n^h(t).$$

By taking into account that for every fixed $(n, m) \in \mathcal{I}_N$, when h tends to zero we have that

$$\begin{aligned} \alpha_{nm}^h &\rightarrow \alpha_{nm}, & \lambda_h^{nm} &\rightarrow \lambda^{nm}, \\ b_m^h(t) &\rightarrow b_m(t), & d_m^h(t) &\rightarrow d_m(t) \quad \text{in } L^2(0, T), \\ h(b_m^h)'(t) &\rightarrow 0, & h(d_m^h)'(t) &\rightarrow 0 \quad \text{in } L^2(0, T), \end{aligned}$$

and by passing to the limit in (6.14), we obtain (2.8). \square

7. Numerical experiments

This section is devoted to present numerical experiments which illustrate the efficiency of scheme (3.4) in controllability problems. This is done by using a fully discrete approximation derived from the semi-discrete scheme (3.4). In Section 7.1, we present the method and in Section 7.2 we consider two examples with different non-smooth initial data and location of controls.

7.1 Description of a fully discrete approximation

We first introduce a fully discrete—in space and time—approximation method associated to system (2.2). This is precisely a classical time discretization of the semi-discrete scheme (3.4).

Given a time interval $[0, T]$, we introduce a uniform mesh $\{t^k = k \Delta t\}_{k=0, \dots, M}$ with time step Δt and $T = M \Delta t$. Let us denote by w_{ij}^k the approximation of the solution w of (2.2) at the point of coordinates (x_i, y_j) and at time $t^k = k \Delta t$, i.e. $w_{ij}^k \approx w(k \Delta t, x_i, y_j)$.

A fully discrete scheme may be obtained by replacing the time derivative $w''_{ij}(t^k)$ by the finite difference $(w_{ij}^{k+1} - 2w_{ij}^k + w_{ij}^{k-1})/\Delta t^2$. If $W^k = (w_{ij}^k)_{1 \leq i, j \leq N} \in \mathbb{R}^{N^2}$ for $0 \leq k \leq M$, the vectorial form (3.6) becomes

$$\begin{cases} M_h \frac{W^{k+1} - 2W^k + W^{k-1}}{\Delta t^2} + K_h W^k = 0, & 0 \leq k \leq M, \\ W^M = w^0, \quad \frac{W^{M+1} - W^{M-1}}{2\Delta t} = w^1. \end{cases} \quad (7.1)$$

The scheme (7.1) is consistent of order 2 in time and space with the continuous system (2.2) and it is stable under the so-called ‘Courant–Friedrichs–Lewy’ condition (see Cohen, 2002)

$$\frac{\Delta t^2}{4} \sup_{W \in \mathbb{R}^{N^2}, W \neq 0} \frac{(K_h W, W)}{(M_h W, W)} < 1 \quad \forall h, \Delta t > 0. \quad (7.2)$$

Moreover, the discrete spectrum $(\lambda_{h, \Delta t}^{mn})_{1 \leq m, n \leq N}$ associated to this scheme is

$$\lambda_{h, \Delta t}^{mn} = \frac{2}{\Delta t} \arcsin\left(\frac{\Delta t}{2} \lambda_h^{mn}\right), \quad 1 \leq m, n \leq N,$$

with λ_h^{mn} defined by (5.3). Therefore, (7.2) implies the following condition:

$$\Delta t \leq Ch^3 \quad (7.3)$$

for some $C > 0$ independent of h .

In order to relax this restrictive stability condition, we use an implicit method replacing the term $K_h W^k$ in (7.1) by $1/4 K_h (W^{k+1} + 2W^k + W^{k-1})$. Note that this corresponds to one of the Newmark methods (with parameters $\gamma = 1/2$ and $\beta = 1/4$, see Cohen, 2002). Thus, we obtain the following scheme:

$$\begin{cases} (M_h + \frac{\Delta t^2}{4} K_h) \frac{W^{k+1} - 2W^k + W^{k-1}}{\Delta t^2} + K_h W^k = 0, & 0 \leq k \leq M, \\ W^M = w^0, \quad \frac{W^{M+1} - W^{M-1}}{2\Delta t} = w^1 \end{cases} \quad (7.4)$$

consistent with the continuous system (2.2) and unconditionally stable for any value of Δt .

Let us now analyse if this fully discrete system conserves the observability properties of the semi-discrete scheme. Following the analysis in Section 4, we study the group velocity of discrete plane waves of the form

$$w_{ij}^k = e^{i(\xi \cdot (x_i, x_j) - \omega t^k)}, \quad \xi = (\xi_1, \xi_2).$$

For the discrete system (7.4), the following relation between the modes ξ and the frequencies ω holds:

$$\omega(\xi) = \frac{2}{\Delta t} \arcsin\left(\frac{\Delta t}{2} \sqrt{\frac{\omega_{\text{mfe}}(\xi)^2}{1 + \frac{\Delta t^2}{4} \omega_{\text{mfe}}(\xi)^2}}\right),$$

where $\omega_{\text{mfe}}(\xi)$ is given by (4.2).

The group velocity associated to a mode ξ in a direction $v = (v_1, v_2)$ is given by $\nabla_{\xi}\omega \cdot v$ and a necessary condition in order to have a uniform (in h and Δt) observability property in finite time is to have a uniform bound from below (in ξ , h and Δt) for $|\nabla_{\xi}\omega| = \sqrt{|\partial_{\xi_1}\omega|^2 + |\partial_{\xi_2}\omega|^2}$, i.e.

$$|\nabla_{\xi}\omega| \geq C > 0 \quad \text{for all } \xi, h \text{ and } \Delta t. \quad (7.5)$$

A straightforward computation shows that the minimum value of $|\nabla_{\xi}\omega|$ is obtained for $\xi = (\pi/h, \pi/h)$ and that

$$|\nabla_{\xi}\omega(\pi/h, \pi/h)| \sim h^{3/2} \Delta t^{-1}.$$

Therefore, this is uniformly bounded from below if

$$\Delta t = Ch^{3/2} \quad \forall C > 0. \quad (7.6)$$

Thus, even if the scheme (7.4) is stable for any discretization step Δt , in order to guarantee a uniform (in h and Δt) controllable scheme, (7.6) should be verified.

Note that the implicit method (7.4)–(7.6) permits to gain a factor $h^{3/2}$ in the ratio $\Delta t/h$ compared with the initial scheme (7.1) for which stability is ensured by (7.3).

7.2 Numerical examples

In this section, we present some numerical experiments for two different initial conditions. The first example is a well-known test proposed by Glowinski *et al.* (1990) for which the initial velocity u^1 is discontinuous. The second example is even more singular, involving a discontinuous initial displacement u^0 . Each one of these examples is defined in the unit square.

The HUM control is obtained by minimizing the functional \mathcal{J} in (5.1) and then by using (5.2). Following Glowinski *et al.* (1990), the iterative conjugate gradient algorithm is used with the initialization $(\hat{W}_h^0, \hat{W}_h^1) = (\mathbf{0}, \mathbf{0})$. We assume that the convergence is obtained when the corresponding relative residual is lower than or equal to $\varepsilon = 10^{-8}$.

7.2.1 Example 1: discontinuity of the initial velocity u^1 . Firstly, we consider the example in Glowinski *et al.* (1990, p. 26). The initial data to be controlled, (u^0, u^1) , is constituted by a Lipschitz continuous function u^0 not belonging to $C^1(\overline{\Omega})$ and a function u^1 belonging to $L^\infty(\Omega)$ but not to $C^0(\overline{\Omega})$. The explicit expressions of (u^0, u^1) may be found in Glowinski *et al.* (1990). The interest of this example is that the analytical solution is known. More precisely, let us consider $T = 15/4\sqrt{2}$ and the solution of the wave equation (2.2) given by

$$\hat{w}(t, x, y) = \sqrt{2} \cos\left(\pi\sqrt{2}\left(T - t - \frac{1}{4\sqrt{2}}\right)\right) \sin(\pi x) \sin(\pi y).$$

Let (\hat{w}^0, \hat{w}^1) be its corresponding initial data. Then, $V = \frac{\partial \hat{w}}{\partial \nu}|_{\partial\Omega}$ is exactly the HUM control acting on the whole boundary $\partial\Omega$ which leads (u^0, u^1) to the rest in time T .

In Glowinski *et al.* (1990), the simplest discretization for the wave equation is considered. It consists in the five-point formula in space for the Laplacian combined with the usual three-point formula for the second derivative in time. This produces an explicit scheme for which condition (7.5) fails. The conjugate gradient algorithm based on this scheme diverges. Several cures have been proposed to

TABLE 1 Results obtained with $\Delta t = h^{3/2}$ in Example 1. The control is active on $\partial\Omega$ and $T = 15/4\sqrt{2}$

	$h = 1/15$	$h = 1/30$	$h = 1/60$	$h = 1/120$
Conjugate gradient CG iterations	5	6	6	6
$\frac{\ \hat{w}^0 - \hat{W}_h^0\ _{L^2(\Omega)}}{\ \hat{w}^0\ _{L^2(\Omega)}}$	2.61×10^{-2}	5.53×10^{-3}	1.43×10^{-3}	5.27×10^{-4}
$\frac{ \hat{w}^0 - \hat{W}_h^0 _{H^1(\Omega)}}{ \hat{w}^0 _{H^1(\Omega)}}$	4.02×10^{-2}	1.80×10^{-2}	7.07×10^{-3}	3.09×10^{-3}
$\frac{\ \hat{w}^1 - \hat{W}_h^1\ _{L^2(\Omega)}}{\ \hat{w}^1\ _{L^2(\Omega)}}$	4.45×10^{-2}	2.13×10^{-2}	9.64×10^{-3}	4.86×10^{-3}
$\frac{\ V - V_h\ _{L^2(\partial\Omega \times (0,T))}}{\ V\ _{L^2(\partial\Omega \times (0,T))}}$	2.31×10^{-1}	1.24×10^{-1}	4.93×10^{-2}	2.08×10^{-2}
$\ V_h\ _{L^2(\partial\Omega \times (0,T))}$	7.4187	7.3782	7.3812	7.3859
$E_h(T)/E_h(0)$	1.55×10^{-3}	4.1×10^{-4}	5.61×10^{-5}	1.01×10^{-6}

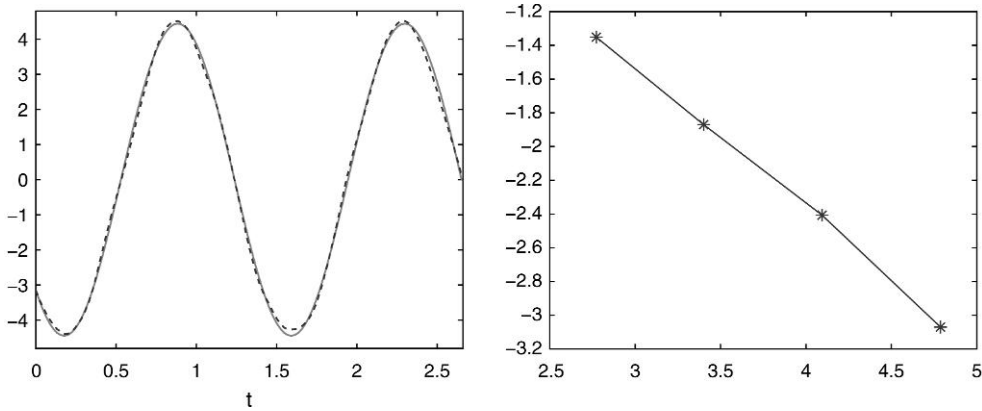


FIG. 2. Left: Exact control $V(t, \bar{x})$ (dashed line) and approximate control V_h (solid line) at the point $\bar{x} = (1, 1/2) \in \partial\Omega$ with $h = 1/15$. Right: $\log(\|(V - V_h)(\cdot, \bar{x})\|_{L^\infty(0,T)})$ versus $\log(1/h)$ (Example 1).

TABLE 2 Results obtained with $\Delta t = h^{3/2}$ in Example 2. The control is active on $\Gamma_1 \subset \partial\Omega$ and $T = 2\sqrt{2}$. The last row indicates that the system is controlled at time T

	$h = 1/15$	$h = 1/30$	$h = 1/60$	$h = 1/120$
CG iterations	13	11	10	10
$\ \hat{W}_h^0\ _{L^2(\Omega)}$	1.50×10^{-1}	1.35×10^{-1}	1.31×10^{-1}	1.30×10^{-4}
$ \hat{W}_h^0 _{H^1(\Omega)}$	1.0990	1.1071	1.1147	1.1169
$\ \hat{W}_h^1\ _{L^2(\Omega)}$	5.871	5.425	5.196	5.164
$\ v_h\ _{L^2((0,1) \times (0,T))} + \ z_h\ _{L^2((0,1) \times (0,T))}$	1.290×10^1	1.243×10^1	1.222×10^1	1.218×10^1
$E_h(T)/E_h(0)$	4.30×10^{-3}	3.68×10^{-4}	1.11×10^{-4}	8.39×10^{-5}

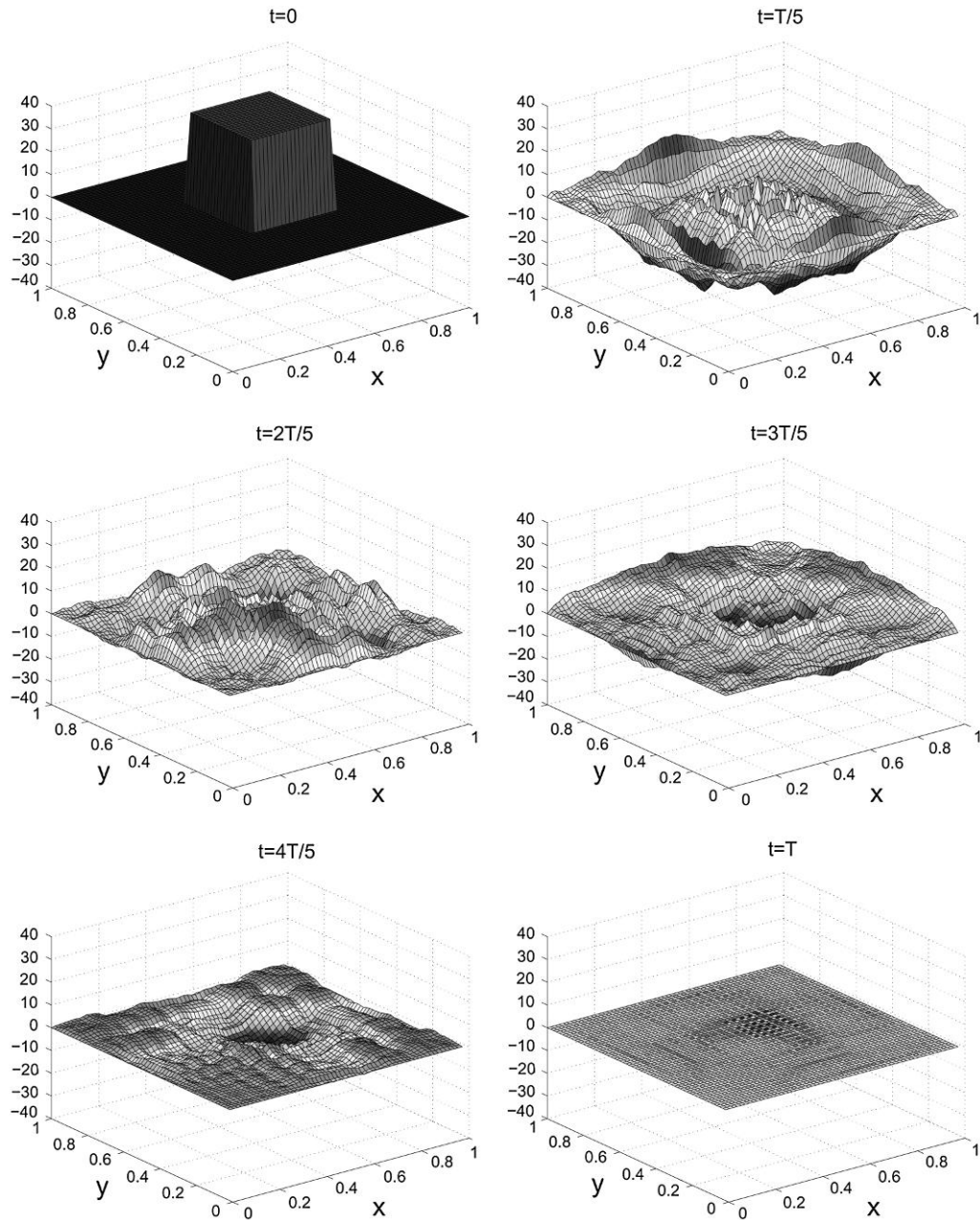


FIG. 3. Controllability of the initial data (7.7) in $\Omega = (0, 1)^2$: approximations $U_h(t)$ of the controlled solution for $t = 0, T/5, 2T/5, 3T/5, 4T/5$ and $T = 2\sqrt{2}$ with $h = 1/60$ in Example 2.

obtain convergence without changing the scheme, such as filtering with a bi-grid strategy or a Tychonoff regularization technique (see Glowinski, 1991; Glowinski *et al.*, 1990).

Table 1 displays the good behaviour of the scheme (7.4) when h and $\Delta t = h^{3/2}$ are decreasing, by comparing the exact and approximate results for the initial data giving the control and for the control itself. $|\hat{w}^0|_{H^1(\Omega)}$ is defined by $|\hat{w}^0|_{H^1(\Omega)} = (\int_{\Omega} |\nabla \hat{w}^0|^2 dx dy)^{1/2}$, whereas the H^{-1} -norm of u^1 in Ω is defined by $\|u^1\|_{H^{-1}} = \|w\|_{H^1(\Omega)}$, where $w \in H_0^1(\Omega)$ is the solution of the Dirichlet problem $-\Delta w = u^1$ in Ω , $w = 0$ on Γ .

REMARK 7.1 An analysis of the results from Table 1 shows that the number of conjugate gradient iterations necessary to achieve convergence is independent of h . Moreover, the approximation errors for (\hat{w}^0, \hat{w}^1) satisfy

$$\|\hat{w}^0 - \hat{W}_h^0\|_{L^2(\Omega)} = O(h^{1.88}), \quad \|\hat{w}^0 - \hat{W}_h^0\|_{H_0^1(\Omega)} = O(h^{1.10}), \quad \|\hat{w}^1 - \hat{W}_h^1\|_{H^1(\Omega)} = O(h^{1.06}),$$

while for the control, we have

$$\|V - V_h\|_{L^2(\partial\Omega \times (0,T))} = O(h^{1.17}).$$

Figure 2 (left) depicts the exact and approximate controls V and V_h at the point $\bar{x} = (1, 1/2) \in \partial\Omega$, obtained with $h = 1/15$ (for $h = 1/30, 1/60, 1/120$, the two curves cannot be distinguished). The approximation error is given in Fig. 2 (right) and satisfies $\|(V - V_h)\|_{L^\infty(0,T)} = O(h^{0.95})$.

At last, some numerical experiments (not reproduced here) highlight the condition (7.6). More precisely, if the unconditionally stable scheme (7.4) is used with $\Delta t = O(h)$, then the conjugate gradient algorithm diverges for h small enough.

7.2.2 *Example 2: discontinuity of the initial position u^0 .* In this second example, we consider a more singular situation with a discontinuous initial displacement u^0 :

$$u^0(x, y) = \begin{cases} 40, & (x, y) \in (\frac{1}{3}, \frac{2}{3})^2, \\ 0, & \text{elsewhere,} \end{cases} \quad u^1(x, y) = 0. \tag{7.7}$$

We assume that the control (v, z) is active on Γ_1 (see (1.1)) and we take $T = 2\sqrt{2}$. As in the previous example, a conjugate gradient algorithm based on the simplest discretization of the wave equation diverges. On the contrary, the use of scheme (7.4) allows to obtain convergence without filtering or regularization techniques. This is displayed in Table 2. The number of iterations to achieve convergence remains low and constant for h small. Moreover, the convergence is slightly affected by the lack of

TABLE 3 $\|T\hat{W}_{Th}^0 - \chi_h^0\|_{H_0^1(\Omega)}$ and $\|T\hat{W}_{Th}^1 - \chi_h^1\|_{L^2(\Omega)}$ with $h = 1/60$ in Example 2

	$T = 3$	$T = 5$	$T = 10$	$T = 20$	$T = 40$
CG iterations	10	9	8	8	5
$\ T\hat{W}_{Th}^0 - \chi_h^0\ _{H_0^1(\Omega)}$	7.15×10^{-1}	3.4×10^{-1}	1.40×10^{-1}	1.11×10^{-1}	3.3×10^{-2}
$\ T\hat{W}_{Th}^1 - \chi_h^1\ _{L^2(\Omega)}$	4.12×10^{-1}	2.21×10^{-1}	1.55×10^{-1}	8.46×10^{-2}	2.47×10^{-2}

regularity of u^0 : we compute for instance that $\|u^1 - U_h^1\|_{H^{-1}(\Omega)} = O(h^{0.71})$ to be compared with $\|u^1 - U_h^1\|_{H^{-1}(\Omega)} = O(h^{1.01})$ for the first example.

For $h = 1/60$, the exact controllability of the wave equation is illustrated on Fig. 3: the approximate controlled solution U_h is drawn in the unit square Ω for six values of time: $t = 0, T/5, 2T/5, 3T/5, 4T/5$ and T . For $t = 0$, U_h coincides with the discontinuous position u^0 , while for $t = T$ the solution is null controlled: the ratio of the energy between the two states is $E_h(T)/E_h(0) \approx 1.11 \times 10^{-4}$. At last, we highlight that the value of T is strictly lower than $2\sqrt{3}$ obtained in Theorem 4.2.

Furthermore, a very useful result to validate our numerical scheme for large values of T is due to Bensoussan (1990) who has shown that when the control is active on the whole boundary,

$$\lim_{T \rightarrow \infty} T(\hat{w}_T^0, \hat{w}_T^1) = (\chi^0, \chi^1), \tag{7.8}$$

where χ^0 and χ^1 are solutions of

$$\Delta \chi^0 = \frac{1}{2}u^1 \text{ in } \Omega, \quad \chi^0 = 0 \text{ on } \partial\Omega; \quad \chi^1 = \frac{1}{2}u^0 \text{ in } \overline{\Omega}$$

and $(\hat{w}_T^0, \hat{w}_T^1)$ are the initial conditions of the backward system (2.2). The numerical results we obtain with the scheme (7.4) (see Table 3) confirm clearly the theoretical property (7.8): $\|T\hat{W}_{Th}^0 - \chi_h^0\|_{H_0^1(\Omega)} = O(T^{-1.10})$ and $\|T\hat{W}_{Th}^1 - \chi_h^1\|_{L^2(\Omega)} = O(T^{-1.0087})$. As advocated in Glowinski *et al.* (1990), these results provide a validation of the numerical methodology introduced here and show that the scheme is particularly robust, accurate and perfectly able to handle very long intervals $[0, T]$.

The numerical results we have presented indicate that the scheme (7.4) under condition (7.6) provides a uniform approximation of the control with respect to the discretization parameters. However, a rigorous proof of the convergence remains to be done.

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REFERENCES

BANKS, H. T., ITO, K. & WANG, C. (1991) Exponentially stable approximations of weakly damped wave equations. *Estimation and Control of Distributed Parameter Systems (Vorau, 1990)* (W. Desch, F. Kappel & K. Kunisch eds). International Series of Numerical Mathematics, vol.100. Basel: Birkhäuser, pp. 1–33.

BENSOUSSAN, A. (1990) On the general theory of exact controllability for skew symmetric operators. *Acta Appl. Math.*, **20**, 197–229.

COHEN, G. C. (2002) *Higher-Order Numerical Methods for Transient Wave Equations*. Scientific Computation. Berlin, Germany: Springer.

GLOWINSKI, R. (1991) Ensuring well-posedness by analogy; Stokes problem and boundary control for the wave equation. *J. Comput. Phys.*, **103**, 189–221.

- GLOWINSKI, R., KINTON, W. & WHEELER, M. F. (1989) A mixed finite element formulation for the boundary controllability of the wave equation. *Int. J. Numer. Methods Eng.*, **27**, 623–636.
- GLOWINSKI, R., LI, C. & LIONS, J.-L. (1990) A numerical approach to the exact boundary controllability of the wave equation (I). Dirichlet controls: description of the numerical methods. *Jpn. J. Appl. Math.*, **7**, 1–76.
- GLOWINSKI, R. & LIONS, J.-L. (1996) Exact and approximate controllability for distributed parameter systems. *Acta Numer.*, **5**, 159–333.
- INFANTE, J. A. & ZUAZUA, E. (1999) Boundary observability for the space semi-discretization of the 1-D wave equation. *Math. Model. Numer. Anal.*, **33**, 407–438.
- KAPPEL, F. & ITO, K. (1998) The Trotter-Kato theorem and approximations of PDE's. *Math. Comput.*, **67**, 21–44.
- LIONS, J.-L. (1988) *Contrôlabilité exacte perturbations et stabilisation de systèmes distribués*, Tome 1. Paris, France: Masson.
- ROBERTS, J. E. & THOMAS, J.-M. (1989) Mixed and hybrid methods. *Handbook of Numerical Analysis* (P. G. Ciarlet & J.-L. Lions eds), vol. 2. Amsterdam, The Netherlands: North-Holland, pp. 523–639.
- TREFETHEN, L. N. (1982) Group velocity in finite difference schemes. *SIAM Rev.*, **24**, 113–136.
- ZUAZUA, E. (1999) Boundary observability for the finite difference space semi-discretizations of the 2-D wave equation in the square. *J. Math. Pures Appl.*, **78**, 523–563.
- ZUAZUA, E. (2005) Propagation, observation, control and numerical approximation of waves. *SIAM Rev.*, **47**, 197–243.

Appendix A

In this section, we prove Theorem 4.2. To simplify the notation, we write

$$\begin{aligned} a_{ij}^{kl} &= w_{ik} + w_{il} + w_{jk} + w_{jl}, & b_{ij}^{kl} &= w'_{ik} + w'_{il} + w'_{jk} + w'_{jl}, \\ c_{ij}^{kl} &= w''_{ik} + w''_{il} + w''_{jk} + w''_{jl}, \\ \Delta_{(1,0)} w_{ij} &= 2w_{ij} - w_{i+1j} - w_{i-1j}, & \Delta_{(0,1)} w_{ij} &= 2w_{ij} - w_{ij+1} - w_{ij-1}, \\ \Delta_{(1,1)} w_{ij} &= 2w_{ij} - w_{i+1j+1} - w_{i-1j-1}, & \Delta_{(1,-1)} w_{ij} &= 2w_{ij} - w_{i+1j-1} - w_{i-1j+1}. \end{aligned}$$

When multiplying the discrete system by the discrete version of the usual continuous multiplier $(x, y) \cdot \nabla u$, i.e.

$$(ih, jh) \cdot \left(\frac{w_{i+1j} - w_{i-1j}}{2h}, \frac{w_{ij+1} - w_{ij-1}}{2h} \right) = i \frac{w_{i+1j} - w_{i-1j}}{2} + j \frac{w_{ij+1} - w_{ij-1}}{2} \equiv \frac{m_{ij}}{2},$$

and summing in i and j , we obtain

$$\begin{aligned} 0 &= \frac{h^2}{32} \int_0^T \underbrace{\sum_{i,j=1}^N (c_{ii+1}^{jj+1} + c_{ii+1}^{j-1j} + c_{i-1i}^{jj+1} + c_{i-1i}^{j-1j}) m_{ij}}_{\equiv C} dt \\ &+ \frac{1}{6} \int_0^T \underbrace{\sum_{i,j=1}^N (\Delta_{(1,0)} w_{ij} + \Delta_{(0,1)} w_{ij} + \Delta_{(1,1)} w_{ij} + \Delta_{(1,-1)} w_{ij}) m_{ij}}_{\equiv D} dt. \end{aligned} \quad (\text{A.1})$$

We study separately C and D . Integration by parts in C allows us to obtain

$$C = \int_0^T C_1 dt + [C_2]_0^T, \quad (\text{A.2})$$

where

$$C_1 = - \sum_{i,j=1}^N (b_{ii+1}^{jj+1} + b_{ii+1}^{j-1j} + b_{i-1i}^{jj+1} + b_{i-1i}^{j-1j}) m'_{ij}, \quad (\text{A.3})$$

$$C_2 = \sum_{i,j=1}^N (b_{ii+1}^{jj+1} + b_{ii+1}^{j-1j} + b_{i-1i}^{jj+1} + b_{i-1i}^{j-1j}) m_{ij}. \quad (\text{A.4})$$

We first consider the term C_1 above. In order to have the common factor b_{ii+1}^{jj+1} , we change the indexes in the last three terms of C_1 above. Then, taking into account that $w_{i0} = w_{i,N+1} = w_{0,j} = w_{j,N+1} = 0$ and after simplification, we obtain

$$C_1 = 2 \sum_{i,j=0}^N (b_{ii+1}^{jj+1})^2 - (N+1) \left[\sum_{i=1}^N (w'_{iN} + w'_{i+1N})^2 + \sum_{j=1}^N (w'_{Nj} + w'_{Nj+1})^2 \right]. \quad (\text{A.5})$$

We now analyse the term D in (A.1). We only make the details for the first term in D since the others can be simplified similarly. It reads

$$\sum_{i,j=1}^N \Delta_{(1,0)} w_{ij} m_{ij} = \sum_{i,j=1}^N \Delta_{(1,0)} w_{ij} [i(w_{i+1j} - w_{i-1j}) + j(w_{ij+1} - w_{ij-1})]. \quad (\text{A.6})$$

We consider separately these two terms. For the second one, we have

$$\begin{aligned} \Delta_{(1,0)} w_{ij} j (w_{ij+1} - w_{ij-1}) &= \sum_{i,j=1}^N j (w_{ij} - w_{i-1j}) w_{ij+1} - \sum_{i,j=1}^N j (w_{i+1j} - w_{ij}) w_{ij+1} \\ &\quad - \left[\sum_{i,j=1}^N j (w_{ij} - w_{i-1j}) w_{ij-1} - \sum_{i,j=1}^N j (w_{i+1j} - w_{ij}) w_{ij-1} \right]. \end{aligned}$$

Changing the indexes to obtain the common factor $(w_{i+1j} - w_{ij})$ in all the terms and taking into account that $w_{i,0} = w_{i,N+1} = w_{0,j} = w_{j,N+1} = 0$, we obtain

$$\begin{aligned} &\sum_{i,j=0}^N [j (w_{i+1j} - w_{ij}) (w_{i+1j+1} - w_{ij+1}) - j (w_{i+1j} - w_{ij}) (w_{i+1j-1} - w_{ij-1})] \\ &= - \sum_{i,j=0}^N (w_{i+1j+1} - w_{ij+1}) (w_{i+1j} - w_{ij}). \end{aligned}$$

An analogous argument allows to simplify the first term in (A.6) and the other three terms in D . We finally have

$$\begin{aligned}
D &= - \sum_{i,j=0}^N [(w_{i+1j+1} - w_{i+1j})(w_{ij+1} - w_{ij}) + (w_{i+1j+1} - w_{ij+1})(w_{i+1j} - w_{ij})] \\
&\quad + \sum_{i,j=0}^N [(w_{i+1j} - w_{ij})^2 + (w_{ij+1} - w_{ij})^2] - (N+1) \sum_{j=0}^N [(w_{Nj})^2 + 2w_{Nj}w_{Nj+1}] \\
&\quad - (N+1) \sum_{i=0}^N [(w_{iN})^2 + 2w_{iN}w_{i+1N}]. \tag{A.7}
\end{aligned}$$

By Young's inequality, we can estimate the first term in this formula

$$\begin{aligned}
&\sum_{i,j=0}^N [(w_{i+1j+1} - w_{i+1j})(w_{ij+1} - w_{ij}) + (w_{i+1j+1} - w_{ij+1})(w_{i+1j} - w_{ij})] \\
&\leq \sum_{i,j=0}^N [(w_{ij+1} - w_{ij})^2 + (w_{i+1j} - w_{ij})^2].
\end{aligned}$$

Therefore,

$$\begin{aligned}
D &\geq -(N+1) \left[\sum_{j=0}^N w_{Nj}^2 + 2 \sum_{j=0}^N w_{Nj}w_{Nj+1} + \sum_{i=0}^N w_{iN}^2 + 2 \sum_{i=0}^N w_{iN}w_{i+1N} \right] \\
&= -(N+1) \left[\sum_{j=1}^N (w_{Nj-1} + w_{Nj} + w_{Nj+1})w_{Nj} + \sum_{i=1}^N (w_{i-1N} + w_{iN} + w_{i+1N})w_{iN} \right]. \tag{A.8}
\end{aligned}$$

Substituting (A.2), (A.5) and (A.8) into (A.1), we obtain

$$\begin{aligned}
h^2 \int_0^T \sum_{i,j=0}^N \left(\frac{b_{ii+1}^{jj+1}}{4} \right)^2 dt &\leq \frac{h}{8} \int_0^T \left[\sum_{i=1}^N \left(\frac{w'_{iN} + w'_{i+1N}}{2} \right)^2 + \sum_{j=1}^N \left(\frac{w'_{Nj} + w'_{Nj+1}}{2} \right)^2 \right] dt \\
&\quad + \frac{1}{2} \int_0^T \left[\sum_{j=1}^N \frac{w_{Nj-1} + w_{Nj} + w_{Nj+1}}{3} \frac{w_{Nj}}{h} + \sum_{i=1}^N \frac{w_{i-1N} + w_{iN} + w_{i+1N}}{3} \frac{w_{iN}}{h} \right] dt \\
&\quad - \frac{h^2}{32} [C_2]_0^T. \tag{A.9}
\end{aligned}$$

We observe that the term in the left-hand side contains only one part of the energy. In order to obtain the full energy, we make an equipartition of the energy. The following lemma is a discrete version of the

well-known equipartition of energy for the continuous wave equation, which reads

$$0 = - \int_0^T \int_{\Omega} (|w_t|^2 + |\nabla u|^2) dx dt + \int_{\Omega} |w_t u|^2 dx \Big|_0^T. \quad (\text{A.10})$$

LEMMA A.1 The following holds:

$$\begin{aligned} 0 = & -h^2 \int_0^T \left[\sum_{i,j=0}^N \left(\frac{b_{ii+1}^{jj+1}}{4} \right)^2 \right] dt + h^2 \left[\sum_{i,j=0}^N \left(\frac{a_{ii+1}^{jj+1}}{4} \right) \left(\frac{b_{ii+1}^{jj+1}}{4} \right) \right]_0^T \\ & + h^2 \sum_{i,j=0}^N \int_0^T \left[\frac{1}{3} \left(\frac{w_{i+1j} - w_{ij}}{h} \right)^2 + \frac{1}{3} \left(\frac{w_{ij+1} - w_{ij}}{h} \right)^2 \right. \\ & \left. + \frac{2}{3} \left(\frac{w_{i+1j+1} - w_{ij}}{h\sqrt{2}} \right)^2 + \frac{2}{3} \left(\frac{w_{i+1j} - w_{ij+1}}{h\sqrt{2}} \right)^2 \right] dt. \end{aligned}$$

The proof of this lemma is straightforward following the idea of the continuous system where (A.10) is obtained multiplying system (2.2) by u and integrating by parts.

When applying Lemma A.1 to the identity (A.9), we obtain

$$\begin{aligned} & \int_0^T E_h(t) dt + \frac{h^2}{32} \left[\sum_{i,j=0}^N a_{ii+1}^{jj+1} b_{ii+1}^{jj+1} + C_2 \right]_0^T \\ & \leq \frac{h}{8} \int_0^T \left[\sum_{i=1}^N \left(\frac{w'_{iN} + w'_{i+1N}}{2} \right)^2 + \sum_{j=1}^N \left(\frac{w'_{Nj} + w'_{Nj+1}}{2} \right)^2 \right] dt \\ & \quad + \frac{1}{2} \int_0^T \left[\sum_{j=1}^N \frac{w_{Nj-1} + w_{Nj} + w_{Nj+1}}{3} \frac{w_{Nj}}{h} + \sum_{i=1}^N \frac{w_{i-1N} + w_{iN} + w_{i+1N}}{3} \frac{w_{iN}}{h} \right] dt. \quad (\text{A.11}) \end{aligned}$$

The following lemma allows us to estimate the second term in the left-hand side of this formula.

LEMMA A.2 The following holds

$$h^2 \left[\sum_{i,j=0}^N a_{ii+1}^{jj+1} b_{ii+1}^{jj+1} + C_2 \right]_0^T \leq 64\sqrt{3} E_h(0). \quad (\text{A.12})$$

Before proving this lemma, we finish the proof of Theorem 4.2.

From Lemma A.2 and the conservation of the discrete energy, stated in Proposition 4.1, we have

$$\int_0^T E_h(t) dt + \frac{h^2}{32} \left[\sum_{i,j=0}^N a_{ii+1}^{jj+1} b_{ii+1}^{jj+1} + C_2 \right]_0^T \geq T E_h(0) - 2\sqrt{3} E_h(0) = (T - 2\sqrt{3}) E_h(0),$$

which combined with (A.11) provides the following:

$$(T - 2\sqrt{3})E_h(0) \leq \frac{h}{8} \int_0^T \left[\sum_{i=1}^N \left(\frac{w'_{iN} + w'_{i+1N}}{2} \right)^2 + \sum_{j=1}^N \left(\frac{w'_{Nj} + w'_{Nj+1}}{2} \right)^2 \right] dt \\ + \frac{1}{2} \int_0^T \left[\sum_{j=1}^N \frac{w_{Nj-1} + w_{Nj} + w_{Nj+1}}{3} \frac{w_{Nj}}{h} + \sum_{i=1}^N \frac{w_{i-1N} + w_{iN} + w_{i+1N}}{3} \frac{w_{iN}}{h} \right] dt.$$

This concludes the proof of Theorem 4.2.

Proof. (Lemma A.2) From (A.4), we have

$$\sum_{i,j=0}^N a_{ii+1}^{jj+1} b_{ii+1}^{jj+1} + C_2 = \sum_{i,j=0}^N a_{ii+1}^{jj+1} b_{ii+1}^{jj+1} + \sum_{i,j=1}^N [b_{ii+1}^{jj+1} + b_{ii+1}^{j-1j} + b_{i-1i}^{jj+1} + b_{i-1i}^{j-1j}] m_{ij}. \quad (\text{A.13})$$

To simplify the notation, we assume that $w_{N+2j} = w_{Nj}$, $w_{iN+2} = w_{iN}$ and $w_{-1j} = w_{i,-1} = 0$ for all $i, j = 0, \dots, N+1$. We change the indexes in each of the terms of the right-hand side of (A.13) in order to have the common factor b_{ii+1}^{jj+1} . Then, we obtain

$$\sum_{i,j=0}^N a_{ii+1}^{jj+1} b_{ii+1}^{jj+1} + C_2 = \sum_{i,j=0}^N (a_{ii+1}^{jj+1} + R_{ij}) b_{ii+1}^{jj+1}, \quad (\text{A.14})$$

where

$$R_{ij} = i[(w_{i+1j} - w_{i-1j}) + (w_{i+1j+1} - w_{i-1j+1})] + (i+1)[(w_{i+2j} - w_{ij}) + (w_{i+2j+1} - w_{ij+1})] \\ + j[(w_{ij+1} - w_{ij-1}) + (w_{i+1j+1} - w_{i+1j-1})] + (j+1)[(w_{ij+2} - w_{ij}) + (w_{i+1j+2} - w_{i+1j})].$$

We estimate the right-hand side in (A.14) using the Schwartz inequality. Thus,

$$\sum_{i,j=0}^N a_{ii+1}^{jj+1} b_{ii+1}^{jj+1} + C_2 \leq \left[\sum_{i,j=0}^N (a_{ii+1}^{jj+1} + R_{ij})^2 \right]^{1/2} \left[\sum_{i,j=0}^N (b_{ii+1}^{jj+1})^2 \right]^{1/2}. \quad (\text{A.15})$$

Now we prove that

$$\sum_{i,j=0}^N (a_{ii+1}^{jj+1} + R_{ij})^2 \leq \sum_{i,j=0}^N R_{ij}^2 + 8 \sum_{i=1}^N (N+1)(w_{iN})^2 + 8 \sum_{j=1}^N (N+1)(w_{Nj})^2. \quad (\text{A.16})$$

Indeed, we have

$$\sum_{i,j=0}^N [(a_{ii+1}^{jj+1} + R_{ij})^2 - R_{ij}^2] = \sum_{i,j=0}^N [(a_{ii+1}^{jj+1})^2 + 2a_{ii+1}^{jj+1} R_{ij}], \quad (\text{A.17})$$

and it is not difficult to see that

$$\sum_{i,j=0}^N a_{ii+1}^{jj+1} R_{ij} = -2 \sum_{i,j=0}^N (a_{ii+1}^{jj+1})^2 + \sum_{i=1}^N (N+1)(w_{iN} + w_{i+1N})^2 + \sum_{j=1}^N (N+1)(w_{Nj} + w_{Nj+1})^2.$$

Therefore, using Young's inequality, the right-hand side in (A.17) reads

$$\begin{aligned} \sum_{i,j=0}^N [(a_{ii+1}^{jj+1})^2 + 2a_{ii+1}^{jj+1} R_{ij}] &= -3 \sum_{i,j=0}^N (a_{ii+1}^{jj+1})^2 + 2 \sum_{i=1}^N (N+1)(w_{iN} + w_{i+1N})^2 \\ &+ 2 \sum_{j=1}^N (N+1)(w_{Nj} + w_{Nj+1})^2 \leq 8 \sum_{i=1}^N (N+1)(w_{iN})^2 + 8 \sum_{j=1}^N (N+1)(w_{Nj})^2. \end{aligned} \quad (\text{A.18})$$

From (A.17) and (A.18), we easily deduce (A.16). Now, we estimate the right-hand side in (A.16). Concerning the first term, we have

$$\sum_{i,j=0}^N R_{ij}^2 \leq \frac{32}{h^2} \sum_{i,j=0}^N [(w_{i+1j} - w_{i-1j})^2 + (w_{ij+1} - w_{ij-1})^2], \quad (\text{A.19})$$

where we have used Young's inequality and the fact that $i, j \leq h^{-1}$. In (A.19), the first term is estimated as follows:

$$\begin{aligned} &\sum_{i,j=0}^N (w_{i+1j} - w_{i-1j})^2 \\ &= \frac{1}{2} \sum_{i,j=0}^N [(w_{i+1j} - w_{ij} + w_{ij} - w_{i-1j})^2 + (w_{i+1j} - w_{ij+1} + w_{ij+1} - w_{i-1j})^2] \\ &\leq \sum_{i,j=0}^N [(w_{i+1j} - w_{ij})^2 + (w_{ij} - w_{i-1j})^2] + \sum_{i,j=1}^N [(w_{i+1j} - w_{ij+1})^2 + (w_{ij+1} - w_{i-1j})^2] \\ &= \sum_{i,j=0}^N [2(w_{i+1j} - w_{ij})^2 + (w_{i+1j} - w_{ij+1})^2 + (w_{i+1j+1} - w_{ij})^2] - 2 \sum_{j=0}^N (w_{Nj})^2, \end{aligned} \quad (\text{A.20})$$

and an analogous formula holds for the second term in (A.19).

Substituting this simplification of (A.19) into (A.16), we easily obtain

$$\begin{aligned} &\sum_{i,j=0}^N (a_{ii+1}^{jj+1} + R_{ij})^2 \\ &\leq \frac{64}{h^2} \sum_{i,j=0}^N [(w_{i+1j} - w_{ij})^2 + (w_{ij+1} - w_{ij})^2 + (w_{i+1j} - w_{ij+1})^2 + (w_{i+1j+1} - w_{ij})^2], \end{aligned} \quad (\text{A.21})$$

which allows us to estimate (A.15). In fact, by Young's inequality, we obtain

$$\begin{aligned}
\sum_{i,j=0}^N a_{ii+1}^{jj+1} b_{ii+1}^{jj+1} + C_2 &\leq \frac{8}{h} \left(\sum_{i,j=0}^N (b_{ii+1}^{jj+1})^2 \right)^{1/2} \left(\sum_{i,j=0}^N [(w_{i+1j} - w_{ij})^2 + (w_{ij+1} - w_{ij})^2 \right. \\
&\quad \left. + (w_{i+1j} - w_{ij+1})^2 + (w_{i+1j+1} - w_{ij})^2] \right)^{1/2} \\
&\leq 16\sqrt{3} \left[\sum_{i,j=0}^N \left(\frac{b_{ii+1}^{jj+1}}{4} \right)^2 + \frac{1}{3} \sum_{i,j=0}^N [(w_{i+1j} - w_{ij})^2 + (w_{ij+1} - w_{ij})^2 \right. \\
&\quad \left. + (w_{i+1j} - w_{ij+1})^2 + (w_{i+1j+1} - w_{ij})^2] \right] = 32\sqrt{3}E_h(t).
\end{aligned}$$

Therefore,

$$\left[\sum_{i,j=0}^N a_{ii+1}^{jj+1} b_{ii+1}^{jj+1} + C_2 \right]_0^T \leq 32\sqrt{3}(E_h(T) + E_h(0)) \leq 64\sqrt{3}E_h(0). \quad (\text{A.22})$$

This concludes the proof of Lemma A.2. □