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AN APPROXIMATION METHOD FOR EXACT CONTROLS OF VIBRATING SYSTEMS*

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Abstract. We propose a new method for the approximation of exact controls of a second order infinite dimensional system with bounded input operator. The algorithm combines Russell's "stabilizability implies controllability" principle with the Galerkin method. The main new feature of this work consists of giving precise error estimates. In order to test the efficiency of the method, we consider two illustrative examples (with the finite element approximations of the wave and the beam equations) and describe the corresponding simulations.

Key words. infinite dimensional systems, exact control, approximation, error estimate

AMS subject classifications. 35L10, 65M60, 93B05, 93B40, 93D15

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1. Introduction.

The numerical study of the exact controls of infinite dimensional systems began in the 1990s with a series of papers by Glowinski and Lions (see [11, 12]) where algorithms for determining the minimal L^2-norm exact controls (sometimes called HUM controls) are provided. Several abnormalities presented in these papers motivated a large number of articles in which a great variety of numerical methods are presented and analyzed (see, for instance, [33, 8] and the references therein). However, except for the recent work [9], where the approximation of the HUM controls for the one dimensional wave equation is considered, to our knowledge, there are no results on the rate of convergence of the approximative controls.

The aim of this work is to provide an efficient numerical method for computing exact controls for a class of infinite dimensional systems modeling elastic vibrations. Our main theoretical result gives the rate of convergence of our approximations to an exact control. Moreover, to illustrate the efficiency of this approach, we apply it to several systems governed by PDEs and describe the associated numerical simulations. Our methodology combines Russell's "stabilizability implies controllability" principle with error estimates for finite element-type approximations of the considered infinite dimensional systems. We focus on the case of bounded input operators which excludes boundary control for systems governed by PDEs. However, the method can be partially extended to the unbounded input operator case; see Remark 2.7.

In order to give the precise statement of our results we need some notation. Let H be a Hilbert space, and assume that $A_0 : \mathcal{D}(A_0) \to H$ is a self-adjoint, strictly positive operator with compact resolvent. Then, according to classical results, the operator A_0 is diagonalizable with an orthonormal basis $(\varphi_k)_{k\geqslant 1}$ of eigenvectors, and the corresponding family of positive eigenvalues $(\lambda_k)_{k\geqslant 1}$ satisfies $\lim_{k\to\infty} \lambda_k = \infty$.

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Moreover, we have

$$\mathcal{D}(A_0) = \left\{ z \in H \;\middle|\; \sum_{k\geqslant 1} \lambda_k^2 \left|\langle z, \varphi_k\rangle\right|^2 < \infty \right\}$$

and

$$A_0 z = \sum_{k\geqslant 1} \lambda_k \langle z, \varphi_k\rangle \varphi_k \qquad (z \in \mathcal{D}(A_0)).$$

For $\alpha \geqslant 0$ the operator A_0^α is defined by

$$\mathcal{D}(A_0^\alpha) = \left\{ z \in H \;\middle|\; \sum_{k\geqslant 1} \lambda_k^{2\alpha} \left|\langle z, \varphi_k\rangle\right|^2 < \infty \right\} \tag{1.1}$$

and

$$A_0^\alpha z = \sum_{k\geqslant 1} \lambda_k^\alpha \langle z, \varphi_k\rangle \varphi_k \qquad (z \in \mathcal{D}(A_0^\alpha)).$$

For every $\alpha \geqslant 0$ we denote by H_α the space $\mathcal{D}(A_0^\alpha)$ endowed with the inner product

$$\langle \varphi, \psi\rangle_\alpha = \langle A_0^\alpha \varphi, A_0^\alpha \psi\rangle \qquad (\varphi,\ \psi \in H_\alpha).$$

The induced norm is denoted by $\|\cdot\|_\alpha$. From the above facts it follows that for every $\alpha \geqslant 0$ the operator A_0 is a unitary operator from $H_{\alpha+1}$ onto H_α, and A_0 is strictly positive on H_α.

Let U be another Hilbert space, and let $B_0 \in \mathcal{L}(U, H)$ be an input operator. Consider the system

$$\ddot{q}(t) + A_0 q(t) + B_0 u(t) = 0 \qquad (t \geqslant 0), \tag{1.2}$$

$$q(0) = q_0, \qquad \dot{q}(0) = q_1. \tag{1.3}$$

The above system is said to be *exactly controllable in time* $\tau > 0$ if for every $q_0 \in H_{\frac{1}{2}}$, $q_1 \in H$ there exists a control $u \in L^2([0,\tau], U)$ such that $q(\tau) = \dot{q}(\tau) = 0$. In order to provide a numerical method to approximate such a control u, we need more assumptions and notation.

Assume that there exists a family $(V_h)_{h>0}$ of finite dimensional subspaces of $H_{\frac{1}{2}}$ and that there exist $\theta > 0$, $h^* > 0$, $C_0 > 0$ such that, for every $h \in (0, h^*)$,

$$\|\pi_h \varphi - \varphi\|_{\frac{1}{2}} \leq C_0\, h^\theta \|\varphi\|_1 \qquad (\varphi \in H_1), \tag{1.4}$$

$$\|\pi_h \varphi - \varphi\| \leq C_0\, h^\theta \|\varphi\|_{\frac{1}{2}} \qquad (\varphi \in H_{\frac{1}{2}}), \tag{1.5}$$

where π_h is the orthogonal projector from $H_{\frac{1}{2}}$ onto V_h. Assumptions (1.4)–(1.5) are, in particular, satisfied when finite elements are used for the approximation of Sobolev spaces. The inner product in V_h is the restriction of the inner product on H and is denoted by $\langle\cdot,\cdot\rangle$. We define the linear operator $A_{0h} \in \mathcal{L}(V_h)$ by

$$\langle A_{0h}\varphi_h, \psi_h\rangle = \langle A_0^{\frac{1}{2}}\varphi_h, A_0^{\frac{1}{2}}\psi_h\rangle \qquad (\varphi_h,\ \psi_h \in V_h). \tag{1.6}$$

The operator A_{0h} is clearly symmetric and strictly positive.

Denote $U_h = B_0^* V_h \subset U$ and define the operators $B_{0h} \in \mathcal{L}(U, H)$ by

$$B_{0h} u = \widetilde{\pi}_h B_0 u \qquad (u \in U), \tag{1.7}$$

where $\widetilde{\pi}_h$ is the orthogonal projection of H onto V_h. Note that $\operatorname{Ran} B_{0h} \subset V_h$. As is well known, since it is an orthogonal projector, the operator $\widetilde{\pi}_h \in \mathcal{L}(H)$ is self-adjoint. Moreover, from (1.5) we deduce that

$$\|\varphi - \widetilde{\pi}_h \varphi\| \leqslant \|\varphi - \pi_h \varphi\| \leqslant C_0\, h^\theta \|\varphi\|_{\frac{1}{2}} \qquad (\varphi \in H_{\frac{1}{2}}). \tag{1.8}$$

The adjoint $B_{0h}^* \in \mathcal{L}(H, U)$ of B_{0h} is

$$B_{0h}^* \varphi = B_0^* \widetilde{\pi}_h \varphi \qquad (\varphi \in H). \tag{1.9}$$

Since $U_h = B_0^* V_h$, from (1.9), it follows that $\operatorname{Ran} B_{0h}^* = U_h$ and that

$$\langle B_{0h}^* \varphi_h, B_{0h}^* \psi_h \rangle_U = \langle B_0^* \varphi_h, B_0^* \psi_h \rangle_U \qquad (\varphi_h,\ \psi_h \in V_h). \tag{1.10}$$

The above assumptions imply that, for every $h^* > 0$, the family $(\|B_{0h}\|_{\mathcal{L}(U,H)})_{h \in (0,h^*)}$ is bounded.

In what follows, we describe an algorithm for computing an approximation $u_h \in C([0,\tau]; U_h)$ of an exact control $u \in C([0,\tau]; U)$, which drives the solution of (1.2)–(1.3) from the initial state $\left[\begin{smallmatrix} q_0 \\ q_1 \end{smallmatrix}\right] \in H_{\frac{3}{2}} \times H_1$ to rest in time τ. We propose the following scheme:

1. Take $\left[\begin{smallmatrix} q_0 \\ q_1 \end{smallmatrix}\right] \in H_{\frac{3}{2}} \times H_{\frac{1}{2}}$.
2. For any $h > 0$ choose $N(h) \in \mathbb{N}$ as in Theorem 1.1.
3. For $n = 1, 2, \ldots, N(h)$ solve the following coupled systems:
   - A forward system

$$\ddot{w}_h^n(t) + A_{0h} w_h^n(t) + B_{0h} B_{0h}^* \dot{w}_h^n(t) = 0 \qquad (t \geqslant 0), \tag{1.11}$$

$$w_h^n(0) = \begin{cases} \pi_h q_0 & \text{if } n = 1, \\ w_{b,h}^{n-1}(0) & \text{if } 1 < n \leqslant N(h), \end{cases} \tag{1.12}$$

$$\dot{w}_h^n(0) = \begin{cases} \pi_h q_1 & \text{if } n = 1, \\ \dot{w}_{b,h}^{n-1}(0) & \text{if } 1 < n \leqslant N(h). \end{cases} \tag{1.13}$$

   - A backward system

$$\ddot{w}_{b,h}^n(t) + A_{0h} w_{b,h}^n(t) - B_{0h} B_{0h}^* \dot{w}_{b,h}^n(t) = 0 \qquad (t \leqslant \tau), \tag{1.14}$$

$$w_{b,h}^n(\tau) = w_h^n(\tau), \qquad \dot{w}_{b,h}^n(\tau) = \dot{w}_h^n(\tau). \tag{1.15}$$

4. Compute $\left[\begin{smallmatrix} w_{0h} \\ w_{1h} \end{smallmatrix}\right]$ as follows:

$$\begin{bmatrix} w_{0h} \\ w_{1h} \end{bmatrix} = \begin{bmatrix} \pi_h q_0 \\ \pi_h q_1 \end{bmatrix} + \sum_{n=1}^{N(h)} \begin{bmatrix} w_{b,h}^n(0) \\ \dot{w}_{b,h}^n(0) \end{bmatrix} = \sum_{n=1}^{N(h)} \begin{bmatrix} w_h^n(0) \\ \dot{w}_h^n(0) \end{bmatrix} + \begin{bmatrix} w_{b,h}^{N(h)}(0) \\ \dot{w}_{b,h}^{N(h)}(0) \end{bmatrix}. \tag{1.16}$$

5. Compute the control u_h,

$$u_h = B_{0h}^* \dot{w}_h + B_{0h}^* \dot{w}_{b,h}, \tag{1.17}$$

where w_h and $w_{b,h}$ are the solution of

$$\ddot{w}_h(t) + A_{0h}w_h(t) + B_{0h}B_{0h}^*\dot{w}_h(t) = 0 \qquad (t \geqslant 0), \tag{1.18}$$

$$w_h(0) = w_{0h}, \quad \dot{w}_h(0) = w_{1h}, \tag{1.19}$$

$$\ddot{w}_{b,h}(t) + A_{0h}w_{b,h}(t) - B_{0h}B_{0h}^*\dot{w}_{b,h}(t) = 0 \qquad (t \leqslant \tau), \tag{1.20}$$

$$w_{b,h}(\tau) = w_h(\tau), \qquad \dot{w}_{b,h}(\tau) = \dot{w}_h(\tau). \tag{1.21}$$

We can now formulate the main result of this paper.

Theorem 1.1. *With the above notation and assumptions, assume furthermore that the system* (1.2), (1.3) *is exactly controllable in some time* $\tau > 0$ *and that* $B_0B_0^* \in \mathcal{L}(H_1, H_{\frac{1}{2}})$. *Then there exists a constant* $m_\tau > 0$ *such that the family* $(u_h)_{h>0}$ *of* $C([0,\tau]; U_h)$, *defined in* (1.17) *with* $N(h) = \left[\theta\, m_\tau \ln(h^{-1})\right]$, *converges when* $h \to 0$ *to an exact control in time* τ *of* (1.2), (1.3), *denoted by* u, *for every* $Q_0 = \left[\begin{smallmatrix} q_0 \\ q_1 \end{smallmatrix}\right] \in H_{\frac{3}{2}} \times H_1$. *Moreover, there exist constants* $h^* > 0$ *and* $C := C_\tau$ *such that we have*

$$\|u - u_h\|_{C([0,\tau];U)} \leqslant Ch^\theta \ln^2(h^{-1})\|Q_0\|_{H_{\frac{3}{2}}\times H_1} \qquad (0 < h < h^*). \tag{1.22}$$

It is known that, if $B_0 \in \mathcal{L}(U, H)$, then any initial data in $H_{\frac{1}{2}} \times H$ can be steered to zero by using controls $u \in C([0,\tau]; U)$ (see also Remark 2.4). However, as in most approximation problems for PDEs, in order to obtain error estimates it is necessary to consider solutions which are more regular than those in the usual energy space. Therefore, we introduce the additional smoothness properties of the initial data, $Q_0 = \left[\begin{smallmatrix} q_0 \\ q_1 \end{smallmatrix}\right] \in H_{\frac{3}{2}} \times H_1$, and of the control operator, $B_0B_0^* \in \mathcal{L}(H_1, H_{\frac{1}{2}})$, assumed in Theorem 1.1. Indeed, under these hypotheses, our control will verify $u \in C^1([0,\tau]; U)$ and $B_0u \in C([0,\tau]; H_{\frac{1}{2}})$. These extra regularity properties of our continuous control not only allow us to give the error estimates (1.22) but also are essential in choosing the truncation parameter $N(h)$ (see also Remark 4.4).

An algorithm based on Russell's principle has been used to compute an exact boundary control for a class of second order evolution equations in [21] (see also [10]). With our notation and after discretizing with respect to the space variable, the method in [21] consists of choosing $N(h) = 1$. This choice is convenient for implementation purposes but it does not yield the convergence of u_h to u. In our work the appropriate choice of $N(h)$ plays a central role in obtaining error estimates.

We prove Theorem 1.1 in section 4. In section 2 we recall some background on exact controllability and stabilizability. Section 3 provides some error estimates. In section 5 we apply our results to the wave equation in two space dimensions and to the Euler–Bernoulli beam equation, providing numerical simulations.

2. Some background on exact controllability and uniform stabilization. In this section we recall, with no claim of originality, some background concerning the exact controllability and uniform stabilizability of the system (1.2), (1.3). We give, in particular, a short proof, adapted to our case, of Russell's "stabilizability implies controllability" principle. This principle has been originally stated in Russell [25, 26] (see also Chen [4]).

Consider the second order differential equation

$$\ddot{w}(t) + A_0w(t) + B_0B_0^*\dot{w}(t) = 0 \qquad (t \geqslant 0), \tag{2.1}$$

$$w(0) = w_0, \quad \dot{w}(0) = w_1. \tag{2.2}$$

It is well known that the above equation defines a well posed dynamical system in the state space $X = H_{\frac{1}{2}} \times H$. More precisely, the solution $\left[\begin{smallmatrix} w \\ \dot w \end{smallmatrix}\right]$ of (2.1), (2.2) is given by

$$\begin{bmatrix} w(t) \\ \dot w(t) \end{bmatrix} = \mathbb{T}_t \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \qquad \left(\begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \in X, \quad t \geqslant 0 \right), \tag{2.3}$$

where $\mathbb{T}$ is the contraction semigroup on X generated by $\mathcal{A} - \mathcal{B}\mathcal{B}^*$, and $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to X$, $\mathcal{B} \in \mathcal{L}(U, X)$ are defined by

$$\mathcal{D}(\mathcal{A}) = H_1 \times H_{\frac{1}{2}}, \quad \mathcal{A} = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} 0 \\ B_0 \end{bmatrix}.$$

We also consider the backwards system

$$\ddot w_b(t) + A_0 w_b(t) - B_0 B_0^* \dot w_b(t) = 0 \qquad (t \leqslant \tau), \tag{2.4}$$

$$w_b(\tau) = w(\tau), \qquad \dot w_b(\tau) = \dot w(\tau). \tag{2.5}$$

It is not difficult to check that the solution $\left[\begin{smallmatrix} w_b \\ \dot w_b \end{smallmatrix}\right]$ of (2.4), (2.5) is given by

$$\begin{bmatrix} w_b(t) \\ \dot w_b(t) \end{bmatrix} = \mathbb{S}_{\tau - t} \begin{bmatrix} w(\tau) \\ \dot w(\tau) \end{bmatrix} \qquad (t \in [0, \tau]), \tag{2.6}$$

where $\mathbb{S}$ is the contraction semigroup in X generated by $-\mathcal{A} - \mathcal{B}\mathcal{B}^*$.

We define $L_\tau \in \mathcal{L}(X)$ by

$$L_\tau \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = \begin{bmatrix} w_b(0) \\ \dot w_b(0) \end{bmatrix} \qquad \left(\begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \in X \right). \tag{2.7}$$

With the above notation, the operator L_τ clearly satisfies $L_\tau = \mathbb{S}_\tau \mathbb{T}_\tau$.

PROPOSITION 2.1. *With the above notation, assume that the system* (1.2), (1.3) *is exactly controllable in some time $\tau > 0$. Then the semigroups $\mathbb{T}$ and $\mathbb{S}$ are exponentially stable, and we have $\|\mathbb{T}_\tau\|_{\mathcal{L}(X)} < 1$ and $\|\mathbb{S}_\tau\|_{\mathcal{L}(X)} < 1$. Moreover, the operator $I - L_\tau$ is invertible, and we have*

$$(I - L_\tau)^{-1} = \sum_{n \geqslant 0} L_\tau^n. \tag{2.8}$$

Proof. The fact that $\mathbb{T}$ and $\mathbb{S}$ are exponentially stable is well known (see, for instance, Haraux [14] and Liu [19]). The more precise facts that $\|\mathbb{T}_\tau\|_{\mathcal{L}(X)} < 1$ and $\|\mathbb{S}_\tau\|_{\mathcal{L}(X)} < 1$ are easy to establish (see, for instance, Lemma 2.2 in Ito, Ramdani, and Tucsnak [15]). Finally, (2.8) follows from $\|L_\tau\|_{\mathcal{L}(X)} < 1$. ∎

The particular case of Russell's principle [26], which we need in this work, is given by the following result.

PROPOSITION 2.2. *Assume that* (1.2), (1.3) *is exactly controllable in time $\tau > 0$. Then a control $u \in C([0, \tau]; U)$ for* (1.2), (1.3) *steering the initial state $\left[\begin{smallmatrix} q_0 \\ q_1 \end{smallmatrix}\right] \in X$ to rest in time τ is given by*

$$u = B_0^* \dot w + B_0^* \dot w_b, \tag{2.9}$$

where w and w_b are the solutions of (2.1)–(2.2) *and* (2.4)–(2.5)*, respectively, with*

$$\begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = (I - L_\tau)^{-1} \begin{bmatrix} q_0 \\ q_1 \end{bmatrix}. \tag{2.10}$$

Remark 2.3. The original assumption of Russell's principle was essentially the exponential stability of the semigroups $\mathbb{T}$ and $\mathbb{S}$, whence came the name "stabilizability implies controllability." Since, according to Proposition 2.1, these stability properties are consequences of the exact controllability of (1.2), (1.3), we made this assumption explicitly in Proposition 2.2. The essential aspect retained from the original Russell's principle is the specific form (2.9) of u, obtained using the "closed loop semigroups" $\mathbb{T}$ and $\mathbb{S}$.

Proof of Proposition 2.2. Denote

$$q(t) = w(t) - w_b(t) \qquad (t \in [0, \tau]).$$

Then q clearly satisfies (1.2) with u given by (2.9). Moreover, from (2.10) it follows that q satisfies the initial conditions (1.3). Finally, from (2.5) it follows that

$$q(\tau) = \dot{q}(\tau) = 0. \qquad \square$$

Remark 2.4. Using the semigroup notation, an alternative way of writing (2.9) is

$$u(t) = \mathcal{B}^* \mathbb{T}_t \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} + \mathcal{B}^* \mathbb{S}_{\tau - t} \mathbb{T}_\tau \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \qquad (t \in [0, \tau]), \tag{2.11}$$

where $w_0,\ w_1$ satisfy (2.10).

Note that the control u given by (2.11) belongs to $C([0,\tau]; U)$. The same property is shared, in the particular case of bounded input operators, by the minimal $L^2(0,\tau;U)$-norm control (the so-called HUM control).

In what follows we need the fact that the restrictions of $\mathbb{T}$ and $\mathbb{S}$ to $H_1 \times H_{\frac{1}{2}}$ and $H_{\frac{3}{2}} \times H_1$, endowed with appropriate norms, are exponentially stable semigroups. Sufficient conditions for this are given in the result below.

Proposition 2.5. *Under the hypothesis of Proposition* 2.1 *assume, in addition, that $B_0 B_0^* \in \mathcal{L}(H_1, H_{\frac{1}{2}})$. Then the restrictions of $\mathbb{T}$ and $\mathbb{S}$ to $H_1 \times H_{\frac{1}{2}}$ and $H_{\frac{3}{2}} \times H_1$ are contraction semigroups on these spaces with generators that are the restrictions of $\mathcal{A} - \mathcal{B}\mathcal{B}^*$ and $-\mathcal{A} - \mathcal{B}\mathcal{B}^*$ to $H_{\frac{3}{2}} \times H_1$ and $H_2 \times H_{\frac{3}{2}}$, respectively. Moreover, there exists a norm $|||\cdot|||$ on $\mathcal{L}(H_{\frac{3}{2}} \times H_1)$, equivalent to the standard norm, such that*

$$|||\mathbb{T}_\tau||| < 1, \qquad |||\mathbb{S}_\tau||| < 1. \tag{2.12}$$

Proof. From a well-known result (see, for instance, [31, Proposition 2.10.4]) it follows that the restriction of $\mathbb{T}$ to $\mathcal{D}(\mathcal{A} - \mathcal{B}\mathcal{B}^*) = H_1 \times H_{\frac{1}{2}}$ is a contraction semigroup on $\mathcal{D}(\mathcal{A} - \mathcal{B}\mathcal{B}^*)$ (endowed with the graph norm) whose generator is the restriction of $\mathcal{A} - \mathcal{B}\mathcal{B}^*$ to $\mathcal{D}((\mathcal{A} - \mathcal{B}\mathcal{B}^*)^2)$. Moreover, it can be easily checked that the graph norm of $\mathcal{A} - \mathcal{B}\mathcal{B}^*$ is equivalent to the standard norm of $H_1 \times H_{\frac{1}{2}}$. Therefore, if we denote by X_1 the space $H_1 \times H_{\frac{1}{2}}$, endowed with the graph norm of $\mathcal{A} - \mathcal{B}\mathcal{B}^*$, and we combine Proposition 2.1 with [31, Proposition 2.10.4], we obtain that

$$\|\mathbb{T}_\tau\|_{\mathcal{L}(X_1)} < 1, \qquad \|\mathbb{S}_\tau\|_{\mathcal{L}(X_1)} < 1. \tag{2.13}$$

Since $B_0 B_0^* \in \mathcal{L}(H_1, H_{\frac{1}{2}})$, it is easy to check that $\mathcal{D}((\mathcal{A} - \mathcal{B}\mathcal{B}^*)^2) = H_{\frac{3}{2}} \times H_1$ and that the graph norm of $(\mathcal{A} - \mathcal{B}\mathcal{B}^*)^2$ is equivalent to the standard norm in $H_{\frac{3}{2}} \times H_1$.

It follows that, indeed, the restriction of $\mathbb{T}$ to $H_1 \times H_{\frac{1}{2}}$ is a contraction semigroup on this space with a generator that is the restriction of $\mathcal{A} - \mathcal{B}\mathcal{B}^*$ to $H_{\frac{3}{2}} \times H_1$. The second assertion on $\mathbb{T}$ can be easily obtained by looking at $\mathcal{A} - \mathcal{B}\mathcal{B}^*$ as an operator in X_1 and repeating the above argument. The corresponding assertions for $\mathbb{S}$ can be proved in a completely similar manner.

Finally, let X_2 be $H_{\frac{3}{2}} \times H_1$ endowed with the graph norm of $(\mathcal{A} - \mathcal{B}\mathcal{B}^*)^2$, and define

$$|||\cdot||| = \|\cdot\|_{\mathcal{L}(X_2)}.$$

It is easily checked that this norm is equivalent to the standard norm in $\mathcal{L}(H_{\frac{3}{2}} \times H_1)$. Moreover, estimates (2.12) follow from (2.13) by using again Proposition 2.10.4 in [31]. □

Remark 2.6. An important property of the control u constructed in (2.9) is that, under appropriate assumptions on B_0, its regularity increases when the initial data are more regular. For instance, if $B_0B_0^* \in \mathcal{L}(H_1, H_{\frac{1}{2}})$ and $\left[\begin{smallmatrix} q_0 \\ q_1 \end{smallmatrix}\right] \in H_{\frac{3}{2}} \times H_1$, then, by Proposition 2.5, $\left[\begin{smallmatrix} w_0 \\ w_1 \end{smallmatrix}\right] = (I - L_\tau)^{-1} \left[\begin{smallmatrix} q_0 \\ q_1 \end{smallmatrix}\right] \in H_{\frac{3}{2}} \times H_1$ so that $u \in C^1([0,\tau]; U)$ and $B_0 u \in C([0,\tau]; H_{\frac{1}{2}})$. This kind of regularity property is important for approximation purposes, and it has been recently investigated for HUM controls. In [9] it is shown that, for the wave equation with boundary control, the HUM controls should be modified to obtain the regularity property. In the case of the wave equation with internal control, it is shown in [5], [17], under assumptions on B_0 which are similar to ours, that the HUM controls are smoother if we increase the regularity of the initial data.

Remark 2.7. Russell's principle can be extended to the case of unbounded input operators $B_0 \in \mathcal{L}(U, H_{-\frac{1}{2}})$, where $H_{-\frac{1}{2}}$ is the dual of $H_{\frac{1}{2}}$ with respect to the pivot space H, so that it can be applied to boundary control problems. In this case the system (2.1)–(2.2) is still well posed and it keeps most of the properties holding for bounded B_0 (see, for instance, [29], [30], and the references therein). For a quite general form of Russell's principle for unbounded input operators we refer to [24]. However, extending our numerical method to boundary control problems would first require showing that the smoothness of the controls given by Russell's principle increases if we increase the regularity of the initial data. This is, for general boundary control problems, an open question (see Remark 2.6 above).

3. An approximation result. The aim of this section is to provide error estimates for the approximations of (2.1) by finite dimensional systems. Using the notation in section 1 for the families of spaces $(V_h)_{h>0}$, $(U_h)_{h>0}$ and the families of operators $(\pi_h)_{h>0}$, $(A_{0h})_{h>0}$, $(B_{0h})_{h>0}$, we consider the family of finite dimensional systems

$$\ddot{w}_h(t) + A_{0h}w_h(t) + B_{0h}B_{0h}^*\dot{w}_h(t) = 0, \tag{3.1}$$

$$w_h(0) = \pi_h w_0, \quad \dot{w}_h(0) = \pi_h w_1. \tag{3.2}$$

In the case in which $B_0 = 0$ and A_0 is the Dirichlet Laplacian, it has been shown by Baker [1] that, given $w_0 \in H_{\frac{3}{2}}$, $w_1 \in H_1$, the solutions of (3.1) converge to the solution of (2.1) when $h \to 0$. Moreover, [1] contains precise estimates of the convergence rate. The result below shows that the same error estimates hold when

A_0 is an arbitrary positive operator and $B_0 \neq 0$. Throughout this section we assume that $B_0 B_0^* \in \mathcal{L}(H_1, H_{\frac{1}{2}})$.

PROPOSITION 3.1. *Let $w_0 \in H_{\frac{3}{2}}, w_1 \in H_1$, and let w, w_h be the corresponding solutions of (2.1), (2.2) and (3.1), (3.2). Moreover, assume that $B_0 B_0^* \in \mathcal{L}(H_1, H_{\frac{1}{2}})$. Then there exist three constants $K_0,\ K_1,\ h^* > 0$ such that, for every $h \in (0, h^*)$, we have*

$$\|\dot{w}(t) - \dot{w}_h(t)\| + \|w(t) - w_h(t)\|_{\frac{1}{2}} \leqslant (K_0 + K_1\, t) h^\theta \left(\|w_0\|_{\frac{3}{2}} + \|w_1\|_1\right) \qquad (t \geqslant 0). \tag{3.3}$$

Proof. We first note that, according to Proposition 2.5, we have

$$w \in C([0,\infty); H_{\frac{3}{2}}) \cap C^1([0,\infty); H_1) \cap C^2([0,\infty); H_{\frac{1}{2}}),$$

$$\|\ddot{w}(t)\|_{\frac{1}{2}} + \|\dot{w}(t)\|_1 + \|w(t)\|_{\frac{3}{2}} \leqslant \widetilde{K} \left(\|w_1\|_1 + \|w_0\|_{\frac{3}{2}}\right) \qquad (t \geqslant 0). \tag{3.4}$$

Equation (2.1) can be written

$$\langle \ddot{w}, v\rangle + \langle A_0^{\frac{1}{2}} w, A_0^{\frac{1}{2}} v\rangle + \langle B_0^* \dot{w}, B_0^* v\rangle_U = 0 \qquad (v \in H_{\frac{1}{2}}),$$

whereas, using (1.6) and (1.10), we see that (3.1) is equivalent to

$$\langle \ddot{w}_h, v_h\rangle + \langle A_0^{\frac{1}{2}} w_h, A_0^{\frac{1}{2}} v_h\rangle + \langle B_0^* \dot{w}_h, B_0^* v_h\rangle_U = 0 \qquad (v_h \in V_h).$$

Taking $v = v_h$ in the first of the above relations and subtracting side by side, it follows that

$$\langle \ddot{w} - \ddot{w}_h, v_h\rangle + \langle A_0^{\frac{1}{2}} (w - w_h), A_0^{\frac{1}{2}} v_h\rangle + \langle B_0^* \dot{w} - B_0^* \dot{w}_h, B_0^* v_h\rangle_U = 0 \qquad (v_h \in V_h),$$

which yields (recall that π_h is the orthogonal projector from $H_{\frac{1}{2}}$ onto V_h) that

$$\begin{aligned} \langle \pi_h \ddot{w} - \ddot{w}_h, v_h\rangle &+ \langle A_0^{\frac{1}{2}} (\pi_h w - w_h), A_0^{\frac{1}{2}} v_h\rangle \\ &= \langle \pi_h \ddot{w} - \ddot{w}, v_h\rangle - \langle B_0^* \dot{w} - B_0^* \dot{w}_h, B_0^* v_h\rangle_U \qquad (v_h \in V_h). \end{aligned} \tag{3.5}$$

We set

$$\mathcal{E}_h(t) = \frac{1}{2}\|\pi_h \dot{w} - \dot{w}_h\|^2 + \frac{1}{2}\|A_0^{\frac{1}{2}} (\pi_h w - w_h)\|^2.$$

Using (3.5) it follows that

$$\begin{aligned} \dot{\mathcal{E}}_h(t) &= \langle \pi_h \ddot{w} - \ddot{w}, \pi_h \dot{w} - \dot{w}_h\rangle - \langle B_0^* (\dot{w} - \dot{w}_h), B_0^* (\pi_h \dot{w} - \dot{w}_h)\rangle_U \\ &= \langle \pi_h \ddot{w} - \ddot{w}, \pi_h \dot{w} - \dot{w}_h\rangle - \|B_0^* (\pi_h \dot{w} - \dot{w}_h)\|_U^2 + \langle B_0 B_0^* (\pi_h \dot{w} - \dot{w}), (\pi_h \dot{w} - \dot{w}_h)\rangle. \end{aligned}$$

We have thus shown that

$$\dot{\mathcal{E}}_h(t) \leqslant M \left(\|\pi_h \ddot{w} - \ddot{w}\| + \|\pi_h \dot{w} - \dot{w}\|\right) \|\pi_h \dot{w} - \dot{w}_h\|,$$

where $M = 1 + \|B_0 B_0^*\|$. It follows that

$$2\, \mathcal{E}_h^{\frac{1}{2}}(t) \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}_h^{\frac{1}{2}}(t) \leqslant M\sqrt{2} \left(\|\pi_h \ddot{w} - \ddot{w}\| + \|\pi_h \dot{w} - \dot{w}\|\right) \mathcal{E}_h^{\frac{1}{2}}(t),$$

which yields

$$\mathcal{E}_h^{\frac{1}{2}}(t) \leqslant \mathcal{E}_h^{\frac{1}{2}}(0) + \frac{M}{\sqrt{2}} \int_0^t \left(\|\pi_h \ddot{w} - \ddot{w}\| + \|\pi_h \dot{w} - \dot{w}\|\right) \mathrm{d}t \qquad (t \geqslant 0).$$

The above estimate, combined with (3.4), with the fact that $\mathcal{E}_h(0) = 0$, and with (1.5), implies that there exist two constants $\widetilde{K}$, $\widetilde{h}^* > 0$ such that, for every $h \in (0, \widetilde{h}^*)$, we have

$$\mathcal{E}_h^{\frac{1}{2}}(t) \leqslant t\,\widetilde{K} h^\theta \left(\|w_0\|_{\frac{3}{2}} + \|w_1\|_1\right) \qquad (t \geqslant 0). \tag{3.6}$$

On the other hand, using (3.4), combined with (1.4) and (1.5), we have that there exists a constant $\widehat{h}^* > 0$ such that, for every $h \in (0, \widehat{h}^*)$,

$$\begin{aligned}\|\dot{w}(t) - \dot{w}_h(t)\| &\leqslant \|\dot{w}(t) - \pi_h \dot{w}(t)\| + \|\pi_h \dot{w}(t) - \dot{w}_h(t)\| \\ &\leqslant K \left[h^\theta \left(\|w_0\|_{\frac{3}{2}} + \|w_1\|_1\right) + \mathcal{E}_h^{\frac{1}{2}}(t)\right],\end{aligned}$$

$$\begin{aligned}\|w(t) - w_h(t)\|_{\frac{1}{2}} &\leqslant \|w(t) - \pi_h w(t)\|_{\frac{1}{2}} + \|\pi_h w(t) - w_h(t)\|_{\frac{1}{2}} \\ &\leqslant K \left[h^\theta \left(\|w_0\|_{\frac{3}{2}} + \|w_1\|_1\right) + \mathcal{E}_h^{\frac{1}{2}}(t)\right]\end{aligned}$$

for some constant $K > 0$. The last two inequalities, combined with (3.6), yield the conclusion (3.3). ∎

For $h > 0$ we denote $X_h = V_h \times V_h$, and we consider the operators

$$\mathcal{A}_h = \begin{bmatrix} 0 & I \\ -A_{0h} & 0 \end{bmatrix}, \quad \mathcal{B}_h = \begin{bmatrix} 0 \\ B_{0h} \end{bmatrix}. \tag{3.7}$$

The discrete analogues of the semigroups $\mathbb{T}$, $\mathbb{S}$ and of the operator L_t, denoted by $\mathbb{T}_h$, $\mathbb{S}_h$, and $L_{h,t}$, respectively, are defined, for every $h > 0$, by

$$\mathbb{T}_{h,t} = \mathrm{e}^{t(\mathcal{A}_h - \mathcal{B}_h \mathcal{B}_h^*)}, \quad \mathbb{S}_{h,t} = \mathrm{e}^{t(-\mathcal{A}_h - \mathcal{B}_h \mathcal{B}_h^*)}, \quad L_{h,t} = \mathbb{S}_{h,t} \mathbb{T}_{h,t} \qquad (t \geqslant 0). \tag{3.8}$$

For every $h > 0$ we define $\Pi_h \in \mathcal{L}(H_{\frac{1}{2}} \times H_{\frac{1}{2}}, X_h)$ by

$$\Pi_h = \begin{bmatrix} \pi_h & 0 \\ 0 & \pi_h \end{bmatrix}. \tag{3.9}$$

The following two results are consequences of Proposition 3.1.

Corollary 3.2. *There exist two constants C_1, $h^* > 0$ such that, for every $h \in (0, h^*)$ and $t > 0$, we have (recall that $L_t = \mathbb{S}_t \mathbb{T}_t$ for every $t \geqslant 0$)*

$$\|\Pi_h \mathbb{T}_t Z_0 - \mathbb{T}_{h,t} \Pi_h Z_0\|_X \leqslant C_1 t h^\theta \|Z_0\|_{H_{\frac{3}{2}} \times H_1} \qquad (Z_0 \in H_{\frac{3}{2}} \times H_1), \tag{3.10}$$

$$\|\Pi_h \mathbb{S}_t Z_0 - \mathbb{S}_{h,t} \Pi_h Z_0\|_X \leqslant C_1 t h^\theta \|Z_0\|_{H_{\frac{3}{2}} \times H_1} \qquad (Z_0 \in H_{\frac{3}{2}} \times H_1), \tag{3.11}$$

$$\|\Pi_h L_t Z_0 - L_{h,t} \Pi_h Z_0\|_X \leqslant C_1 t h^\theta \|Z_0\|_{H_{\frac{3}{2}} \times H_1} \qquad (Z_0 \in H_{\frac{3}{2}} \times H_1). \tag{3.12}$$

Proof. The estimate (3.10) is nothing else but (3.6) rewritten in semigroup terms.

We next notice that

$$P\mathbb{S}_t \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = \mathbb{T}_t P \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \qquad \left(\begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \in H_1 \times H_{\frac{1}{2}} \right), \tag{3.13}$$

where

$$P \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = \begin{bmatrix} w_0 \\ -w_1 \end{bmatrix} \qquad \left(\begin{bmatrix} w_0 \\ w_1 \end{bmatrix} \in X \right).$$

Indeed, denoting

$$\mathbb{S}_t \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = \begin{bmatrix} w(t) \\ z(t) \end{bmatrix} \qquad (t \geqslant 0),$$

formula (3.13) follows from

$$\frac{\mathrm{d}}{\mathrm{d}t} P\mathbb{S}_t \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = P \frac{\mathrm{d}}{\mathrm{d}t} \mathbb{S}_t \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = P \begin{bmatrix} -z(t) \\ A_0 w(t) - B_0 B_0^* z(t) \end{bmatrix} = (\mathcal{A} - \mathcal{B}\mathcal{B}^*) P\mathbb{S}_t \begin{bmatrix} w_0 \\ w_1 \end{bmatrix}.$$

To prove (3.11), it suffices to use (3.13), its discrete analogues, and (3.10) to obtain that

$$\|\Pi_h \mathbb{S}_t Z_0 - \mathbb{S}_{h,t} \Pi_h Z_0\|_X = \|P(\Pi_h \mathbb{S}_t Z_0 - \mathbb{S}_{h,t} \Pi_h Z_0)\|_X = \|\Pi_h P\mathbb{S}_t Z_0 - P\mathbb{S}_{h,t} \Pi_h Z_0\|_X$$

$$= \|\Pi_h \mathbb{T}_t P Z_0 - \mathbb{T}_{h,t} P \Pi_h Z_0\|_X = \|\Pi_h \mathbb{T}_t P Z_0 - \mathbb{T}_{h,t} \Pi_h P\, Z_0\|_X \leqslant C_1 t h^\theta \|P Z_0\|_{H_{\frac{3}{2}} \times H_1}.$$

Finally, estimate (3.12) can be easily obtained from (3.10) and (3.11). ∎

COROLLARY 3.3. *There exist three constants* $C_0,\ C_1,\ h^* > 0$ *such that, for every* $t > 0$, $h \in (0, h^*)$, *and* $k \in \mathbb{N}$, *we have*

$$\|L_t^k Z_0 - L_{h,t}^k \Pi_h Z_0\|_X \leqslant (C_0 + kC_1 t) h^\theta \|Z_0\|_{H_{\frac{3}{2}} \times H_1} \qquad (Z_0 \in H_{\frac{3}{2}} \times H_1).$$

Proof. We have

$$\|L_t^k Z_0 - L_{h,t}^k \Pi_h Z_0\|_X \leqslant \|L_t^k Z_0 - \Pi_h L_t^k Z_0\|_X + \|\Pi_h L_t^k Z_0 - L_{h,t}^k \Pi_h Z_0\|_X. \tag{3.14}$$

From Proposition 2.5 it follows that, for every $t \geqslant 0$, $H_{\frac{3}{2}} \times H_1$ is an invariant space for L_t. Using this fact combined with (1.4) and (1.5), we obtain that there exists a constant $C_0 > 0$ such that the first term on the right-hand side of the above inequality satisfies

$$\|L_t^k Z_0 - \Pi_h L_t^k Z_0\|_X \leqslant C_0 h^\theta \|Z_0\|_{H_{\frac{3}{2}} \times H_1}. \tag{3.15}$$

For the second term on the right-hand side of (3.14) we have

$$\begin{aligned} &\|\Pi_h L_t^k Z_0 - L_{h,t}^k \Pi_h Z_0\|_X \\ &\leqslant \|\Pi_h L_t^k Z_0 - L_{h,t} \Pi_h L_t^{k-1} Z_0\|_X + \|L_{h,t} \Pi_h L_t^{k-1} Z_0 - L_{h,t}^k \Pi_h Z_0\|_X \\ &= \|\Pi_h L_t (L_t^{k-1} Z_0) - L_{h,t} \Pi_h L_t^{k-1} Z_0\|_X + \|L_{h,t} (\Pi_h L_t^{k-1} Z_0 - L_{h,t}^{k-1} \Pi_h Z_0)\|_X. \end{aligned}$$

Applying (3.12), we obtain

$$\|\Pi_h L_t^k Z_0 - L_{h,t}^k \Pi_h Z_0\|_X \leqslant C_1 t h^\theta \|Z_0\|_{H_{\frac{3}{2}} \times H_1} + \|\Pi_h L_t^{k-1} Z_0 - L_{h,t}^{k-1} \Pi_h Z_0\|_X.$$

By an obvious induction argument it follows that

$$\|\Pi_h L_t^k Z_0 - L_{h,t}^k \Pi_h Z_0\|_X \leqslant C_1 t k h^\theta \|Z_0\|_{H_{\frac{3}{2}} \times H_1} \qquad (Z_0 \in H_{\frac{3}{2}} \times H_1). \tag{3.16}$$

Finally, combining (3.14)–(3.16), we obtain the conclusion of the corollary. ∎

4. Proof of the main result. In this section we continue to use the notation from (3.7)–(3.9) for $\mathcal{A}_h$, $\mathcal{B}_h$, $\mathbb{T}_h$, $\mathbb{S}_h$, L_h, and Π_h. We first give the following result.

LEMMA 4.1. *Suppose that the system* (1.2), (1.3) *is exactly controllable in time* $\tau > 0$ *and that* $B_0 B_0^* \in \mathcal{L}(H_1, H_{\frac{1}{2}})$. *Let* $Q_0 = \left[\begin{smallmatrix} q_0 \\ q_1 \end{smallmatrix}\right] \in H_{\frac{3}{2}} \times H_1$, *and let* u *be the control given by* (2.11), *where* $W_0 = \left[\begin{smallmatrix} w_0 \\ w_1 \end{smallmatrix}\right]$ *is as given by* (2.10). *Let* $v_h : [0,\tau] \to U_h$ *be defined by*

$$v_h(t) = \mathcal{B}_h^* \mathbb{T}_{h,t} \Pi_h W_0 + \mathcal{B}_h^* \mathbb{S}_{h,\tau-t} \mathbb{T}_{h,\tau} \Pi_h W_0 \qquad (t \in [0,\tau]). \tag{4.1}$$

Then there exist three constants $C_2,\ C_3,\ h^* > 0$ *such that, for every* $h \in (0, h^*)$, *we have*

$$\|(u - v_h)(t)\|_U \leqslant \frac{C_2 + tC_3}{1 - |||L_\tau|||} h^\theta \|Q_0\|_{H_{\frac{3}{2}} \times H_1} \qquad (t \in [0,\tau]), \tag{4.2}$$

where $|||\cdot|||$ *is the norm introduced in Proposition* 2.5.

Proof. We first note that from Proposition 2.5 and the fact that $Q_0 \in H_{\frac{3}{2}} \times H_1$ it follows that W_0 given by (2.10) still belongs to $H_{\frac{3}{2}} \times H_1$. Using (2.11), (4.1), (1.9), and (3.7) we see that for every $t \in [0,\tau]$ we have

$$\begin{aligned} \|(u - v_h)(t)\|_U &= \|\mathcal{B}^* \mathbb{T}_t W_0 + \mathcal{B}^* \mathbb{S}_{\tau-t} \mathbb{T}_\tau W_0 - \mathcal{B}_h^* \mathbb{T}_{h,t} \Pi_h W_0 - \mathcal{B}_h^* \mathbb{S}_{h,\tau-t} \mathbb{T}_{h,\tau} \Pi_h W_0\|_U \\ &\leqslant \left\|\mathcal{B}^* \mathbb{T}_t W_0 - \mathcal{B}_h^* \mathbb{T}_t W_0\right\|_U + \left\|\mathcal{B}_h^* \left(\mathbb{T}_t W_0 - \mathbb{T}_{h,t} \Pi_h W_0\right)\right\|_U \\ &+ \left\|\mathcal{B}^* \mathbb{S}_{\tau-t} \mathbb{T}_\tau W_0 - \mathcal{B}_h^* \mathbb{S}_{\tau-t} \mathbb{T}_\tau W_0\right\|_U + \left\|\mathcal{B}_h^* (\mathbb{S}_{\tau-t} \mathbb{T}_\tau W_0 - \mathbb{S}_{h,\tau-t} \mathbb{T}_{h,\tau} \Pi_h W_0)\right\|_U . \end{aligned} \tag{4.3}$$

Let $h^* > 0$ be chosen as in Proposition 3.1. To bound the first term in the right-hand side of (4.3) we note that since $\mathcal{B}^* = \begin{bmatrix} 0 & B_0^* \end{bmatrix}$ and $\mathcal{B}_h^* = \begin{bmatrix} 0 & B_0^* \widetilde{\pi}_h \end{bmatrix}$ we have that

$$\left\|\mathcal{B}^* \mathbb{T}_t W_0 - \mathcal{B}_h^* \mathbb{T}_t W_0\right\|_U = \|B_0^* \left(\dot{w}(t) - \widetilde{\pi}_h \dot{w}(t)\right)\|_U \leqslant \|B_0^*\|_{\mathcal{L}(H,U)} \|\dot{w}(t) - \widetilde{\pi}_h \dot{w}(t)\|,$$

where we have denoted $\mathbb{T}_t W_0 = \left[\begin{smallmatrix} w(t) \\ \dot{w}(t) \end{smallmatrix}\right]$. Using next (1.8) and Proposition 2.5 we obtain that there exists a constant $C_0 > 0$ such that

$$\left\|\mathcal{B}^* \mathbb{T}_t W_0 - \mathcal{B}_h^* \mathbb{T}_t W_0\right\|_U \leqslant C_0 h^\theta \left\|\dot{w}(t)\right\|_{\frac{1}{2}} \leqslant C_0 h^\theta \|W_0\|_{H_{\frac{3}{2}} \times H_1}. \tag{4.4}$$

Similarly we show that the third term on the right-hand side of (4.3) satisfies

$$\left\|\mathcal{B}^* \mathbb{S}_{\tau-t} \mathbb{T}_\tau W_0 - \mathcal{B}_h^* \mathbb{S}_{\tau-t} \mathbb{T}_\tau W_0\right\|_U \leqslant C_0 h^\theta \|W_0\|_{H_{\frac{3}{2}} \times H_1}. \tag{4.5}$$

To bound the second term on the right-hand side of (4.3), we use the uniform boundedness of the family of operators $(B_{0h}^*)_{h \in (0,h^*)}$ in $\mathcal{L}(H,U)$ and Proposition 3.1 to get

$$\left\|\mathcal{B}_h^* \left(\mathbb{T}_t W_0 - \mathbb{T}_{h,t} \Pi_h W_0\right)\right\|_U \leqslant (K_0 + K_1\, t) h^\theta \|W_0\|_{H_{\frac{3}{2}} \times H_1}. \tag{4.6}$$

The fourth term on the right-hand side of (4.3) can be estimated similarly to get

$$\left\|\mathcal{B}_h^* (\mathbb{S}_{\tau-t} \mathbb{T}_\tau W_0 - \mathbb{S}_{h,\tau-t} \mathbb{T}_{h,\tau} \Pi_h W_0)\right\|_U \leqslant (K_0 + K_1\, t) h^\theta \|W_0\|_{H_{\frac{3}{2}} \times H_1}. \tag{4.7}$$

Using (4.4)–(4.7), relation (4.3) yields

$$\|(u - v_h)(t)\|_U \leqslant (C_2' + tC_3') h^\theta \|W_0\|_{H_{\frac{3}{2}} \times H_1}$$

for some constants $C_2',\ C_3' > 0$, and $h \in (0, h^*)$. In the above estimate, using the fact following from Proposition 2.5 and (2.10) that

$$\|W_0\|_{H_{\frac{3}{2}}\times H_1} \leqslant \frac{C}{1-|||L_\tau|||}\|Q_0\|_{H_{\frac{3}{2}}\times H_1}, \tag{4.8}$$

we obtain the conclusion of this lemma. ☐

We are now in a position to prove the main result of this work.

Proof of Theorem 1.1. Using the semigroup notation introduced in section 2 we can write u_h given by (1.17) as

$$u_h(t) = \mathcal{B}_h^*\mathbb{T}_{h,t}\begin{bmatrix} w_{0h} \\ w_{1h}\end{bmatrix} + \mathcal{B}_h^*\mathbb{S}_{h,\tau-t}\mathbb{T}_{h,\tau}\begin{bmatrix} w_{0h} \\ w_{1h}\end{bmatrix} \qquad (t\in[0,\tau]), \tag{4.9}$$

where

$$\begin{bmatrix} w_{0h} \\ w_{1h}\end{bmatrix} = \sum_{n=0}^{N(h)} L_{h,\tau}^n \Pi_h \begin{bmatrix} q_0 \\ q_1\end{bmatrix}. \tag{4.10}$$

Let u be the control given by (2.11), where $W_0 = \left[\begin{smallmatrix} w_0 \\ w_1\end{smallmatrix}\right]$ is given by (2.10). We evaluate $||u - u_h||_{C([0,\tau];U)}$ and show that (1.22) is verified. Since

$$\|u-u_h\|_{C([0,\tau],U)} \leqslant \|u-v_h\|_{C([0,\tau],U)} + \|v_h-u_h\|_{C([0,\tau],U)}, \tag{4.11}$$

it suffices to evaluate the two terms from the right, where v_h is as given by (4.1).

To estimate the second term in the right-hand side of (4.11) we first note that

$$\begin{aligned}
&(v_h-u_h)(t)\\
&= \mathcal{B}_h^*\mathbb{T}_{h,t}\Pi_h W_0 + \mathcal{B}_h^*\mathbb{S}_{h,\tau-t}\mathbb{T}_{h,\tau}\Pi_h W_0 - \mathcal{B}_h^*\mathbb{T}_{h,t}\Pi_h\begin{bmatrix} w_{0h} \\ w_{1h}\end{bmatrix} - \mathcal{B}_h^*\mathbb{S}_{h,\tau-t}\mathbb{T}_{h,\tau}\Pi_h\begin{bmatrix} w_{0h} \\ w_{1h}\end{bmatrix}.
\end{aligned}$$

It follows that there exists a positive constant C such that, for any $t\in[0,\tau]$,

$$\begin{aligned}
\|(v_h-u_h)(t)\|_U &\leqslant \left\|\mathcal{B}_h^*\mathbb{T}_{h,t}\Pi_h W_0 - \mathcal{B}_h^*\mathbb{T}_{h,t}\Pi_h\begin{bmatrix} w_{0h} \\ w_{1h}\end{bmatrix}\right\|_U \\
&\quad + \left\|\mathcal{B}_h^*\mathbb{S}_{h,\tau-t}\mathbb{T}_{h,\tau}\Pi_h W_0 - \mathcal{B}_h^*\mathbb{S}_{h,\tau-t}\mathbb{T}_{h,\tau}\Pi_h\begin{bmatrix} w_{0h} \\ w_{1h}\end{bmatrix}\right\|_U \\
&\leqslant C\left\|W_0 - \begin{bmatrix} w_{0h} \\ w_{1h}\end{bmatrix}\right\|_X = C\left\|\sum_{n=0}^{\infty} L_\tau^n Q_0 - \sum_{n=0}^{N(h)} L_{h,\tau}^n\Pi_h Q_0\right\|_X \\
&\leqslant C\sum_{n=N(h)+1}^{\infty}\|L_\tau\|_{\mathcal{L}(X)}^n\|Q_0\|_X + C\sum_{n=0}^{N(h)}\|(L_\tau^n - L_{h,\tau}^n\Pi_h)Q_0\|_X.
\end{aligned}$$

The above estimate and Corollary 3.3 imply that there exists $\widetilde{C} > 0$ such that

$$\begin{aligned}\|v_h - u_h\|_{C([0,\tau];U)} &\leqslant C\frac{\|L_\tau\|_{\mathcal{L}(X)}^{N(h)+1}}{1-\|L_\tau\|_{\mathcal{L}(X)}}\|Q_0\|_X + Ch^\theta\sum_{n=0}^{N(h)}(C_0+nC_1\tau)\|Q_0\|_{H_{\frac{3}{2}}\times H_1}\\ &= C\frac{\|L_\tau\|_{\mathcal{L}(X)}^{N(h)+1}}{1-\|L_\tau\|_{\mathcal{L}(X)}}\|Q_0\|_X\\ &\quad + C(N(h)+1)\left(C_0 + C_1\frac{N(h)}{2}\tau\right)h^\theta\|Q_0\|_{H_{\frac{3}{2}}\times H_1}\\ &\leqslant C\frac{\|L_\tau\|_{\mathcal{L}(X)}^{N(h)+1}}{1-\|L_\tau\|_{\mathcal{L}(X)}}\|Q_0\|_X + \widetilde{C}N^2(h)(1+\tau)h^\theta\|Q_0\|_{H_{\frac{3}{2}}\times H_1}\\ &\leqslant \frac{\widetilde{C}(1+\tau)}{1-\|L_\tau\|_{\mathcal{L}(X)}}\left(\|L_\tau\|_{\mathcal{L}(X)}^{N(h)} + N^2(h)h^\theta\right)\|Q_0\|_{H_{\frac{3}{2}}\times H_1}.\end{aligned}$$

By choosing $N(h) = [\frac{\theta}{\ln(\|L_\tau\|_{\mathcal{L}(X)})}\ln(h)]$ we deduce that

$$\|v_h - u_h\|_{C([0,\tau];U)} \leqslant \frac{\widetilde{C}(1+\tau)}{(1-\|L_\tau\|_{\mathcal{L}(X)})\ln^2(\|L_\tau\|_{\mathcal{L}(X)}^{-1})}\ln^2(h^{-1})\,h^\theta\|Q_0\|_{H_{\frac{3}{2}}\times H_1}.$$

Combining this last estimate with (4.2) and taking $m_\tau = \frac{1}{\ln(\|L_\tau\|_{\mathcal{L}(X)}^{-1})}$ we obtain the conclusion (1.22). ∎

Remark 4.2. The functions u_h given by (1.17) and v_h from (4.1) do not coincide with the exact control ζ_h, obtained by applying Russell's principle to the finite dimensional system

$$\ddot{q}_h(t) + A_{0h}q_h(t) + B_{0h}\zeta_h = 0, \tag{4.12}$$

$$q_h(0) = \pi_h q_0, \qquad \dot{q}_h(0) = \pi_h q_1. \tag{4.13}$$

Indeed, this control ζ_h is given by the formula

$$\zeta_h(t) = \mathcal{B}_h^*\mathbb{T}_{h,t}Z_{h0} + \mathcal{B}_h^*\mathbb{S}_{h,\tau-t}\mathbb{T}_{h,\tau}Z_{h0}, \qquad Z_{h0} = (I-L_{h,\tau})^{-1}\Pi_h\left[\begin{smallmatrix}q_0\\ q_1\end{smallmatrix}\right], \tag{4.14}$$

so that u_h is obtained by "filtering" (in an appropriate sense) ζ_h. Note that, since $\mathbb{T}_h$ and $\mathbb{S}_h$ are not, in general, uniformly exponentially stable (with respect to h), the control ζ_h does not, in general, converge to u (see, for instance, [33]).

If we modify the discretization of (2.1)–(2.2) in order to ensure that the semigroups $\mathbb{T}_h$ are uniformly (with respect to h) exponentially stable (see, for instance, [7, 23, 28]), the structure of (3.1) would be altered. For example, if we introduce a numerical viscosity, we have to add a term of the form $h^\theta A_{0h}\dot{v}_h$ in (4.12), which would represent a spurious distributed control in the discretized problem. In this case, by performing the corresponding modifications in the definition of $L_{h,\tau}$ we can prove the convergenceof the family $\left((I-L_{h,\tau})^{-1}\Pi_h Q_0\right)_{h>0}$ to $(I-L_\tau)^{-1}Q_0$, i.e., without truncating the Neumann series. Indeed, we have

$$\left\|(I-L_{h,\tau})^{-1}\Pi_h Q_0-(I-L_\tau)^{-1}Q_0\right\|=\left\|\sum_{n=0}^{\infty}L_{h,\tau}^n\Pi_h Q_0-L_\tau^n Q_0\right\|$$

$$\leqslant \sum_{n=0}^{N-1}\left\|L_{h,\tau}^n\Pi_h Q_0-L_\tau^n Q_0\right\|+\sum_{n=N}^{\infty}\left\|L_{h,\tau}^n\Pi_h Q_0-L_\tau^n Q_0\right\|$$

$$\leqslant \sum_{n=0}^{N-1}\left\|L_{h,\tau}^n\Pi_h Q_0-L_\tau^n Q_0\right\|+\sum_{n=N}^{\infty}\left(\|L_{h,\tau}\|^n+\|L_\tau\|^n\right)\|Q_0\|.$$

Now, we can easily see that, given any $\varepsilon>0$, there exists $N=N(\varepsilon)$ such that

$$\sum_{n=N}^{\infty}\left(\|L_{h,\tau}\|^n+\|L_\tau\|^n\right)\|Q_0\|\leqslant \max\left\{\frac{\|L_\tau\|^N}{1-\|L_\tau\|},\frac{\|L_{h,\tau}\|^N}{1-\|L_{h,\tau}\|}\right\}\|Q_0\|<\frac{\varepsilon}{2}.$$

On the other hand, since for each $n\geqslant 0$ we have that $L_{h,\tau}^n\Pi_h Q_0\to L_\tau^n Q_0$ when h goes to zero, there exists h sufficiently small such that

$$\sum_{n=0}^{N-1}\left\|L_{h,\tau}^n\Pi_h Q_0-L_\tau^n Q_0\right\|<\frac{\varepsilon}{2}.$$

This gives precisely the convergence we mentioned above. However, from a practical viewpoint, inverting exactly $I-L_{h,\tau}$ might be a difficult issue (note that the condition number of this matrix can be quite large, depending on τ).

Even though u_h given by (1.17) does not drive the solution of (4.1) exactly to zero in time τ, it is an approximate control for (4.1). In fact, estimate (1.22) allows us to tell how far we are from the target, as in related problems which have been investigated in [6, 16, 20, 32]. We have the following property.

Corollary 4.3. *For each $h<h^*$, let u_h be the discrete control given by Theorem 1.1 corresponding to the initial data $Q_0=\left[\begin{smallmatrix}q_0\\ q_1\end{smallmatrix}\right]\in H_{\frac{3}{2}}\times H_1$, and let $(q_h,\dot q_h)$ be the solution of the equation*

$$\ddot q_h(t)+A_{0h}q_h(t)+B_{0h}u_h=0, \tag{4.15}$$

$$q_h(0)=\pi_h q_0,\qquad \dot q_h(0)=\pi_h q_1. \tag{4.16}$$

There exists a positive constant $M>0$ independent of h such that we have

$$\|(q_h(\tau),\dot q_h(\tau))\|_X\leqslant Mh^\theta\ln^2(h^{-1})\|Q_0\|_{H_{\frac{3}{2}}\times H_1}\qquad (0<h<h^*). \tag{4.17}$$

Proof. Let $(q,\dot q)$ be the controlled solution of (1.2)–(1.3) with the exact control given by (2.9). Since, from Remark 2.6, $B_0u\in C([0,\tau];H_{\frac{1}{2}})$ and

$$\|B_0u\|_{C([0,\tau];H_{\frac{1}{2}})}\leqslant M\|Q_0\|_{H_{\frac{3}{2}}\times H_1},$$

we deduce that

$$q\in C([0,\infty);H_{\frac{3}{2}})\cap C^1([0,\infty);H_1)\cap C^2([0,\infty);H_{\frac{1}{2}}),$$

$$\|\ddot q(t)\|_{\frac{1}{2}}+\|\dot q(t)\|_1+\|q(t)\|_{\frac{3}{2}}\leqslant M\left(\|q_1\|_1+\|q_0\|_{\frac{3}{2}}\right)\qquad (t\geqslant 0). \tag{4.18}$$

Here and in what follows, M denotes a positive constant, which may change from one line to another but remains independent of h. By arguing as in the first part of the proof of Proposition 3.1, we deduce that

$$\dot{\mathcal{E}}_h(t) = \langle \pi_h \ddot{q}(t) - \ddot{q}(t), \pi_h \dot{q}(t) - \dot{q}_h(t)\rangle - \langle B_0 u(t) - B_{0h} u_h(t), \pi_h \dot{q}(t) - \dot{q}_h(t)\rangle$$

$$\leqslant M \left(\|\pi_h \ddot{q}(t) - \ddot{q}(t)\| + \|B_0 u(t) - B_{0h} u_h(t)\|\right) \mathcal{E}_h^{\frac{1}{2}}(t)$$

$$\leqslant M \left(h^\theta \|Q_0\|_{H_{\frac{3}{2}}\times H_1} + \|u(t) - u_h(t)\|_U\right) \mathcal{E}_h^{\frac{1}{2}}(t)$$

$$\leqslant Mh^\theta \left(1 + \ln^2(h^{-1})\right) \|Q_0\|_{H_{\frac{3}{2}}\times H_1} \mathcal{E}_h^{\frac{1}{2}}(t) \leqslant Mh^\theta \ln^2(h^{-1}) \|Q_0\|_{H_{\frac{3}{2}}\times H_1} \mathcal{E}_h^{\frac{1}{2}}(t),$$

where

$$\mathcal{E}_h(t) = \frac{1}{2}\|\pi_h \dot{q}(t) - \dot{q}_h(t)\|^2 + \frac{1}{2}\|A_0^{\frac{1}{2}}(\pi_h q(t) - q_h(t))\|^2.$$

We deduce that

$$\mathcal{E}_h^{\frac{1}{2}}(t) \leqslant Mth^\theta \ln^2(h^{-1}) \|Q_0\|_{H_{\frac{3}{2}}\times H_1}. \tag{4.19}$$

Now, by taking into account that u is an exact control for (1.2)–(1.3), we deduce that

$$\|(q_h(\tau), \dot{q}_h(\tau))\|_X = \|\Pi_h(q(\tau), \dot{q}(\tau)) - (q_h(\tau), \dot{q}_h(\tau))\|_X \leqslant \sqrt{2}\mathcal{E}_h^{\frac{1}{2}}(t).$$

Relation (4.17) follows immediately from the last inequality and (4.19). ∎

Remark 4.4. It would be interesting to have a counterpart to our scheme in the case of initial data in $H_{\frac{1}{2}} \times H$. We cannot hope to obtain error estimates in this case, but obtaining convergence seems an attainable objective. Indeed, as shown in Remark 4.2 the discretization procedure can be slightly modified to ensure this property.

However, it is not clear how to extend our scheme to the case $Q_0 \in H_{\frac{1}{2}} \times H$ without modification. Indeed, the choice of $N(h)$ in our method is essentially based on the error estimates from Proposition 3.1, which are not valid for initial data in $H_{\frac{1}{2}} \times H$.

Remark 4.5. When applying our method to simulations we do not have the exact solutions w_h and $w_{b,h}$. More precisely, we need to discretize equations (1.11)–(1.21) with respect to the time variable. Using error estimates for full space-time discretizations corresponding to those in Proposition 3.1 it would be possible, after a careful numerical analysis, to obtain convergence rates for the controls based on full discretizations. Such an analysis is outside the scope of this work, and we refer to [13] for a similar analysis for a duality related reconstruction of initial states problem.

5. Examples and numerical results. In this section we apply our numerical method to approximate exact controls for the two dimensional wave equation and for the Euler–Bernoulli beam equation. For both examples we consider distributed controls.

5.1. The wave equation. In this subsection we consider the approximation of an internal distributed exact control for the wave equation with homogeneous Dirichlet boundary condition.

Let $\Omega \subset \mathbb{R}^2$ be an open connected set with boundary of class C^2, or let Ω be a rectangular domain. Let $\mathcal{O} \subset \Omega$, $\mathcal{O} \neq \Omega$ be an open set. We consider the control problem

$$\ddot{q}(x,t) - \Delta q(x,t) + \chi_{\mathcal{O}}(x) u(x,t) = 0, \quad (x,t) \in \Omega \times [0,\tau], \tag{5.1}$$

$$q(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,\tau], \tag{5.2}$$

$$q(x,0) = q_0(x), \quad \dot{q}(x,0) = q_1(x), \quad x \in \Omega, \tag{5.3}$$

where $\chi_{\mathcal{O}} \in \mathcal{D}(\Omega)$ is such that $\chi_{\mathcal{O}}(x) = 1$ for $x \in \mathcal{O}$ and $\chi_{\mathcal{O}}(x) \geqslant 0$ for $x \in \Omega$.

In order to apply the method described in (1.11)–(1.21) to this case we need appropriate choices of spaces and operators. We take $H = L^2(\Omega)$, $U = H$, and $A_0 : \mathcal{D}(A_0) \to H$ with

$$\mathcal{D}(A_0) = \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega), \quad A_0\varphi = -\Delta\varphi \qquad (\varphi \in \mathcal{D}(A_0)),$$

where we use the notation $\mathcal{H}^m(\Omega)$, with $m \in \mathbb{N}$, for the standard Sobolev spaces. It is well known that A_0 is a self-adjoint, strictly positive operator with compact resolvents. The corresponding spaces $H_{\frac{3}{2}}$, H_1, and $H_{\frac{1}{2}}$ introduced in section 1 are in this case given by

$$H_{\frac{3}{2}} = \left\{\varphi \in \mathcal{H}^3(\Omega) \cap \mathcal{H}_0^1(\Omega) \;\middle|\; \Delta\varphi = 0 \ \text{ on } \ \partial\Omega\right\},$$

$$H_1 = \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega), \qquad H_{\frac{1}{2}} = \mathcal{H}_0^1(\Omega).$$

The control operator $B_0 \in \mathcal{L}(H)$ is defined by

$$B_0 u = \chi_{\mathcal{O}} u \qquad (u \in H).$$

The operator B_0 is clearly self-adjoint and $B_0 B_0^* \in \mathcal{L}(H_1, H_{\frac{1}{2}})$. Moreover, we assume that τ and $\mathcal{O}$ are such that the system (5.1)–(5.3) is exactly controllable in time τ, i.e., that for every $\left[\begin{smallmatrix} q_0 \\ q_1 \end{smallmatrix}\right] \in H_0^1(\Omega) \times L^2(\Omega)$ there exists a control $u \in L^2([0,\tau], U)$ such that $q(\tau) = \dot{q}(\tau) = 0$. Sufficient conditions in which this assumption holds are give in various works; see Lions [18]; Bardos, Lebeau, and Rauch [2]; and Liu [19].

To construct an approximating family of spaces $(V_h)_{h>0}$ we consider a quasi-uniform triangulation $\mathcal{T}_h$ of Ω of diameter h, as defined, for instance, in [3, p. 106]. For each $h > 0$ we define V_h by

$$V_h = \left\{\varphi \in C(\overline{\Omega}) \;\middle|\; \varphi_{|T} \in P_1(T) \ \text{ for every } \ T \in \mathcal{T}_h, \ \varphi_{|\partial\Omega} = 0\right\},$$

where $P_1(T)$ is the set of affine functions on T. It is well known (see, for instance, [22, pp. 96–97]) that the orthogonal projector π_h from $H_{\frac{1}{2}} = \mathcal{H}_0^1(\Omega)$ onto V_h satisfies (1.4) and (1.5) for $\theta = 1$.

We define $U_h = \{\chi_{\mathcal{O}} v_h \ |v_h \in V_h\} \subset U$ and let $B_{0h} \in \mathcal{L}(H)$ be given by $B_{0h}\varphi = \widetilde{\pi}_h(\chi_{\mathcal{O}}\varphi)$ for every $\varphi \in H$. Note that $B_{0h}^*\varphi_h = \chi_{\mathcal{O}}\varphi_h$ and $\langle B_{0h} B_{0h}^* \varphi_h, \psi_h\rangle = \langle \chi_{\mathcal{O}}^2 \varphi_h, \psi_h\rangle$ for every $\varphi_h, \psi_h \in V_h$, where $\langle\cdot,\cdot\rangle$ denotes the inner product in $L^2(\Omega)$.

With the above choice of spaces and operators, denoting by $N(h) = [\frac{\theta}{\ln \|L_\tau\|} \ln h]$ the first part of the general method described in (1.11)–(1.21) reduces to the computation of the families of functions $(w_h^n)_{1 \leqslant n \leqslant N(h)+1}$, $(w_{b,h}^n)_{1 \leqslant n \leqslant N(h)}$ satisfying, for every $v_h \in V_h$,

$$\langle \ddot{w}_h^n(t), v_h \rangle + \langle \nabla w_h^n(t), \nabla v_h \rangle + \langle \chi_{\mathcal{O}}^2 \dot{w}_h^n(t), v_h \rangle = 0 \qquad (t \in [0, \tau]), \tag{5.4}$$

$$w_h^n(0) = \begin{cases} \pi_h q_0 & \text{if } n = 1, \\ w_{b,h}^{n-1}(0) & \text{if } 1 < n \leqslant N(h) + 1, \end{cases} \tag{5.5}$$

$$\dot{w}_h^n(0) = \begin{cases} \pi_h q_1 & \text{if } n = 1, \\ \dot{w}_{b,h}^{n-1}(0) & \text{if } 1 < n \leqslant N(h) + 1, \end{cases} \tag{5.6}$$

and

$$\langle \ddot{w}_{b,h}^n(t), v_h \rangle + \langle \nabla w_{b,h}^n(t), \nabla v_h \rangle - \langle \chi_{\mathcal{O}}^2 \dot{w}_{b,h}^n(t), v_h \rangle = 0 \qquad (t \in [0, \tau]), \tag{5.7}$$

$$w_{b,h}^n(\tau) = w_h^n(\tau), \qquad \dot{w}_{b,h}^n(\tau) = \dot{w}_h^n(\tau). \tag{5.8}$$

The second part of the method described in (1.11)–(1.21) reduces to the computation of w_{0h} and w_{1h} defined by

$$\begin{bmatrix} w_{0h} \\ w_{1h} \end{bmatrix} = \begin{bmatrix} \pi_h q_0 \\ \pi_h q_1 \end{bmatrix} + \sum_{n=1}^{N(h)} \begin{bmatrix} w_{b,h}^n(0) \\ \dot{w}_{b,h}^n(0) \end{bmatrix}. \tag{5.9}$$

Finally, the approximation u_h of the exact control u is given by

$$u_h = \chi_{\mathcal{O}} \dot{w}_h + \chi_{\mathcal{O}} \dot{w}_{b,h}, \tag{5.10}$$

where w_h and $w_{b,h}$ are the solution of

$$\langle \ddot{w}_h(t), v_h \rangle + \langle \nabla w_h(t), \nabla v_h \rangle + \langle \chi_{\mathcal{O}}^2 \dot{w}_h(t), v_h \rangle = 0 \qquad (v_h \in V_h,\ t \in [0, \tau]), \tag{5.11}$$

$$w_h(0) = w_{0h}, \qquad \dot{w}_h(0) = w_{1h}, \tag{5.12}$$

$$\langle \ddot{w}_{b,h}(t), v_h \rangle + \langle \nabla w_{b,h}(t), \nabla v_h \rangle - \langle \chi_{\mathcal{O}}^2 \dot{w}_{b,h}(t), v_h \rangle = 0 \qquad (v_h \in V_h,\ t \in [0, \tau]), \tag{5.13}$$

$$w_{b,h}(\tau) = w_h(\tau), \qquad \dot{w}_{b,h}(\tau) = \dot{w}_h(\tau). \tag{5.14}$$

Since we checked above all the necessary assumptions, we can apply Theorem 1.1 to obtain that (u_h) converges in $C([0, \tau]; L^2(\Omega))$ to an exact control u such that

$$\|u - u_h\|_{C([0,\tau];L^2(\Omega))} \leqslant Ch \ln^2(h^{-1})(\|q_0\|_{\mathcal{H}^3(\Omega)} + \|q_1\|_{\mathcal{H}^2(\Omega)}) \qquad (0 < h < h^*) \tag{5.15}$$

for some constants $h^*,\ C > 0$.

The efficiency of the algorithm has been tested in the case $\Omega = [0, 1]^2$ and $\mathcal{O} = [(x_1, x_2) \times (0, 1)] \cup [(0, 1) \times (y_1, y_2)]$, where $x_1, x_2, y_1, y_2 \in (0, 1)$ are such that $x_1 < x_2$ and $y_1 < y_2$. The initial data that we want to steer to zero are the "bubble" functions

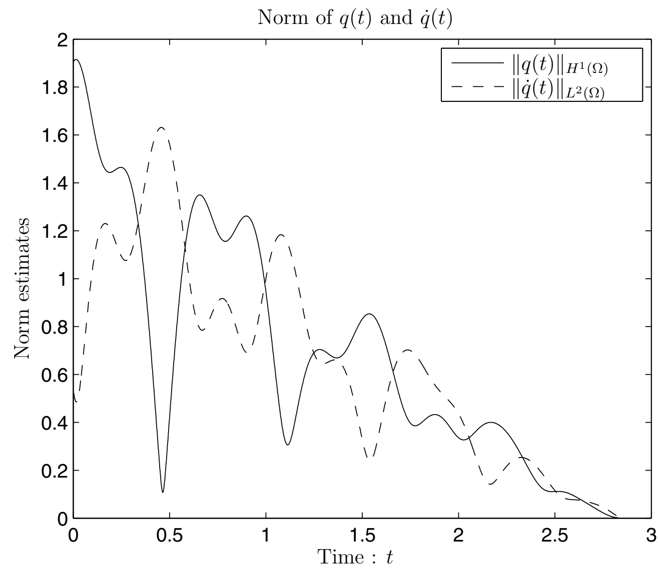

FIG. 1. *The norms of the solution of the controlled wave equation with the control u_h given by (1.17). The solid line is the norm H_0^1 of $q(t)$, and the dashed line is the norm L^2 of $\dot{q}(t)$.*

$q_0(x,y) = q_1(x,y) = x^3y^3(1-x)^3(1-y)^3$, and the control time is $\tau = 2\sqrt{2}$. Note that $\left[\begin{smallmatrix} q_0 \\ q_1 \end{smallmatrix}\right] \in H_{\frac{3}{2}} \times H_{\frac{1}{2}}$. We use 60 points of discretization in each space direction. For the time discretization we used a classical centered-difference implicit scheme, and the CFL number is $\alpha = 1/20$.

Figure 1 shows the norm decay of the solution of the discretized wave equation corresponding to (5.1)–(5.3), with the control u_h given by (1.17).

Figure 2 displays the norm of the solution of the controlled discretized wave equation, corresponding to (5.1)–(5.3), at the time τ for different values of N used in calculus of $\left[\begin{smallmatrix} w_{0h} \\ w_{1h} \end{smallmatrix}\right]$.

5.2. The Euler–Bernoulli beam equation. This subsection is dedicated to the problem of the approximation of an internal distributed exact control for the Euler–Bernoulli beam equation.

Let $\Omega = (0,1)$, and let $\mathcal{O} \subset \Omega$ be an open and nonempty interval included in Ω. We consider the problem

$$\ddot{q}(x,t) + \frac{\partial^4 q}{\partial x^4}(x,t) + \chi_{\mathcal{O}}(x)u(x,t) = 0, \quad (x,t) \in \Omega \times [0,\tau], \tag{5.16}$$

$$q(0,t) = \frac{\partial^2 q}{\partial x^2}(0,t) = q(1,t) = \frac{\partial^2 q}{\partial x^2}(1,t) = 0, \quad t \in [0,\tau], \tag{5.17}$$

$$q(x,0) = q_0(x), \quad \dot{q}(x,0) = q_1(x), \quad x \in \Omega, \tag{5.18}$$

modeling a beam hinged at both ends with a control u applied in an internal region. We denote by $\chi_{\mathcal{O}} \in \mathcal{D}(\Omega)$ a positive function which satisfies $\chi_{\mathcal{O}}(x) = 1$ for every $x \in \mathcal{O}$. It is well known (see, for instance, [31, Example 6.8.3]) that the system (5.16)–(5.18) is exactly controllable in any time $\tau > 0$.

In order to apply the method described in this paper we need to choose appropriate spaces and operators. Let $H = L^2(\Omega)$, $U = H$, and consider the operator

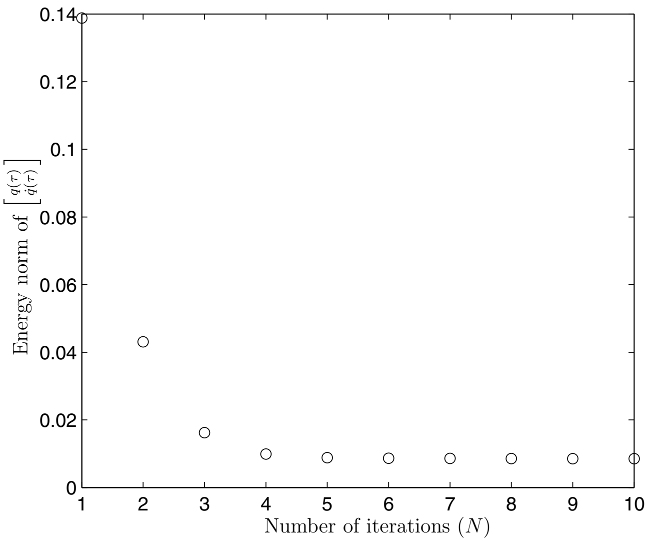

FIG. 2. *The energy of the controlled wave equation solution at time τ versus the number of terms N in the approximation of the control u_h.*

$A_0 : \mathcal{D}(A_0) \to H$, defined by

$$\mathcal{D}(A_0) = \left\{ \varphi \in \mathcal{H}^4(\Omega) \;\middle|\; \varphi(0) = \frac{d^2\varphi}{dx^2}(0) = \varphi(1) = \frac{d^2\varphi}{dx^2}(1) = 0 \right\},$$

$$A_0\varphi = \frac{d^4\varphi}{dx^4} \qquad (\varphi \in \mathcal{D}(A_0)).$$

It is well known that A_0 is a self-adjoint, strictly positive operator with compact resolvents. The corresponding spaces $H_{\frac{3}{2}}$, H_1, and $H_{\frac{1}{2}}$ introduced in section 1 are now given by

$$H_{\frac{3}{2}} = \left\{ \varphi \in \mathcal{H}^6(\Omega) \;\middle|\; \varphi(0) = \varphi(1) = \frac{d^2\varphi}{dx^2}(0) = \frac{d^2\varphi}{dx^2}(1) = \frac{d^4\varphi}{dx^4}(0) = \frac{d^4\varphi}{dx^4}(1) = 0 \right\},$$

$$H_1 = \left\{ \varphi \in \mathcal{H}^4(\Omega) \;\middle|\; \varphi(0) = \frac{d^2\varphi}{dx^2}(0) = \varphi(1) = \frac{d^2\varphi}{dx^2}(1) = 0 \right\},\ H_{\frac{1}{2}} = \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega).$$

As in the case of the wave equation, the control operator $B_0 \in \mathcal{L}(H)$ is defined by $B_0 u = \chi_{\mathcal{O}} u$ for every $u \in H$. Clearly B_0 is self-adjoint and $B_0 \in \mathcal{L}(H_1, H_{\frac{1}{2}})$.

To construct an approximating family of spaces $(V_h)_{h>0}$ we consider a uniform discretization $\mathcal{I}_h$ of the interval $(0, 1)$ formed by $\mathcal{N}$ points and $h = 1/(\mathcal{N} - 1)$. For each $h > 0$ we define V_h by

$$V_h = \{\varphi \in C^1([0, 1]) \mid \varphi_{|I} \in P_3(T) \text{ for every } I \in \mathcal{I}_h, \quad \varphi(0) = \varphi(1) = 0\},$$

where $P_3(I)$ is the set of polynomial functions of degree 3 on I. Note that V_h is the cubic Hermite finite element space. Denoting by π_h the orthogonal projector from $H_{\frac{1}{2}}$

to V_h and applying Theorem 3.3 from Strang and Fix [27, p. 144] we obtain estimates (1.4) and (1.5) with $\theta = 2$.

The method described by (1.11)–(1.21) reduces to the computation of the families of functions $(w_h^n)_{1\leqslant n\leqslant N(h)+1}$, $(w_{b,h}^n)_{1\leqslant n\leqslant N(h)}$ satisfying, for every $v_h \in V_h$,

$$\langle \ddot{w}_h^n(t), v_h\rangle + \left\langle \frac{\partial^2 w_h^n}{\partial x^2}(t), \frac{\mathrm{d}^2 v_h}{\mathrm{d}x^2}\right\rangle + \langle \chi_{\mathcal{O}}^2 \dot{w}_h^n(t), v_h\rangle = 0 \qquad (t\in[0,\tau]), \tag{5.19}$$

$$w_h^n(0) = \begin{cases} \pi_h q_0 & \text{if } n = 1,\\ w_{b,h}^{n-1}(0) & \text{if } 1 < n \leqslant N(h)+1,\end{cases} \tag{5.20}$$

$$\dot{w}_h^n(0) = \begin{cases} \pi_h q_1 & \text{if } n = 1,\\ \dot{w}_{b,h}^{n-1}(0) & \text{if } 1 < n \leqslant N(h)+1,\end{cases} \tag{5.21}$$

and

$$\langle \ddot{w}_{b,h}^n(t), v_h\rangle + \left\langle \frac{\partial^2 w_{b,h}^n}{\partial x^2}(t), \frac{\mathrm{d}^2 v_h}{\mathrm{d}x^2}\right\rangle - \langle \chi_{\mathcal{O}}^2 \dot{w}_{b,h}^n(t), v_h\rangle = 0 \qquad (t\in[0,\tau]), \tag{5.22}$$

$$w_{b,h}^n(\tau) = w_h^n(\tau), \qquad \dot{w}_{b,h}^n(\tau) = \dot{w}_h^n(\tau). \tag{5.23}$$

The second part of the method described in (1.11)–(1.21) reduces to the computation of w_{0h} and w_{1h} defined by

$$\begin{bmatrix} w_{0h}\\ w_{1h}\end{bmatrix} = \begin{bmatrix} \pi_h q_0\\ \pi_h q_1\end{bmatrix} + \sum_{n=1}^{N(h)} \begin{bmatrix} w_{b,h}^n(0)\\ \dot{w}_{b,h}^n(0)\end{bmatrix}. \tag{5.24}$$

Finally, the approximation u_h of the exact control u is given by

$$u_h = \chi_{\mathcal{O}} \dot{w}_h + \chi_{\mathcal{O}} \dot{w}_{b,h}, \tag{5.25}$$

where w_h and $w_{b,h}$ are the solution of

$$\langle \ddot{w}_h(t), v_h\rangle + \left\langle \frac{\partial^2 w_h}{\partial x^2}(t), \frac{\mathrm{d}^2 v_h}{\mathrm{d}x^2}\right\rangle + \langle \chi_{\mathcal{O}}^2 \dot{w}_h(t), v_h\rangle = 0 \qquad (t\in[0,\tau]), \tag{5.26}$$

$$w_h(0) = w_{0h}, \qquad \dot{w}_h(0) = w_{1h}, \tag{5.27}$$

$$\langle \ddot{w}_{b,h}(t), v_h\rangle + \left\langle \frac{\partial^2 w_{b,h}}{\partial x^2}(t), \frac{\mathrm{d}^2 v_h}{\mathrm{d}x^2}\right\rangle - \langle \chi_{\mathcal{O}}^2 \dot{w}_{b,h}(t), v_h\rangle = 0 \qquad (t\in[0,\tau]), \tag{5.28}$$

$$w_{b,h}(\tau) = w_h(\tau), \qquad \dot{w}_{b,h}(\tau) = \dot{w}_h(\tau). \tag{5.29}$$

From Theorem 1.1 we obtain that (u_h) converges in $C([0,\tau]; L^2(\Omega))$ to an exact control u such that

$$\|u - u_h\|_{C([0,\tau];L^2(\Omega))} \leqslant C h^2 \ln^2(h^{-1}) (\|q_0\|_{\mathcal{H}^6(\Omega)} + \|q_1\|_{\mathcal{H}^4(\Omega)}) \qquad (0 < h < h^*) \tag{5.30}$$

for some constants h^*, $C > 0$.

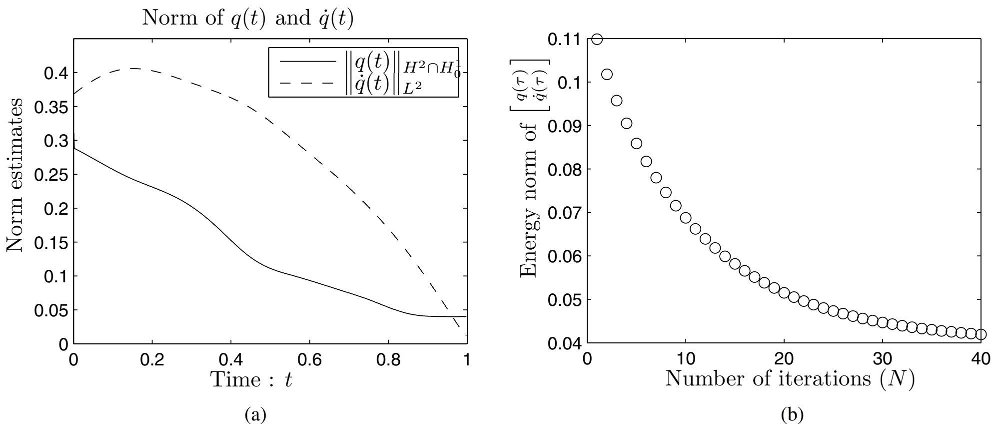

FIG. 3. (a) *The norm of the solution of the controlled beam equation, with* $u = u_h$ *and initial state* $q_0(x) = x^5(1-x)^5$, $q_1(x) = -q_0(x)$. (b) *The energy of the solution of the controlled beam at time* τ *versus the number of terms* N *in the approximation of* u_h.

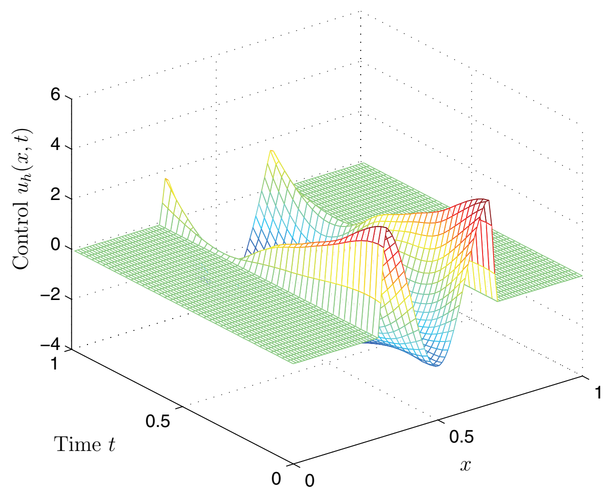

FIG. 4. *The approximation* u_h *with initial state* $q_0(x) = x^5(1-x)^5$, $q_1(x) = -q_0(x)$ *and control time* $\tau = 1$.

We tested the algorithm in the case $\mathcal{O} = (\frac{1}{3}, \frac{2}{3})$, and the initial data that we want to steer to zero are $q_0(x) = x^5(1-x)^5$, $q_1(x) = -q_0(x)$ and the control time is $\tau = 1$. Note that $\left[\begin{smallmatrix} q_0 \\ q_1 \end{smallmatrix}\right] \in H_{\frac{3}{2}} \times H_1$. We used $\mathcal{N} = 100$ discretization points in space, and in time we used an implicit centered-difference scheme with the CFL number equal to 0.1.

Figure 3(a) shows the norm decay of the solution of the discretized beam equation corresponding to (5.16)–(5.18), with the control u_h given by (1.17). Figure 3(b) displays the dependence of the norm of the solution of (5.16)–(5.18), at time τ, on the number N of terms used in the calculus of $\left[\begin{smallmatrix} w_{0h} \\ w_{1h} \end{smallmatrix}\right]$.

Figure 4 gives the form of the approximate control u_h corresponding to the initial data $\left[\begin{smallmatrix} q_0 \\ q_1 \end{smallmatrix}\right]$ given above.

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