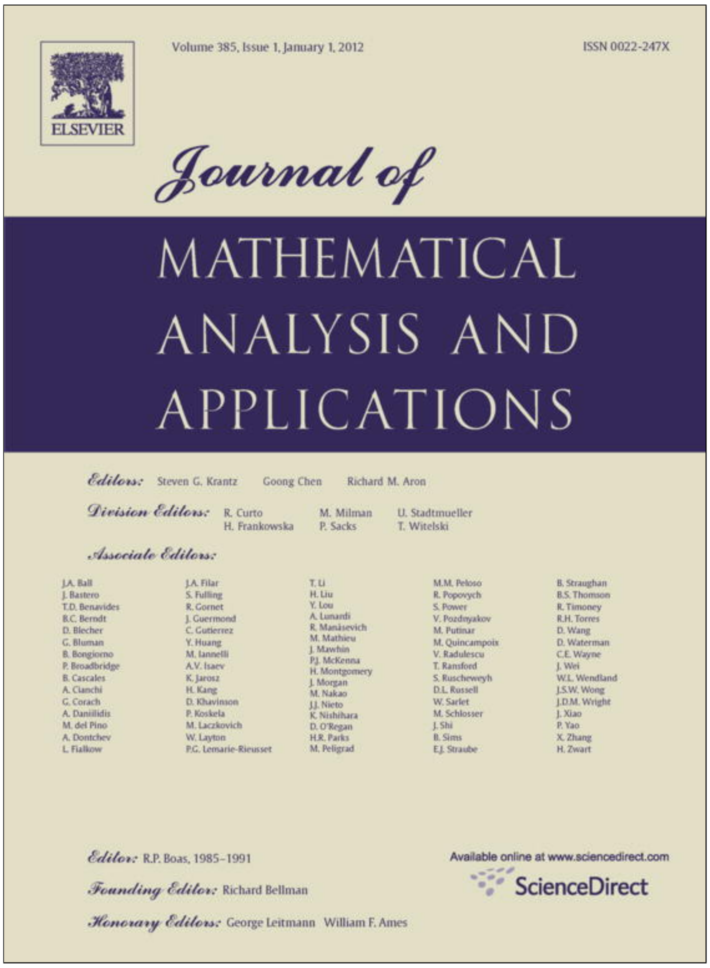
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Periodic solutions for a weakly dissipated hybrid system

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ABSTRACT

We consider the motion of a stretched string coupled with a rigid body at one end and we study the existence of periodic solution when a periodic force f acts on the body. The main difficulty of the study is related to the weak dissipation that characterizes this hybrid system, which does not ensure a uniform decay rate of the energy. Under additional regularity conditions on f, we use a perturbation argument in order to prove the existence of a periodic solution. In the last part of the paper we present some numerical simulations based on the theoretical results.

1. Introduction

In this paper we consider the following hybrid system

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = 0 & \text{in } (0,1)\times(0,\infty), \\ u(0,t) = 0 & \text{on } (0,\infty), \\ u_{tt}(1,t) + u_x(1,t) + \alpha u_t(1,t) = f(t) & \text{on } (0,\infty) \end{cases} \tag{1.1}$$

where α is a positive constant.

System (1.1) models the coupling between a vibrating string located on the interval $(0,1)$ and a rigid body attached at the right extremity $x=1$. Here $u(x,t)$ describes the position of the string at each moment $t>0$ and point $x \in (0,1)$ and verifies the linear wave equation. The left extremity $x=0$ of the string is supposed to be fixed. In the third equation of (1.1), $u(1,t)$ gives the movement of the body and satisfies the Newton law. When $\alpha>0$, the term $\alpha u_t(1,t)$ represents a friction proportional with the velocity of the body. Note that the dissipation of system (1.1) acts only through the ordinary differential equation representing the movement of the rigid body. For simplicity, we have chosen the length of the string, the wave velocity and the mass of the solid to be equal to one.

Model (1.1) has been introduced and analyzed in [12] (see also [11,13,17]) and represents the simplest example of hybrid system which couples two different types of differential equations. In [1,14,19,20] similar models in which the string is replaced by a beam with different boundary conditions are considered. Most of the results obtained in this paper can be extended to these models with similar statements. Many more complex one or multi dimensional hybrid systems have been introduced and analyzed (see, for instance, [2–4,6,7,15,16,22]). Some of these models are characterized by a weak dissipation acting on one component of the system only. This does not ensure a uniform decay rate of the corresponding energy and makes more difficult the study of such properties as asymptotic behavior, compactness of trajectories, existence of periodic

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or almost periodic functions, etc. To establish these properties may be as a delicate problem as the corresponding one for some nonlinear models.

The non-homogeneous term f in (1.1) represents an external force acting on the body and it is supposed to be not zero and periodic in time with period T

$$f(t+T) = f(t), \quad \forall t \geqslant 0. \tag{1.2}$$

The aim of this article is to study whether there exists a periodic solution of (1.1) under hypothesis (1.2) for f. In the positive case, any other solution of (1.1) is bounded and converges to the periodic one as t goes to infinity. On the other hand, the non-existence of a periodic solution implies that all the solutions are unbounded in the finite energy space when t tends to infinity. In this case, we are dealing with a *resonance phenomenon* in which a bounded perturbation f of the system leads to unbounded solutions.

In this paper we show that, if the non-homogeneous term f is such that $f_{|(0,T)} \in H^1(0,T)$, then there exists a unique periodic solution of (1.1). To prove this result we introduce a perturbed system, depending on a small parameter ε, devoted to go to zero, for which we have an exponential decay rate of the corresponding energy. This property allows us to apply a fixed point argument and to show that the perturbed system has a unique periodic solution for each $\varepsilon > 0$. If $f_{|(0,T)} \in H^1(0,T)$, these periodic solutions are uniformly bounded. Hence, we can pass to the limit to obtain a periodic solution for the initial system (1.1).

On the other hand, note that finite energy solutions for (1.1) exist even if $f_{|(0,T)}$ belongs to $L^2(0,T)$. In this case we use the Fourier decomposition method in order to characterize the periods T for which there exists a periodic solution. Due to the complexity of this characterization, it is not easy to say if this property holds for any T. Thus, in the case $f_{|(0,T)} \in L^2(0,T)$, we cannot guarantee the absence of the resonance phenomenon but we mathematically show that its possible occurrence is precisely related to the weak dissipation of the system. In [15] a more complex hybrid system, coupling two partial differential equations, is considered and the existence of periodic solutions is studied. Due to simpler nature of the model, the results we obtain in the present paper are much sharper.

In the last part of the article we use the perturbation argument and the fixed point method mentioned above in order to give some numerical approximation for the periodic solution. From these computations we can see that the extra dissipation introduced in the perturbed system ensures, in many cases, a better numerical behavior of the approximation scheme. Also, we provide some criteria for the optimal choice of the perturbed parameter ε with respect to the discretization step.

The remaining part of the paper is organized as follows. In Section 2 we present some basic properties of system (1.1). Section 3 is devoted to the analysis of the perturbed system and the proof of existence of periodic solutions for it. Their uniform boundedness is proved in Section 4 supposing that $f_{|(0,T)} \in H^1(0,T)$. The existence of a periodic solution for the initial system (1.1) is a consequence of those uniform estimates. In Section 5 we analyze the case $f_{|(0,T)} \in L^2(0,T)$ and we give characterization of the periods T for which there exists a periodic solution for (1.1). Finally, in the last section we present the numerical simulations which are based on and confirm the theoretical results.

2. Preliminaries

In this section we present some basic properties of system (1.1). Most of these results are well known (see, for instance, [1,11,12,17]) but we prefer to present them in order to make our paper self contained and easier to read. Firstly, we need to introduce some notation and to rewrite (1.1) in an abstract form. We define the Hilbert spaces

$$V = \{u \in H^1(0,1)\colon u(0) = 0\}, \qquad H = V \times L^2(0,1) \times \mathbb{R}$$

endowed with their natural inner product. We shall denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the inner product and the corresponding norm in H, respectively.

By introducing the new variables $w = u_t$ and $z = u_t(1,t)$, we equivalently write (1.1) as

$$\frac{dU(t)}{dt} = AU(t) + F(t), \quad t > 0 \tag{2.1}$$

and we add the initial condition

$$U(0) = U_0, \tag{2.2}$$

where

$$U = \begin{bmatrix} u \\ w \\ z \end{bmatrix}, \qquad AU = \begin{bmatrix} w \\ u_{xx} \\ -u_x(1) - \alpha w(1) \end{bmatrix}, \qquad F = \begin{bmatrix} 0 \\ 0 \\ f \end{bmatrix} \quad \text{and} \quad U_0 = \begin{bmatrix} u_0 \\ w_0 \\ z_0 \end{bmatrix}.$$

The operator A is an unbounded operator in H, with domain

$$D(A) = \{(u,w,z) \in H^2(0,1) \cap V \times V \times \mathbb{R}\colon w(1) = z\}.$$

With this notation, we have the following result.

Theorem 2.1. *The operator $(D(A), A)$ is a maximal-dissipative operator which generates a contraction semigroup $(S(t))_{t\geqslant 0}$ in H. If $U_0 \in H$ and $f \in \mathcal{C}[0,\infty)$, there exists a unique mild solution $U \in \mathcal{C}([0,\infty); H)$ of (2.1)–(2.2) given by the variation of constant formula*

$$U(t) = S(t)U_0 + \int_0^t S(t-s)F(s)\,ds. \tag{2.3}$$

The total energy associated to (1.1) is given by

$$E(t) = \frac{1}{2}\int_0^1 \left(u_t^2(x,t) + u_x^2(x,t)\right)dx + \frac{1}{2}u_t^2(1,t) \tag{2.4}$$

and under the above boundary conditions we can, formally, deduce that

$$\frac{dE(t)}{dt} = -\alpha u_t^2(1,t) + f(t)u_t(1,t). \tag{2.5}$$

When $f = 0$, (2.5) indicates that the energy is decreasing which has been shown sufficient for the strong stability, but as we have said before, does not ensure the uniform stability. The next theorem describes this asymptotic behavior of solutions of the homogeneous system (2.1)–(2.2).

Theorem 2.2. *For any $U_0 \in H$ we have that*

$$\lim_{t\to\infty}\left\|S(t)U_0\right\| = 0. \tag{2.6}$$

Moreover, there are no positive constants M and ω such that

$$\left\|S(t)U_0\right\| \leqslant M\exp(-\omega t)\|U_0\|, \quad \forall U_0 \in H,\ \forall t \geqslant 0. \tag{2.7}$$

Remark 2.3. The asymptotic properties described by Theorem 2.2 show that the dissipation $\alpha u_t(1,t)$ of (1.1), which is concentrated on the rigid body, cannot ensure the exponential decay of the energy (2.5). As it is shown in [13], this is equivalent to the existence of solutions with arbitrarily slow decay rate.

A spectral analysis of the operator A confirms the asymptotic behavior described in the previous theorem. More precisely, we have the following result.

Theorem 2.4. *The operator $(D(A), A)$ has a sequence of eigenvalues $(\nu_n)_{n\geqslant 1}$ with the following behavior*

$$\nu_n = n\pi i + \frac{i}{n\pi} - \frac{\alpha}{n^2\pi^2} + o\left(\frac{1}{n^2}\right) \quad \text{as } n \to \infty. \tag{2.8}$$

Remark 2.5. From (2.8) we see that $\mathcal{R}e\,(\nu_n) < 0$ but $\lim_{n\to\infty}|\mathcal{R}e\,(\nu_n)| = 0$. This confirms the non-uniform exponential decay rate stated in Theorem 2.2.

As we have mentioned before, our interest is to prove the existence of a periodic solution for (2.1) when f is *T-periodic*, i.e. satisfies

$$f(t+T) = f(t), \quad \forall t \geqslant 0. \tag{2.9}$$

A first result in this direction is the following simple one.

Proposition 2.6. *Suppose that $f \in \mathcal{C}[0,\infty)$ and verifies (2.9). If system (2.1) has one T-periodic solution $U \in \mathcal{C}([0,\infty); H)$, then any other solution $V \in \mathcal{C}([0,\infty); H)$ verifies*

$$\lim_{t\to\infty}\left\|U(t) - V(t)\right\| = 0. \tag{2.10}$$

As a consequence, system (2.1) has at most one T-periodic solution in $\mathcal{C}([0,\infty); H)$.

Proof. Suppose that (2.1) has a periodic solution U. Let V be any other solution and $W = U - V$. It follows that $W \in \mathcal{C}([0,\infty); H)$ and verifies

$$W(t) = S(t)\big(U(0) - V(0)\big), \quad \forall t \geqslant 0.$$

Now, (2.10) follows from (2.6) in Theorem 2.2.

If V is a T-periodic solution we deduce that $W = U - V$ is T-periodic too. From (2.10) we obtain that $U \equiv V$ and the proof ends. □

3. Existence of periodic solution for a perturbed system

As we have said before, the main difficulty in the study of existence of periodic solutions for (2.1) comes from the non-uniform decay rate of the corresponding energy (2.4). To overcome this difficulty, we introduce an extra dissipative term, depending on a small parameter ε, which ensures the uniform decay of the energy. More precisely, for any $\varepsilon > 0$, we consider the system

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + \varepsilon u_t = 0 & \text{in } (0,1) \times (0,\infty), \\ u(0,t) = 0 & \text{on } (0,\infty), \\ u_{tt}(1,t) + u_x(1,t) + \alpha u_t(1,t) = f(t) & \text{on } (0,\infty) \end{cases} \tag{3.1}$$

where f is a T-periodic function verifying (1.2).

The aim of this section is to prove that, for each $\varepsilon > 0$, there exists a unique T-periodic solution of (3.1). As shown in [8], if the corresponding linear system is uniformly stable, the existence of a periodic solution may be obtained from a fixed point argument.

The total energy associated to (3.1) is given by

$$E_\varepsilon(t) = \frac{1}{2}\int_0^1 \big(u_t^2(x,t) + u_x^2(x,t)\big)\,dx + \frac{1}{2}u_t^2(1,t). \tag{3.2}$$

It is easy to see that, at least formally,

$$\frac{dE_\varepsilon(t)}{dt} = -\alpha u_t^2(1,t) - \varepsilon\int_0^1 u_t^2(x,t)\,dx + f(t)u_t(1,t). \tag{3.3}$$

By comparing (2.5) and (3.3) we see that the term εu_t introduced in (3.1) reinforces the dissipation of the system. In fact, as we shall prove latter on, this term ensures the uniform stability of system (3.1) and allows to prove the existence of a periodic solution for (3.1) as in [8].

As in the previous section, introducing the new variables $w = u_t$ and $z = u_t(1,t)$, we write (3.1) as

$$\frac{dU(t)}{dt} = A_\varepsilon U(t) + F(t) \tag{3.4}$$

and we add the initial condition

$$U(0) = U_0, \tag{3.5}$$

where

$$U = \begin{bmatrix} u \\ w \\ z \end{bmatrix}, \qquad A_\varepsilon U = \begin{bmatrix} w \\ u_{xx} - \varepsilon w \\ -u_x(1) - \alpha w(1) \end{bmatrix}, \qquad F = \begin{bmatrix} 0 \\ 0 \\ f \end{bmatrix} \quad \text{and} \quad U_0 = \begin{bmatrix} u_0 \\ w_0 \\ z_0 \end{bmatrix}.$$

The operator A_ε is an unbounded operator in H, with domain

$$D(A_\varepsilon) = \big\{(u,w,z) \in H^2(0,1) \cap V \times V \times \mathbb{R}:\ w(1) = z\big\}$$

which is independent of ε. In fact, $D(A_\varepsilon) = D(A)$ for each $\varepsilon > 0$.

With this notation, we have the following result.

Theorem 3.1. *If $U_0 \in H$ and $f \in \mathcal{C}[0,\infty)$, there exists a unique weak solution $U \in \mathcal{C}([0,\infty); H)$ of (3.4)–(3.5) with the property that, for any $t_0 > 0$,*

$$\|U\|_{\mathcal{C}([0,t_0];H)} \leqslant \|U_0\| + \sqrt{t_0}\,\|f\|_{L^2(0,t_0)}. \tag{3.6}$$

Moreover, if $U_0 \in D(A_\varepsilon)$ and $f_{|(0,a)} \in H^1(0,a)$, for all $a > 0$, there exists a unique weak solution $U \in \mathcal{C}^1([0,\infty); H) \cap \mathcal{C}([0,\infty); D(A_\varepsilon))$ of (3.4)–(3.5).

Proof. Performing as in [17] we can prove that $(D(A_\varepsilon), A_\varepsilon)$ is the infinitesimal generator of a semigroup $(S_\varepsilon(t))_{t\geqslant 0}$ of contractions in H. Now, the result follows from the classical theory of linear equations of evolution (see, for instance, [5]). □

The following theorem shows that system (3.4)–(3.5) is exponentially stable if $f \equiv 0$.

Theorem 3.2. *Let $(S_\varepsilon(t))_{t\geqslant 0}$ be the semigroup of contractions generated by the operator $(D(A_\varepsilon), A_\varepsilon)$. There exist two positive constants $M > 0$ and $\omega = \omega(\varepsilon) > 0$ such that, for any $U_0 \in H$,*

$$\left\| S_\varepsilon(t)U_0 \right\| \leqslant Me^{-\omega t}\|U_0\|, \quad t > 0. \tag{3.7}$$

Moreover, for any $U_0 \in D(A_\varepsilon)$,

$$\left\| S_\varepsilon(t)U_0 \right\|_{D(A_\varepsilon)} \leqslant Me^{-\omega t}\|U_0\|_{D(A_\varepsilon)}, \quad t > 0. \tag{3.8}$$

Proof. Multiplying the first equation in (3.1) by u it is easy to see that the solution u satisfies

$$\frac{d}{dt}\left\{ \int_0^1 uu_t\,dx + \frac{\varepsilon}{2}\int_0^1 u^2\,dx + u(1,t)u_t(1,t) + \frac{\alpha}{2}u^2(1,t) \right\} = -\int_0^1 u_x^2\,dx + \int_0^1 u_t^2\,dx + u_t^2(1,t). \tag{3.9}$$

For any $\delta > 0$, from (3.3) and (3.9) we deduce that

$$\frac{d\mathcal{L}(t)}{dt} = -\alpha u_t^2(1,t) - \varepsilon\int_0^1 u_t^2\,dx - \delta\int_0^1 u_x^2\,dx + \delta\int_0^1 u_t^2\,dx + \delta u_t^2(1,t)$$

where $\mathcal{L} = \mathcal{L}(t)$ is a suitable perturbation of the energy given by

$$\mathcal{L}(t) = E_\varepsilon(t) + \delta\left\{ \int_0^1 uu_t\,dx + \frac{\varepsilon}{2}\int_0^1 u^2\,dx + u(1,t)u_t(1,t) + \frac{\alpha}{2}u^2(1,t) \right\}. \tag{3.10}$$

Moreover, for $0 < \delta < \varepsilon$, sufficiently small, we have

$$\frac{d\mathcal{L}(t)}{dt} \leqslant -cE_\varepsilon(t), \tag{3.11}$$

where $c = \frac{1}{2}\min\{\alpha - \delta, \varepsilon - \delta, \delta\}$. On the other hand, (3.10) and the Sobolev embedding theorem allow us to conclude that

$$\left|\mathcal{L}(t) - E_\varepsilon(t)\right| \leqslant c_1\delta E_\varepsilon(t),$$

for some c_1 independent of ε as well. Then, choosing δ sufficiently small the above inequality allow to deduce that

$$\frac{1}{2}E_\varepsilon(t) \leqslant \mathcal{L}(t) \leqslant \frac{3}{2}E_\varepsilon(t).$$

Since $\delta > 0$ is small enough, the above inequality combined with (3.11) leads to the differential inequality

$$\frac{dE_\varepsilon(t)}{dt} \leqslant -c_2E_\varepsilon(t),$$

for some $c_2 > 0$, which gives us (3.7).

In order to prove (3.8), note that $\|S_\varepsilon(t)U_0\|_{D(A_\varepsilon)} = \|S_\varepsilon(t)A_\varepsilon U_0\| + \|S_\varepsilon(t)U_0\|$ and use two times (3.7). □

Remark 3.3. Since Theorem 2.2 shows that (3.8) does not hold for the homogeneous system (2.1)–(2.2), we have that

$$\lim_{\varepsilon\to 0}\omega(\varepsilon) = 0.$$

Now we have all the ingredients needed to prove the existence of a periodic solution for (3.4).

Theorem 3.4. *If $f \in \mathcal{C}[0,\infty)$ is a T-periodic function, for any $\varepsilon > 0$, there exists a unique weak solution $U_\varepsilon \in \mathcal{C}([0,\infty); H)$ of (3.4) which is T-periodic. Moreover, if $f_{|(0,T)} \in H^1(0,T)$ then there exists a unique classical T-periodic solution $U_\varepsilon \in \mathcal{C}([0,\infty); D(A_\varepsilon))$ of (3.4).*

Proof. The proof uses the ideas from [8] and it is done by using a fixed point argument. We introduce the map

$$\Lambda : H \to H$$

given by

$$\Lambda U_0 = S_\varepsilon(T)U_0 + \int_0^T S_\varepsilon(T-s)F(s)\,ds$$

where $(S_\varepsilon(t))_{t\geqslant 0}$ is the semigroup generated by A_ε (see Theorem 3.1). Then, from Theorem 3.2 we get

$$\left\| \Lambda^n U_0 - \Lambda^n U_1 \right\| \leqslant \left\| S_\varepsilon(nT)(U_0 - U_1) \right\| \leqslant Me^{-nT\omega}\|U_0 - U_1\|.$$

Then, for n sufficiently large Λ^n is a contraction and, therefore, there exists a unique $U_0 \in H$ such that

$$\Lambda^n U_0 = U_0.$$

Consequently,

$$\Lambda^n(\Lambda U_0) = \Lambda\big(\Lambda^n U_0\big) = \Lambda U_0.$$

From the uniqueness of the fixed point of Λ^n it follows that

$$\Lambda U_0 = U_0$$

which says that Λ has a fixed point. The solution U of (3.4)–(3.5) with initial data U_0 belongs to $\mathcal{C}([0,\infty); H)$ and verifies $U(0) = U(T)$. This gives us a T-periodic weak solution of (3.4).

To treat the case $f_{|(0,T)} \in H^1(0,T)$, we use Theorem 3.1 and we repeat the above fixed point argument in $D(A_\varepsilon)$ instead of H. From estimate (3.8), we deduce that there exists a unique fixed point U_0 of Λ in $D(A_\varepsilon)$. This gives us a periodic classical solution $U_\varepsilon \in \mathcal{C}([0,\infty); D(A_\varepsilon))$ of (3.4). □

4. Existence of periodic solution when $f \in H^1(0,T)$

In this section we prove the existence of a periodic solution for our system (2.1) under the additional condition $f_{|(0,T)} \in H^1(0,T)$. Theorem 3.4 gives us a family $(U_\varepsilon)_{\varepsilon>0}$ of T-periodic functions. Any limit point of this family is a periodic solution for (2.1). However, in order to prove that this family has a limit, we need some uniform estimates with respect to ε. These estimates require additional regularity conditions for f mentioned above.

Theorem 4.1. *Let $f \in \mathcal{C}[0,\infty)$ be a T-periodic function such that $f_{|(0,T)} \in H^1(0,T)$ and $\varepsilon < 1/2$. If U_ε is the T-periodic solution of* (3.4)*, then there exists a positive constant $C > 0$, independent of ε, such that*

$$\|U_\varepsilon\|_{\mathcal{C}([0,\infty);H)} \leqslant C\|f\|_{H^1(0,T)}. \tag{4.1}$$

Proof. In the sequel C denotes a positive constant which may change from one line to another but it remains independent of ε. Multiplying the first in (3.1) equation by xu_x and integrating by parts over $(0,1)\times(0,T)$, we deduce that

$$\frac{1}{2}\int_0^T\int_0^1 \big(u_t^2 + u_x^2\big)\,dx\,dt = \frac{1}{2}\int_0^T u_t^2(1,t)\,dt + \frac{1}{2}\int_0^T u_x^2(1,t)\,dt - \varepsilon\int_0^T\int_0^1 u_t x u_x\,dx\,dt$$

since, from the periodicity, $\int_0^1 u_t x u_x|_0^T\,dx = 0$. Consequently,

$$(1-\varepsilon)\frac{1}{2}\int_0^T\int_0^1 \big(u_t^2 + u_x^2\big)\,dx\,dt \leqslant \frac{1}{2}\int_0^T u_t^2(1,t)\,dt + \frac{1}{2}\int_0^T u_x^2(1,t)\,dt. \tag{4.2}$$

In order to bound the terms $\int_0^T u_t^2(1,t)\,dt$ and $\int_0^T u_x^2(1,t)\,dt$ we proceed as follows. Integrating (3.3) in $(0,T)$ and taking into account the periodicity of the solutions U_ε, the following holds

$$0 = \varepsilon\int_0^T\int_0^1 u_t^2\,dx\,dt + \int_0^T \big(\alpha u_t^2(1,t) - f(t)u_t(1,t)\big)\,dt$$

which allows to conclude that

$$\frac{\alpha}{2}\int_0^T u_t^2(1,t)\,dt \leqslant \frac{1}{2\alpha}\int_0^T f^2(t)\,dt. \tag{4.3}$$

Now, differentiating (3.1) with respect to t and proceeding in the same way we can bound $\int_0^T u_{tt}^2(1,t)\,dt$. Indeed, letting $v = u_t$, we get

$$\begin{cases} v_{tt}(x,t) - v_{xx}(x,t) + \varepsilon v_t = 0 & \text{in } (0,1)\times(0,\infty),\\ v(0,t) = 0 & \text{on } (0,\infty),\\ v_{tt}(1,t) + v_x(1,t) + \alpha v_t(1,t) = f'(t) & \text{on } (0,\infty),\\ v(x,0) = u_1(x), \quad v_t(x,0) = v_1(x) & \text{on } (0,1), \end{cases} \tag{4.4}$$

where $v_1(x) = u_{tt}(x,0) = u_{0,xx} - \varepsilon u_1(x)$. Since $f \in H^1(0,T)$, the same argument as before allows us to obtain that

$$\frac{\alpha}{2}\int_0^T u_{tt}^2(1,t)\,dt \leqslant \frac{1}{2\alpha}\int_0^T \left(f'\right)^2(t)\,dt. \tag{4.5}$$

By using the third equation from (3.1), we deduce that

$$\int_0^T u_x^2(1,t)\,dt \leqslant \frac{1}{\alpha^2}\int_0^T \left(f'\right)^2(t)\,dt + \left(1+\frac{1}{\alpha}\right)\int_0^T f^2(t)\,dt.$$

Now, from (4.2), (4.3) and (4.5), we obtain that

$$\int_0^T E_\varepsilon(s)\,ds \leqslant C\|f\|^2_{H^1(0,T)}. \tag{4.6}$$

Finally, integrating (3.3) from 0 to $t \in (0,T)$, we have

$$E_\varepsilon(0) = E_\varepsilon(t) + \varepsilon\int_0^t\int_0^1 u_t^2\,dx\,ds + \alpha\int_0^t u_s^2(1,s)\,ds - \int_0^t f(s)u_s(1,s)\,ds.$$

Consequently,

$$\begin{aligned} E_\varepsilon(0) &\leqslant E_\varepsilon(t) + 2\varepsilon\int_0^t E_\varepsilon(s)\,ds + \left(\alpha+\frac{1}{2}\right)\int_0^t u_s^2(1,s)\,ds + \frac{1}{2}\int_0^t f^2(s)\,ds \\ &\leqslant E_\varepsilon(t) + 2\varepsilon\int_0^T E_\varepsilon(s)\,ds + \left(\alpha+\frac{1}{2}\right)\int_0^T u_s^2(1,s)\,ds + \frac{1}{2}\int_0^T f^2(s)\,ds. \end{aligned} \tag{4.7}$$

By integrating (4.7) on $(0,T)$ we deduce that

$$T E_\varepsilon(0) \leqslant (1+2\varepsilon T)\int_0^T E_\varepsilon(t)\,dt + T\left(\alpha+\frac{1}{2}\right)\int_0^T u_t^2(1,t)\,dt + \frac{T}{2}\int_0^T f^2(t)\,dt. \tag{4.8}$$

From (4.3), (4.6) and (4.8) it follows that

$$\left\|U_\varepsilon(0)\right\| = \sqrt{2E_\varepsilon(0)} \leqslant C\|f\|_{H^1(0,T)}. \tag{4.9}$$

Now, (4.1) is a consequence of (4.9), (3.6) and the T-periodicity of U_ε. □

The uniform estimate from Theorem 4.1 allows us to deduce the existence of a periodic solution for our initial problem (2.1).

Theorem 4.2. *Let $f \in C[0,\infty)$ be a T-periodic function such that $f_{|(0,T)} \in H^1(0,T)$. Then* (2.1) *has a unique T-periodic solution $U \in C([0,\infty); H)$.*

Proof. Following [5, Proposition 2.3.1], we introduce the completion $\tilde{H}$ of the space H with respect to the norm $|||\Phi||| = \|(I - A)^{-1}\Phi\|$ and the unbounded operator in $\tilde{H}$ given by $(H, \tilde{A})$ which extends $(D(A), A)$. Also, let $(\tilde{S}(t))_{t\geqslant 0}$ be the semigroup generated by $(H, \tilde{A})$ in $\tilde{H}$.

According to Theorem 4.1, the family of periodic functions $(U_\varepsilon)_{\varepsilon\in(0,1)}$ is uniformly bounded in $\mathcal{C}([0,T];H)$ and verifies

$$U_\varepsilon(t) = S(t)U_\varepsilon(0) + \int_0^t S(t-s)\big(F(s) + \varepsilon P_2(s)\big)\,ds, \quad t \in [0,T] \tag{4.10}$$

where $P_2(s) = (0, -u_t(s), 0)^*$.

Since the inclusion $H \subset \tilde{H}$ is compact, it follows that $(U_\varepsilon)_{\varepsilon\in(0,1)}$ is relatively compact in $\mathcal{C}([0,T];\tilde{H})$. Thus, there exist $\tilde{U} \in \mathcal{C}([0,T];\tilde{H})$ and a subsequence that we denote in the same way, $(U_\varepsilon)_{\varepsilon\in(0,1)}$, which converges to $\tilde{U}$ in $\mathcal{C}([0,T];\tilde{H})$. Let us denote $\tilde{U}_0 = U(0) \in \tilde{H}$. It is easy to see that

$$\tilde{U}(t) = \tilde{S}(t)\tilde{U}_0 + \int_0^t \tilde{S}(t-s)F(s)\,ds, \quad t \in [0,T]. \tag{4.11}$$

On the other hand, since $(U_\varepsilon(0))_{\varepsilon\in(0,1)}$ is a bounded sequence in H, it follows that there exist $U_0 \in H$ and a subsequence that we denote in the same way, $(U_\varepsilon(0))_{\varepsilon\in(0,1)}$, which converges weakly to U_0 in H. From the uniqueness of the limit in $\tilde{H}$, we deduce that $\tilde{U}(0) = \tilde{U}_0 = U_0 \in H$.

Since $\tilde{S}(t)\Phi = S(t)\Phi$ for any $\Phi \in H$, it follows from (4.11) that

$$\tilde{U}(t) = S(t)U_0 + \int_0^t S(t-s)F(s)\,ds, \quad t \in [0,T]. \tag{4.12}$$

Consequently, $\tilde{U}$ is a weak solution in $\mathcal{C}([0,T];H)$ of (2.1) which may be extended to a T-periodic solution of the same equation. □

5. Existence of periodic solution when $f \in L^2(0,T)$

Theorem 4.2 shows that $f_{|(0,T)} \in H^1(0,T)$ is a sufficient condition for the existence of a periodic solution of (1.1). If this condition is necessary remains an open question. In this section we use Fourier decomposition to give a characterization of the periods T for which periodic solutions exist even when $f \in L^2(0,T)$. Firstly, we consider the Fourier expansion of f,

$$f(t) = \sum_{n\in\mathbb{Z}} a_n e^{i\lambda_n t} \tag{5.1}$$

where $\lambda_n = \frac{2n\pi}{T}$ and $a_n \in \mathbb{C}$. Note that f given by (5.1) belongs to $L^2(0,T)$ if and only if $\sum_{n\in\mathbb{Z}} |a_n|^2 < \infty$ and belongs to $H^1(0,T)$ if and only if $\sum_{n\in\mathbb{Z}} n^2|a_n|^2 < \infty$.

It is easy to check that, at least formally, the corresponding periodic solution of (1.1) is given by

$$u(x,t) = \sum_{n\in\mathbb{Z}} a_n e^{i\lambda_n t} u_n(x) \tag{5.2}$$

where, for each $n \in \mathbb{Z}$, the functions $u_n = u_n(x)$ satisfy the system

$$\begin{cases} -\lambda_n^2 u_n(x) - u_{n,xx}(x) = 0, \\ u_n(0) = 0, \\ -\lambda_n^2 u_n(1) + u_{n,x}(1) + i\alpha\lambda_n u_n(1) = 1. \end{cases} \tag{5.3}$$

From (5.3) it follows that $u_0(x) = x$ and, for any $n \in \mathbb{Z}^*$,

$$u_n(x,t) = \frac{\sin(\lambda_n x)}{\gamma_n} \tag{5.4}$$

where

$$\gamma_n = -\lambda_n^2 \sin\lambda_n + \lambda_n \cos\lambda_n + i\alpha\lambda_n \sin\lambda_n.$$

Now, from (5.2), we deduce that

$$\|u\|^2_{L^2(0,T;V)} = \frac{T}{2}\sum_{n\in\mathbb{Z}^*}\frac{|\lambda_n|^2}{|\gamma_n|^2}\left(1-\frac{\sin(2\lambda_n)}{2\lambda_n}\right)|a_n|^2 + T|a_0|^2$$

and consequently $u(x,t)$ gives a finite energy periodic solution if and only if

$$\sum_{n\in\mathbb{Z}^*}\frac{|\lambda_n|^2}{|\gamma_n|^2}|a_n|^2 < \infty. \tag{5.5}$$

Remark 5.1. When $f \in H^1(0,T)$, which is equivalent to the convergence of de series $\sum_{n\in\mathbb{Z}} n^2|a_n|^2$, (5.5) does hold. In fact, we can prove that $|\frac{\lambda_n}{\gamma_n}|^2 \leqslant cn^2$ for some positive constant $c > 0$. Indeed, we have the following cases:

- If $|\lambda_n \sin\lambda_n| > \frac{1}{2}$, we have $|\sin\lambda_n| > \frac{1}{2|\lambda_n|} = \frac{T}{4n\pi}$ and, therefore,

$$\left|\frac{\lambda_n}{\gamma_n}\right|^2 = \frac{1}{\alpha^2\sin^2\lambda_n + (\lambda_n\sin\lambda_n - \cos\lambda_n)^2} \leqslant \frac{1}{\alpha^2\sin^2\lambda_n} < n^2\left(\frac{4\pi}{\alpha T}\right)^2.$$

- $|\lambda_n \sin\lambda_n| \leqslant \frac{1}{2}$, we have $|\cos\lambda_n|^2 \geqslant 1 - \frac{1}{4\lambda_n^2}$. Then there exists $c > 0$ such that

$$\left|\frac{\lambda_n}{\gamma_n}\right|^2 \leqslant \frac{1}{(\lambda_n\sin\lambda_n - \cos\lambda_n)^2} \leqslant \frac{1}{(|\cos\lambda_n| - |\lambda_n\sin\lambda_n|)^2} \leqslant \frac{1}{(\sqrt{1-\frac{1}{4\lambda_n^2}} - \frac{1}{2})^2} \leqslant c.$$

This is precisely the case we have studied in the previous section, when we have assumed that $f_{|(0,T)} \in H^1(0,T)$.

If (5.5) does not hold, no periodic solution for Eq. (1.1) exists. As we have mentioned in the introduction, the non-existence of periodic solution is equivalent to the presence of the resonance phenomenon. Note that (5.5) does not hold if and only if $\lambda_n = \frac{2n\pi}{T}$ verifies

$$\sin\lambda_{k_n} \to 0 \quad \text{as } n \to \infty, \tag{5.6}$$

$$\lambda_{k_n}\sin\lambda_{k_n} - \cos\lambda_{k_n} \to 0 \quad \text{as } n \to \infty, \tag{5.7}$$

for some (strictly) increasing sequence of natural numbers $(k_n)_{n\geqslant 1} \subset \mathbb{N}$.

Let us recall that $\nu_n = n\pi i + \frac{i}{n\pi} - \frac{\alpha}{n^2\pi^2} + o(\frac{1}{n^2})$ are the eigenvalues of (2.1) while the eigenvalues of the corresponding conservative system with $\alpha = 0$ are of the form iz_n where

$$z_n = n\pi + \frac{1}{n\pi} + o\left(\frac{1}{n}\right) \quad \text{as } n \to \infty \tag{5.8}$$

and verifies, for each $n \geqslant 1$, the equation

$$t\sin t - \cos t = 0. \tag{5.9}$$

Remark 5.2. As it follows form (5.5), the existence of a periodic solution in the case $f \in L^2(0,T)$ is equivalent to the boundedness of the sequence $(\frac{\lambda_n}{\gamma_n})_{n\geqslant 1}$. Thus, the resonance phenomenon occurs if, and only if, a subsequence of $(\lambda_n)_{n\geqslant 1}$ is close to the roots of the equation

$$-z\sin z + \cos z + i\alpha\sin z = 0. \tag{5.10}$$

Note that $-i\nu_n$ are exactly the roots of (5.10). Since, as follows from Remark 2.5, the real parts of $(\nu_n)_{n\geqslant 1}$ are smaller and smaller as n tends to infinity, some of the real numbers $(\lambda_n)_{n\geqslant 1}$ may be close to $(-i\nu_n)_{n\geqslant 1}$. Consequently, the possible occurrence of the resonance phenomenon is precisely related to the weak dissipation of system (1.1).

With the notation above, we have the following properties, which characterize the periods T for which the resonance phenomenon may occur.

Property 5.3. *The resonance phenomenon occurs for the T-periodic function f given by* (5.1) *if, and only if, there exist two sequences of* (*strictly*) *increasing natural numbers $(k_n^i)_{n\geqslant 1}$, $i = 1, 2$, such that*

$$|\lambda_{k_n^1} - z_{k_n^2}||z_{k_n^2}| \to 0 \quad \text{as } n \to \infty. \tag{5.11}$$

Proof. First, we suppose that (5.11) holds and we prove (5.6)–(5.7). From (5.11) we deduce that $\lim_{n\to\infty}|\lambda_{k_n^1}-z_{k_n^2}|=0$, which together with (5.8) gives (5.6). In order to show (5.7), let $g(z)=z\sin z-\cos z$. We have that there exists ξ_n between $\lambda_{k_n^1}$ and $z_{k_n^2}$ such that

$$\left|g(\lambda_{k_n^1})\right|=\left|g(\lambda_{k_n^1})-g(z_{k_n^2})\right|\leqslant\left|g'(\xi_n)\right||\lambda_{k_n^1}-z_{k_n^2}|\leqslant 2|\xi_n||\lambda_{k_n^1}-z_{k_n^2}|$$

from which we immediately deduce (5.7).

Now suppose that (5.6)–(5.7) holds, which implies that $g(\lambda_{k_n^1})\to 0$ as $n\to\infty$. Let $\varepsilon>0$ small and k sufficiently large. In each interval $I_k=((2k-1)\frac{\pi}{2}+\varepsilon,(2k+1)\frac{\pi}{2}-\varepsilon)$ the function g is invertible since $g'(z)=z\cos z+2\sin(z)\neq 0$. Then, in each interval I_k there is exactly one number z_k for which $g(z_k)=0$. Let k_n^2 the index of the interval I_k for which $\lambda_{k_n^1}\in I_{k_n^2}$. We have that there exists ζ_n between $g(\lambda_{k_n^1})$ and $g(z_{k_n^2})$ such that

$$\begin{aligned}|\lambda_{k_n^1}-z_{k_n^2}|&=\left|g^{-1}\left(g(\lambda_{k_n^1})\right)-g^{-1}\left(g(z_{k_n^2})\right)\right|\leqslant\left|\left(g^{-1}\right)'(\zeta_n)\right|\left|g(\lambda_{k_n^1})-g(z_{k_n^2})\right|\\&=\left|\left(g^{-1}\right)'(\zeta_n)\right|\left|g(\lambda_{k_n^1})\right|=\left|\frac{1}{g'(t_n)}\right|\left|g(\lambda_{k_n^1})\right|=\frac{|g(\lambda_{k_n^1})|}{|t_n\cos t_n+2\sin t_n|}\leqslant\left|g(\lambda_{k_n^1})\right|\frac{c}{|t_n|}\end{aligned}$$

for some constant $c>0$, where $t_n=g^{-1}(\zeta_n)$ belongs to $I_{k_n^2}$ and tends to $z_{k_n^2}$ as n goes to infinity. Consequently, $|\lambda_{k_n^1}-z_{k_n^2}||z_{k_n^2}|\leqslant c|g(\lambda_{k_n^1})|$ and (5.11) holds. □

The following characterization is a direct consequence of Property 5.3.

Property 5.4. *The resonance phenomenon occurs for the T-periodic function f given by* (5.1) *if, and only if, there exist two* (*strictly*) *increasing sequences of natural numbers $(k_n^i)_{n\geqslant 1}$, $i=1,2$ such that*

$$\left(\frac{k_n^1}{k_n^2}-\frac{T}{2}\right)\left(k_n^2\right)^2\to\frac{T}{2\pi^2}\quad\text{as } n\to\infty.\tag{5.12}$$

Proof. Firstly, observe that $|\lambda_{k_n^1}-z_{k_n^2}||z_{k_n^2}|\to 0$ as n tends to infinity if and only if

$$\left|\frac{2k_n^1\pi}{T}-k_n^2\pi-\frac{1}{k_n^2\pi}+o\left(\frac{1}{k_n^2}\right)\right|\left|k_n^2\pi+\frac{1}{k_n^2\pi}+o\left(\frac{1}{k_n^2}\right)\right|\to 0\quad\text{as } n\to\infty.$$

It follows that (5.11) is equivalent to

$$\left|\frac{k_n^1}{k_n^2}-\frac{T}{2}-\frac{T}{2(k_n^2)^2\pi^2}\right|\left|k_n^2\right|^2\to 0\quad\text{as } n\to\infty$$

and the proof ends. □

Remark 5.5. Property 5.4 gives the conditions for resonance phenomenon's occurrence in (2.1). However, it is not easy to say whether periods T verifying (5.12) exist or not. For instance, if T is a rational number (5.12) cannot hold. Indeed, if we suppose that $T=\frac{p}{q}$, with $p,q\in\mathbb{N}^*$, from (5.12) we deduce that

$$\lim_{n\to\infty}\left|2qk_n^1-pk_n^2\right|=0$$

and, consequently, $T=\frac{2k_n^1}{k_n^2}$ for any n sufficiently large. However, this contradicts (5.12).

As mentioned before, the existence of irrational numbers T which verify (5.12) and produce the resonance phenomenon remains an open question.

Remark 5.6. Let us compare our results with the corresponding ones for the classical boundary dissipated wave equation. If $f\in C[0,\infty)$ is a periodic function of period T, then there exists a unique T-periodic solution of the non-homogeneous wave equation (α is a positive constant)

$$\begin{cases}u_{tt}(x,t)-u_{xx}(x,t)=0 & \text{in } (0,1)\times(0,\infty),\\ u(0,t)=0 & \text{on } (0,\infty),\\ u_x(1,t)+\alpha u_t(1,t)=f(t) & \text{on } (0,\infty).\end{cases}\tag{5.13}$$

Indeed, this is a consequence of the exponential decay of the energy corresponding to (5.13) in the homogeneous case (see, for instance, [9,10,23]) and the fixed point argument used in Theorem 3.4. Note that the regularity assumption $f_{|(0,T)}\in$

$H^1(0,T)$ is not required in the case of Eq. (5.13), where the energy of the equation is directly dissipated through the boundary.

The presence of the point mass in $x = 1$ introduces a qualitatively new type of boundary condition for (1.1) which determines the lack of uniform decay of the corresponding energy. Roughly speaking, the extra regularity assumption $f_{|(0,T)} \in H^1(0,T)$ from Theorem 4.2 compensates the energy's weak dissipation.

6. Numerical approximation

In this section we approximate numerically the periodic solutions of the hybrid system (2.1), using the perturbation argument presented in the theoretical part. More precisely, for a given small value of $\varepsilon > 0$, we use an iterative algorithm to approximate the fixed point of the contractive map Λ introduced in Theorem 3.4. Once we have obtained this fixed point with a given precision, we can compute the periodic solution of the perturbed system (3.4). The proof of Theorem 4.2 shows that taking smaller and smaller values of $\varepsilon > 0$ the periodic solutions of the perturbed system converge weakly to the periodic solution of the initial hybrid system, if $f_{|(0,T)} \in H^1(0,T)$.

In order to preserve the precision of the numerical scheme for (2.1), the parameter ε should be small enough. On the other hand, the contractive property of the map Λ gets better when ε increases. This analysis allows us to conclude that the optimal value of ε is of order of the precision of the numerical scheme (see the conclusions from Section 6.2).

First of all, we need to approximate the solution of system (3.4)–(3.5). We choose to semi-discretize in space using P_1 finite elements, paying attention to the special boundary condition in $x = 1$. More precisely, we denote by $N \in \mathbb{N}^*$ the number of discretization points in $(0,1)$ and by $h = \frac{1}{N+1}$ the discretization step. The vector $u_h(t) = (u_1(t), \ldots, u_{N+1}(t))^*$ will be the solution of the following differential system

$$\begin{cases} Mu_{h,tt}(t) + Ku_h(t) + Lu_{h,t}(t) = F_h(t), \quad t > 0, \\ u_h(0) = u_{0h}, \quad u_{h,t}(0) = u_{1h}, \end{cases} \tag{6.1}$$

where

$$M = \frac{h}{6}\begin{pmatrix} 4 & 1 & 0 & \cdots & 0 \\ 1 & 4 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 4 & 1 \\ 0 & 0 & \cdots & 0 & \frac{6}{h} \end{pmatrix}, \qquad K = \frac{1}{h}\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & \cdots & -1 & 1 \end{pmatrix},$$

$$L = \frac{h\varepsilon}{6}\begin{pmatrix} 4 & 1 & 0 & \cdots & 0 \\ 1 & 4 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 4 & 1 \\ 0 & 0 & \cdots & 0 & \frac{6\alpha}{h\varepsilon} \end{pmatrix}, \qquad F_h(t) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f(t) \end{pmatrix}.$$

In the sequel U^* denotes the transposed vector.

Denoting $U_h = \binom{u_h}{u_{h,t}}$, we write (6.1) as a first order system

$$\begin{cases} U_{h,t}(t) = A_h U_h(t) + \tilde{F}_h(t), \quad t > 0, \\ U_h(0) = U_{0h}, \end{cases} \tag{6.2}$$

where

$$A = \begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}L \end{pmatrix}, \qquad \tilde{F}_h(t) = \begin{pmatrix} 0 \\ F_h(t) \end{pmatrix}, \qquad U_{0h} = \begin{pmatrix} u_{0h} \\ u_{1h} \end{pmatrix}.$$

To discretize in time (6.2) we use the following implicit mid-point scheme

$$\begin{cases} \dfrac{U_h^{k+1} - U_h^k}{\Delta t} = A_h \dfrac{U_h^k + U_h^{k+1}}{2} + \dfrac{\tilde{F}_h^k + \tilde{F}_h^{k+1}}{2}, \quad k \in \mathbb{N}^*, \\ U_h^0 = U_{0h}, \end{cases} \tag{6.3}$$

where $\Delta t > 0$ is the time discretization step and U_h^k and $\tilde{F}_h^k$ approximate $U_h(k\Delta t)$ and $\tilde{F}_h(k\Delta t)$, respectively. We define the energy associated to (6.3) by the discrete equivalent of (2.4)

$$E_h(U_h^k) = \frac{1}{2}\left((U_h^k)^* \begin{pmatrix} K_e & 0 \\ 0 & M_e \end{pmatrix} U_h^k + (U_{h,2N+2}^k)^2 \right), \tag{6.4}$$

Table 1
Values of $\max\{E_h(u_{h,\varepsilon}(t) - u(t)):\ t \in [0, 10]\}$ for different choices of h and ε.

Value of h	$\varepsilon = 1$	$\varepsilon = 0.1$	$\varepsilon = 0.01$	$\varepsilon = 0.001$	$\varepsilon = 0.0001$	$\varepsilon = 0$
0.02	3.2384	0.1043	0.0012	0.0021	0.0023	0.0023
0.01	4.7285	0.1686	0.0015	0.0005	0.0006	0.0006
0.005	7.6631	0.2859	0.0033	0.0004	0.0005	0.0004

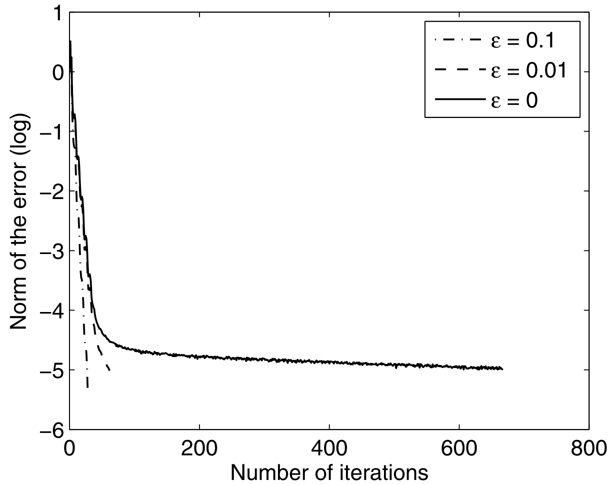

Fig. 1. Decay of the error in the fixed point algorithm for different values of ε, $N = 100$ and f given by (6.5).

where $M_e, K_e \in \mathcal{M}_{N+1}$ are defined by

$$M_e = \frac{h}{6}\begin{pmatrix} 4 & 1 & 0 & \cdots & 0 \\ 1 & 4 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 4 & 1 \\ 0 & 0 & \cdots & 1 & 4 \end{pmatrix}, \qquad K_e = \frac{1}{h}\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & \cdots & -1 & 2 \end{pmatrix}.$$

Once we have a solver for (3.4)–(3.5), the second step is to compute the fixed point of the application Λ introduced in Theorem 3.4, for a given T-periodic function f. Remark that Λ is nothing else than the application which associates $U(T)$ to U_0, where U is the solution of (3.4)–(3.5). Assuming that there exists $l \in \mathbb{N}^*$ such that $T = l\Delta t$ we define Λ_h the application which associates U_h^l to U_{0h}, where U_h^k verifies (6.3). To compute then an approximation of the fixed point of Λ we use the following algorithm:

1. Let U_{0h} be arbitrarily chosen.
2. Let $e > 0$ be a given tolerance and $n = 1$.
3. Repeat
   - $V_h^n = \Lambda_h^n U_{0h}$,
   - $n = n + 1$,

   until $E_h(V_h^n - V_h^{n-1}) < e$.
4. The approximation of the fixed point is V_h^n.

6.1. *Numerical experiments*

We shall present three different choices of periodic functions f and the corresponding numerical results. In all the computations the parameter α is set to 1 and the tolerance e in the fixed point algorithm is 10^{-5}.

6.1.1. *Case of an explicit periodic solution*

It is easy to verify that considering the 2-periodic function

$$f(t) = -\pi\cos(\pi t) \quad \text{for } t \in [0, 10], \tag{6.5}$$

the corresponding periodic solution of (1.1) is given by $u(x,t) = \cos(\pi t)\sin(\pi x)$. We take the number of discretization points $N \in \{50, 100, 200\}$ and $\Delta t = h = \frac{1}{N+1}$.

Table 1 displays the values of $\max\{E_h(u_{h,\varepsilon}(t) - u(t)):\ t \in [0, 10]\}$ for different choices of h and ε. In Fig. 1 we show the decay of the error in the fixed point algorithm for different values of ε and $N = 100$.

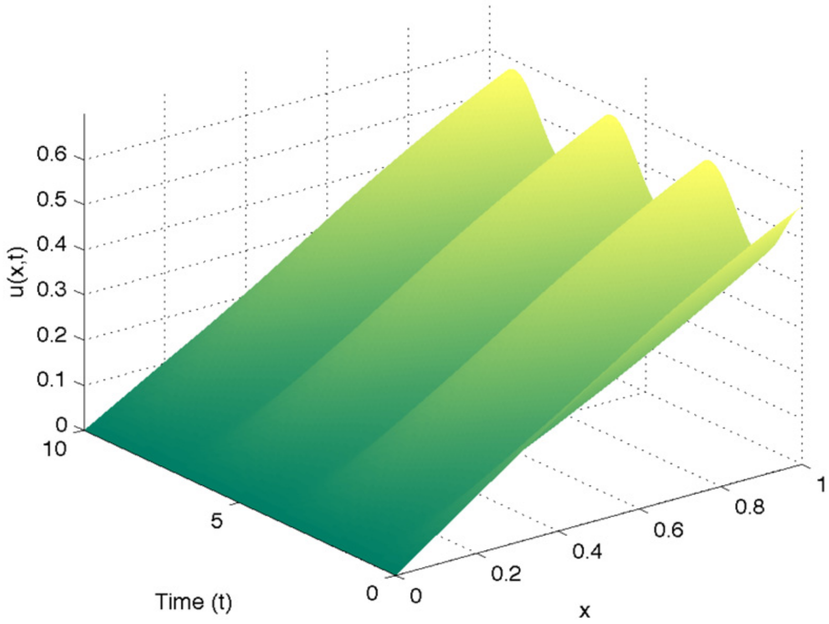

Fig. 2. Periodic solution of (6.3) for $\varepsilon = 0$ and f given by (6.6).

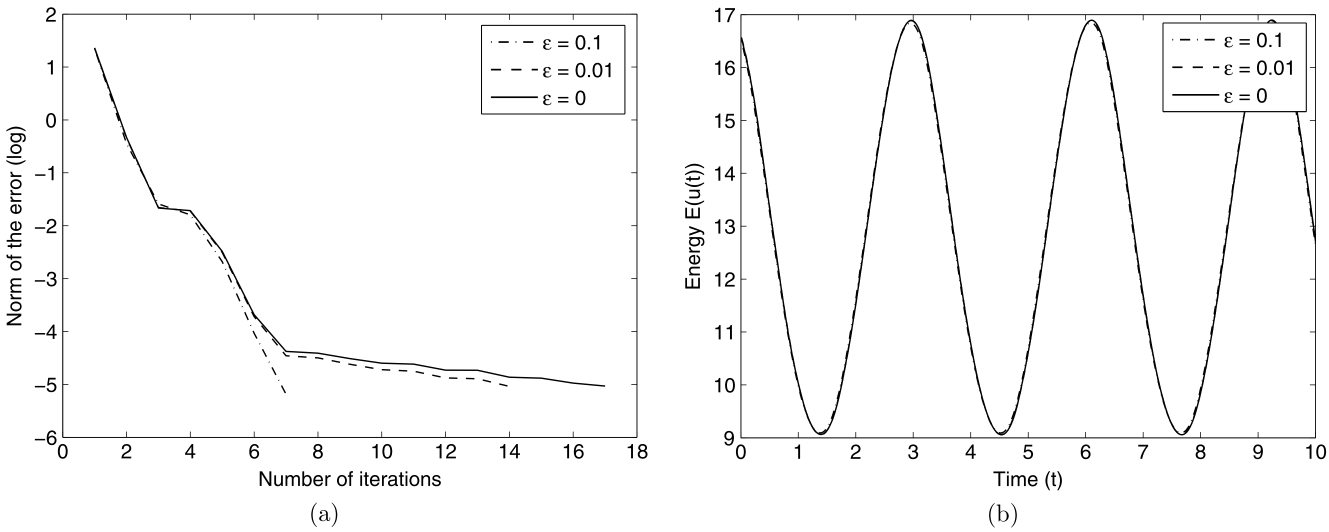

Fig. 3. The function f is given by (6.6). (a) Norm of the error in the fixed point algorithm. (b) Energy of the solution of (6.3) with the initial data being the fixed point of Λ_h.

6.1.2. Case $f \in H^1(0, T)$

Let $T = \pi$ and consider the following T-periodic function

$$f(t) = 1 - \frac{2}{T}\left|t - \frac{T}{2}\right| \quad \text{for } t \in [0, T]. \tag{6.6}$$

By periodicity, we extend f to the interval $[0, 10]$. We take $N = 100$ and $\Delta t = h = \frac{1}{N+1}$. The corresponding periodic solution for $\varepsilon = 0$ is displayed in Fig. 2. In Fig. 3 we show the norm of the error in the fixed point algorithm (a) and the energy of the solution of (6.3) with the initial data being the fixed point of Λ_h, (b) for different values of ε.

6.1.3. Case $f \in L^2(0, T)$

Let $T = \sqrt{2}$ and consider the following T-periodic function

$$f(t) = \begin{cases} 1 & \text{if } x \in (T/3, 2T/3), \\ 0 & \text{if } x \in (0, T/3) \cup (2T/3, T). \end{cases} \tag{6.7}$$

By periodicity, we extend f to the interval $[0, 10]$. As in the case $f \in H^1(0, T)$ we take $N = 100$ and $\Delta t = h = \frac{1}{N+1}$. A very similar behavior of the solutions as in the case $f \in H^1$ can be observed in Figs. 4 and 5.

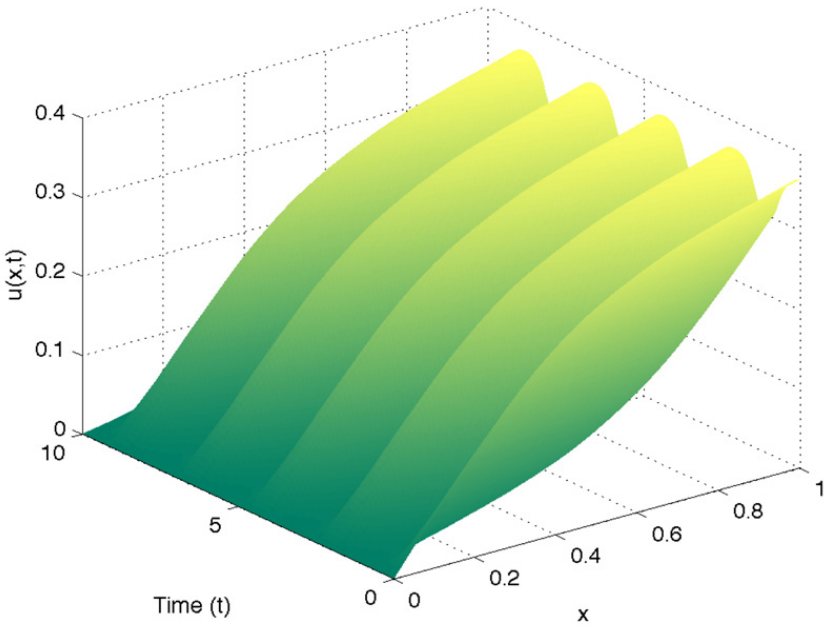

Fig. 4. Periodic solution of (6.3) for $\varepsilon = 0$ and f given by (6.7).

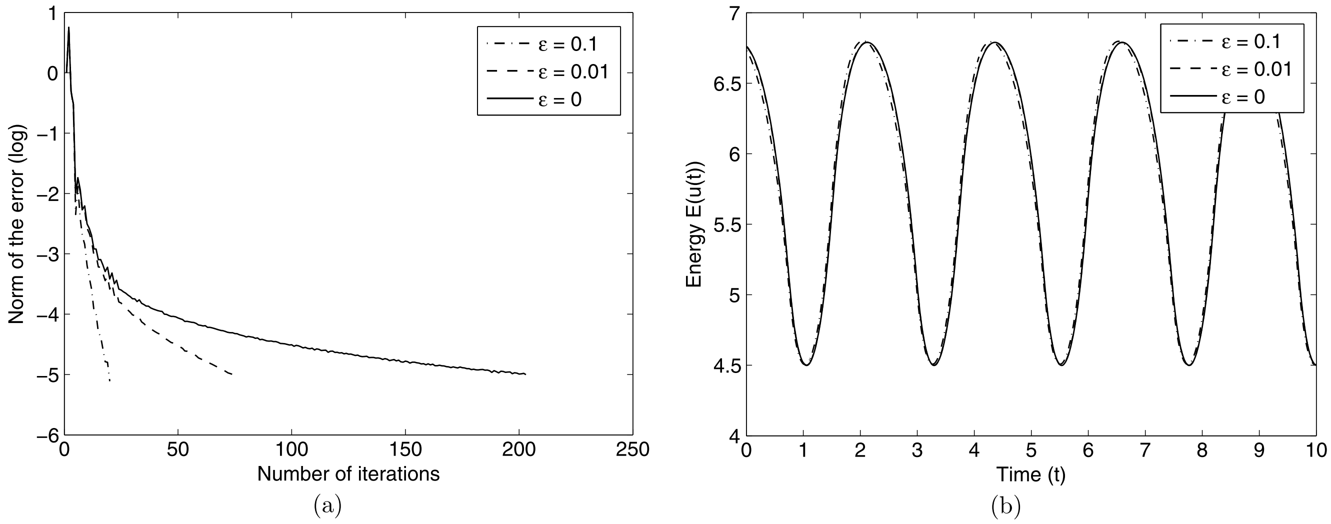

Fig. 5. Function f is given by (6.7). (a) Norm of the error in the fixed point algorithm. (b) Energy of the solution of (6.3) with the initial data being the fixed point of Λ_h.

Table 2
Values of the energy of the periodic solution at the time $t = 1$ and different values of ε.

Function	$\varepsilon = 1$	$\varepsilon = 0.1$	$\varepsilon = 0.01$	$\varepsilon = 0.001$	$\varepsilon = 0$
f given by (6.6)	10.1315	10.0268	10.0679	10.0671	10.0674
f given by (6.7)	4.2197	4.5134	4.5245	4.5232	4.5232

In Table 2 we summarize the values of the energy of the computed periodic solution of (6.2) at the time $t = 1$ for different periodic functions. We remark that the values of the energy for $\varepsilon = 1$ and $\varepsilon = 0.1$ are clearly different from the values obtained for smaller ε.

6.2. Conclusion and comments

First of all, we should remark from Table 1 that the solutions of the perturbed systems converge to the exact solution when $\varepsilon \to 0$. The results displayed in Table 2 confirm this fact, the difference between the solutions decreasing as ε tends to zero.

Analyzing Fig. 1, Fig. 3(a) and Fig. 5(a) we remark that the number of iterations necessary for the fixed point algorithm increases when ε decreases with different velocities depending on f. This is a consequence of the fact that the application Λ_h has better contractive properties when ε is larger. At the numerical level, this is related to Remark 3.3 for the continuous problem. Note that taking $\Delta t = h$, the order of convergence of the approximation scheme using P_1 finite elements for the wave equation is $O(h)$ (see, for instance, [21, p. 213] and [18, pp. 96–97]). Coupling the wave equation with the rigid body at one end, and still using a P_1 finite elements approximation of this hybrid system, we maintain the same order of convergence as for the wave equation with homogeneous Dirichlet boundary condition. Therefore, the perturbation εu_t added in system (3.4) does not affect the precision of the numerical approximation if ε is at most of the same order as h. Since large values of ε ensure better convergence properties of the fixed point algorithm, we deduce that the optimal choice is ε of order h.

In conclusion, the numerical simulations above are coherent with our theoretical results concerning the existence of periodic solutions for the hybrid system (2.1) if the T-periodic source term f is in $H^1(0,T)$. For all T-periodic function $f_{|(0,T)} \in L^2(0,T)$ that we have tested, a periodic solution has been numerically found.

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