Proceedings of the European Control Conference 2007
Kos, Greece, July 2-5, 2007

On the Approximation of the Boundary Control of the Wave Equation with Numerical Viscosity

Sorin Micu

Abstract— **This article deals with the approximation of the boundary control of the 1-D linear wave equation. Due to the spurious high frequencies, the semi-discrete models obtained with finite difference or classical finite element methods are not uniformly controllable as the discretization parameter h goes to zero (see [8]). We propose a new strategy for the approximation of the boundary control based on the addition of a numerical vanishing viscous term. This will damp out the spurious high frequencies and will ensure the existence of a convergent sequence of approximate controls. We present an approximation algorithm and some numerical experiments.**

Keywords: boundary controllability, wave equation, semi-discrete model, viscosity.

I. INTRODUCTION

This article is dealing with the boundary exact controllability property for the 1-D linear wave equation: given $T \geq 2$ and $(u^0, u^1) \in L^2(0,1) \times H^{-1}(0,1)$ there exists a control function $v \in L^2(0,T)$ such that the solution of the equation

$$\begin{cases} u'' - u_{xx} = 0 & x \in (0,1), t > 0, \\ u(t,0) = 0 & t > 0, \\ u(t,1) = v(t) & t > 0, \\ u(0,x) = u^0(x) & x \in (0,1), \\ u'(0,x) = u^1(x) & x \in (0,1), \end{cases} \tag{1}$$

satisfies

$$u(T,\cdot) = u'(T,\cdot) = 0. \tag{2}$$

By $'$ we denote the time derivative.

For the study of this controllability problem the moments theory has been successfully used (see, for instance, [1] and [14]). This approach is based on the construction of a control for each initial data equal to an eigenfunction of the wave operator.

The Hilbert Uniqueness Method (HUM) (see [9]) has offered a different and general way to study this and similar multi-dimensional problems. It provides the minimal L^2-norm control and reduces the problem to minimization of a coercive convex functional. The control with minimal L^2-norm is unique and it will be referred in the sequel as the HUM control.

In the last years there was an increasing interest for the numerical approximations of the controls. For instance HUM was used in [4], [6] and [7] to deduce numerical algorithms with finite differences in the context of the two dimensional wave equation. A one-dimensional correspondent of the problem studied in the above references may be obtained in the following way: let $N \in \mathbb{N}^*$, $h = \frac{1}{N+1}$ and an equidistant mesh of the interval $(0,1)$, $0 = x_0 < x_1 < \ldots < x_N < x_{N+1} = 1$ with $x_j = jh$, $0 \leq j \leq N+1$. A finite-difference approximation of the space derivatives leads to the following semi-discretization (space discretization) of (1)

$$\begin{cases} u_j''(t) - \frac{u_{j+1}(t)+u_{j-1}(t)-2u_j(t)}{h^2} = 0 & t > 0 \\ u_0(t) = 0, u_{N+1}(t) = v_h(t) & t > 0 \\ u_j(0) = u_j^0(x), u_j' = u_j^1(x) & 1 \leq j \leq N. \end{cases} \tag{3}$$

System (3) consists of N linear differential equations with N unknowns $u_1, u_2, \ldots, u_N$. Roughly speaking, $u_j(t)$ approximates $u(t, x_j)$, the solution of (1), provided that $(u_j^0, u_j^1)_{0 \leq j \leq N+1}$ is an approximation for the initial datum in (1). In fact we shall choose $u_j^0 = u^0(jh)$ and $u_j^1 = u^1(jh)$ for $0 \leq j \leq N+1$. In (3) the control v_h acts on the $(N+1)-$th component of the solution.

The following controllability property may be addressed for (3): *given $T > 2$, and $(u_j^0, u_j^1)_{1 \leq j \leq N} \in \mathbb{C}^{2N}$, there exists a control function $v_h \in H^1(0,T)$ such that the corresponding solution $(u_j, u_j')_{1 \leq j \leq N}$ of (3) satisfies*

$$u_j(T) = u_j'(T) = 0, \quad 1 \leq j \leq N. \tag{4}$$

If this holds for any $(u_j^0, u_j^1)_{1 \leq j \leq N} \in \mathbb{C}^{2N}$ we say that (3) is *exactly controllable in time T*.

It is not difficult to see that the controllability problem we have just addressed has a positive answer and a sequence of discrete controls $(v_h)_{h>0}$ may be found easily. What it is considerably more difficult is to show that the sequence $(v_h)_{h>0}$ converges in some way to a control v of the continuous wave equation (1). In fact, a bad behavior of the approximate controls may be readily seen. These negative numerical results are due to the fact that, in general, any semi-discrete dynamics generates spurious high-frequency oscillations that do not exist at the continuous level. Moreover, a dispersion phenomenon appears and the velocity of propagation of these high frequency numerical waves may converge to zero when the mesh size tends to zero. Note that these spurious oscillations correspond to the high frequencies of the discrete model and therefore, they weakly converge to zero when the discretization parameter h does. Consequently, their existence is compatible with the convergence of the numerical scheme. However, when we are dealing with the exact controllability problem, an uniform time for the control of all numerical waves is needed. Since the velocity of propagation of some high frequency numerical waves may tend to zero with the mesh size, the

Partially Supported by Grant MTM2005-00714 of MCYT (Spain) and Grant CEEX-05-D11-36/2005(Romania).

S. Micu is with Faculty of Mathematics and Computer Science, University of Craiova, Craiova 200497, Romania sd_micu@yahoo.com

ISBN: 978-960-89028-5-5

uniform controllability properties of the semi-discrete model may eventually disappear for a fixed time $T > 0$. As it was shown in [8] and [15], this is indeed the case when finite differences are used like in (3). The conclusion is that, with this and other similar semi-discrete models, the controllability property is not uniform as the discretization parameter h goes to zero which means that there are initial data of the wave equation (even very regular ones) for which the corresponding controls of the semi-discrete model will diverge in the L^2-norm.

Since the main problem are the spurious high frequencies generated by the discretization process, the idea of eliminating them in one way or another is natural. How to do that in an optimal and general way and how to show mathematically the uniform controllability results is not so clear. At the beginning there were numerical experiments such as a Tychonoff regularization technique ([6] and [7]), a bi-grid algorithm ([4] and [7]) and a mixed finite element approximation ([5]). In the last years many theoretical results were obtained, too. In [10] the high frequency modes of the discrete initial data are filtered out in an appropriate manner and a convergent sequence of discrete controls is constructed. In [2] a mixed finite elements method is analyzed. It consists in a discretization scheme with different basis functions: while the classical piecewise linear polynomials are used for the position, discontinuous elements approximate the velocity. With this method an explicit sequence of discrete controls which tends to the HUM control of the limit wave equation (1) is constructed. Analysis of a bi-grid method is presented in [12] where uniform results were also proved.

II. A SCHEME WITH NUMERICAL VISCOSITY

This paper considers a different method to achieve the uniform controllability as the discretization parameter h goes to zero. The idea is to introduce in the discrete equation a numerical viscous term vanishing in the limit. The dissipation has the role to damp out the bad spurious high frequencies, ensuring the uniform controllability of the system. More precisely, we consider the following alternative to (3)

$$\begin{cases} u_j''(t) - \frac{u_{j+1}(t)+u_{j-1}(t)-2u_j(t)}{h^2} - \\ \qquad -\varepsilon\frac{u_{j+1}'(t)+u_{j-1}'(t)-2u_j'(t)}{h^2} = 0 & t>0 \\ u_0(t)=0,\ u_{N+1}(t)=v_h(t) & t>0 \\ u_j(0)=u_j^0(x),\ u_j'=u_j^1(x) & 1\le j\le N \end{cases} \tag{5}$$

and we address the same controllability problem as before.

The parameter ε which multiplies the viscous term $\frac{1}{h^2}\left(u_{j+1}'(t)+u_{j-1}'(t)-2u_j'(t)\right)$ depends on the step size h and tends to zero as $h\to 0$,

$$\lim_{h\to 0}\varepsilon(h)=0. \tag{6}$$

Hence, in (5), the term $\varepsilon\frac{u_{j+1}'(t)+u_{j-1}'(t)-2u_j'(t)}{h^2}$ represents a vanishing numerical viscosity and it has the role to damp out the spurious high frequencies introduced by the numerical discretization. This will eventually ensure the boundedness of the sequence $(v_h)_{h>0}$. More precisely, we have the following result

Theorem 1: *For any $h>0$ let $\varepsilon=h$ in (5). There exists a control v_h of the semi-discrete problem (5) with the property that the sequence $(v_h)_{h>0}$ is bounded in $L^2(0,T)$. If $v\in L^2(0,T)$ is a weak limit of the sequence $(v_h)_{h>0}$ then v is a control for the continuous problem (1).*

The proof of this result is based on moments theory and biorthogonals technique (see [11] for details). Roughly speaking, in our context, an element of a biorthogonal sequence is a control for a particular eigenmode. The norms of those elements which correspond to high modes may be very large and may give an unbounded sequence of discrete controls. However, the dissipation is stronger precisely on the high modes and it acts as a compensation for the norm increasing. This mechanism leads to a uniform controllability result and therefore produces a sequence of discrete controls weakly convergent to a control of the continuous wave equation (1) when $h\to 0$.

The situation is similar to the heat equation where the biorthogonals have large norms but the dissipation fully compensates that (see [3]). The main difficulty here consists on the fact that the norm increasing of the elements of the biorthogonal sequence and the decay rate of the energy due to the dissipative term are of the same order and some very precise estimates are needed in order to compare them.

III. THE HUM APPROACH

In this section we transform the controllability problem in a minimization problem. Let us first write (5) in an equivalent vectorial form. We define the following matrix from $\mathscr{M}_{N\times N}(\mathbb{R})$

$$A_h=\frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & 0 & \dots\dots & 0 & 0\\ -1 & 2 & -1 & 0 & \dots\dots & 0 & 0\\ 0 & -1 & 2 & -1 & \dots\dots & 0 & 0\\ \dots & \dots & \dots & \dots & \dots\dots & \dots & \dots\\ 0 & 0 & 0 & 0 & \dots\dots & 2 & -1\\ 0 & 0 & 0 & 0 & \dots\dots & -1 & 2 \end{pmatrix}.$$

If we denote the unknown of (5) by $U(t)=(u_1(t),u_2(t),\dots,u_N(t))^T$ system (5) can be written as

$$\begin{cases} U''(t)+A_hU(t)+\varepsilon A_hU'(t)=F_h & t>0\\ U(0)=U_h^0,\ U'(0)=U_h^1, \end{cases} \tag{7}$$

where $U_h^0=(u_j^0)_{1\le j\le N}$ and $U_h^1=(u_j^1)_{1\le j\le N}$ and the vector F_h is given by

$$F_h(t)=\begin{pmatrix} 0\\ 0\\ .\\ .\\ .\\ 0\\ \frac{1}{h^2}\left(v_h+\varepsilon v_h'\right) \end{pmatrix}.$$

In (7) we have taken into account that $u_{N+1}(t)=v_h(t)$ and $u_0(t)=0$ for all $t>0$.

Our aim is to study the following controllability property for (7): *given $T>2$, and $(U_h^0,U_h^1)\in\mathbb{C}^{2N}$, there exists a* *control function $v_h \in H^1(0,T)$ such that the solution (U_h, U_h') of (7) satisfies*

$$U_h(T) = U_h'(T) = 0. \tag{8}$$

If this holds for any $(U_h^0, U_h^1) \in \mathbb{C}^{2N}$ we say that (7) is *exactly controllable*.

Let us define in $\mathbb{C}^N$ the canonic inner product

$$(f,g) = h\sum_{k=1}^{N} f_k \overline{g}_k \tag{9}$$

where $f = (f_k)_{1\le k\le N}$ and $g = (g_k)_{1\le k\le N}$ belong to $\mathbb{C}^N$.

Also, we consider in $\mathbb{C}^{2N}$ the inner product defined by

$$(f,g)_1 = (A_h f^1, g^1) + (f^2, g^2), \tag{10}$$

where $f = (f_k)_{1\le k\le 2N}$ and $g = (g_k)_{1\le k\le 2N}$ are two vectors from $\mathbb{C}^{2N}$ with $f^1 = (f_k)_{1\le k\le N}$, $f^2 = (f_k)_{N+1\le k\le 2N}$, $g^1 = (g_k)_{1\le k\le N}$, $g^2 = (g_k)_{N+1\le k\le 2N}$. The corresponding norm will be denoted by $||\cdot||_1$.

The following discrete duality product will be needed in the study of the control problem

$$\langle (f^1, f^2), (g^1, g^2)\rangle_D = -(f^1, g^2) + (f^2 + \varepsilon A_h f^1, g^1) \tag{11}$$

where $f = (f^1, f^2)$ and $g = (g^1, g^2)$ are as above.

Firstly, we deduce a variational characterization of the controllability property for the system (7). Let (ϕ_h, ϕ_h') be the solution of the adjoint backward homogeneous system

$$\begin{cases} \phi''(t) + A_h\phi(t) - \varepsilon A_h \phi'(t) = 0 \quad t \in (0,T), \\ \phi(T) = \phi_h^0,\ \phi'(T) = \phi_h^1 \end{cases} \tag{12}$$

The unknown of (12) is the vector-valued function $\phi_h(t) = (\phi_1(t), \phi_2(t), ..., \phi_N(t))^T$. Note that, to simplify the notation, we do not make explicit the dependence in h of the components $\phi_j(t)$.

Multiplying system (12) by the solution U_h of system (7) and integrating in time we obtain the following characterization of the controllability property.

Theorem 2: *Given $T > 0$, system (7) is exactly controllable if, for any $(U_h^0, U_h^1) \in \mathbb{C}^{2N}$, there exists $v_h \in H_0^1(0,T)$ which satisfies*

$$\begin{aligned} &\int_0^T v_h(t)\overline{\frac{\phi_N(t) - \varepsilon\phi_N'}{h}}dt + \\ &\qquad + \langle (U_h^0, U_h^1), (\phi_h(0), \phi_h'(0))\rangle_D = 0, \end{aligned} \tag{13}$$

for any $(\phi_h^0, \phi_h^1) \in \mathbb{C}^{2N}$, (ϕ_h, ϕ_h') being the corresponding solution of (12).

Given $T > 2$, let us also consider a cut-off function $\rho \in C^\infty[0,T]$ with the property that there exists a positive number $\varepsilon > 0$ such that $T - 2\varepsilon > 2$ and

$$\begin{array}{rl} \text{(i)} & \operatorname{supp}(\rho) \subset (\varepsilon/2, T - \varepsilon/2), \\ \text{(ii)} & 0 \le \rho(t) \le 1 \text{ for all } t \in [0,T], \\ \text{(iii)} & \rho(t) \ge 1/2 \text{ for all } t \in [\varepsilon, T-\varepsilon]. \end{array} \tag{14}$$

Finally, let $J : \mathbb{C}^{2N} \to \mathbb{C}$ be a functional defined by

$$\begin{aligned} J(\phi_h^0, \phi_h^1) = \frac{1}{2}\int_0^T \rho(t)\left|\frac{\phi_N(t) - \varepsilon\phi_N'(t)}{h}\right|^2 dt + \\ + \langle (U_h^0, U_h^1), (\phi_h(0), \phi_h'(0))\rangle_D, \end{aligned} \tag{15}$$

where (ϕ_h, ϕ_h') is the solution of the adjoint homogeneous system (12).

Theorem 3: *Given any $T > 2$ and $(U_h^0, U_h^1) \in \mathbb{C}^{2N}$, the functional J defined by (15) has a unique minimizer $(\widehat{\phi}_h^0, \widehat{\phi}_h^1) \in \mathbb{C}^{2N}$. If $v_h \in H_0^1(0,T)$ is given by*

$$v_h = \rho\frac{\widehat{\phi}_N - \varepsilon\widehat{\phi}_N'}{h}, \tag{16}$$

where $(\widehat{\phi}_h, \widehat{\phi}_h')$ is the solution of (12) with initial data $(\widehat{\phi}_h^0, \widehat{\phi}_h^1)$, then v_h is a control for (7).

We define

$$\begin{aligned} &a : \mathbb{C}^{2N} \times \mathbb{C}^{2N} \to \mathbb{C}, \\ &a((\phi_h^0, \phi_h^1), (\xi_h^0, \xi_h^1)) = \\ &= \int_0^T \rho(t)\left(\frac{\phi_N - \varepsilon\phi_N'}{h}\right)\left(\frac{\xi_N - \varepsilon\xi_N'}{h}\right)dt, \end{aligned} \tag{17}$$

$$\begin{aligned} &L : \mathbb{C}^{2N} \to \mathbb{C}, \\ &L(\phi_h^0, \phi_h^1) = \langle (U_h^0, U_h^1), (\phi_h(0), \phi_h'(0))\rangle_D \end{aligned} \tag{18}$$

where (ϕ, ϕ') and (ξ, ξ') are the solution of (12) with initial data (ϕ_h^0, ϕ^1, h) and (ξ_h^0, ξ^1, h) respectively.

With these notations we have that

$$J(\phi_h^0, \phi_h^1) = \frac{1}{2}a((\phi_h^0, \phi_h^1), (\phi_h^0, \phi_h^1)) + L(\phi_h^0, \phi_h^1).$$

In order to describe the algorithm we use to approximate the minimizer of J let us first make some remarks concerning the quantities which define J.

i) *Computation of $L(\phi_h^0, \phi_h^1)$.*
If (τ, τ') is the solution of the equation

$$\begin{cases} \tau''(t) + A_h\tau(t) + \varepsilon A_h\tau'(t) = 0, \quad t \in (0,T), \\ \tau(0) = U_h^0,\ \tau'(0) = U_h^1 \end{cases} \tag{19}$$

and (ϕ, ϕ') is the solution of (12) with initial data (ϕ_h^0, ϕ_h^1) then

$$\begin{aligned} &\langle (U_h^0, U_h^1), (\phi'(0), \phi'(0))\rangle_D = \\ &= \langle (\tau(T), \tau'(T)), (\phi_h^0, \phi_h^1)\rangle_D. \end{aligned}$$

Moreover, for any (ξ_h^0, ξ_h^1), we have that

$$\langle (\tau(T), \tau'(T)), (\xi_h^0, \xi_h^1)\rangle_D = \left((f_h^0, f_h^1), (\xi_h^0, \xi_h^1)\right)_1$$

where (f_h^0, f_h^1) are given by

$$\begin{cases} f_h^1 = -\tau(T), \\ A_h f_h^0 = \tau'(T) + \varepsilon A_h\tau(T). \end{cases} \tag{20}$$

Hence,

$$L(\phi_h^0, \phi_h^1) = \left((f_h^0, f_h^1), (\phi_h^0, \phi_h^1)\right)_1 \tag{21}$$

where (f_h^0, f_h^1) is given by (19)-(20).

ii) *Computation of* $a((\phi_h^0,\phi_h^1),(\xi_h^0,\xi_h^1))$.
For any (ϕ_h^0,ϕ_h^1) and (ξ_h^0,ξ_h^1), we have that

$$\int_0^T \rho(t)\left(\frac{\phi_N-\varepsilon\phi_N'}{h}\right)\left(\frac{\xi_N-\varepsilon\xi_N'}{h}\right)dt =$$

$$=\langle (w(T),w'(T)),(\xi_h^0,\xi_h^1)\rangle_D$$

where (w,w') is the solution of

$$\left\{\begin{array}{l} w''+A_h w+\varepsilon A_h w'=F,\\ w(0)=0,\ w'(0)=0,\end{array}\right. \tag{22}$$

and the vector F is given by

$$F(t)=\begin{pmatrix} 0\\ 0\\ .\\ .\\ .\\ 0\\ \frac{1}{h^3}\left[\rho\left(\phi_N-\varepsilon\phi_N'\right)+\varepsilon(\rho\left(\phi_N-\varepsilon\phi_N'\right))'\right]\end{pmatrix}.$$

Moreover,

$$\langle (\tau(T),\tau'(T)),(\xi_h^0,\xi_h^1)\rangle_D=\left((\tilde f_h^0,\tilde f_h^1),(\xi_h^0,\xi_h^1)\right)_1$$

where $(\tilde f_h^0,\tilde f_h^1)$ is given by

$$\left\{\begin{array}{c}\tilde f_h^1=-w(T),\\ A_h\tilde f_h^0=w'(T)+\varepsilon A_h w(T)\end{array}\right. \tag{23}$$

Hence,

$$a((\phi_h^0,\phi_h^1),(\xi_h^0,\xi_h^1))=\left((\tilde f_h^0,\tilde f_h^1),(\xi_h^0,\xi_h^1)\right)_1 \tag{24}$$

where $(\tilde f_h^0,\tilde f_h^1)$ is given by (22)-(23).

iii) *Computation of the gradient of* J.
We have that

$$\nabla J(\phi_h^0,\phi_h^1)(\xi_h^0,\xi_h^1)=$$

$$=a((\phi_h^0,\phi_h^1),(\xi_h^0,\xi_h^1))+L(\xi_h^0,\xi_h^1)=$$

$$=\left((f_h^0+\tilde f_h^0,f_h^1+\tilde f_h^1),(\xi_h^0,\xi_h^1)\right)_1.$$

Hence,

$$\nabla J(\phi_h^0,\phi_h^1)=(g^0,g^1) \tag{25}$$

where (g^0,g^1) are given by

$$\left\{\begin{array}{c} g^1=-\tau(T)-w(T),\\ A_h g^0=\tau'(T)+w'(T)+\varepsilon A_h(\tau(T)+w(T)).\end{array}\right. \tag{26}$$

with (τ,τ') and (w,w') solutions of (19) and (22) respectively.

IV. Conjugate Gradient Method

For the sake of completeness, we present in this section the steps of the conjugate gradient method applied to a general variational problem in a Hilbert space (see [6]).

Let H be a Hilbert space with the inner product $(\cdot,\cdot)$ and the norm $||\ ||$ and consider the following general variational problem: find $u\in H$ such that

$$a(u,\varphi)+L(\varphi)=0,\quad \forall\varphi\in H. \tag{27}$$

We suppose that $a:H\times H\to\mathbb{R}$ is a bilinear, continuous, symmetric and coercive form in H and $L:H\to\mathbb{R}$ is a linear and continuous form in H. These hypothesis ensure the existence of a unique solution $u\in H$ of problem (27).

Now, define the functional $J:H\to\mathbb{R}$,

$$J(\varphi)=\frac{1}{2}a(\varphi,\varphi)+L(\varphi).$$

Under the above conditions, it follows that the problem

$$J(\widehat\varphi)=\min_{\varphi\in H}J(\varphi) \tag{28}$$

has a unique solution $\widehat\varphi\in H$ which is the solution u of (27).

In order to approximate the solution $\widehat\varphi$ of the minimization problem (28) we may use the following conjugate gradient method:

0. Initialization: take $\varphi_0\in H$.
1. Compute the gradient $g_0=\nabla J(\varphi_0)\in H$ by using
$$(g_0,\psi)=(\nabla J(\varphi_0),\psi)=\lim_{h\to 0}\frac{J(\varphi_0+h\psi)-J(\varphi_0)}{h}=$$
$$=a(\varphi_0,\psi)+L(\psi),\quad\forall\psi\in H.$$
2. If $||g_0||\le\varepsilon$ take $\widehat\varphi=\varphi_0$ and finish.
3. If $||g_0||>\varepsilon$ take the descent direction $d_0=-g_0$.

Suppose that we have φ_n, $g_n=\nabla J(\varphi_n)$ and d_n and we compute φ_{n+1}, g_{n+1} and d_{n+1} as it follows:

4. Compute the step descent
$$\rho_n=-\frac{(g_n,d_n)}{a(d_n,d_n)}$$
Since $(g_n,g_j)=0$, $0\le j\le n-1$, we also have
$$\rho_n=\frac{(g_n,g_n)}{a(d_n,d_n)}.$$
5. Compute the next approximation
$$\varphi_{n+1}=\varphi_n+\rho_n d_n.$$
6. Compute the new gradient $g_{n+1}=\nabla J(\varphi_{n+1})$ by using
$$(g_{n+1},\psi)=a(\varphi_{n+1},\psi)+L(\psi),\quad\forall\psi\in H$$
or, taking into account that $\varphi_{n+1}=\varphi_n+\rho_n d_n$
$$(g_{n+1},\psi)=(g_n,\psi)+\rho_n a(d_n,\psi),\quad\forall\psi\in H.$$
7. If $||g_{n+1}||\le\varepsilon$ take $\widehat\varphi=\varphi_{n+1}$ and finish.
8. If $||g_{n+1}||>\varepsilon$ compute the new descent direction
$$d_{n+1}=-g_{n+1}+\frac{||g_{n+1}||^2}{||g_n||^2}d_n.$$

9. Make $n = n + 1$ and go to 4.

Remark 1: In order to calculate $a(\varphi, \psi)$ we may use the fact that, for any $\varphi \in H$ there exists a unique $\zeta = \zeta(\varphi) \in H$ such that

$$a(\varphi, \psi) = (\zeta(\varphi), \psi), \quad \forall \psi \in H. \tag{29}$$

Hence, to compute ρ_n at step 4, we use the formula

$$\rho_n = \frac{(g_n, g_n)}{(d_n, \zeta(d_n))}$$

and to compute g_{n+1} at step 6 we use the formula

$$g_{n+1} = g_n + \rho_n \zeta(d_n).$$

Remark 2: The gradient $\nabla J(\varphi_0)$ is a linear and continuous map from H to $\mathbb{R}$. However, since H is a Hilbert space we have used the Riesz identification and considered that $\nabla J(\varphi_0) \in H$ and $\nabla J(\varphi_0)(\psi) = (\nabla J(\varphi_0), \psi)$.

V. A NUMERICAL ALGORITHM

The following numerical algorithm is a version of that introduced in [6] and uses conjugate gradient method in order to find the minimizer of J defined by (15) and consequently the approximate control v_h.

0. Initialization: take $(\varphi_0^0, \varphi_0^1) \in \mathbb{R}^N \times \mathbb{R}^N$.
1. Compute the gradient

$$(g_0^0, g_0^1) = \nabla J(\varphi_0^0, \varphi_0^1) \in \mathbb{R}^N \times \mathbb{R}^N.$$

To do that, solve the equations:

$$\begin{cases} \varphi'' + A_h \varphi - \varepsilon A_h \varphi' = 0 \text{ in } (0,T) \\ \varphi(T,\cdot) = \varphi_0^0, \quad \varphi'(T,\cdot) = \varphi_0^1, \end{cases} \tag{30}$$

$$\begin{cases} w'' + A_h w + \varepsilon A_h w' = F \text{ in } (0,T) \\ w(0,\cdot) = w'(0,\cdot) = 0, \end{cases} \tag{31}$$

$$\begin{cases} \tau'' + A_h \tau + \varepsilon A_h \tau = 0 \text{ in } (0,T) \\ \tau(0,\cdot) = u^0, \quad \tau'(0) = u^1, \end{cases} \tag{32}$$

$$\begin{cases} g_0^1 = -\tau(T) - w(T), \\ A_h g_0^0 = \tau'(T) + w'(T) + \varepsilon A_h(\tau(T) + w(T) \end{cases} \tag{33}$$

where

$$F = \begin{pmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 0 \\ \frac{1}{h^3}\left[\rho\left(\phi_N - \varepsilon \phi_N'\right) + \varepsilon(\rho\left(\phi_N - \varepsilon \phi_N'\right))'\right] \end{pmatrix}.$$

2. If $||(g_0^0, g_0^1)||_1 \leq \varepsilon$ take $(\widehat{\varphi}^0, \widehat{\varphi}^1) = (\varphi_0^0, \varphi_0^1)$ and finish.
3. If $||(g_0^0, g_0^1)||_1 > \varepsilon$ take the descent direction

$$(d_0^0, d_0^1) = -(g_0^0, g_0^1).$$

Suppose that we have $(\varphi_n^0, \varphi_n^1)$, $(g_n^0, g_n^1) = \nabla J(\varphi_n^0, \varphi_n^1)$ and (d_n^0, d_n^1). Compute $(\varphi_{n+1}^0, \varphi_{n+1}^1)$, (g_{n+1}^0, g_{n+1}^1) and (d_{n+1}^0, d_{n+1}^1) as it follows:

4. Solve the equations:

$$\begin{cases} \varphi'' + A_h \varphi - \varepsilon A_h \varphi' = 0 \text{ in } (0,T) \\ \varphi(T,\cdot) = d_n^0, \quad \varphi'(T,\cdot) = d_n^1, \end{cases} \tag{34}$$

$$\begin{cases} w'' + A_h w + \varepsilon A_h w' = F \text{ in } (0,T) \\ w(0,\cdot) = w'(0,\cdot) = 0, \end{cases} \tag{35}$$

$$\begin{cases} \tilde{g}_n^1 = -w(T), \\ A_h \tilde{g}_n^0 = w'(T) + \varepsilon A_h w(T), \end{cases} \tag{36}$$

where

$$F = \begin{pmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 0 \\ \frac{1}{h^3}\left[\rho\left(\phi_N - \varepsilon \phi_N'\right) + \varepsilon(\rho\left(\phi_N - \varepsilon \phi_N'\right))'\right] \end{pmatrix}.$$

5. Compute the step descent,

$$\rho_n = -\frac{\left((g_n^0, g_n^1), (d_n^0, d_n^1)\right)}{a((d_n^0, d_n^1), (d_n^0, d_n^1))} =$$

$$= \frac{||(g_n^0, g_n^1)||^2}{a((d_n^0, d_n^1), (d_n^0, d_n^1))} = \frac{||(g_n^0, g_n^1)||^2}{((\tilde{g}_n^0, \tilde{g}_n^1), (d_n^0, d_n^1))}.$$

6. Compute the next approximation

$$(\varphi_{n+1}^0, \varphi_{n+1}^1) = (\varphi_n^0, \varphi_n^1) + \rho_n (d_n^0, d_n^1).$$

7. Compute the new gradient $(g_{n+1}^0, g_{n+1}^1) = \nabla J(\varphi_{n+1}^0, \varphi_{n+1}^1)$ by using the relation

$$(g_{n+1}^0, g_{n+1}^1) = (g_n^0, g_n^1) + \rho_n (\tilde{g}_n^0, \tilde{g}_n^1).$$

8. If $||(g_{n+1}^0, g_{n+1}^1)|| \leq \varepsilon$ take $(\widehat{\varphi}^0, \widehat{\varphi}^1) = (\varphi_{n+1}^0, \varphi_{n+1}^1)$ and finish.
9. If $||(g_{n+1}^0, g_{n+1}^1)|| > \varepsilon$ compute the new descent direction

$$(d_{n+1}^0, d_{n+1}^1) = -(g_{n+1}^0, g_{n+1}^1) +$$

$$+ \frac{||(g_{n+1}^0, g_{n+1}^1)||^2}{||(g_n^0, g_n^1)||^2}(d_n^0, d_n^1).$$

10. Make $n = n + 1$ and go to 4.

VI. NUMERICAL EXPERIMENTS

In this section we present some numerical experiments in which we approximate the control HUM v of (1) by using the discretization (5). More precisely, the algorithm presented in the previous section is implemented.

In order to solve the differential equations in t we use a time discretization: given a time interval $[0,T]$ we introduce a uniform mesh $\{t^k = k\Delta t\}_{k=0,\dots,M}$ with time-step Δt and $T = M\Delta t$. A fully-discrete scheme may be obtained by replacing the time derivative $w''(t^k)$ by the central finite difference $(w(t^{k+1}) - 2w(t^k) + w(t^{k-1}))/\Delta t^2$. It is known that, if $\Delta t = h$ and $\varepsilon = 0$, the exact control is obtained. This very special situation is due to the exact resolution of the 1-D wave equation by finite differences with these numerical parameters. Note that this correspond to (3). However, as

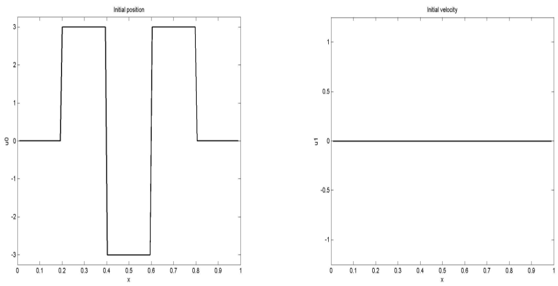

Fig. 1. Initial data (u^0, u^1) to be controlled.

we have said before, if $\Delta t \neq h$, (3) no longer provides good approximations for the control.

In order to obtain good approximations in the case $\Delta t \neq h$ we use (5). In our experiments with (5) we shall chose

$$\varepsilon = h, \quad \Delta t = lh, \quad l = 7/8.$$

Other values of the Courant number l do not alter significantly the numerical results. Hence, scheme (5) is more robust and may be applied successfully to other types of equations.

Numerical example: We consider the initial data depicted in Figure 1,

$$u^0(x) = \begin{cases} 3 & \text{if } x \in [1/5, 2/5] \cup [3/5, 4/5] \\ -3 & \text{if } x \in (2/5, 3/5) \\ 0 & \text{if } x \in [0, 1/5) \cup (4/5, 0] \end{cases}$$

$$u^1(x) = 0 \text{ if } x \in [0,1].$$

As it may be immediately seen, u^0 belongs to $L^2(0,1)$ and it is discontinuous. Note that both initial data have compact support.

Since we were not interested in the optimal time, we have taken $T = 4$. The exact value of the HUM control, v, is obtained by considering the algorithm presented in Section V with $\varepsilon = 0$ (i.e. based on (3)), $\Delta t = h$ and the usual central finite difference scheme in time. In Fig. 2 it is drawn with solid line. We note that $||v||_{L^2} = 1.1874$.

As we have said before, the algorithm based on the finite difference scheme (3) fails to converge when $\Delta t/h < 1$. On the contrary, (5) provides satisfactory approximations of the control. Indeed, four approximations of the control with different values of h are given in Fig. 2 with dashed line. These are obtained by considering the algorithm presented in Section V and based on (5), $\Delta t = 7/8\,h$ and the Newmark scheme in time with parameters $\beta = 1/2$ and $\gamma = 1/4$ which is unconditionally stable (see [13]). However, we note the slow convergency of the algorithm in this singular case.

TABLE I
NUMERICAL RESULTS OBTAINED WITH $\Delta t = 7/8h$.

h	1/100	1/500	1/1000	1/5000
$\|v_h\|_{L^2}$	0.3506	0.7529	0.8999	1.0693
$\|v_h - v\|_{L^2}$	0.8817	0.5432	0.4429	0.1580

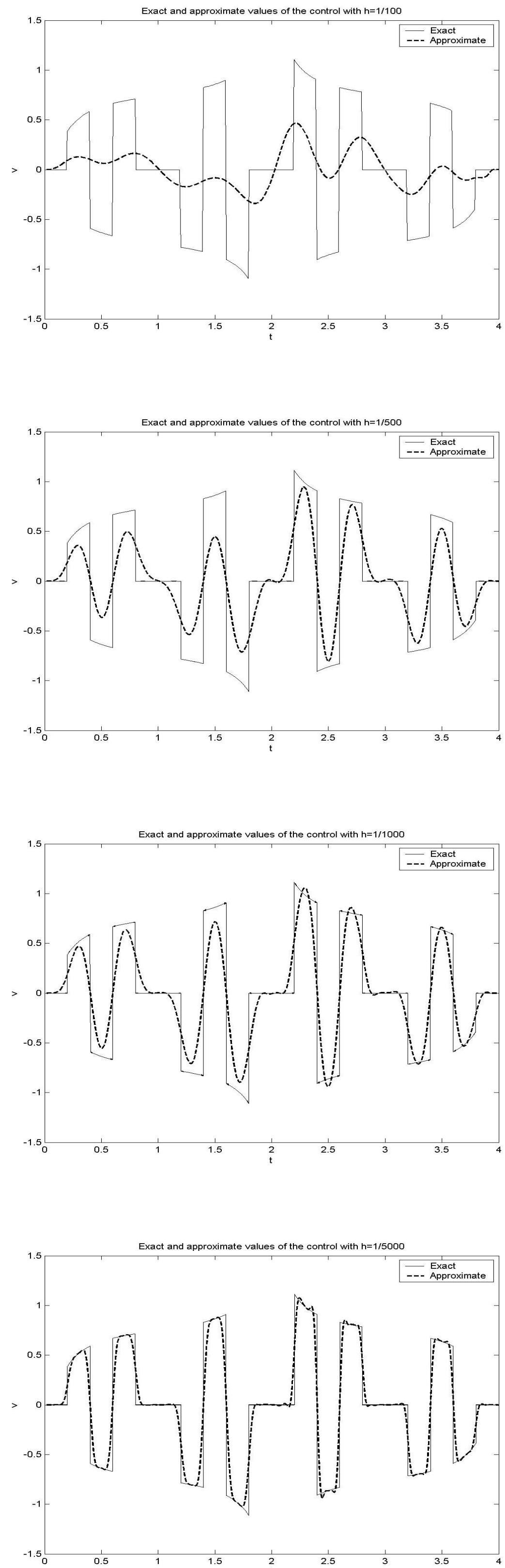

Fig. 2. Approximations of the control with four different values of h, $h = 1/100, 1/500, 1/1000, 1/5000$.

VII. Comments

Acknowledgements: The author is grateful to Enrique Zuazua for several discussions and suggestions related to this work and for encouraging him to study this problem. Also, the author thanks Carlos Castro for invaluable help concerning the programm and numerical experiments.

References

[1] S.A. Avdonin and S.A. Ivanov, *Families of exponentials. The method of moments in controllability problems for distributed parameter systems*, Cambridge University Press, 1995.

[2] C. Castro and S. Micu, *Boundary controllability of a linear semi-discrete 1-D wave equation derived from a mixed finite element method*, Numer. Math., vol. 102 (3), 2006, pp. 413-462.

[3] H.O. Fattorini and D.L. Russell, *Exact controllability theorems for linear parabolic equations in one space dimension*, Arch. Rat. Mech. Anal., vol. 4, 1971, pp. 272-292.

[4] R. Glowinski, *Ensuring well-posedness by analogy; Stokes problem and boundary control for the wave equation*, J. Comput. Phys., vol. 103, 1992, pp. 189-221.

[5] R. Glowinski, W. Kinton and M.F. Wheeler, *A mixed finite element formulation for the boundary controllability of the wave equation*, Int. J. Numer. Methods Eng., vol. 27 (3), 1989, pp. 623-636.

[6] R. Glowinski, C.H. Li and J.-L. Lions, *A numerical approach to the exact boundary controllability of the wave equation (I). Dirichlet controls: Description of the numerical methods*, Jap. J. Appl. Math., vol. 7, 1990, pp. 1-76.

[7] R. Glowinski and J.-L. Lions, *Exact and approximate controllability for distributed parameter systems*, Acta Numerica 1996, pp. 159-333.

[8] J.A. Infante and E. Zuazua, *Boundary observability for the space semi-discretization of the 1-D wave equation*, M2AN, vol. 33 (2), 1999, pp. 407-438.

[9] J.-L. Lions, *Contrôlabilité exacte perturbations et stabilisation de systèmes distribués*, Tome 1, Masson, Paris, 1988.

[10] S. Micu, *Uniform boundary controllability of a semi-discrete 1-D wave equation*, Numer. Math., vol. 91, 2002, pp. 723-768.

[11] S. Micu, *Uniform boundary controllability of a semi-discrete 1-D wave equation with vanishing viscosity*, submitted for publication, 2006.

[12] M. Negreanu and E. Zuazua, *Uniform boundary controllability of a discrete 1-D wave equation*, System and Control Letters, vol. 48 (3-4), 2003, pp. 261-280.

[13] P.A. Raviart and J.-M. Thomas, *Introduction à l'analyse numérique des équations aux dérivées partielles*, Masson, Paris, 1983.

[14] D.L. Russell, *Controllability and stabilizaton theory for linear partial differential equations: recent progress and open questions*, SIAM Review, vol. 20 (4), 1978, pp. 639-737.

[15] E. Zuazua, *Boundary observability for the finite difference space semi-discretizations of the 2-D wave equation in the square*, J. Math. Pures Appl., vol. 78, 1999, pp. 523-563.

[16] E. Zuazua, *Propagation, Observation, Control and Numerical Approximation of Waves approximated by finite difference methods*, SIAM Review, vol. 47(2), 2005, 197-243.