ESTIMATES OF THE CONSTANTS IN GENERALIZED INGHAM'S INEQUALITY AND APPLICATIONS TO THE CONTROL OF THE WAVE EQUATION

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Abstract

We are considering Ingham's inequality (see [8]) for a family of exponential functions $\{e^{i\lambda_n t}\}_{n\geq 1}$, in the case in which the distance $|\lambda_{n+1}-\lambda_n|$ between two consecutive exponents becomes smaller and smaller for $|n|\leq N$ but there still exists an asymptotic gap sufficiently large. We give explicit estimates for the two constants appearing in the inequality and we analyze how does the small gap between the first exponents affect the constants. These results are applied to a control problem for the wave equation in a case in which the geometric condition for controllability, deduced in [4], are not satisfied.

1 Introduction

It is well known that, if $\theta_n = \frac{2n\pi}{T}$ and $f = f(t) = \sum_{n\in\mathbb{Z}} a_n e^{i\theta_n t}$, the following relationship between the coefficients a_n and the function f holds for any sequence $(a_n)_n \in \ell^2$

$$(1) \qquad \frac{1}{T}\int_0^T \left|\sum_{n\in\mathbb{Z}} a_n e^{i\theta_n t}\right|^2 dt = \sum_{n\in\mathbb{Z}} \mid a_n \mid^2 .$$

Formula (1) is the well-known Parseval's identity and plays a fundamental role in orthogonal expansions. In fact $\{e^{i\theta_n t}\}_{n\in\mathbb{Z}}$ forms an orthogonal basis in $L^2(0,T)$.

Parseval's identity has been generalized in many ways for different exponential families. For instance, if $\{e^{i\lambda_n t}\}_{n\geq 1}$ forms a Riesz basis in $L^2(0,T)$, there exist two positive constants C_1 and C_2 such that

$$(2) \qquad C_1\sum_{n\geq 1} \mid a_n \mid^2 \leq \int_0^T \left|\sum_{n\geq 1} a_n e^{i\lambda_n t}\right|^2 dt \leq C_2 \sum_{n\geq 1} \mid a_n \mid^2,$$

for all $(a_n)_n \in l^2$.

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We recall that a Riesz basis is a basis obtained from an orthonormal basis by means of a bounded invertible operator (see [17]). The Riesz basis constitute the most tractable class of bases known.

A Riesz basis is a complete sequence but (2) is also true for incomplete sequences of exponentials. For instance, if the exponential family $\{e^{i\lambda_n t}\}_{n\geq 1}$ has the property that

$$|\lambda_{n+1}-\lambda_n| > \frac{2\pi}{T}, \quad \forall n \geq 1, \tag{3}$$

then (2) is still true. This result was proved by Ingham in [8] and was intensively used in the last time (see [7], [12] and [14]).

Condition (3) ensures that the sequence $\{e^{i\lambda_n t}\}_{n\geq 1}$ has infinite deficiency in $L^2(0,T)$. If we add any finite number of different exponentials, say N, (2) still holds. In fact, a sufficient condition for (2) is

$$\limsup_n |\lambda_{n+1}-\lambda_n| > \frac{2\pi}{T}. \tag{4}$$

These results are included in Kahane's paper [11]. Later on Ball and Slemrod in [3] studied a similar problem and Haraux gave in [7] a proof by recurrence on the number N of the exponents which do not satisfy (3). So, one can show that there are two constants $C_1(N,T)$ and $C_2(N,T)$ such that

$$C_1(N,T)\sum_{n\geq 1} | a_n |^2 \leq \int_0^T \left| \sum_{n\geq 1} a_n e^{i\lambda_n t}\right|^2 dt \leq C_2(N,T)\sum_{n\geq 1} | a_n |^2 . \tag{5}$$

Many generalizations of Ingham's inequality have been proved in the last years for different distributions of the eigenvalues λ_n. For instance, the "gap" condition has been weakened in [16], [5] and [10]. A generalization of these results was proved in [2] where, instead of (4), it is assumed that there exists an integer $M\geq 1$ such that

$$\lambda_{n+M}-\lambda_n \geq M\gamma \text{ for all } n. \tag{6}$$

It is proved that, for $T > \frac{2\pi}{\gamma}$, there exist two constants $C_1(M,T)$ and $C_2(M,T)$ such that

$$C_1(M,T)\sum_{n\geq 1} | a_n |^2 \leq \int_0^T \left| \sum_{n\geq 1} a_n e_n(t)\right|^2 dt \leq C_2(M,T)\sum_{n\geq 1} | a_n |^2, \tag{7}$$

where $e_n(t) = e^{i\lambda_n t}$ only if λ_n is a separated exponent. In the other cases e_n is obtained as divided difference of the exponential functions. We also remark that (6) allows to have repeated exponents.

Most of these results were used to solve interesting control applications.

In the second section of the paper we use the ideas of the proof of Haraux from [7] to obtain explicit estimates for the constants $C_1 = C_1(N,T)$ and $C_2 = C_2(N,T)$. Although the method we use is well known now (for instance it is also used in [2]), we include it in order to make our paper self-contained. Moreover, we obtain a recurrence formula for the constant $C_1 = C_1(N,T)$ which allows us to say easily how does this constant change with N. Generally, the constants C_1 and C_2 degenerate when N tends to infinity, the first one much more rapidly than the second. We shall estimate the behavior of C_1 and C_2 as N goes to infinity in some particular cases.

Exponential families with the property (4) appear in many problems from PDE. Let us consider, for instance, the eigenvalues of the wave equation in the unit square, $i\,\lambda_n^k$, with $\lambda_n^k =$

$\mathrm{sgn}(n)\sqrt{n^2+k^2}\pi$. For each $k\geq 0$, the sequence $(\lambda_n^k)_{n\in\mathbb{Z}^*}$ does not satisfy (3) but satisfies (4), if $T>2$. Therefore, for each $k\geq 0$, inequality (5) is verified with constants depending on k. To know how do the constant C_1 and C_2 change with k turns to be an important question when we want to characterize classes of initial data, by using some properties of the coefficients of their Fourier decomposition. By using the recurrence formula, we prove that, in this particular case, $C_1(k,T)$ decays exponentially fast with k whereas $C_2(k,T)$ increases polynomially with k.

In [1] a method based on the moments theory was used to study the behavior of the constants C_1 and C_2 from (5) for the case in which the exponents λ_n are the eigenvalues of a wave equation in a domain with some symmetry which allows separation of variables. Explicit estimates where also obtained for the dependence of the two constants on T. The results we obtain are somehow more general and can be applied to a larger class of exponential families.

An interesting question, which completes the study of the inequality (5), is the following: does the constant $C_1(N,T)$ in (5) (which is generally very small) really multiply all the coefficients a_n? The main result of the paper is given in the third section where we show that there are a constant c_1, not depending on N, and a range $I_1(N)$ such that

$$C_1(N,T)\sum_{n\leq I_1(N)}|a_n|^2+c_1\sum_{n>I_1(N)}|a_n|^2\leq\int_0^T\left|\sum_{n\geq 1}a_ne^{i\lambda_nt}\right|^2dt. \tag{8}$$

This result (see Theorem 4, Section 3) indicates that the high eigenmodes are not sensibly affected by the distribution of the low ones. A similar property is also true for the second inequality of (5).

To prove (8) we use a perturbation technique introduced in [9] which allows to estimate the elements of the inverse of a matrix whose entries decay exponentially away from the diagonal.

In the fourth section we shall apply the previous results to the problem of controllability of the wave equation in the square $\Omega=(0,1)\times(0,1)$ by acting only on the face of the boundary $\Gamma_0=\{0\}\times(0,1)$. More precisely, given $T>2$ we are interested in characterizing the space of initial data (u^0,u^1) with the property that there exists a control $v\in L^2(\Gamma_0\times(0,T))$ such that the solution u of

$$\begin{cases} u_{tt}-\Delta u=0 & \text{in } (0,\infty)\times\Omega\\ u=0 & \text{on } (0,\infty)\times(\partial\Omega\setminus\Gamma_0)\\ u=v & \text{on } (0,\infty)\times\Gamma_0\\ u(0)=u^0,\ u_t(0)=u^1 & \text{in } \Omega\end{cases} \tag{9}$$

satisfies $u(T)=u_t(T)=0$.

In this case the geometric condition for controllability, deduced in [4], are not satisfied. Hence, no Sobolev space of initial data is controllable to zero.

The eigenvalues of this problem are $\lambda_n^k=\mathrm{sgn}(n)\sqrt{n^2+k^2}\pi$ which where analyzed above and for which an inequality like (8) can be proven. Moreover, the corresponding eigenfunctions are of the type $\varphi_n^k(x)\sin(k\pi y)$ where $\varphi_n^k(x)$ is a vectorial function of two components.

By using separation of variables, inequality (8) and Hilbert Uniqueness Method (see [13]) we show that the space of of controllable initial data contains the space

$$\mathcal{X}=\left\{(u^0,u^1)\in L^2(\Omega)\times H^{-1}(\Omega):\quad (u^0,u^1)=\sum_{k,n}a_{k,n}\varphi_n^k\sin(k\pi y),\ \text{such that}\right.$$

$$\sum_k \left(\frac{k^2}{C_1(k,T)} \sum_{1 \le |n| \le \widehat{I}_1(k)} \frac{1}{|\lambda_n^k|^2} |a_{k,n}|^2 + \sum_{|n| > \widehat{I}_1(k)} \frac{1}{|\lambda_n^k|^2} |a_{k,n}|^2 \right) < \infty \Bigg\}$$

where $C_1(k,T)$ is exponentially decreasing and $\widehat{I}_1(k)$ is exponentially increasing with k.

Note that the constant $(C_1(k,T))^{-1}$ multiplies only the Fourier coefficients with small n. The constant which multiplies the coefficients with n large enough ($n > \widehat{I}(k)$) does not depend of k. This is a direct consequence of inequality (8). Hence, we found a space of controllable functions larger that the one given in [1] in which a constant which increases with k multiplies all the coefficients $(a_{n,k})_n$.

We finally remark that similar results can be obtained for problems in several dimensions and with a symmetry that allows separation of variables.

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2 A first estimate

In this section we start from a family of exponentials $\{e^{i\lambda_n t}\}_n$ with the property that there exist two constants C_1 and C_2 such that

$$C_1 \sum_{n\ge1} |\, a_n \,|^2 \le \int_0^T \left| \sum_{n\ge1} a_n e^{i\lambda_n t} \right|^2 dt \le C_2 \sum_{n\ge1} |\, a_n \,|^2 \tag{10}$$

and we want to see how do the constants C_1 and C_2 change when N new eigenmodes $e^{i\lambda_n t}$ are introduced.

First of all let us recall two classical results due to A.E. Ingham (see [8]).

THEOREM 1 *(Ingham [8], Theorem 1) Let* $f = f(t) = \sum_{n\ge1} a_n e^{i\lambda_n t}$ *where* $(\lambda_n)_n$ *is an increasing sequence of real numbers. We assume that there exists* $\gamma_0 > 0$ *such that*

$$\lambda_{n+1} - \lambda_n \ge \gamma_0, \ \forall n \ge 1. \tag{11}$$

Let $J = [0,T]$ *with* $T\gamma_0 - 2\pi = \alpha > 0$*. Then, there exists a positive constant* $C_1^0 = C_1^0(\alpha) = \dfrac{2\alpha(4\pi+\alpha)}{\pi(2\pi+\alpha)^2} T$ *such that, for all* $(a_n)_n \in \ell^2$,

$$C_1^0 \sum_{n\ge1} |\, a_n \,|^2 \le \int_J |\, f(t) \,|^2 \, dt. \tag{12}$$

THEOREM 2 *(Ingham [8], Theorem 2) Let* $f = f(t) = \sum_{n\ge1} a_n e^{i\lambda_n t}$ *where* $(\lambda_n)_n$ *is an increasing sequence of real numbers. We assume that there exists* $\gamma_0 > 0$ *such that*

$$\lambda_{n+1} - \lambda_n \ge \gamma_0, \ \forall n \ge 1. \tag{13}$$

Let $J = [0, T]$ with $T\gamma_0 = \tau > 0$. Then, there exists a positive constant $C_2^0 = C_2^0(\gamma_0, T) = \frac{10}{min\{\pi,\tau\}} T$ such that, for all $(a_n)_n \in \ell^2$,

$$\int_J | f(t) |^2 \, dt \leq C_2^0 \sum_{n\geq 1} | a_n |^2 . \tag{14}$$

Remark 1 *Observe that the constant C_1^0 depends on $T - \frac{2\pi}{\gamma_0}$ and T while C_2^0 depends on γ_0 and T. For the first inequality T has to be greater than $\frac{2\pi}{\gamma_0}$ whereas for the second one there is no restriction on T.*

We prove now the following refined version of a result by A. Haraux (see [7]) on non-harmonic Fourier series.

THEOREM 3 *Let $f = f(t) = \sum_{n\geq 1} a_n e^{i\lambda_n t}$ where $(\lambda_n)_n$ is an increasing sequence of real numbers. We assume that there exist $N \geq 1$, $\gamma > 0$ and $\gamma_\infty > 0$ such that*

$$\lambda_{n+1} - \lambda_n \geq \gamma_\infty > 0 \text{ if } n > N, \tag{15}$$

$$\lambda_{n+1} - \lambda_n \geq \gamma > 0 \text{ for any } n \geq 1. \tag{16}$$

Let $J = [0, T] \subset \mathbb{R}$ be a finite interval with $T > \frac{2\pi}{\gamma_\infty}$. Then, there exist two positive constants $C_1, C_2 > 0$ such that, for all $(a_n)_n \in \ell^2$,

$$C_1 \sum_{n\geq 1} | a_n |^2 \leq \int_J | f(t) |^2 \, dt \leq C_2 \sum_{n\geq 1} | a_n |^2 . \tag{17}$$

More precisely $C_2 = C_2(\gamma) = \dfrac{10\,T}{min\{\pi, T\gamma\}}$ and $C_1 = C_1(N)$ is given by the following recurrent formula:

$$C_1(j) = \left[\left(\frac{2C_2(r_j)}{T} + 1 \right) \frac{4}{C_1(j-1)p_j} + \frac{2}{T} \right]^{-1}, \quad 1 \leq j \leq N. \tag{18}$$

where $r_j = min\{\lambda_{m+1} - \lambda_m : m \geq N - j + 1\}$, $p_j = min\left\{1, \frac{4r_j^2}{\pi^4}\left(T - \frac{2\pi}{\gamma_\infty}\right)^2\right\}$ with $0 \leq j \leq N$.

Remark 2 *When $\gamma_\infty = \gamma$, the sequence of Theorem 3 satisfies $\lambda_{n+1} - \lambda_n \geq \gamma_\infty > 0$, $\forall n \geq 1$ and we are in the case of Theorems 1 and 2.*

Theorem 3 allows to deduce that (29) holds when the length of the interval J is smaller. Indeed, it suffices $| J | > 2\pi/\gamma_\infty$, γ_∞ being the "asymptotic gap" of the sequence $\{\lambda_n\}$, which is in general larger than γ.

Note that the existence of constants C_1 and C_2 is a consequence of Kahane's theorem (see [11]). However, our purpose is to have an explicit control on the size of C_1 in terms of γ, γ_∞ and N.

In order to do this a constructive argument of Haraux (see [7]) will be used.

Remark 3 *Observe that $r_N = \gamma$ and $r_0 = \gamma_\infty$. In most interesting cases, coming from PDE, the sequence $(r_n)_{0\le n\le N}$ is decreasing and $C_2(r_n)$ is increasing.*

Proof of Theorem 3: To prove (17) we follow the ideas of Haraux [7], paying special attention to the evaluation of the constants appearing there.

The second inequality of (17) results immediately by using Theorem 2 with $\gamma_0 = \gamma$. We remark that $C_2 = C_2(r_N) = \dfrac{10\,T}{\min\{\pi, T\,r_N\}}$ depends only on r_N, the gap of the family $(\lambda_n)_n$ and the length T of the interval J.

We pass now to prove the first inequality of (17).

We begin with the function $f_0(t) = \sum_{n>N} a_n e^{i\lambda_n t}$ and we add one by one the exponentials $e^{i\,\lambda_n t}$ for $n = N, N-1, ..., 1$.

Firstly, since (16) holds, it follows by Theorems 1 and 2 that

$$C_1^0 \sum_{n>N} \mid a_n \mid^2 \le \int_J \mid f_0(t) \mid^2 dt \le C_2^0 \sum_{n>N} \mid a_n \mid^2 . \tag{19}$$

Let now $f_1(t) = f_0 + a_N e^{i\,\lambda_N t} = \sum_{n>N} a_n e^{i\lambda_n t} + a_N e^{i\,\lambda_N t}$. Without loss of generality we may suppose that $\lambda_N = 0$ (since we can consider the function $f_1(t)e^{-i\lambda_N t}$ instead of $f_1(t)$).

Let $\varepsilon > 0$ be such that $T' = T - \varepsilon > \frac{2\pi}{\gamma_\infty}$. We have

$$\int_0^\varepsilon (f_1(t+\eta) - f_1(t))\, d\eta = \sum_{n>N} a_n \left(\frac{e^{i\lambda_n\varepsilon} - 1}{i\lambda_n} - \varepsilon \right) e^{i\lambda_n t}, \quad \forall t \in [0, T'].$$

Applying now Theorem 1 to the function $h(t) = \displaystyle\int_0^\varepsilon (f_1(t+\eta) - f_1(t))\, d\eta$ we obtain that:

$$C_1^0 \sum_{n>N} \left| \frac{e^{i\lambda_n\varepsilon} - 1}{i\lambda_n} - \varepsilon \right|^2 |a_n|^2 \le \int_0^{T'} \left| \int_0^\varepsilon (f_1(t+\eta) - f_1(t))\, d\eta \right|^2 dt. \tag{20}$$

We evaluate now the coefficients $\dfrac{e^{i\lambda_n\varepsilon} - 1}{i\lambda_n} - \varepsilon$. We have:

$$\left| e^{i\lambda_n\varepsilon} - 1 - i\lambda_n\varepsilon \right|^2 = |\cos(\lambda_n\varepsilon) - 1|^2 + |\sin(\lambda_n\varepsilon) - \lambda_n\varepsilon|^2 =$$

$$= 4\sin^4\left(\frac{\lambda_n\varepsilon}{2}\right) + (\sin(\lambda_n\varepsilon) - \lambda_n\varepsilon)^2 \ge \begin{cases} 4\left(\frac{\lambda_n\varepsilon}{\pi}\right)^4, & \text{if } |\lambda_n|\varepsilon \le \pi \\ (\lambda_n\varepsilon)^2, & \text{if } |\lambda_n|\varepsilon > \pi. \end{cases}$$

Finally, taking into account that $|\lambda_n| \ge r_1$, we obtain that,

$$\left| \frac{e^{i\lambda_n\varepsilon} - 1}{i\lambda_n} - \varepsilon \right|^2 \ge p_1\varepsilon^2.$$

We return now to (20) and we get that:

$$p_1\varepsilon^2 C_1^0 \sum_{n>N} |a_n|^2 \le \int_0^{T'} \left| \int_0^\varepsilon (f_1(t+\eta) - f_1(t))\, d\eta \right|^2 \tag{21}$$

On the other hand

$$\int_0^{T'} \left| \int_0^{\varepsilon} (f_1(t+\eta) - f_1(t))\, d\eta \right|^2 \leq \int_0^{T'} \varepsilon \int_0^{\varepsilon} |f_1(t+\eta) - f_1(t)|^2 \, d\eta \leq$$

$$\leq 2\varepsilon \int_0^{T'} \int_0^{\varepsilon} \left(|f_1(t+\eta)|^2 + |f_1(t)|^2 \right) d\eta \leq 2\varepsilon^2 \int_0^{T} |f_1(t)|^2 +$$

$$+2\varepsilon \int_0^{\varepsilon} \int_0^{T'} |f_1(t+\eta)|^2 \, dt \, d\eta = 2\varepsilon^2 \int_0^{T} |f_1(t)|^2 + 2\varepsilon \int_0^{\varepsilon} \int_{\eta}^{T'+\eta} |f_1(\tau)|^2 \, d\tau \, d\eta \leq$$

$$\leq 2\varepsilon^2 \int_0^{T} |f_1(t)|^2 + 2\varepsilon \int_0^{\varepsilon} \int_0^{T} |f_1(\tau)|^2 \, d\tau \, d\eta \leq 4\varepsilon^2 \int_0^{T} |f_1(t)|^2 \, .$$

From (21) it follows that

$$\sum_{n>N} |a_n|^2 \leq \frac{4}{p_1 C_1^0} \int_0^T |f_1(t)|^2 \, . \tag{22}$$

Observe that:

$$|a_N|^2 = \left| f_1(t) - \sum_{n>N} a_n e^{i\lambda_n t} \right|^2 = \frac{1}{T} \int_0^T \left| f_1(t) - \sum_{n>N} a_n e^{i\lambda_n t} \right|^2 dt \leq$$

$$\leq \frac{2}{T} \left(\int_0^T |f_1(t)|^2 + \int_0^T \left| \sum_{n>N} a_n e^{i\lambda_n t} \right|^2 \right) \leq \frac{2}{T} \left(\int_0^T |f_1(t)|^2 + C_2^0 \sum_{n>N} |a_n|^2 \right) \leq$$

$$\leq \left(\frac{2}{T} + \frac{8 C_2^0}{T p_1 C_1^0} \right) \int_0^T |f_1(t)|^2 \, .$$

From (22) we get that

$$\sum_{n \geq N} |a_n|^2 \leq \left[\frac{4}{p_1 C_1^0} \left(\frac{2 C_2^0}{T} + 1 \right) + \frac{2}{T} \right] \int_0^T |f_1(t)|^2 \, .$$

We obtain the desired result and a recurrent formula to compute the constant $C_1(1)$:

$$C_1(1) = \left[\frac{4}{p_1 C_1^0} \left(\frac{2 C_2^0}{T} + 1 \right) + \frac{2}{T} \right]^{-1} .$$

Let us remark that the following inequality holds for f_1:

$$\int_0^T |f_1(t)|^2 \leq C_2^1 \sum_{n \geq N} |a_n|^2$$

where $C_2^1 = C_2(r_1) = \dfrac{10\,T}{\min\{\pi, T\, r_1\}}$. Observe that, if $(r_n)_n$ is decreasing, $C_2^1 \leq C_2 = C_2(r_N) = \dfrac{10\,T}{\min\{\pi, T\, r_N\}}$.

Next step consists of introducing the term $a_{N-1} e^{i\lambda_{N-1} t}$.

Let now $f_2(t) = f_1 + a_{N-1}e^{i\,\lambda_{N-1}t} = \sum_{n>N-1} a_n e^{i\lambda_n t} + a_{N-1}e^{i\,\lambda_{N-1}t}$.
We repeat the whole argument and obtain that

$$\sum_{n\geq N-1} |a_n|^2 \leq \left[\frac{4}{p_2 C_1(1)}\left(\frac{2C_2^1}{T}+1\right)+\frac{2}{T}\right]\int_0^T |f_2(t)|^2\,,$$

where $C_2^1 = C_2(r_1) = \dfrac{10\,T}{\min\{\pi, T\,r_1\}}$.

Step by step we finish introducing all the terms of f. Remark that, in each step, a different constant C_2 is used.

■

Remark 4 *In order to see how does C_1 change with N some new estimates are necessary. We have*

$$(23)\qquad \begin{aligned}(C_1(N))^{-1} &\leq \frac{2C_2^N}{T}\frac{4}{p_N}(C_1(N-1))^{-1} \leq \ldots \leq\\ &\leq \left(\frac{8}{T}\right)^N\left(\prod_{1\leq n\leq N}\frac{C_2^{n-1}}{p_n}\right)(C_1^0)^{-1}.\end{aligned}$$

Recall that $C_2^n = C_2(r_n) = \dfrac{10\,T}{min\{\pi, T\,r_n\}}$ and $p_n = min\left\{1, \dfrac{4r_n^2}{\pi^4}\left(T-\dfrac{2\pi}{\gamma_\infty}\right)^2\right\}$.

Let us remark that, if $C_2^n = \dfrac{10}{r_n}$ and $p_n = \dfrac{4r_n^2}{\pi^4}\left(T-\dfrac{2\pi}{\gamma_\infty}\right)^2$, then

$$(24)\qquad (C_1(N))^{-1} \leq \left(\frac{20\pi^4}{T\left(T-\frac{2\pi}{\gamma_\infty}\right)^2}\right)^N\left(\prod_{1\leq n\leq N}\frac{1}{(r_n)^3}\right)(C_1^0)^{-1}C_2^0(C_2^N)^{-1}.$$

Remark 5 *Let us evaluate the constant $C_1(N)$ in the case $\lambda_n = \sqrt{n^2+k^2}\pi$ with $k>0$ case which will be especially important in the last section.*

If $0<\delta<\pi$ is an arbitrary number then we can choose $\gamma_\infty = \pi-\delta$. Indeed we have

$$|\lambda_{n+1}-\lambda_n| = \frac{(2n+1)\pi}{\sqrt{(n+1)^2+k^2}+\sqrt{n^2+k^2}} \geq \frac{(2n+1)\pi}{2k+2n+1} \geq \frac{n\pi}{n+k} \geq \pi-\delta = \gamma_\infty$$

when $n > N = \left[k\dfrac{\pi-\delta}{\delta}\right]$. Observe that $k\dfrac{\pi-\delta}{\delta}-1 \leq N \leq k\dfrac{\pi-\delta}{\delta}$.

In order to obtain more explicit results let us consider $T>2$ small enough such that $C_2^n = C_2(r_n) = \dfrac{10\,T}{T\,r_n}$ and $p_n = \dfrac{4r_n^2}{\pi^4}\left(T-\dfrac{2\pi}{\gamma_\infty}\right)^2$, $n = 0, 1, ..., N$.

We have that $r_n = \lambda_{n+1}-\lambda_n \geq \dfrac{n\pi}{n+k}$. It follows that

$$(C_1(N))^{-1} \leq \left(\frac{8}{T}\right)^N\left(\prod_{1\leq n\leq N}\frac{C_2^{n-1}}{p_n}\right)(C_1^0)^{-1} =$$

$$= \left(\frac{8}{T}\right)^N \left(\prod_{1\le n\le N} \frac{C_2^n}{p_n}\right) (C_1^0)^{-1} C_2^0 (C_2^N)^{-1} =$$

$$= \left(\frac{8}{T}\right)^N \left(\prod_{1\le n\le N} \frac{10\,\pi^4}{4(r_n)^3 \left(T-\frac{2\pi}{\gamma_\infty}\right)^2}\right) (C_1^0)^{-1} C_2^0 \frac{r_N}{10}.$$

Since $\sum_{k=0}^{N} \left(\begin{array}{c} N \\ k \end{array}\right) = 2^N$ *a rough estimate gives*

$$\prod_{1\le n\le N} (r_n)^{-1} \le \prod_{1\le n\le N} \frac{n+k}{\pi\, n} = \frac{1}{\pi^N} \left(\begin{array}{c} N \\ k \end{array}\right) \le \frac{2^N}{\pi^N}.$$

We obtain

$$(C(N))^{-1} \le \left(\frac{160\pi}{T\left(T-\frac{2\pi}{\gamma_\infty}\right)^2}\right)^N \frac{r_N}{10} (C_1^0)^{-1} C_2^0.$$

Hence we have proved that, in this case, $C_1(N)$ *decreases exponentially with* N. *A detailed discussion of the behaviour of* $C_1(N)$ *can be found in [6]*

The same argument can be used to prove the following result:

PROPOSITION 1 *Let* $(\lambda_n)_{n\ge1}$ *be a real sequence such that there exist two positive constants* C_1 *and* C_2 *with the property that*

$$C_1^0 \sum_{n\ge1} |a_n|^2 \le \int_0^T \left|\sum_{n\ge1} a_n\, e^{i\,\lambda_n t}\right|^2 dt \le C_2^0 \sum_{n\ge1} |a_n|^2 \tag{25}$$

for any sequence $(a_n)_{n\ge1} \in \ell^2$.

Consider also N *real values* $(\mu_m)_{1\le m\le N}$ *such that* $\mu_m \ne \lambda_n$, *for all* $n\ge1$ *and* $1\le m\le N$.

Then, for each $\varepsilon>0$ *there exist two positive constants* C_1 *and* C_2 *such that*

$$C_1 \left(\sum_{n\ge1} |a_n|^2 + \sum_{1\le m\le N} |b_m|^2\right) \le$$

$$\int_0^{T+\varepsilon} \left|\sum_{n\ge1} a_n\, e^{i\,\lambda_n t} + \sum_{1\le m\le N} b_n\, e^{i\,\mu_n t}\right|^2 dt \le \tag{26}$$

$$\le C_2 \left(\sum_{n\ge1} |a_n|^2 + \sum_{1\le m\le N} |b_m|^2\right)$$

for any sequence $(a_n)_{n\ge1} \in \ell^2$ *and* $(b_m)_{1\le m\le N} \subset \mathbb{C}^N$.

Moreover, we can choose

$$C_2 = \frac{10\,(T+\varepsilon)}{min\,\{\pi, (T+\varepsilon)\,min\,\{r_j,\, j=0,1,...,N\}\}}$$

$$(C_1)^{-1} = \left(\frac{8}{T+\varepsilon}\right)^N \left(\prod_{1\le j\le N} \frac{10(T+\varepsilon)}{min\,\{\pi, (T+\varepsilon) r_{j-1}\} p_j}\right) (C_1^0)^{-1}$$

where $r_j = dist\{(\lambda_n)_{n\ge 1} \cup (\mu_m)_{1\le m\le j-1},\ \mu_j\}$ *and* $p_j = min\left\{1, \dfrac{4r_j^2\,\varepsilon^2}{\pi^4}\right\}$ *with* $0 \le j \le N$.

3 A second estimate

In this section we consider again a family of exponentials $\{e^{i\lambda_n t}\}_{n\ge 1}$ such that $(\lambda_n)_{n\ge 1}$ is an increasing sequence of real numbers with the property that there exist $N \ge 1,\ \gamma > 0$ and $\gamma_\infty > 0$ such that

(27) $$\lambda_{n+1} - \lambda_n \ge \gamma_\infty > 0 \text{ if } n > N,$$

(28) $$\lambda_{n+1} - \lambda_n \ge \gamma > 0 \text{ for any } n \ge 1.$$

Let also $f = \sum_{n\ge 1} a_n e^{i\lambda_n t}$ and suppose that $f(-t) = \overline{f(t)}$.

In this section $C_1 = C_1(N,T)$ and $C_2 = C_2(N,2T)$ are the constants given by Theorem 3. We shall suppose that $C_2(N,2T) \to \infty$ and $C_1(N,T) \to 0$ as $N \to \infty$. These conditions are fulfilled when γ tends to zero as N goes to infinity and it is the most interesting case.

Our aim is to prove the following Theorem:

Theorem 4 *Let* $\frac{2\pi}{\gamma_\infty} < T$, $p \in \mathbb{N}$, $p > 1$ *and* $C_3(N,T) = \left(\frac{C_2(N,2T)}{C_1(N,T)} \ln\left(\frac{C_2(N,2T)}{C_1(N,T)}\right)\right)^{\frac{p}{p-1}}$. *If the exponential family* $\{e^{i\lambda_n t}\}_{n\ge 1}$ *satisfies (27) and (28), then there are two constants* c_1 *and* δ*, depending only on* γ_∞ *and* T *but not depending on* N*, such that, for any range* $I_1(N) \ge \delta C_3(N,T)$*, the following inequality is verified*

(29) $$C_1(N,T) \sum_{n\le I_1(N)} |\, a_n\,|^2 + c_1 \sum_{n>I_1(N)} |\, a_n\,|^2 \le \int_0^T \left|\sum_{n\ge 1} a_n e^{i\lambda_n t}\right|^2 dt,$$

for all $(a_n)_n \in \ell^2$.

Remark 6 *In Theorem 3 we have shown that (29) is true with* $c_1 = C_1(N,T)$*. Theorem 4 shows that even if the gap between the first exponents becomes small the sufficiently high modes are not affected. From this point of view (29) improves the similar inequalities obtained in [1] and [14] where the same constant, depending on* N*, multiplies all the coefficients* a_n.

The proof of Theorem 4 will be given in the last paragraph of this section after some necessary developments. Let us explain now briefly the main ideas of the proof. First remark that

$$\int_0^T |f(t)|^2\,dt = \frac{1}{2}\int_{-T}^T |f(t)|^2\,dt = \frac{1}{2}\sum_{n,m\ge 1} a_n \overline{a}_m \int_{-T}^T e^{i(\lambda_n-\lambda_m)t}\,dt.$$

If we denote by $\mathcal{A}$ the infinite matrix with elements $\mathcal{A}(n,m) = \frac{1}{2}\int_{-T}^{T} e^{i(\lambda_n-\lambda_m)t}\, dt$ and by $\mathcal{D}$ the infinite diagonal matrix with positive diagonal elements $\mathcal{D}(n,m) = \sqrt{c_1}\delta_{nm}$ if $n > I_1(N)$ and $\mathcal{D}(n,m) = \sqrt{C_1(N)}\delta_{nm}$ if $n \leq I_1(N)$, then (29) can be written as

$$\langle \mathcal{D}v, \mathcal{D}v\rangle \leq \langle \mathcal{A}v, v\rangle, \forall v \in \ell^2. \tag{30}$$

We denote by $\langle\cdot,\cdot\rangle$ the usual inner product in ℓ^2.

Since $\mathcal{D}$ is an one by one operator in ℓ^2 (it is an infinite diagonal matrix with constant coefficients), (30) is equivalent to

$$\langle v, v\rangle \leq \langle \mathcal{D}^{-1}\mathcal{A}\mathcal{D}^{-1}v, v\rangle, \forall v \in \ell^2.$$

The last relation takes place if and only if the smallest eigenvalues of the self-adjoint matrix $\mathcal{D}^{-1}\mathcal{A}\mathcal{D}^{-1}$ is bigger than 1 or, equivalently, the norm of the matrix $\mathcal{D}\mathcal{A}^{-1}\mathcal{D}$ is less than 1.

Hence the problem consists of proving that there is a diagonal matrix $\mathcal{D}$, with positive diagonal elements, such that $\mathcal{D}(n,n)$ does not depend on N for $n \geq I_1(N)$ and $||\mathcal{D}\mathcal{A}^{-1}\mathcal{D}|| \leq 1$. In order to do this we need more information about the behavior of the elements $\mathcal{A}^{-1}(n,m)$ for $n, m > I_1(N)$. More precisely, we need to show that the elements $\mathcal{A}^{-1}(n,m)$ are uniformly bounded in N for $n, m > I_1(N)$. In order to do this we use a result proved by S. Jaffard in [9] which gives estimates for $\mathcal{A}^{-1}(n,m)$ by using the fact that the elements of the matrix $\mathcal{A}$ decay exponentially fast far away from the diagonal. This property implies that inversion is a local transformation of the matrix: a change in the coefficients is not felt far away from the changed area. Hence estimates for $\mathcal{A}^{-1}(n,m)$ can be obtained by perturbing the matrix $\mathcal{A}$ with another matrix $\mathcal{B}$ of known inverse.

Firstly we introduce a kernel function Φ and we evaluate $\int_{-T}^{T}\Phi(t)\,|f(t)|^2\, dt$ instead of $\int_{-T}^{T}|f(t)|^2\, dt$ in order to make the contribution of the terms $e^{i(\lambda_n-\lambda_m)t}$ smaller when $|n-m|$ becomes large. In the next paragraph we introduce the kernel Φ and we study its properties.

3.1 A special function

In this section we give a non-negative function, with compact support and with fast decay Fourier transform. This function will be used as kernel to evaluate the constant c_1 in (29).

Let $R > 0$ and $p \in \mathbb{N}^*$. We define the function $\Phi : \mathbb{R} \to \mathbb{R}$ by

$$\Phi(t) = \begin{cases} \exp\left[R\left(\frac{R^2}{t^2-R^2}\right)^{2p-1}\right] & \text{if} \quad |t| < R \\ 0 & \text{if} \quad |t| \geq R. \end{cases} \tag{31}$$

The function Φ belongs to $\mathcal{C}^\infty(\mathbb{R})$, is non-negative and his support is $[-R, R]$. Moreover, Φ can be extended to a function which is analytic in the ball $D = \{z \in \mathbb{C} : |z| < R\}$ and continuous in $\overline{D}$. The following property of the function Φ will be used intensively in the next paragraphs.

Proposition 2 *If $\widehat{\Phi}(x) = \displaystyle\int_{\mathbb{R}} \Phi(t)\exp(-i\,xt)\, dt$ is the Fourier transform of Φ then there exist two positive constants M and ω such that*

$$\left|\widehat{\Phi}(x)\right| \leq M\exp\left(-\omega x^q\right) \text{ for all } x \geq 0, \tag{32}$$

where $q = 1 - \frac{1}{p}$.

Proof: Let us first remark that it is sufficient to prove (32) for x large enough (say $x > x_0 > 1$, where x_0 will be fixed later on). Indeed, for $x \leq x_0$, we have

$$|\widehat{\Phi}(x)| = \left| \int_{\mathbb{R}} \Phi(t) \exp(-i\,xt)\, dt \right| \leq \int_{\mathbb{R}} \Phi(t)\, dt \leq M_1 \exp(-\omega \sqrt[p]{x^{p-1}}),$$

where $M_1 = \exp(\omega \sqrt[p]{(x_0)^{p-1}}) \int_{\mathbb{R}} \Phi(t)\, dt$.

Hence, we can suppose that $x > x_0 > 1$ where x_0 will be chosen later on. We evaluate $\widehat{\Phi}$ by changing the contour of integration.

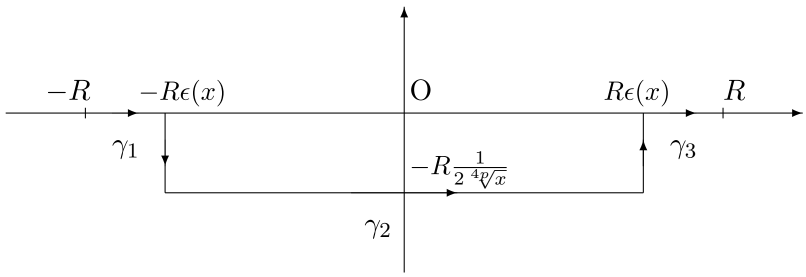

Fig. 1

Let $\epsilon(x) = \sqrt{1 - \frac{1}{\sqrt[2p]{x}}} < 1$ and define, in the complex plane, the curves (see Fig. 1):

$$\gamma_1 : \left[0, \frac{1}{2\sqrt[4p]{x}}\right] \longrightarrow \mathbb{C}, \quad \gamma_1(s) = -R\epsilon(x) - R\,s\,i, \quad s \in \left[0, \frac{1}{2\sqrt[4p]{x}}\right]$$

$$\gamma_2 : [-\epsilon(x), \epsilon(x)] \longrightarrow \mathbb{C}, \quad \gamma_2(s) = R\,s - R\frac{1}{2\sqrt[4p]{x}}\,i, \quad s \in [-\epsilon(x), \epsilon(x)]$$

$$\gamma_3 : \left[-\frac{1}{2\sqrt[4p]{x}}, 0\right] \longrightarrow \mathbb{C}, \quad \gamma_3(s) = R\epsilon(x) + R\,s\,i, \quad s \in \left[-\frac{1}{2\sqrt[4p]{x}}, 0\right].$$

Remark that, for any $x > 1$,

$$(R\varepsilon(x))^2 + \left(\frac{R}{2\sqrt[4p]{x}}\right)^2 = R^2\left(1 - \frac{1}{\sqrt[2p]{x}} + \frac{1}{4\sqrt[2p]{x}}\right) = R^2\left(1 - \frac{3}{4\sqrt[2p]{x}}\right) < R^2.$$

Hence, the curves γ_1, γ_2 and γ_3 are contained in D. Moreover, we have

$$\widehat{\Phi} = \int_{\mathbb{R}} \Phi(t) \exp(-i\,xt)\, dt = \int_{-R}^{R} \Phi(t) \exp(-i\,xt)\, dt =$$

$$= \int_{-R}^{-R\epsilon(x)} \Phi(t) \exp(-i\,xt)\, dt + \int_{\gamma_1} \Phi(z) \exp(-i\,xz)\, dz + \int_{\gamma_2} \Phi(z) \exp(-i\,xz)\, dz +$$

$$+ \int_{\gamma_3} \Phi(z) \exp(-i\,xz)\, dz + \int_{R\epsilon(x)}^{R} \Phi(t) \exp(-i\,xt)\, dt.$$

We shall evaluate each of these five integrals. We have

$$|I_1| = \left| \int_{-R}^{-R\epsilon(x)} \Phi(t) \exp(-i\,xt)\, dt \right| \leq \int_{-R}^{-R\epsilon(x)} \Phi(t)\, dt.$$

Since, for $t \le 0$, Φ is an increasing function, we obtain

$$|I_1| \le (1-\epsilon(x)) R \exp\left[R\left(\frac{R^2}{R^2(\epsilon^2(x)-1)}\right)^{2p-1}\right] \le R\exp\left[R\left(-x^{\frac{2p-1}{2p}}\right)\right] \le R\exp\left(-Rx^{\frac{p-1}{p}}\right).$$

We also have

$$|I_5| = \left|\int_{R\epsilon(x)}^{R} \Phi(t)\exp(-i\,xt)\,dt\right| \le \int_{R\epsilon(x)}^{R} \Phi(t)\,dt \le R\,\exp\left(-Rx^{\frac{p-1}{p}}\right).$$

We evaluate now I_2 and I_4. For $x \ge x_1 = 2^{\frac{4p}{3}}$ we obtain

$$|I_2| = \left|\int_{\gamma_1} \Phi(z)\exp(-i\,xz)\,dz\right| =$$

$$= \left|-R\,i\int_0^{\frac{1}{2\sqrt[4p]{x}}} \Phi(-Rsi - R\epsilon(x))\exp(-Rsx + Rx\epsilon(x)\,i)\,ds\right| \le$$

$$\le R\int_0^{\frac{1}{2\sqrt[4p]{x}}} |\Phi(-Rsi - R\epsilon(x))|\exp(-sRx)\,ds =$$

$$= R\int_0^{\frac{1}{\sqrt[p]{x}}} \exp\left[R\,\mathrm{Re}\,\left(\frac{R^2}{(Rs\,i + R\epsilon(x))^2 - R^2}\right)^{2p-1}\right]\exp(-sRx)\,ds +$$

$$+ R\int_{\frac{1}{\sqrt[p]{x}}}^{\frac{1}{2\sqrt[4p]{x}}} |\Phi(-Rsi - R\epsilon(x))|\exp(-sRx)\,ds.$$

Let us first remark that, since Φ is continuous in $\overline{D}$, there exists a constant $C > 0$ such that

(33) $$|\Phi(z)| \le C, \quad \forall z \in \overline{D}.$$

Hence,

$$R\int_{\frac{1}{\sqrt[p]{x}}}^{\frac{1}{2\sqrt[4p]{x}}} |\Phi(-Rsi - R\epsilon(x))|\exp(-sRx)\,ds \le RC\exp\left(-Rx^{\frac{p-1}{p}}\right)$$

On the other hand, remark that

$$\mathrm{Re}\,\left(\frac{R^2}{(Rs\,i + R\epsilon(x))^2 - R^2}\right)^{2p-1} = -x^{\frac{2p-1}{2p}}\,\mathrm{Re}\,\left(\frac{1}{1 + \sqrt[2p]{x}s^2 - 2\sqrt[2p]{x}\epsilon(x)si}\right)^{2p-1}.$$

Let us denote by $z = z(x,s) = \sqrt[2p]{x}s^2 - 2\sqrt[2p]{x}\epsilon(x)si$. It follows that, for $s \in \left[0, \frac{1}{\sqrt[p]{x}}\right]$, $\lim_{x\to\infty}|z(x,s)| = 0$ uniformly in s. We obtain that

$$\lim_{x\to\infty}\mathrm{Re}\,\left(\frac{1}{1 + \sqrt[2p]{x}s^2 - 2\sqrt[2p]{x}\epsilon(x)si}\right)^{2p-1} = \lim_{x\to\infty}\mathrm{Re}\,\left(\frac{1}{1+z}\right)^{2p-1} = s$$

uniformly in s. Hence, there exists $x_2 > 1$ such that, for all $x > x_2$ and $s \in \left[0, \frac{1}{\sqrt[p]{x}}\right]$,

$$\mathrm{Re}\,\left(\frac{1}{1 + \sqrt[2p]{x}s^2 - 2\sqrt[2p]{x}\epsilon(x)si}\right)^{2p-1} > \frac{1}{2}.$$

It follows that, for $x > x_2$,

$$R\int_0^{\frac{1}{\sqrt[p]{x}}} \exp\left[R\mathrm{Re}\ \left(\frac{1}{(Rs\,i+R\epsilon(x))^2-R^2}\right)^{2p-1}\right]\exp(-sRx)\,ds \le$$

$$R\int_0^{\frac{1}{\sqrt[p]{x}}} \exp\left(-\frac{R}{2}x^{\frac{2p-1}{2p}}\right)\,ds \le R\exp\left(-\frac{R}{2}x^{\frac{2p-1}{2p}}\right) \le R\exp\left(-\frac{R}{2}x^{\frac{p-1}{p}}\right).$$

Hence, for $x > \max\{x_1, x_2\}$ we obtain that

$$|I_2| \le \max\{R, RC\}\exp\left(-\frac{R}{2}x^{\frac{p-1}{p}}\right).$$

In a similar way

$$|I_4| = \left|\int_{\gamma_3}\Phi(z)\exp(-i\,xz)\,dz\right| \le \max\{R, RC\}\exp\left(-\frac{R}{2}x^{\frac{p-1}{p}}\right).$$

We pass now to evaluate

$$|I_3| = \left|\int_{\gamma_2}\Phi(z)\exp(-i\,xz)\,dz\right| = \left|R\int_{-\epsilon(x)}^{\epsilon(x)}\Phi\left(Rs - R\frac{1}{2\sqrt[4p]{x}}\,i\right)\exp\left(-Rx\frac{1}{2\sqrt[4p]{x}} - i\,Rsx\right)\,ds\right| \le$$

$$\le R\int_{-\epsilon(x)}^{\epsilon(x)} C\exp\left(-Rx\frac{1}{2\sqrt[4p]{x}}\right)\,ds \le 2RC\exp\left(-\frac{R}{2}x^{\frac{p-1}{p}}\right).$$

We obtain that there exists a constant $M_2 = 5\max\{R, 2RC\}$ such that, for $x \ge x_0 = \max\{x_1, x_2\}$,

$$|\widehat{\Phi}(x)| = \left|\int_{\mathbb{R}}\Phi(t)\exp(-i\,xt)\,dt\right| = |I_1+I_2+I_3+I_4+I_5| \le M\exp\left(-\frac{R}{2}x^{\frac{p-1}{p}}\right).$$

Finally, it results that, for all $x \ge 0$,

$$|\widehat{\Phi}(x)| = \left|\int_{\mathbb{R}}\Phi(t)\exp(-i\,xt)\,dt\right| \le M\exp(-\omega\sqrt[p]{x}),$$

where $M = \max\{M_1, M_2\}$ and $\omega = \frac{R}{2}$.

■

Remark 7 *The result of Proposition 2 is, in some sense, optimal. Indeed, one can prove that any analytic function of exponential type, φ, such that $\varphi(x) = \mathcal{O}(\exp(-\omega|x|))$ as $|x| \to \infty$ is identically zero (see Titchmarsh [15], p. 185). But our function $\widehat{\Phi}$ is the Fourier transform of an $L^2(-R,R)$ function and, by Paley-Wiener Theorem, is analytic and of exponential type. Moreover, we have prove that, for any $p \in \mathbb{N}^*$, $\widehat{\Phi}(x) = \mathcal{O}\left(\exp(-\omega|x|^{\frac{p-1}{p}})\right)$ as $|x| \to \infty$.*

Let us now go back to the family of exponentials $\{e^{i\lambda_n t}\}_{n\ge1}$ satisfying (27) and (28). Let also $p \in \mathbb{N}^*$ and $q = \frac{p-1}{p}$.

We define the following metrics in $\mathbb{N}^* \times \mathbb{N}^*$

(34) $$d : \mathbb{N}^* \times \mathbb{N}^* \longrightarrow \mathbb{R}, \quad d(n,m) = |\lambda_n - \lambda_m|^q\,.$$

The following property of the metrics d will be used in the next sections.

Lemma 1 *The metrics d defined in $\mathbb{N}^* \times I\!N^*$ satisfies*

$$\forall \varepsilon > 0 \ \exists c_0(\varepsilon, N) \text{ such that } \sup_{n \in \mathbb{N}^*} \sum_{m \in \mathbb{N}^*} \exp(-\varepsilon d(n,m)) \leq c_0(\varepsilon, N) \tag{35}$$

where $c_0(\varepsilon, N) = N + c_0(\varepsilon)$ with $c_0(\varepsilon)$ depending only on ε and γ_∞.

Proof: We have

$$\sum_{m \geq 1} \exp(-\varepsilon d(n,m)) = \sum_{m \geq 1} \exp\left(-\varepsilon |\lambda_m - \lambda_n|^q\right) =$$

$$= \sum_{1 \leq m \leq N} \exp\left(-\varepsilon |\lambda_m - \lambda_n|^q\right) + \sum_{m > N} \exp\left(-\varepsilon |\lambda_m - \lambda_n|^q\right) \leq N + \sum_{m > N} \exp\left(-\varepsilon |\lambda_m - \lambda_n|^q\right).$$

But, for $m > N$, we have

$$|\lambda_m - \lambda_n| \geq \begin{cases} (m-N)\gamma_\infty & \text{if } n \leq N \\ (m-n)\gamma_\infty & \text{if } n > N. \end{cases}$$

It follows that

$$\sum_{m > N} \exp\left(-\varepsilon |\lambda_m - \lambda_n|^q\right) \leq 2 \sum_{m \geq 1} \exp\left(-\varepsilon (m\gamma_\infty)^q\right).$$

Finally, we obtain that $\sum_{m \geq 1} \exp(-\varepsilon d(n,m)) \leq N + \beta$ where

$$\beta = 2 \sum_{m \geq 1} \exp(-\varepsilon (\gamma_\infty m)^q) = 2 \sum_{m \geq 1} \left(\exp(-(\gamma_\infty)^q)\right)^{\varepsilon m^q} \leq$$

$$\leq c(r) \sum_{m \geq 1} \left(\frac{1}{\varepsilon m^q}\right)^r \leq \frac{c(r)}{\varepsilon^r} \sum_{m \geq 1} \frac{1}{m^{qr}} < \infty$$

for any $r > q$, $c(r)$ being a positive constant.

■

3.2 Two matrices

Let $R > 0$ and $T > 0$ such that $\frac{2\pi}{\gamma_\infty} < T < 2R < 2T$.

Let $C_1(N) = C_1(N, T)$ and $C_2(N) = C_2(\gamma, 2T)$ be the constants given by Theorem 3 and let Φ be the function defined by (31) with $p \in \mathbb{N}$, $p > 1$ and $q = 1 - \frac{1}{p}$.

Observe that

$$\int_{-R}^{R} \Phi(t) |f(t)|^2 \, dt = \int_{-R}^{R} \Phi(t) \left| \sum_{n \geq 1} a_n e^{\lambda_n t i} \right|^2 dt = \sum_{n,m \geq 1} a_n \overline{a}_m \int_{-R}^{R} \Phi(t) e^{(\lambda_n - \lambda_m) t i} \, dt.$$

It turns out that the proof of the inequality (29) is related to the study of the behavior of the quantities $\int_{-R}^{R} \Phi(t) e^{(\lambda_n - \lambda_m) t i} \, dt$.

We define the infinite matrix $A = (A(n,m))_{n,m \in \mathbb{N}^*}$ by

$$A(n,m) = \frac{1}{C_2(N)} \int_{\mathbb{R}} \Phi(t) \exp(i\, t(\lambda_n - \lambda_m)) \, dx \tag{36}$$

where Φ is the function defined in Proposition 2 with $p \in \mathbb{N}$, $p > 1$.

The matrix A can be considered as an operator from ℓ^2 to ℓ^2. In fact A is a bounded operator from ℓ^2 to ℓ^2. In order to see this let us first introduce, as in [9], a new space.

Definition 5 *A matrix $E = (E(n,m))_{n,m\in\mathbb{N}^*}$ belongs to $\mathcal{E}_\gamma$ if the coefficients of E satisfy*

$$\forall \gamma' < \gamma, \quad |E(n,m)| \le c(\gamma') \exp(-\gamma' \, d(n,m)), \forall n, m \in \mathbb{N}^*. \tag{37}$$

The following result is a direct consequence of the definition of the matrix A and the properties of the function Φ.

Lemma 2 *The matrix A belongs to $\mathcal{E}_\omega$ where ω is the constant given by Proposition 2.*

Proof: Indeed we have

$$A(n,m) = \frac{1}{C_2(N)} \int_{\mathbb{R}} \Phi(x) \exp(-i\, x(\lambda_n - \lambda_m))\, dx.$$

From Proposition 2 it follows that

$$|A(n,m)| \le \frac{M}{C_2(N)} \exp\left(-\omega |\lambda_n - \lambda_m|^q\right) = \frac{M}{C_2(N)} \exp(-\omega\, d(n,m)),$$

where ω and M are the constants given by Proposition 2.

Hence A belongs to $\mathcal{E}_\omega$.

■

Remark 8 *Let us remark that, from the proof of Lemma 2, there is a constant M (the one given by Proposition 2), not depending on N, such that*

$$|A(n,m)| \le \frac{M}{C_2(N)} \exp(-\omega\, d(n,m)), \ \forall (n,m) \in \mathbb{N}^* \times \mathbb{N}^*. \tag{38}$$

Note that M and ω do not depend at all on the exponential family.

Remark 9 *We recall that, from Schur's Lemma (see [9]), if a matrix E satisfies*

$$\sup_{n\ge1} \sum_{m\ge1} |E(n,m)| \le e \ \textit{and} \ \sup_{m\ge1} \sum_{n\ge1} |E(n,m)| \le e$$

then E is bounded on ℓ^2 and $||E|| \le e$.

In our case, from (38), it follows that

$$\sup_{m\ge1} \sum_{n\ge1} |A(n,m)| = \sup_{n\ge1} \sum_{m\ge1} |A(n,m)| \le \frac{M}{C_2(N)} \sup_{n\ge1} \sum_{m\ge1} \exp(-\omega\, d(n,m)).$$

Hence, from Schur's Lemma and Lemma 1, we obtain that

$$||A|| \le \sup_{m\ge1} \sum_{n\ge1} |A(n,m)| = \sup_{n\ge1} \sum_{m\ge1} |A(n,m)| \le \frac{M\, c_0(\omega,N)}{C_2(N)}.$$

It follows that the matrix A is a bounded operator from ℓ^2 to ℓ^2 and $||A|| \le \frac{M\, c_0(\omega,N)}{4C_2(N)}$.

Other properties of the matrix A are given in the following Lemma.

LEMMA 3 *The matrix A has the following properties:*

i) A is a self-adjoint and positive defined matrix.

ii) There is a constant c, not depending on the exponential family (hence, not depending on N), such that the norm of A satisfies

$$\frac{c\,C_1(N)}{C_2(N)} \leq ||A|| \leq 1.$$

iii) If ν_1 is the first eigenvalue of A then $\nu_1 > \frac{c\,C_1(N)}{C_2(N)}$. Moreover, the matrix A is invertible and

$$||A^{-1}|| \leq \frac{C_2(N)}{c\,C_1(N)}.$$

Proof: i) It follows immediately from the fact that $A(n,m) = \overline{A(n,m)}$ that A is self-adjoint. To show that A is positive defined take $v = (v_n)_{n\geq 1} \in \ell^2$ and observe that

$$\langle Av, v\rangle = \frac{1}{C_2(N)} \int_{\mathbb{R}} \Phi(t) \left| \sum_{n\geq 1} v_n\, e^{i\,\lambda_n t} \right|^2 dt \geq 0.$$

Let us show that $\langle Av, v\rangle = 0$ implies $v = 0$.

Indeed, from $\langle Av, v\rangle = 0$ it follows that $\sum_{n=1}^{\infty} v_n\, e^{i\,\lambda_n t} = 0$.

Since $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = \gamma_\infty > \frac{\pi}{R}$ the family of exponentials $(e^{i\,\lambda_n t})_n$ is incomplete in $L^2(-R,R)$ (in fact it has infinite deficiency). Therefore $(e^{i\,\lambda_n t})_n$ is minimal and no exponential function can be expressed as a linear combination of the others. Hence $v_n = 0$ for all $n \geq 1$. It results that A is positive defined.

ii) If we denote by $f(t) = \sum_{n=1}^{\infty} v_n\, e^{i\,\lambda_n t}$, by using Theorem 3, we have

$$||A|| = \rho(A) = \sup_{||v||=1} \langle Av, v\rangle = \frac{1}{C_2(N)} \sup_{||v||=1} \int_{-R}^{R} \Phi(t)|f(t)|^2 dt \leq$$

$$\leq \frac{1}{C_2(N)} \sup_{||v||=1} \int_{-R}^{R} |f(t)|^2 dt \leq \frac{C_2(N)}{C_2(N)} \sup_{||v||=1} \sum_{n\geq 1} |v_n|^2 = 1.$$

On the other hand, if $2R > T$, there is a constant c such that

$$||A|| \geq \frac{c}{C_2(N)} \sup_{||v||=1} \int_{-\frac{T}{2}}^{\frac{T}{2}} |f(t)|^2 dt.$$

From Theorem 3 it follows that

$$||A|| \geq \frac{c}{C_2(N)} \sup_{||v||=1} \int_{-\frac{T}{2}}^{\frac{T}{2}} |f(t)|^2 dt \geq \frac{cC_1(N)}{C_2(N)} \sup_{||v||=1} \sum_{n\geq 1} |v_n|^2 = \frac{cC_1(N)}{C_2(N)}.$$

iii) If ν_1 is the first eigenvalue of A, by using Theorem 3, we obtain

$$\nu_1 = \inf_{||v||=1} \langle Av, v\rangle = \inf_{||v||=1} \frac{1}{C_2(N)} \int_{\mathbb{R}} \Phi(t) \left| \sum_{n\geq 1} v_n\, e^{i\,\lambda_n t} \right|^2 dt \geq$$

$$\geq \inf_{||v||=1} \frac{c}{C_2(N)} \int_{-\frac{T}{2}}^{\frac{T}{2}} \left| \sum_{n\geq 1} v_n \, e^{i\,\lambda_n t} \right|^2 dt \geq \frac{c\,C_1(N)}{C_2(N)}.$$

On the other hand, since A is positive defined, $\nu = 0$ does not belongs to the (punctual) spectrum of A. Hence A is invertible. Moreover,

$$||A^{-1}|| = \rho(A^{-1}) = \frac{1}{\nu_1} \leq \frac{C_2(N)}{c\,C_1(N)}.$$

■

Let us now define the matrix B by

$$B(n,m) = \begin{cases} \dfrac{1}{C_2(N)}\delta_{n,m} & \text{if } 1 \leq n \leq N \text{ or } 1 \leq m \leq N \\ A(n,m) & \text{if } n > N \text{ and } m > N. \end{cases} \tag{39}$$

The main properties of the matrix B are given in the following Lemma.

LEMMA 4 *The matrix B has the following properties:*

i) B is a self-adjoint and positive defined matrix.

ii) There exists a positive constant b, depending only on R and γ_∞ (hence, independent of N), such that

$$\frac{b}{C_2(N)} < ||B|| \leq 1.$$

iii)If μ_1 is the first eigenvalue of B then $\mu_1 > \frac{b}{C_2(N)}$. Moreover, B is invertible and

$$||B^{-1}|| \leq \frac{C_2(N)}{b}.$$

Proof: i) Since A is self-adjoint and positive defined B has the same properties.

ii) Let $v \in \ell^2$. We have

$$\langle Bv, v\rangle = \frac{1}{C_2(N)} \sum_{m=1}^{N} |v_m|^2 + \sum_{n,m>N} v_n \overline{v_m} A(n,m) =$$

$$= \frac{1}{C_2(N)} \sum_{m=1}^{N} |v_m|^2 + \frac{1}{C_2(N)} \sum_{n,m>N} v_n \overline{v_m} \int_{-R}^{R} \Phi(t) e^{i\,(\lambda_n - \lambda_m)t} dt =$$

$$= \frac{1}{C_2(N)} \sum_{m=1}^{N} |v_m|^2 + \frac{1}{C_2(N)} \int_{-R}^{R} \Phi(t) |f_1(t)|^2 dt$$

where $f_1(t) = \sum_{m>N} v_m e^{i\,\lambda_m t}$.

By using the fact that $R > \frac{T}{2}$ and Theorem 2 we obtain that there is a constant $c' > 1$, independent of N, such that

$$\int_{-R}^{R} \Phi(t)|f_1(t)|^2 dt \leq c' \sum_{m>N} |v_m|^2.$$

It follows that

$$\langle Bv, v\rangle \leq \frac{1}{C_2(N)}\sum_{m=1}^{N}|v_m|^2 + \frac{c'}{C_2(N)}\sum_{m>N}|v_m|^2 \leq \frac{c'}{C_2(N)}\sum_{m\geq 1}|v_m|^2.$$

Hence, for N large enough, $||B|| = \sup_{||v||=1}\langle Bv, v\rangle < 1$.

On the other hand, by using the fact that $R > T$ and Theorem 1 we obtain that there is a constant b' such that

$$\int_{-R}^{R}\Phi(t)|f_1(t)|^2 dt \geq b'\sum_{m>N}|v_m|^2.$$

It follows that

$$\langle Bv, v\rangle \geq \frac{1}{C_2(N)}\sum_{m=1}^{N}|v_m|^2 + \frac{b'}{C_2(N)}\sum_{m>N}|v_m|^2 \geq \frac{b}{C_2(N)}\sum_{m\geq 1}|v_m|^2$$

where $b = \min\{1, b'\}$.

Hence,

$$||B|| = \sup_{||v||=1}\langle Bv, v\rangle \geq \frac{b}{C_2(N)}.$$

iii) By Rayleigh's Theorem

$$\mu_1 = \inf_{||v||=1}\langle Bv, v\rangle \geq \frac{b}{C_2(N)}\sum_{m\geq 1}|v_m|^2.$$

Moreover,

$$\frac{1}{\mu_1} = \rho(B^{-1}) = ||B^{-1}||.$$

Finally, we obtain $||B^{-1}|| \leq \frac{C_2(N)}{b}$.

■

3.3 Some estimates for A^{-1}

We have the following decomposition of the two matrices

$$A = \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right), \quad B = \left(\begin{array}{c|c} \frac{1}{C_2(N)}I & 0 \\ \hline 0 & A_{22} \end{array}\right)$$

where the first blocks have dimension $N \times N$.

The matrix $A - B$ has the following decomposition

$$A - B = \left(\begin{array}{c|c} A_{11} - \frac{1}{C_2(N)}I & A_{12} \\ \hline A_{21} & 0 \end{array}\right).$$

We compare now the two matrices A and B.

LEMMA 5 *Let $\Omega = \{1, 2, ..., N\} \subset \mathbb{N}^*$ and let ω and M be the constants from Proposition 2. Then*

$$|A(n,m) - B(n,m)| \leq \frac{M+1}{C_2(N)} \exp(-\omega\,(d(n,\Omega) + d(m,\Omega))), \quad \forall n, m \in \mathbb{N}^*. \tag{40}$$

Proof: Let $(n,m) \in \mathbb{N}^* \times \mathbb{N}^*$. We consider the following cases:

(i) $n \in \Omega$ or $m \in \Omega$. In this case $A(n,m) - B(n,m) = A(n,m) - \frac{1}{C_2(N)}\delta_{nm}$ and, from Lemma 2, we obtain that

$$|A(n,m) - B(n,m)| \leq \frac{M+1}{C_2(N)} \exp(-\omega\, d(n,m)) \leq \frac{M+1}{C_2(N)} \exp(-\omega\,(d(n,\Omega) + d(m,\Omega))).$$

(ii) $n \notin \Omega$ and $m \notin \Omega$. Now

$$|A(n,m) - B(n,m)| = 0 \leq \frac{M+1}{C_2(N)} \exp(-\omega\,(d(n,\Omega) + d(m,\Omega))).$$

■

Remark 10 *The previous result tells us that the difference between matrix A and B decays exponentially away from Ω. This property allows us to apply a result of S. Jaffard (see [9]) to compare the elements of the inverses of the matrices A and B. In fact we shall prove that the difference between A^{-1} and B^{-1} decays also exponentially away from Ω.*

PROPOSITION 3 *Let $\omega' < \omega$. Then, for all n, $m \geq 1$,*

$$|A^{-1}(n,m) - B^{-1}(n,m)| \leq \frac{1}{1-\rho} \exp\left(-\omega'(1-\alpha)(d(n,\Omega) + d(m,\Omega))\right), \tag{41}$$

for all N sufficiently large, where $\rho = 1 - \frac{cC_1(N)}{C_2(N)}$, $L > 1$ is a number which does not depend on the exponential family (hence, not depending on N) and $\alpha = \alpha(N) = \frac{\ln L}{\ln L - \ln \rho}$.

Proof: We give the proof in several steps.

First step: we define two new matrices which will help us to express the inverses of A and B.

From Lemmas 3 and 4 we know that A and B are two self-adjoint and positive defined matrices such that $||A|| \leq 1$ and $||B|| \leq 1$.

It follows that there exist two self-adjoint and positive defined matrices V and W such that $A = I - V$ and $B = I - W$. Moreover, $\max\{||V||,\, ||W||\} = \rho < 1$.

Indeed, since V is self-adjoint and $||A|| \leq 1$, we have

$$||V|| = r(V) = \sup_{||v||=1} (v, Rv) = \sup_{||v||=1} (1 - (v, Av)) = 1 - \inf_{||v||=1} (v, Av) = 1 - \nu_1$$

where $r(V)$ is the spectral radius of V and ν_1 is the first eigenvalue of A. From Lemma 3 we obtain that

$$||V|| \leq 1 - \frac{c\,C_1(N)}{C_2(N)} < 1. \tag{42}$$

By using a similar argument it follows that $||W|| \leq 1 - \frac{c\,C_1(N)}{C_2(N)} < 1$.

Therefore, $\rho \leq 1 - \frac{cC_1(N)}{C_2(N)} < 1$.

We have $A^{-1} = \sum_{k=0} V^k$ and $B^{-1} = \sum_{k=0} W^k$.

We evaluate now V^k and W^k. First of all, remark that

$$\max\{|V^k(n,m)|, |W^k(n,m)|\} \leq ||\rho||^k. \tag{43}$$

Step two: we evaluate the difference between $V^k(n,m)$ and $W^k(n,m)$.

Lemma 6 *There exists a constant $L > 1$ such that*

$$|V^k(n,m) - W^k(n,m)| \leq L^k \exp(-\omega' (d(n,\Omega) + d(m,\Omega))), \quad \forall n,m \in \mathbb{N}^*, \quad \forall k \in \mathbb{N}. \tag{44}$$

Proof: Let us first remark that, if $n \in \Omega$ and $m \in \Omega$, (44) follows from (43). So, we can suppose that $n \notin \Omega$ or $m \notin \Omega$. Moreover, since V and W are self-adjoint it is sufficient to prove (44) only for the case $n \in \mathbb{N}^*$ and $m > N$.

We give a proof by recurrence on k.

Let first $k = 1$. We have from (40)

$$|V(n,m) - W(n,m)| = |A(n,m) - B(n,m)| \leq$$

$$\leq \frac{M+1}{C_2(N)} \exp(-\omega (d(n,\Omega) + d(m,\Omega))) \leq \frac{M+1}{C_2(N)} \exp(-\omega' (d(n,\Omega) + d(m,\Omega)))$$

and (44) follows with $L = \max\left\{\frac{M+1}{C_2(N)}, 1\right\}$.

Let us now suppose that, for some $k \in \mathbb{N}$,

$$|V^k(n,m) - W^k(n,m)| \leq L(k) \exp(-\omega' (d(n,\Omega) + d(m,\Omega))), \quad \forall n,m \in \mathbb{N}^*. \tag{45}$$

We have

$$|V^{k+1}(n,m) - W^{k+1}(n,m)| =$$

$$= \left| \sum_u (V^k(n,u) - W^k(n,u))V(u,m) + \sum_u W^k(n,u)(V(u,m) - W(u,m)) \right|.$$

But, for $u > N$, $V(u,m) - W(u,m) = B(u,m) - A(u,m) = 0$. Moreover, for $1 \leq u \leq N$, $W^k(n,u) = \frac{1}{(C_2(N))^k}\delta_{nu}$. It follows that

$$\left| \sum_u W^k(n,u)(V(u,m) - W(u,m)) \right| \leq \frac{1}{(C_2(N))^k} |V(n,m) - W(n,m)| = \frac{1}{(C_2(N))^k} |A(n,m)| \leq$$

$$\leq \frac{M}{(C_2(N))^k} \exp(-\omega (d(n,\Omega) + d(m,\Omega))) \leq L(K) \frac{M}{(C_2(N))^k} \exp(-\omega (d(n,\Omega) + d(m,\Omega))).$$

On the other hand

$$\left| \sum_u (V^k(n,u) - W^k(n,u))V(u,m) \right| \leq$$

$$\leq L(k)|V(m,m)| \exp(-\omega' (d(n,\Omega) + d(m,\Omega))) + \sum_{u \neq m} L(k) \exp(-\omega' (d(n,\Omega) + d(u,\Omega)))|V(u,m)|.$$

But, for $u \neq m$, $|V(u,m)| = |A(u,m)| \leq \frac{M}{C_2(N)} \exp(-\omega\, d(u,m))$. It follows that

$$|V^{k+1}(n,m) - W^{k+1}(n,m)| \leq L(k)|V(m,m)| \exp(-\omega' (d(n,\Omega) + d(m,\Omega)))+$$

$$+L(k)\frac{M}{C_2(N)} \exp(-\omega' (d(n,\Omega) + d(m,\Omega))) \sum_u \exp(-\omega' (d(u,\Omega) - d(m,\Omega))) \exp(-\omega'' d(u,m))$$

where $\omega' < \omega'' < \omega$.

Since $d(u,m) + d(u,\Omega) \geq d(m,\Omega)$ it follows that

$$\sum_u \exp(-\omega' (d(u,\Omega) - d(m,\Omega))) \exp(-\omega'' d(u,m)) \leq$$

$$\leq \sum_u \exp(\omega' d(u,m)) \exp(-\omega'' d(u,m)) =$$

$$= \sum_u \exp(-(\omega'' - \omega') d(u,m)) = c_0(\omega'' - \omega', N).$$

Remark also that $|V(m,m)| = |1 - A(m,m)| = 1 - A(m,m) \leq 1$.

We obtain that

$$|V^{k+1}(n,m) - W^{k+1}(n,m)| \leq$$

$$L(k) \left(\frac{(M+1)\, c_0(\omega'' - \omega', N)}{C_2(N)} + \frac{M}{(C_2(N))^k} + 1 \right) \exp(-\omega' (d(n,\Omega) + d(m,\Omega))).$$

Hence

$$|V^k(n,m) - W^k(n,m)| \leq L^k \exp(-\omega' (d(n,\Omega) + d(m,\Omega))),$$

where $L = \sup_k \left\{ \frac{(M+1)\, c_0(\omega''-\omega', N)}{C_2(N)} + \frac{M}{(C_2(N))^k} + 1 \right\}$.

Remark that, since $C_2(N) \to \infty$, L can be chosen independent of N and, more generally, of the exponential family.

∎

Step Three: we pass now to prove the estimate (41).

We have

$$|A^{-1}(n,m) - B^{-1}(n,m)| \leq \left| \sum_{k=0}^{\infty} (V^k(n,m) - W^k(n,m)) \right| \leq$$

$$\leq \left| \sum_{k=0}^{K_0} (V^k(n,m) - W^k(n,m)) \right| + \left| \sum_{k=K_0+1}^{\infty} (V^k(n,m) - W^k(n,m)) \right| \leq$$

$$\leq \sum_{k=0}^{K_0} L^k \exp(-\omega' (d(n,\Omega) + d(m,\Omega))) + 2 \sum_{k=K_0+1}^{\infty} \rho^k$$

by (40) and (44). It follows that

$$|A^{-1}(n,m) - B^{-1}(n,m)| \leq 2 \left(L^{K_0} \exp(-\omega' (d(n,\Omega) + d(m,\Omega))) + \frac{\rho^{K_0}}{1-\rho} \right).$$

We choose $K_0 = K_0(N, n, m)$ such that

$$L^{K_0} \exp(-\omega' (d(n,\Omega) + d(m,\Omega))) = \frac{\rho^{K_0}}{1-\rho}.$$

Hence

$$K_0 = \frac{\ln\left(\frac{1}{1-\rho}\right) + \omega'(d(n,\Omega) + d(m,\Omega))}{\ln\left(\frac{L}{\rho}\right)}. \tag{46}$$

For this K_0 we obtain that

$$|A^{-1}(n,m) - B^{-1}(n,m)| \le 4L^{K_0} \exp(-\omega' (d(n,\Omega) + d(m,\Omega))) =$$

$$= \frac{4}{(1-\rho)^{\alpha}} \exp\left(-\omega'(1-\alpha)(d(n,\Omega) + d(m,\Omega))\right) \le$$

$$= \frac{4}{1-\rho} \exp\left(-\omega'(1-\alpha)(d(n,\Omega) + d(m,\Omega))\right)$$

■

Remark 11 *The quantities $\frac{4}{1-\rho}$ and $\omega'(1-\alpha)$ depend on N. Proposition 3 tells us that far away from Ω the elements of A^{-1} are uniformly bounded in N, but, as N goes to infinity, we have to go further and further from Ω.*

First of all we consider the following decomposition of the matrices A^{-1}, B^{-1}:

$$A^{-1} = \left(\begin{array}{c|c} A_{11}^{-1} & A_{12}^{-1} \\ \hline A_{21}^{-1} & A_{22}^{-1} \end{array}\right), \quad B^{-1} = \left(\begin{array}{c|c} B_{11}^{-1} & B_{12}^{-1} \\ \hline B_{21}^{-1} & B_{22}^{-1} \end{array}\right)$$

where the first blocks have dimension $I_1(N) \times I_1(N)$ and $I_1(N)$ will be chosen conveniently.

Since, from the previous proposition we know that the elements of $A^{-1} - B^{-1}$ are uniformly bounded in N far away from Ω, we can deduce the following result.

Proposition 4 *There exists $\delta > 0$ such that, if $I_1(N) \ge \delta \left(\frac{C_2(N,2T)}{C_1(N,T)} \ln\left(\frac{C_2(N,2T)}{C_1(N,T)}\right)\right)^{\frac{p}{p-1}}$ and N is sufficiently large, then*

$$\max\left\{||A_{22}^{-1} - B_{22}^{-1}||,\ ||A_{12}^{-1} - B_{12}^{-1}||,\ ||A_{21}^{-1} - B_{21}^{-1}||\right\} \le 1. \tag{47}$$

Proof: From Schur's Lemma

$$||A_{22}^{-1} - B_{22}^{-1}|| \le \sup_{n > I_1(N)} \sum_{m > I_1(N)} |(A^{-1} - B^{-1})(n,m)|.$$

From Proposition 3 it follows that

$$||A_{22}^{-1} - B_{22}^{-1}|| \le \sup_{n > I_1(N)} \sum_{m > I_1(N)} \frac{4}{1-\rho} \exp\left(-\omega'(1-\alpha)(d(n,\Omega) + d(m,\Omega))\right) \le$$

$$\le \frac{4}{1-\rho} \sup_{n > I_1(N)} \exp\left(-\omega'(1-\alpha) d(n,\Omega)\right) \sum_{m > I_1(N)} \exp\left(-\omega' (1-\alpha)\, d(m,\Omega)\right) \le$$

$$\leq \frac{4}{1-\rho} \exp\left(-\omega'\,(1-\alpha)\, d(I_1(N),\Omega)\right) \sum_{m>I_1(N)} \exp\left(-\omega'\,(1-\alpha)\, d(m,\Omega)\right).$$

Since $|\lambda_{m+1}-\lambda_m| \geq \gamma_\infty$ if $m>N$ it results that

$$||A_{22}^{-1}-B_{22}^{-1}|| \leq \frac{4}{1-\rho} \exp\left(-2\omega'(1-\alpha)d(I_1(N),\Omega)\right) \sum_{m\geq 1} \left(\exp\left(-\omega'(1-\alpha)(\gamma_\infty\, m)^q\right)\right) \leq$$

$$\leq \frac{4}{1-\rho} \exp\left(-2\omega'(1-\alpha)d(I_1(N),\Omega)\right) \sum_{m\geq 1} \left(\exp(-2\omega'(\gamma_\infty)^q)\right)^{(1-\alpha)\,m^q} \leq$$

$$\leq \frac{4}{1-\rho} \exp\left(-2\omega'\,(1-\alpha)\, d(I_1(N),\Omega)\right) c(r) \sum_{m\geq 1} \left(\frac{1}{(1-\alpha)m^q}\right)^r$$

$$= \frac{4c(r)}{(1-\rho)(1-\alpha)^r} \exp\left(-2\omega'\,(1-\alpha)\, d(I_1(N),\Omega)\right) \sum_{m\geq 1} \frac{1}{m^{rq}},$$

where r is a number such that $rq>1$.

We shall prove that the quantity

$$\frac{1}{(1-\rho)(1-\alpha)^r} \exp\left(-2\omega'(1-\alpha)d(I_1(N),\Omega)\right)$$

tends to zero as N tends to infinity.

Let us first remark that $\frac{1}{1-\rho} \leq \frac{C_2}{cC_1}$. Moreover, since $\rho \leq 1-\frac{cC_1}{C_2}$, we have

$$1-\alpha = -\frac{\ln \rho}{\ln L - \ln \rho} \geq -\frac{\ln\left(1-\frac{cC_1}{C_2}\right)}{\ln L - \ln\left(1-\frac{cC_1}{C_2}\right)}.$$

Since $\frac{cC_1}{C_2}$ tends to zero as N tends to infinity, there is a constant $\delta_1>0$, not depending on N, such that

$$1-\alpha \geq \delta_1 \left(\frac{C_1}{C_2}\right).$$

It follows that, there exists a constant δ such that, if $I_1(N) \geq \delta \left(\frac{C_2(N,2T)}{C_1(N,T)} \ln\left(\frac{C_2(N,2T)}{C_1(N,T)}\right)\right)^{\frac{1}{q}}$,

$$\frac{1}{(1-\rho)(1-\alpha)^r} \exp\left(-2\omega'(1-\alpha)d(I_1(N),\Omega)\right) \leq$$

$$\delta_1^r \left(\frac{C_2}{C_1}\right)^{r+1} \exp\left(-2\omega'\delta_1 \left(\frac{C_2}{C_1}\right) d(I_1(N),N)\right) \longrightarrow 0 \text{ as } N\to\infty.$$

Hence, for N large enough,

$$||A_{22}^{-1}-B_{22}^{-1}|| \leq 1.$$

We pass now to prove that $||A_{12}^{-1}-B_{12}^{-1}||<1$. Let $v\in\ell^2$. We have

$$||(A_{12}^{-1}-B_{12}^{-1})v||^2 = \sum_{1\leq n\leq I_1(N)} \left| \sum_{m>I_1(N)} (A_{12}^{-1}-B_{12}^{-1})(n,m)v_m \right|^2 \leq$$

$$\leq I_1(N) \sup{}_{1\leq n\leq I_1(N)} \sum_{m>I_1(N)} \left|(A_{12}^{-1} - B_{12}^{-1})(n,m)v_m\right|^2.$$

By using Proposition 3 we obtain

$$||(A_{12}^{-1} - B_{12}^{-1})v||^2 \leq I_1(N)\frac{1}{1-\rho} \sup{}_{1\leq n\leq I_1(N)} \sum_{m>I_1(N)} \exp\big(-\omega'(1-\alpha)(d(n,\Omega)+d(m,\Omega))\big) \leq$$

$$\leq I_1(N)\frac{1}{1-\rho} \sum_{m>I_1(N)} \exp\big(-\omega'(1-\alpha)d(m,\Omega)\big) =$$

$$= I_1(N)\frac{1}{1-\rho} \exp\big(-\omega'(1-\alpha)d(I_1(N),N)\big) \sum_{m>I_1(N)} \exp\big(-\omega'(1-\alpha)d(m,I_1(N))\big) \leq$$

$$\leq I_1(N)\frac{1}{1-\rho} \exp\big(-\omega'(1-\alpha)d(I_1(N),N)\big) \sum_{m>I_1(N)} \exp\big(-\omega'(1-\alpha)\sqrt[p]{\gamma_\infty m}\big) \leq$$

$$\leq I_1(N)\frac{c(r)}{(1-\rho)(1-\alpha)^r} \exp\big(-\omega'(1-\alpha)d(I_1(N),N)\big) \sum_{m\geq 1} \frac{1}{\sqrt[p]{m^r}},$$

where $r > 1$.

But, for $I_1(N)$ chosen as before,

$$I_1(N)\frac{1}{(1-\rho)(1-\alpha)^r} \exp\big(-\omega'(1-\alpha)d(I_1(N),N)\big) \longrightarrow 0 \text{ as } N \to \infty.$$

It follows that, for N large enough,

$$||A_{12}^{-1} - B_{12}^{-1}|| \leq 1.$$

For the other estimate the same technique can be applied and the proof is complete.

■

We have now all the instruments we need to prove Theorem 4.

3.4 Proof of Theorem 4:

Since $\frac{2\pi}{\gamma_\infty} < T < 2R < 2T$ we have

$$\int_0^T |f(t)|^2 dt \geq \int_0^R |f(t)|^2 dt = \frac{1}{2}\int_{-R}^R |f(t)|^2 dt \geq \frac{1}{2}\int_{-R}^R \Phi(t)|f(t)|^2 dt =$$

$$= \frac{1}{2}\sum_{n,m\geq 1} a_n\bar{a}_m \int_{-R}^R \Phi(t)\exp(i(\lambda_n-\lambda_m)t)\,dt = \frac{C_2(N)}{2}\sum_{n,m\geq 1} a_n\bar{a}_m A(n,m) = \frac{C_2(N)}{2}\langle a, Aa\rangle.$$

We define now the diagonal matrix D by

$$D(n,m) = \begin{cases} \frac{1}{2}\sqrt{\frac{b}{1+b}} & \text{if } n = m > I_1(N) \\ \frac{1}{2}\sqrt{c\,C_1(N)} & \text{if } n = m \leq I_1(N) \\ 0 & \text{if } n \neq m, \end{cases}$$

where b and c are given in Lemmas 3 and 4.

We want to show that

$$\frac{C_2(N)}{2}\langle a, Aa\rangle \geq \langle Da, Da\rangle, \quad \forall a \in \ell^2. \tag{48}$$

But the last relation takes place if and only if the smallest eigenvalue of the matrix $D^{-1}AD^{-1}$ is bigger than $\frac{2}{C_2(N)}$ or, equivalently, the norm of the matrix $DA^{-1}D$ is less than $\frac{C_2(N)}{2}$.

In order to evaluate the norm of the matrix $DA^{-1}D$ let us consider the following decomposition of the two matrices A^{-1} and D

$$A^{-1} = \left(\begin{array}{c|c} A_{11}^{-1} & A_{12}^{-1} \\ \hline A_{21}^{-1} & A_{22}^{-1} \end{array}\right), \quad D = \left(\begin{array}{c|c} D_{11} & 0 \\ \hline 0 & D_{22} \end{array}\right)$$

where the first blocks have dimension $I_1(N) \times I_1(N)$.

We have

$$DA^{-1}D = \left(\begin{array}{c|c} D_{11}A_{11}^{-1}D_{11} & D_{11}A_{12}^{-1}D_{22} \\ \hline D_{22}A_{21}^{-1}D_{11} & D_{22}A_{22}^{-1}D_{22} \end{array}\right).$$

We obtain that

$$||DA^{-1}D|| \leq$$
$$\leq \sqrt{2}(||D_{11}A_{11}^{-1}D_{11}|| + ||D_{11}A_{12}^{-1}D_{22}|| + ||D_{22}A_{21}^{-1}D_{11}|| + ||D_{22}A_{22}^{-1}D_{22}||) \leq$$
$$\leq \sqrt{2}(||D_{11}||^2\,||A_{11}^{-1}|| + ||D_{11}||\,||D_{22}||\,(||A_{12}^{-1}|| + ||A_{21}^{-1}||) + ||D_{22}||^2\,||A_{22}^{-1}||).$$

But $||A_{11}^{-1}|| \leq ||A^{-1}||$. Hence, from Lemma 3,

$$||A_{11}^{-1}|| \leq \frac{1}{c\,C_1(N)}C_2(N).$$

Moreover, from Proposition 4 and Lemma 4,

$$||A_{22}^{-1}|| \leq 1 + ||B_{22}^{-1}|| \leq 1 + ||B^{-1}|| \leq 1 + \frac{1}{b}C_2(N) \leq \frac{b+1}{b}C_2(N).$$

Again from Proposition 4 and the definition of matrix B it follows that

$$||A_{12}^{-1}|| = ||A_{12}^{-1} - B_{12}^{-1}|| + ||B_{12}^{-1}|| \leq 1 + ||B_{12}^{-1}|| \leq \frac{b+1}{b}C_2(N)$$

$$||A_{21}^{-1}|| = ||A_{21}^{-1} - B_{21}^{-1}|| \leq 1 + ||B_{21}^{-1}|| \leq 1 + ||B_{21}^{-1}|| \leq \frac{b+1}{b}C_2(N).$$

Finally we obtain that

$$||DA^{-1}D|| \leq \frac{C_2(N)}{2}$$

and, by taking $c_1 = \frac{b}{b+1}$, the proof finishes.

■

Remark 12 *Theorem 4 says that the high eigenmodes* $e^{i\,\lambda_n\,t}$*, for* $n \geq I_1(N)$*, are not affected when* N *new eigenmodes are introduced.*

Remark 13 *As it follows from the proof, the constants* c_1 *and* δ *in (29) only depend on* T *and* γ_∞ *(the asymptotic gap of the exponential family) but not on other properties of the exponential family* $\{e^{i\lambda_n t}\}_{n\geq 1}$.

4 A controllability result for the wave equation

Let Ω be the unit square $\Omega = (0,1) \times (0,1)$, $\Gamma_0 = \{0\} \times (0,1)$ and $\Gamma_1 = \partial\Omega \setminus \Gamma_0$.

We want to solve the following control problem: Given $T > 2$ find the initial data $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ with the property that there exists a function $v \in L^2((0,T) \times \Gamma_0)$ such that the solution u of the equation

$$\left\{ \begin{array}{ll} u_{tt} - \Delta u = 0 & \text{in } (0,\infty) \times \Omega \\ u = 0 & \text{on } (0,\infty) \times \Gamma_1 \\ u = v & \text{on } (0,\infty) \times \Gamma_0 \\ u(0) = u^0, \ u_t(0) = u^1 & \text{in } \Omega \end{array} \right. \tag{49}$$

satisfies $u(T) = u_t(T) = 0$.

Remark 14 *Let us remark that the control v acts only on a face of the boundary. As we shall point out in Remark 15, given $T > 0$, we can not control a Sobolev space of initial data in time T. Our aim is to give a space of controllable initial data larger that the one found in [1]. To do this, inequality (29) from Theorem 4 will be used.*

To solve this problem we shall use separation of variables. Indeed, let us decompose the control v, the solution u and the initial data in the following way

$$\left\{ \begin{array}{l} v = \displaystyle\sum_{k=1}^{\infty} v_k(t) \sin(k\pi y), \\ u = \displaystyle\sum_{k=1}^{\infty} u_k(t,x) \sin(k\pi y), \\ (u^0, u^1) = \displaystyle\sum_{k=1}^{\infty} (u_k^0(x), u_n^1(x)) \sin(k\pi y). \end{array} \right. \tag{50}$$

With this decomposition, system (49) can be split into the following sequence of one-dimensional controlled systems for $k = 1, 2, \ldots$:

$$\left\{ \begin{array}{ll} u_{k,tt} - u_{k,xx} + k^2\pi^2 u_k = 0 & \text{for } (t,x) \in (0,\infty) \times (0,1) \\ u_k(t,0) = v_k & \text{for } t \in (0,\infty) \\ u_k(t,1) = 0 & \text{for } t \in (0,\infty) \\ u_k(0) = u_k^0, u_{k,t}(0) = u_k^1 & \text{in } (0,1) \end{array} \right. \tag{51}$$

We study the controllability of system (51) by using classical methods that combine HUM and Ingham type inequalities. Combining these one-dimensional results with the Fourier decomposition (50), the controllability result for system (49) will be proved.

Let us consider first the following homogeneous equation

$$\left\{ \begin{array}{ll} z_{tt} - z_{xx} + k^2\pi^2 z = 0 & \text{in } (0,\infty) \times (0,1) \\ z(t,0) = z(t,1) = 0 & \text{for } t \in (0,\infty) \\ z(0) = z^0, \ z_t(0) = z^1 & \text{in } \Omega \end{array} \right. \tag{52}$$

The eigenvalues of this equation are, for each $k \geq 0$,

$$\lambda_n^k = \left\{ \begin{array}{ll} \sqrt{n^2 + k^2}\pi & \text{if } n > 0 \\ -\sqrt{n^2 + k^2}\pi & \text{if } n < 0. \end{array} \right.$$

The corresponding eigenfunctions are

$$\varphi_n^k(x) = \frac{2}{\sqrt{(\lambda_n^k)^2 + n^2\pi^2}} \begin{pmatrix} \sin(n\pi x) \\ i\,\lambda_n^k \sin(n\pi x) \end{pmatrix}.$$

Remark 15 *We can show that, given any $T > 2$, there are analytic initial data of the two-dimensional problem (49) which are not exactly controllable in time T. Like in [6], it is easy to show that the initial datum $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$,*

$$(u^0, u^1) = \sum_{k,n} a_{k,n}\varphi_n^k \sin(k\pi y),$$

is exactly controllable in time T iff there exists $v \in L^2\left((0,T) \times \Gamma_0\right)$ such that

$$v(t,x) = \sum_{k=1}^{\infty} v_k(t)\sin(k\pi y)$$

and the following moments problem is satisfied

$$\int_0^T v_k(t)e^{-i\lambda_n^k t}dt = \frac{\lambda_n^k i}{n\pi\sqrt{(\lambda_n^k)^2 + n^2\pi^2}}a_{k,n}, \quad \forall k \geq 1,\ \forall n \neq 0. \tag{53}$$

Let us now consider an initial datum of the form

$$(u^0, u^1) = \sum_{k\geq 1} a_{k,1}\varphi_1^k(x) \sin(k\pi y)$$

where $e^{-d_1 k} < |a_{k,1}| < e^{-d_2 k}$ for all $k \geq 1$ and $0 < d_2 < d_1$. The constant d_1 will be chosen later on. Note that (u^0, u^1) is analytic.

On the other hand (u^0, u^1) is controllable to zero in time T iff there exists $v \in L^2\left((0,T) \times \Gamma_0\right)$ such that

$$v(t,x) = \sum_{k=1}^{\infty} v_k(t)\sin(k\pi y)$$

and

$$\int_0^T v_k(t)e^{-i\lambda_n^k t}dt = \frac{i}{\pi}\sqrt{\frac{k^2+1}{k^2+2}}a_{k,1}\delta_{1,n}, \quad \forall k \geq 1,\ \forall n \neq 0. \tag{54}$$

From (54) it follows that, for each k, the function

$$\theta_k(t) = -\frac{i\pi}{a_{k,1}}\sqrt{\frac{k^2+2}{k^2+1}}v_k(t)$$

satisfies

$$\int_0^T \theta_k(t)e^{-i\lambda_n^k t} = \delta_{n,1}, \quad \forall n \neq 0$$

and therefore it is an element of a biorthogonal sequence for the exponential family $\left(e^{-i\lambda_n^k t}\right)_{n\neq 0}$.

But, as it is proved in [6], the norms of the biorthogonal sequences for this exponential family increase exponentially with k. Hence, there exists $d_0 > 0$ such that

$$\frac{\pi}{|a_{k,1}|}||v_k||_{L^2(0,T)} \geq ||\theta_k||_{L^2(0,T)} \geq e^{\delta_0 k}, \quad \forall k \geq 1.$$

Since $v \in L^2((0,T)\times\Gamma_0)$ we obtain that (u^0,u^1) is controllable in time T only if

$$|a_{k,1}| \leq 4e^{-\delta_0 k}||v||_{L^2}, \quad \forall k \geq 1. \tag{55}$$

Hence, for $T>2$ fixed and $d_1<d_0$, we obtain an analytic initial datum (u^0,u^1) which does not satisfy (55) and therefore it is not controllable in time T. As we have mentioned before, this phenomenon is due to the fact that the control v acts only on a face of the boundary.

Finally, let us remark that in [1] it is proved that any analytic initial datum can be controlled in a time sufficiently large (which depends on the amount of analyticity of the initial datum).

Remark that, for each $k \geq 0$, the sequence $(\lambda_n^k)_{n\geq 1}$ satisfies $\limsup_n |\lambda_{n+1}-\lambda_n| > \frac{2\pi}{T}$ if $T>2$. Therefore we can apply the previous results to this particular case.

Let $T>2$ and $0<\delta<\pi$. We have

$$|\, \lambda_{n+1}^k - \lambda_n^k \,| \geq \pi - \delta \tag{56}$$

for any n with $|\, n\,| \geq N(k,\delta)$ where

$$N(k,\delta) = \max\left[\frac{\pi-\delta}{\delta}k\right]. \tag{57}$$

By taking $\gamma_\infty = \pi - \delta$ and δ small enough we obtain that

$$\begin{cases} \dfrac{2\pi}{T} > \gamma_\infty, \\ |\, \lambda_{n+1}^k - \lambda_n^k \,| \geq \gamma_\infty, \quad \forall n > N(k,\delta), \\ |\, \lambda_{n+1}^k - \lambda_n^k \,| \geq \gamma = \frac{\pi}{k+1}, \quad \forall n \geq 1. \end{cases} \tag{58}$$

Hence the hypothesis of Theorem 4 are satisfied for the exponential family $\{e^{i\lambda_n^k t}\}_{n\geq 1}$. We shall apply Theorem 4 to prove the following estimates known as the observation inequality.

Theorem 6 *Let c_1, $C_1(N)$ and $I_1(N)$ be the constants given by Theorems 3 and 4. Suppose that $(z^0,z^1)\in H_0^1(0,1)\times L^2(0,1)$ have the following decomposition*

$$(z^0(x), z^1(x)) = \sum_{n\in\mathbb{Z}^*} b_n \varphi_n^k(x).$$

If z is the solution of (52), the following inequality is true

$$\frac{\widehat{C}_1(k,T)}{k^2} \sum_{1\leq|n|\leq \widehat{I}_1(k)} |b_n|^2 + \widehat{c}_1 \sum_{|n|>\widehat{I}_1(k)} |b_n|^2 \leq \int_0^T |z_x(t,0)|^2\, dt. \tag{59}$$

where $\widehat{c}_1$ does not depend on N, $\widehat{C}_1(k,T) = C_1(2N(k,\delta),T)$ and $\widehat{I}_1(k) = I_1(2N(k,\delta))$.

Proof: We only have to remark that

$$(z(t,x), z_t(t,x)) = \sum_{n\in\mathbb{Z}^*} b_n\, e^{i\lambda_n^k t}\varphi_n^k(x).$$

Therefore

$$z_x(t,0) = \sum_{n\in\mathbb{Z}^*} \frac{2n\, b_n\, \pi}{\sqrt{(\lambda_n^k)^2 + n^2\pi^2}} e^{i\lambda_n^k t}.$$

We next apply Theorem 4 and the proof finishes.

■

Now, by using Theorem 6 and HUM we deduce that for any $(u_k^0, u_k^1) \in L^2(0,1) \times H^{-1}(0,1)$,

$$(u_k^0(x), u_k^1(x)) = \sum_{n\in\mathbb{Z}^*} a_{k,n}\varphi_n^k(x),$$

there exists a control $v_k(t)$ for (51) such that

$$\int_0^T |v_k(t)|^2\, dt \le C \left[\frac{k^2}{\widehat{C}_1(k,T)} \sum_{1\le |n|\le \widehat{I}_1(k)} \frac{1}{|\lambda_n^k|^2}|a_{k,n}|^2 + \frac{1}{\widehat{c}_1} \sum_{|n|>\widehat{I}_1(k)} \frac{1}{|\lambda_n^k|^2}|a_{k,n}|^2 \right]$$

where C is a positive constant independent of k and n.

By adding all the controls v_k, $k \ge 1$, we finally obtain that

THEOREM 7 *For any initial data from the space*

$$\mathcal{X} = \left\{ (u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega) : \quad (u^0, u^1) = \sum_{k,n} a_{k,n}\varphi_n^k(x)\sin(k\pi y), \text{ such that} \right.$$

$$\left. \sum_k \left(\frac{k^2}{\widehat{C}_1(k,T)} \sum_{1\le |n|\le \widehat{I}_1(k)} \frac{1}{|\lambda_n^k|^2}|a_{k,n}|^2 + \frac{1}{\widehat{c}_1} \sum_{|n|>\widehat{I}_1(k)} \frac{1}{|\lambda_n^k|^2}|a_{k,n}|^2 \right) < +\infty \right\}$$

problem (49) is exactly controllable with controls $v \in L^2((0,T)\times\Gamma_0)$.

Remark 16 *In this particular case*

$$\widehat{I}_1(k) = I_1(2N(k,\delta)) \sim \left(\frac{C_2(2N(k,\delta))}{C_1(2N(k,\delta))} \ln \left(\frac{C_2(2N(k,\delta))}{C_1(2N(k,\delta))} \right) \right)^{\frac{p-1}{p}}$$

for any $p \in \mathbb{N}$, $p > 1$.

By taking into account the estimates we have obtained for the constants C_1 *and* C_2 *in the second section (see Remark 5) we obtain that*

$$\widehat{I}_1(k) \sim e^{ck}, \quad as N \to \infty.$$

Remark 17 *Let us remark that if inequality (17) is used instead of (29) we obtain that the space of initial data*

$$\mathcal{X}_1 = \left\{ (u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega) : \quad (u^0, u^1) = \sum_{k,n} a_{k,n}\varphi_n^k(x)\sin(k\pi y), \text{ such that} \right.$$

$$\left. \sum_k \left(\frac{k^2}{\widehat{C}_1(k,T)} \sum_{n\neq 0} \frac{1}{|\lambda_n^k|^2}|a_{k,n}|^2 \right) < +\infty \right\}$$

is L^2-*controllable to zero in time* T.

On the other hand, in [1] it was proved that the space of initial data

$$\mathcal{X}_2 = \Bigg\{ (u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega) : \quad (u^0, u^1) = \sum_{k,n} a_{k,n} \sin(n\pi x) \sin(k\pi y), \text{ such that}$$

$$\sum_k \left(e^{\frac{c_\epsilon}{T^{1-\epsilon}} k} \sum_{0<|n|\le k} |a_{k,n}|^2 + \sum_{|n|>k} (1+|n|+|k|)^{N+2} |a_{k,n}|^2 \right) < +\infty \Bigg\}$$

is L^2-controllable in time T. Here $N \in \mathbb{N}^*$, $c_\epsilon > 0$ and $\epsilon \in (0,1)$.

Remark that, since the constant $\frac{k^2}{\widehat{C}_1(k,T)}$ increases exponentially with k, the space $\mathcal{X}_1$ contains only analytic functions in $y-$variable whereas the space $\mathcal{X}_2$ contains only H^{N+2} functions in $y-$variable.

The space $\mathcal{X}$ given by Theorem 7 contains initial data which belongs to $L^2 \times H^{-1}$ but not to $H_0^1 \times L^2$. Indeed, let us consider $(a_k)_{k\ge 1} \notin \ell^2$ such that $\sum_{k\ge1} \frac{|a_k|^2}{k^2} < \infty$ and let

$$(u^0, u^1) = \sum_{k\ge 1} a_k \varphi^k_{n(k)}(x) \sin(k\pi y)$$

where $n(k) > \widehat{I}_1(k)$ for all $k \ge 1$. It follows that $(u^0, u^1) \in \mathcal{X}$ (are controllable in time T) but $(u^0, u^1) \notin H_0^1 \times L^2$.

Hence, $\mathcal{X}_1 \subsetneq \mathcal{X}$ and $\mathcal{X}_2 \subsetneq \mathcal{X}$.

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