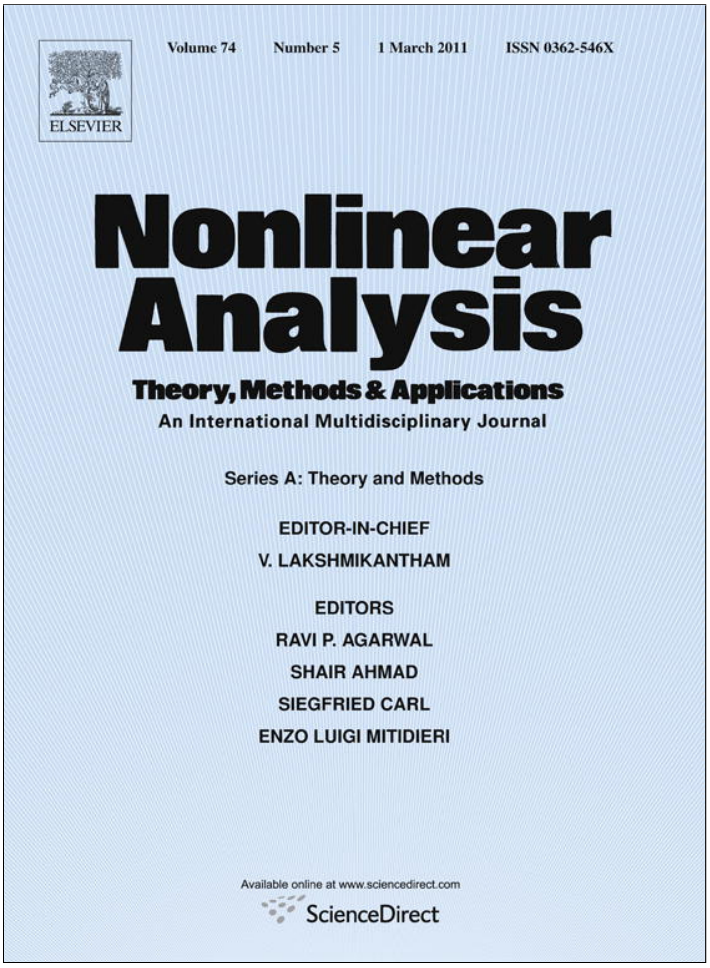
Volume 74 Number 5 1 March 2011 ISSN 0362-546X
ELSEVIER
Nonlinear Analysis
Theory, Methods & Applications
An International Multidisciplinary Journal
Series A: Theory and Methods
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Nonlinear Analysis 74 (2011) 1641–1652

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Boundary optimal control for nonlinear antiplane problems

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ARTICLE INFO

Article history:
Received 16 June 2010
Accepted 18 October 2010

MSC:
74M10
74M15
49J20
49K20

Keywords:
Slip dependent frictional contact
Quasi-variational inequalities
Optimal control
Regularization

ABSTRACT

We consider a nonlinear antiplane problem which models the deformation of an elastic cylindrical body in frictional contact with a rigid foundation. The contact is modelled with Tresca's law of dry friction in which the friction bound is slip dependent.

The aim of this article is to study an optimal control problem which consists of leading the stress tensor as close as possible to a given target, by acting with a control on the boundary of the body. The existence of at least one optimal control is proved. Next we introduce a regularized problem, depending on a small parameter ρ, and we study the convergence of the optimal controls when ρ tends to zero. An optimality condition is delivered for the regularized problem.

1. Introduction

The purpose of the present paper is to study an optimal control problem related to a static mechanical model describing a bilateral frictional contact between a deformable body and a rigid foundation under the small deformation hypothesis. The envisaged mechanical model is an antiplane one. We recall that the antiplane shear deformation is the expected deformation of a very long cylinder loaded in the direction of its generators. In such a model the displacement vectorial field is parallel to the generators of the cylinder and it is independent of the axial coordinate. An excellent reference on this topic is the review article [1]. Due to their simplicity in the writing of the equations without loss of physical relevance, antiplane models have enjoyed special attention in recent years (see, for instance [2–11]). The antiplane models appear in the technical literature in engineering, describing the functioning of various mechanisms, and in geophysics, focusing on the deformation of the tectonic plates, in particular on earthquakes.

In this work we consider an antiplane model whose nonlinear feature comes from the frictional contact condition. Using Tresca's law of dry friction in which the friction bound prescribed on the contact zone depends on the slip, the weak formulation of the model consists of an elliptic quasi-variational inequality, (PS1). For details on the model we refer the interested reader to [11]. The optimal control problem related to the proposed model, denoted by (POC1), consists of minimizing a quadratic functional which is dependent on the deformation's norm and is penalized by the control's norm. Firstly, the existence of at least one solution of (POC1) is proved. The main difficulties of this minimization problem are related to the non convex dependence of the functional we want to minimize with respect to its argument. Therefore, the standard approach based on convexity cannot be used and has to be replaced by an indirect method which takes advantage of the particular form of our quasi-variational inequality and the compactness of the trace operator. According to our study, we can act on a part of the boundary in order to minimize the distance between the stress σ and a given target σ_d.

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doi:10.1016/j.na.2010.10.034

Once we have proved the existence of a solution for (POC1), we introduce a regularized variational state problem (PS2), depending on a parameter ρ, whose solution converges to the solution of the state problem (PS1) when ρ tends to zero. The optimal control problem corresponding to this regularization, denoted by (POC2), has at least one solution. Related to the regularized problem, we deliver an optimality condition and we show that any solution of (POC2) is weakly convergent to a solution of (POC1), as ρ tends to zero. Both the regularization technique and the optimality condition are extremely useful in the numerical approximation processes. Moreover, the results about (POC2) may help the investigation of the optimal control for the original (POC1) in the sense of the approximation to the optimal control in the weak topology. To obtain an optimality condition (a Pontryagin's maximum principle) for the original problem (POC1) is an open problem.

The optimal control of variational inequalities has a long history, see for instance [12–18]. Moreover, the recent book [19] is devoted to the optimal control of linear or nonlinear elliptic problems, including variational inequalities. Despite their mechanical relevance, optimal control problems for contact models are not so frequent in the literature. A notable exception is [20], where a quasistatic frictional bilateral contact problem is considered. Unlike [20], in our problem the friction bound is slip dependent, a fact that enforces the nonlinear character of the mathematical problem, leading us to a more difficult mathematical problem even in the case of the simplified antiplane setting, for static processes. More importantly, the standard convexity arguments used in [20], cannot be applied in our case due to the lack of convexity.

The rest of the paper is structured as follows. Section 2 introduces some notation and useful results. In Section 3 we describe the mechanical problem, recall its weak solvability and state the optimal control problem (POC1). The main result of Section 3 is the existence of at least one optimal control. In Section 4 we discuss the regularized problem (POC2) and a necessary optimality condition is provided for it. In Section 5 we prove two convergence results.

2. Preliminaries

In this section we introduce notations and present some results used throughout the paper. Let $(X, (\cdot,\cdot)_X, \|\cdot\|_X)$ be a Hilbert space.

Definition 2.1. A map $\phi : X \to \mathbb{R}$ is *Gâteaux differentiable at* $u \in X$ if there exists an element $\nabla\phi(u) \in X$ such that

$$\lim_{t\to 0} \frac{\phi(u+tv)-\phi(u)}{t} = (\nabla\phi(u), v)_X \quad \forall v \in X.$$

The element $\nabla\phi(u)$ which satisfies the relation above is called the *gradient of* ϕ *at* u. The function $\phi : X \to \mathbb{R}$ is said to be *Gâteaux differentiable* if it is Gâteaux differentiable at every point of X. In this case, the operator $\nabla\phi : X \to X$ that maps every element $u \in X$ into the element $\nabla\phi(u)$ is called the *gradient operator* of ϕ.

Definition 2.2. Let $\phi : X \to (-\infty, \infty]$ and $u \in X$. The *subdifferential* of ϕ at u is the set

$$\partial\phi(u) = \{f \in X \mid \phi(v) - \phi(u) \geq (f, v-u)_X \quad \forall v \in X\}.$$

Denote

$$D(\partial\phi) = \{u \in X \mid \partial\phi(u) \neq \emptyset\}.$$

A function ϕ is said to be *subdifferentiable at* $u \in X$ if $u \in D(\partial\phi)$. A function ϕ is called *subdifferentiable* if it is subdifferentiable at each point $u \in X$.

Lemma 2.3. *Let $\phi : X \to (-\infty,\infty]$ be a convex and Gâteaux differentiable functional. Then, ϕ is subdifferentiable, $\partial\phi$ is a single-valued operator on X and $\partial\phi(u) = \{\nabla\phi(u)\}$ for all $u \in X$.*

The proof of Lemma 2.3 is standard and can be found in many books (see, for instance, [11]).

For the convenience of the reader, we recall below a result in the theory of quasi-variational inequalities. Given $f \in X$, we consider the problem of finding an element $u \in X$ such that

$$a(u, v-u) + j(u, v) - j(u, u) \geq (f, v-u)_X \quad \forall v \in X. \tag{2.1}$$

Assuming that

$$\left.\begin{array}{l} a : X \times X \to \mathbb{R} \text{ is a bilinear, symmetric form, such that} \\ \text{there exists } M > 0 : |a(u, v)| \leq M\|u\|_X\|v\|_X \quad \forall u, v \in X, \\ \text{there exists } m > 0 : a(v, v) \geq m\|v\|_X^2 \quad \forall v \in X, \end{array}\right\} \tag{2.2}$$

and

$$\left.\begin{array}{l} j : X \times X \to \mathbb{R} \text{ is a functional such that} \\ \text{for every } \eta \in X, j(\eta,\cdot) \text{ is convex and lower semi-continuous on } X, \\ \text{there exists } \alpha \geq 0 \text{ such that} \\ j(\eta_1, v_2) - j(\eta_1, v_1) + j(\eta_2, v_1) - j(\eta_2, v_2) \\ \leq \alpha\|\eta_1-\eta_2\|_X\|v_1-v_2\|_X \quad \forall \eta_1, \eta_2, v_1, v_2 \in X, \end{array}\right\} \tag{2.3}$$

the following existence and uniqueness result takes place.

Theorem 2.4. *Assume that (2.2) and (2.3) hold. Moreover, assume that $m > \alpha$. Then, for each $f \in X$, the quasi-variational elliptic inequality (2.1) has a unique solution which depends Lipschitz continuously on f.*

The proof of this theorem can be found for example in [11].
Finally, we recall a theorem that will be used in Section 4.3.

Theorem 2.5. *Let $\mathcal{B}$ be a Banach space, X and Y two reflexive Banach spaces. Let also be given two $\mathcal{C}^1$ functions*

$$F : \mathcal{B} \times X \to Y, \quad L : \mathcal{B} \times X \to Y.$$

We suppose that, for all $\beta \in \mathcal{B}$,

(i) *There exists a unique $\tilde{u}(\beta)$ such that $F(\beta, \tilde{u}(\beta)) = 0$,*
(ii) *$\partial_2 F(\beta, \tilde{u}(\beta))$ is an isomorphism from X onto Y.*

Then, $J(\beta) = L(\beta, \tilde{u}(\beta))$ is differentiable and, for every $\zeta \in \mathcal{B}$,

$$\frac{\mathrm{d}J}{\mathrm{d}\beta}(\beta)\zeta = \partial_1 L\left(\beta, \tilde{u}(\beta)\right)\zeta - \left\langle p(\beta), \partial_1 F\left(\beta, \tilde{u}(\beta)\right)\zeta\right\rangle_{Y',Y}, \tag{2.4}$$

where $p(\beta) \in Y'$ is the adjoint state, unique solution of

$$\left[\partial_2 F\left(\beta, \tilde{u}(\beta)\right)\right]^* p(\beta) = \partial_2 L\left(\beta, \tilde{u}(\beta)\right) \quad \text{in } X'. \tag{2.5}$$

For the proof of Theorem 2.5, we refer the reader to [15].

3. A frictional contact problem

In the first part of this section we state the mechanical problem, list the assumptions on the data and recall an existence and uniqueness result for the weak solution. In a second part we introduce and solve an optimal control problem related to this mechanical model.

3.1. The model and its weak solvability

Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, connected set, with Lipschitz continuous boundary Γ partitioned in three measurable parts $\Gamma_1, \Gamma_2, \Gamma_3$ such that the Lebesgue measures of Γ_i are strictly positive, for each $i \in \{1, 2, 3\}$.
We consider the following mechanical problem: *find a displacement field $u : \bar{\Omega} \to \mathbb{R}$ such that*

$$\mathrm{div}(\mu(x)\nabla u(x)) + f_0(x) = 0 \quad \text{in } \Omega, \tag{3.6}$$
$$u(x) = 0 \quad \text{on } \Gamma_1, \tag{3.7}$$
$$\mu(x)\partial_\nu u(x) = f_2(x) \quad \text{on } \Gamma_2, \tag{3.8}$$
$$\left.\begin{aligned} &|\mu(x)\partial_\nu u(x)| \le g(x, |u(x)|), \\ &\mu(x)\partial_\nu u(x) = -g(x, |u(x)|)\frac{u(x)}{|u(x)|} \quad \text{if } u(x) \neq 0 \end{aligned}\right\} \text{on } \Gamma_3. \tag{3.9}$$

The proposed problem models the antiplane shear deformation of an elastic, isotropic, nonhomogeneous cylindrical body, in frictional contact on Γ_3 with a rigid foundation. Referring the body to a cartesian coordinate system $Ox_1x_2x_3$ such that the generators of the cylinder are parallel with the axis Ox_3, the domain $\Omega \subset Ox_1x_2$ denotes the cross section of the cylinder. The function $\mu = \mu(x_1, x_2) : \bar{\Omega} \to \mathbb{R}$ denotes a coefficient of the material (one of *Lamé's coefficients*), the functions $f_0 = f_0(x_1, x_2) : \Omega \to \mathbb{R}, f_2 = f_2(x_1, x_2) : \Gamma_2 \to \mathbb{R}$ are related to the density of the body forces and the density of the surface traction, respectively and $g : \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}_+$ is a given function called *friction bound*. Here $\boldsymbol{\nu} = (\nu_1, \nu_2)$, $\nu_i = \nu_i(x_1, x_2)$, for each $i \in \{1, 2\}$, represents the outward unit normal vector to the boundary of Ω and $\partial_\nu u = \nabla u \cdot \boldsymbol{\nu}$. The unknown of the problem is the function $u = u(x_1, x_2) : \bar{\Omega} \to \mathbb{R}$ that represents the third component of the displacement vector $\boldsymbol{u}$. We recall that, in the antiplane physical setting, the displacement vectorial field has the particular form $\boldsymbol{u} = (0, 0, u(x_1, x_2))$. Once the displacement field u is determined, we compute the stress tensor $\boldsymbol{\sigma}$ as follows,

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & 0 & \mu\dfrac{\partial u}{\partial x_1} \\ 0 & 0 & \mu\dfrac{\partial u}{\partial x_2} \\ \mu\dfrac{\partial u}{\partial x_1} & \mu\dfrac{\partial u}{\partial x_2} & 0 \end{pmatrix}. \tag{3.10}$$

The structure of the mechanical problem is the following: (3.6) represents the equilibrium equation, (3.7) is the displacement boundary condition, (3.8) is the traction boundary condition and (3.9) is a frictional contact condition. To write (3.9) we used Tresca's law of dry friction with slip dependent friction bound g. To give an example of such a function g we consider

$$g : \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}_+ \quad g(x, r) = k\left(1 + \delta e^{-r}\right); \quad k, \delta > 0. \tag{3.11}$$

It is easy to observe that, in this case, the friction bound decreases from the value $k(1+\delta)$ to the value k. The slip dependent friction law (3.9) with the friction bound given by (3.11) describes the slip weakening phenomenon which appears in the study of geophysical problems, see for example [3,4,21]. More details concerning the physical significance of the problem (3.6)–(3.9) can be found in [11].

The weak solvability of this model is based on Theorem 2.4.

Let us assume that

$$\mu \in L^\infty(\Omega), \quad \mu(x) \geq \mu^* > 0 \text{ a.e. in } \Omega, \tag{3.12}$$

$$f_0 \in L^2(\Omega), \quad f_2 \in L^2(\Gamma_2), \tag{3.13}$$

and

$$\left.\begin{array}{l} \text{(a) } g : \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}_+; \\ \text{(b) There exists } L_g > 0 : |g(x, r_1) - g(x, r_2)| \leq L_g |r_1 - r_2| \\ \qquad \forall r_1, r_2 \in \mathbb{R}_+, \text{ a.e. } x \in \Gamma_3; \\ \text{(c) The mapping } x \mapsto g(x, r) \\ \qquad \text{is Lebesgue measurable on } \Gamma_3, \forall r \in \mathbb{R}_+; \\ \text{(d) The mapping } x \mapsto g(x, 0) \text{ belongs to } L^2(\Gamma_3). \end{array}\right\} \tag{3.14}$$

Furthermore, we consider the Hilbert space

$$V = \{v \in H^1(\Omega) \mid \gamma v = 0 \text{ a.e. on } \Gamma_1\};$$

γ denotes here the Sobolev trace operator. We endow the space V with the following inner product,

$$(\cdot, \cdot)_V : V \times V \to \mathbb{R}, \quad (u, v)_V = \int_\Omega \nabla u(x) \cdot \nabla v(x) \mathrm{d}x.$$

The corresponding norm will be denoted by $\|\cdot\|_V$. We recall that $\gamma : H^1(\Omega) \to L^2(\Gamma)$ is a linear, continuous and compact operator. Moreover, there exists $c_0 > 0$ such that

$$\|\gamma v\|_{L^2(\Gamma)} \leq c_0 \|v\|_V \quad \forall v \in V. \tag{3.15}$$

Finally, we assume

$$\mu^* > c_0^2 L_g. \tag{3.16}$$

Let us define the operator

$$A : V \to V; \quad (Au, v)_V = \int_\Omega \mu(x) \nabla u(x) \cdot \nabla v(x) \mathrm{d}x \quad \forall u, v \in V, \tag{3.17}$$

and the functional

$$j : V \times V \to \mathbb{R}; \quad j(u, v) = \int_{\Gamma_3} g(x, |\gamma u(x)|) |\gamma v(x)| \mathrm{d}s \quad \forall u, v \in V. \tag{3.18}$$

Next, using Riesz's representation theorem, we define $f \in V$ as follows,

$$(f, v)_V = \int_\Omega f_0(x) v(x) \mathrm{d}x + \int_{\Gamma_2} f_2(x) \gamma v(x) \mathrm{d}\Gamma \quad \forall v \in V. \tag{3.19}$$

We are led to the following weak formulation of the problem (3.6)–(3.9): *Given* $f_0 \in L^2(\Omega)$ and $f_2 \in L^2(\Gamma_2)$, *find* $u \in V$ *such that*

$$(Au, v - u)_V + j(u, v) - j(u, u) \geq (f, v - u)_V \quad \forall v \in V, \tag{3.20}$$

where f is given by (3.19).

By the hypothesis (3.12), it is straightforward to verify that

$$a : V \times V \to \mathbb{R}, \quad a(u, v) = (Au, v)_V,$$

is a continuous, V-elliptic, symmetric bilinear form. In addition, we note that for every $\eta \in V$, $j(\eta, \cdot) : V \to \mathbb{R}$ is a convex and continuous functional. Furthermore, by the definition of the functional j, taking into account (3.14) and (3.15), it can be proved (see [11]) that for any $\eta_1, \eta_2, v_1, v_2 \in V$,

$$j(\eta_1, v_2) - j(\eta_1, v_1) + j(\eta_2, v_1) - j(\eta_2, v_2) \leq L_g c_0^2 \|\eta_1 - \eta_2\|_V \|v_1 - v_2\|_V. \tag{3.21}$$

Thus, all the hypotheses of Theorem 2.4 are fulfilled and the following result takes place.

Theorem 3.6. *Assume (3.12), (3.13), (3.14) and (3.16). Then, the problem (3.20) has a unique solution $u \in V$ which depends Lipschitz continuously on f.*

3.2. The optimal control problem

For a fixed function $f_0 \in L^2(\Omega)$, we consider the following *state problem*.
(PS1) *Let* $f_2 \in L^2(\Gamma_2)$ *(called control) be given. Find* $u \in V$ *such that*

$$(Au, v-u)_V + j(u,v) - j(u,u) \geq \int_\Omega f_0(x)(v(x)-u(x))\mathrm{d}x + \int_{\Gamma_2} f_2(x)(\gamma v(x) - \gamma u(x))\mathrm{d}\Gamma \quad \forall v \in V. \tag{3.22}$$

Using Theorem 3.6, we deduce that, for every control $f_2 \in L^2(\Gamma_2)$, the *state problem* (PS1) has a unique solution $u \in V$, $u = u(f_2)$. In addition, taking into account the properties of the operator A and of the functional j, choosing in (3.22) $v = 0_V$, we deduce that

$$\|u\|_V \leq \frac{1}{\mu^*}\left(\|f_0\|_{L^2(\Omega)} + c_0\|f_2\|_{L^2(\Gamma_2)}\right), \tag{3.23}$$

where μ^* is the constant in (3.12) and c_0 is the constant in (3.15). Now, we would like to act a control on Γ_2 such that the resulting stress $\boldsymbol{\sigma}$ be as close as possible to a given target

$$\boldsymbol{\sigma}_d = \begin{pmatrix} 0 & 0 & \mu\dfrac{\partial u_d}{\partial x_1} \\ 0 & 0 & \mu\dfrac{\partial u_d}{\partial x_2} \\ \mu\dfrac{\partial u_d}{\partial x_1} & \mu\dfrac{\partial u_d}{\partial x_2} & 0 \end{pmatrix}$$

where u_d is a given function. Note that, since

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_d\|_{L^2(\Omega)^{3\times 3}} = \sqrt{2}\|\mu\nabla(u-u_d)\|_{L^2(\Omega)} \leq \sqrt{2}\|\mu\|_{L^\infty(\Omega)}\|u-u_d\|_V,$$

$\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}_d$ will be close to one another if the difference between the functions u and u_d is small in the sense of V-norm. To give an example of a target of interest, u_d, we can consider $u_d = 0$. In this situation, by acting a control f_2 on Γ_2, the tension $\boldsymbol{\sigma}$ is small in the sense of L^2-norm, even if f_0 does not vanish in Ω.

Let $\alpha, \beta > 0$ be two positive constants and let us define the following functional

$$L : L^2(\Gamma_2) \times V \to \mathbb{R}, \quad L(f_2, u) = \frac{\alpha}{2}\|u-u_d\|_V^2 + \frac{\beta}{2}\|f_2\|^2_{L^2(\Gamma_2)}. \tag{3.24}$$

Furthermore, we denote

$$\mathcal{V}_{ad} = \left\{[u, f_2] \mid [u, f_2] \in V \times L^2(\Gamma_2), \text{ such that (3.22) is verified}\right\}$$

and we introduce the following *optimal control problem*,
(POC1) *Find* $[u^*, f_2^*] \in \mathcal{V}_{ad}$ *such that* $L(f_2^*, u^*) = \min_{[u,f_2]\in\mathcal{V}_{ad}}\left\{L(f_2, u)\right\}$.
The following result holds.

Theorem 3.7. *Assume* (3.12)–(3.14) *and* (3.16). *Then, (POC1) has at least one solution* (u^*, f_2^*).

Proof. From definition (3.24) of the functional L, we deduce that

$$\inf_{[u,f_2]\in\mathcal{V}_{ad}}\{L(f_2,u)\} \in [0, \infty). \tag{3.25}$$

There exists a minimizing sequence $([u^n, f_2^n])_n \subset \mathcal{V}_{ad}$ such that

$$\lim_{n\to\infty} L\left(f_2^n, u^n\right) = \inf_{[u,f_2]\in\mathcal{V}_{ad}}\{L(f_2,u)\}.$$

Due to (3.25), the sequences $(f_2^n)_n$ and $(u^n)_n$ are bounded in $L^2(\Gamma_2)$ and V, respectively. Thus, there exists $[u^*, f_2^*] \in V \times L^2(\Gamma_2)$ such that, passing eventually to a subsequence, but keeping the notation to simplify the writing, we have

$$\begin{aligned} &u^n \rightharpoonup u^* \text{ in } V \quad \text{as } n\to\infty, \\ &f_2^n \rightharpoonup f_2^* \text{ in } L^2(\Gamma_2) \quad \text{as } n\to\infty. \end{aligned} \tag{3.26}$$

In fact, it can be proved that

$$u^n \to u^* \text{ in } V \quad \text{as } n\to\infty. \tag{3.27}$$

Indeed, from the V-ellipticity of the bilinear form a,

$$\mu^*\|u^n - u^*\|_V^2 \leq \left(Au^n, u^n - u^*\right)_V + \left(Au^*, u^* - u^n\right)_V. \tag{3.28}$$

On the other hand, due to (3.22), we can write

$$\left(Au^n, u^* - u^n\right)_V + j\left(u^n, u^*\right) - j\left(u^n, u^n\right) \geq \left(f_0, u^* - u^n\right)_{L^2(\Omega)} + \left(f_2^n, \gamma u^* - \gamma u^n\right)_{L^2(\Gamma_2)}. \tag{3.29}$$

Next, taking into account the definition of the functional j, the hypotheses (3.14) and the compactness of the trace operator γ, we deduce that

$$\lim_{n\to\infty}\left[j\left(u^n, u^*\right) - j\left(u^n, u^n\right)\right] = 0.$$

Moreover, since $(f_2^n)_n$ is bounded in $L^2(\Gamma_2)$ and γ is a compact operator, we obtain

$$\lim_{n\to\infty}\left(f_2^n, \gamma u^* - \gamma u^n\right)_{L^2(\Gamma_2)} = 0.$$

Consequently, from (3.29) we deduce that

$$\limsup_{n\to\infty}\left(Au^n, u^n - u^*\right)_V \leq 0.$$

Using now (3.28), we obtain (3.27).

Since for every $n \in \mathbb{N}^*$, $[u^n, f_2^n] \in \mathcal{V}_{ad}$, we pass to the limit as $n \to \infty$ in (3.22) and we deduce that $[u^*, f_2^*] \in \mathcal{V}_{ad}$ and it is a solution of (POC1). □

A solution of (POC1) will be called an *optimal pair*. The second component of the optimal pair is called an *optimal control*.

Remark 3.8. Any optimal control f_2^* minimizes the functional

$$J : L^2(\Gamma_2) \to \mathbb{R}, \quad J(f_2) = \frac{\alpha}{2}\|u(f_2) - u_d\|_V^2 + \frac{\beta}{2}\|f_2\|_{L^2(\Gamma_2)}^2 \tag{3.30}$$

where $u = u(f_2)$ is the solution of (3.22).

Generally, the functional J defined by (3.30) is not convex. This is the reason why we cannot use a convexity argument in the proof of Theorem 3.7 and we have introduced the functional L where the control and the state can be viewed as independent variables. It is only in the set $\mathcal{V}_{ad}$ where these two quantities are related.

Remark 3.9. Let f_2^* be a minimizer of J and suppose that

$$\text{there exists } f_{2d} \in L^2(\Gamma_2) \text{ such that } u_d = u(f_{2d}). \tag{3.31}$$

We have that

$$\frac{\alpha}{2}\|u(f_2^*) - u_d\|_V^2 + \frac{\beta}{2}\|f_2^*\|_{L^2(\Gamma_2)}^2 \leq J(f_{2d}) = \frac{\beta}{2}\|f_{2d}\|_{L^2(\Gamma_2)}^2$$

and consequently

$$\|u(f_2^*) - u_d\|_V^2 \leq \frac{\beta}{\alpha}\left(\|f_{2d}\|_{L^2(\Gamma_2)}^2 - \|f_2^*\|_{L^2(\Gamma_2)}^2\right).$$

Thus, by taking β arbitrarily small, we deduce that we may lead the displacement u as close as we want to the prescribed valued $u_d = u(f_{2d})$. The same conclusion would hold without condition (3.31) if the set $\{u(f_2) : f_2 \in L^2(\Gamma_2)\}$ would be dense in V. To our knowledge, this property is an open problem. For a similar density argument see [22].

Remark 3.10. Due to the lack of convexity of $\mathcal{V}_{ad}$ and J, the uniqueness of the optimal control is not ensured. Nevertheless, it is easy to see that if an optimal pair $[u^*, f_2^*] \in \mathcal{V}_{ad}$ also minimizes the functional L in $L^2(\Gamma_2) \times V$, then the uniqueness of the solution of (POC1) does hold. Indeed, this is a consequence of the strict convexity of L in $L^2(\Gamma_2) \times V$. Note that, in this very particular case, $[u^*, f_2^*] = [u_d, 0]$.

4. A regularized problem

In this section we introduce a regularized variational problem by replacing the functional j, given by (3.18), with a more regular one. As in the previous section, we investigate first the solvability of the regularized problem and we introduce an optimal control problem for it. In addition, in this section we obtain a necessary optimality condition.

4.1. A regularized state problem

Let $\rho > 0$. We define a functional $j_\rho : V \times V \to \mathbb{R}$ as follows,

$$j_\rho(u, v) = \int_{\Gamma_3} g\left(x, \sqrt{(\gamma u(x))^2 + \rho^2} - \rho\right)\left(\sqrt{(\gamma v(x))^2 + \rho^2} - \rho\right) \mathrm{d}\Gamma \quad \forall u, v \in V. \tag{4.32}$$

Taking into account (3.14), we deduce for the functional j_ρ the following properties:

$$j_\rho(u,\cdot)\geq 0 \quad \forall u\in V; \qquad j_\rho(u,0_V)=0 \quad \forall u\in V;$$

$$j_\rho(\eta_1,v_2)-j_\rho(\eta_1,v_1)+j_\rho(\eta_2,v_1)-j_\rho(\eta_2,v_2)\leq L_g c_0^2\|\eta_1-\eta_2\|_V\|v_1-v_2\|_V \quad \forall \eta_1,\eta_2,v_1,v_2\in V. \tag{4.33}$$

In addition, in the second argument, the functional j_ρ is convex, lower semi-continuous and Gâteaux differentiable. More precisely, for every $(u,v)\in V\times V$,

$$\lim_{h\to 0}\frac{j_\rho(u,v+hw)-j_\rho(u,v)}{h}=\big(\nabla_2 j_\rho(u,v),w\big)_V \quad \forall w\in V,$$

where

$$\big(\nabla_2 j_\rho(u,v),w\big)_V=\int_{\Gamma_3} g\left(x,\sqrt{(\gamma u(x))^2+\rho^2}-\rho\right)\frac{\gamma v(x)\gamma w(x)}{\sqrt{(\gamma v(x))^2+\rho^2}}\mathrm{d}\Gamma \quad \forall w\in V. \tag{4.34}$$

Replacing in the problem (3.20) the functional j with the functional j_ρ, we state the following problem: *Given* $\rho>0$, $f_0\in L^2(\Omega)$ *and* $f_2\in L^2(\Gamma_2)$, *find the displacement field* $u^\rho\in V$ *such that*

$$\begin{aligned}(Au^\rho,v-u^\rho)_V+j_\rho(u^\rho,v)-j_\rho(u^\rho,u^\rho) \ \geq\ & \int_\Omega f_0(x)\,(v(x)-u^\rho(x))\,\mathrm{d}x\\ &+\int_{\Gamma_2} f_2(x)\,(\gamma v(x)-\gamma u^\rho(x))\,\mathrm{d}\Gamma \quad \forall v\in V.\end{aligned} \tag{4.35}$$

Theorem 4.11. *Assume (3.12)–(3.14) and (3.16). Then, problem (4.35) has a unique solution $u^\rho\in V$ which depends Lipschitz continuously on f.*

The proof of this theorem follows again from Theorem 2.4. To simplify the writing, in the remaining parts of this section we shall denote the solution u^ρ of (4.35) by u, dropping out the index ρ.

4.2. The regularized optimal control problem

Let us fix $\rho>0$ and $f_0\in L^2(\Omega)$. We introduce the following regularized problem

(PS2) *Let $f_2\in L^2(\Gamma_2)$ (called control). Find $u\in V$ such that*

$$(Au,v-u)_V+j_\rho(u,v)-j_\rho(u,u)\geq (f_0,v-u)_{L^2(\Omega)}+(f_2,\gamma v-\gamma u)_{L^2(\Gamma_2)} \quad \forall v\in V. \tag{4.36}$$

By Theorem 4.11, for every $f_2\in L^2(\Gamma_2)$, the problem (PS2) has a unique solution $u\in V$, $u=u(f_2)$. In addition, by similar arguments with those used in order to get (3.23), we deduce

$$\|u\|_V\leq \frac{1}{\mu^*}\left(\|f_0\|_{L^2(\Omega)}+c_0\|f_2\|_{L^2(\Gamma_2)}\right). \tag{4.37}$$

Remark 4.12. By Riesz's representation theorem, there exists an unique $z\in V$ such that

$$(z,v)_V=\int_\Omega f_0(x)v(x)\mathrm{d}x \quad \forall v\in V. \tag{4.38}$$

Furthermore, there exists an unique $y(f_2)\in V$ such that

$$(y(f_2),v)_V=\int_{\Gamma_2} f_2(x)\gamma v(x)\mathrm{d}\Gamma \quad \forall v\in V. \tag{4.39}$$

Let $u\in V$ be the unique solution of (PS2). Denoting

$$\partial_2 j_\rho(u,u)=\left\{\zeta\in V \mid j_\rho(u,v)-j_\rho(u,u)\geq(\zeta,v-u)_V\ \forall v\in V\right\},$$

the inequality (4.36) is equivalent with the following inclusion

$$z+y(f_2)-Au\in\partial_2 j_\rho(u,u).$$

Since, in the second argument, j_ρ is convex and Gâteaux differentiable, see Lemma 2.3 in Section 2, we have

$$\partial_2 j_\rho(u,u)=\left\{\nabla_2 j_\rho(u,u)\right\}.$$

Thus, (4.36) is equivalent with the nonlinear equation

$$Au+\nabla_2 j_\rho(u,u)=z+y(f_2). \tag{4.40}$$

Let us define the following admissible set,

$$\mathcal{V}_{ad}^{\rho} = \left\{[u, f_2] \mid [u, f_2] \in V \times L^2(\Gamma_2), \text{ such that (4.36) is verified}\right\}.$$

Using the functional L, given by (3.24), we introduce an *optimal control problem* as follows.

(POC2) *Find* $[\bar{u}, \bar{f}_2] \in \mathcal{V}_{ad}^{\rho}$ such that $L(\bar{f}_2, \bar{u}) = \min_{[u,f_2]\in\mathcal{V}_{ad}^{\rho}} \left\{L(f_2, u)\right\}$.

With arguments similar to those used in Theorem 3.7, the following result can be proved.

Theorem 4.13. *Assume (3.12)–(3.14) and (3.16). Then, (POC2) has at least one solution* $(\bar{u}, \bar{f}_2)$.

A solution of (POC2) is called a *regularized optimal pair* and the second component $\bar{f}_2$ is called a *regularized optimal control*. Like for (POC1), we have that $\bar{f}_2$ is a minimizer of the functional

$$J : L^2(\Gamma_2) \to \mathbb{R}, \quad J(f_2) = \frac{\alpha}{2}\|u(f_2) - u_d\|_V^2 + \frac{\beta}{2}\|f_2\|_{L^2(\Gamma_2)}^2 \tag{4.41}$$

where $u = u(f_2)$ is the solution of the state problem (PS2). Again, the functional J is not convex in general.

4.3. Optimality condition

In this last part of Section 3 we obtain an optimality condition for the problem (POC2), replacing the hypotheses (b) and (d) in (3.14), with the following stronger ones,

$$\left.\begin{array}{l} g(x,\cdot) \in C^1 \text{ a.e. } x \in \Gamma_3, \\ \text{there exists } L_g > 0 : |\partial_2 g(x,r)| \le L_g \forall r \in \mathbb{R}_+, \text{ a.e } x \in \Gamma_3, \\ \text{there exists } M > 0 : |g(x,r)| \le M \forall r \in \mathbb{R}_+, \text{ a.e. } x \in \Gamma_3. \end{array}\right\} \tag{4.42}$$

Here and below, by ∂_i, $i \in \{1, 2\}$, we denote a partial derivative.

Note that the function g in the example (3.11) verifies (4.42) and, in addition, $\partial_2 g(x, r) < 0$, for all $x \in \Gamma_3$ and $r \in \mathbb{R}_+$.

We have the following result.

Theorem 4.14 (*Optimality Condition*). *Any optimal control* $\bar{f}_2$ *of the state problem (PS2) verifies*

$$\bar{f}_2 = -\frac{1}{\beta}\gamma\left(p(\bar{f}_2)\right), \tag{4.43}$$

where $p(\bar{f}_2)$ *is the unique solution of the variational equation*

$$\alpha\,(\bar{u} - u_d, w)_V = \left(p(\bar{f}_2), Aw + D_2^2 j_\rho(\bar{u}, \bar{u})w\right)_V \quad \forall w \in V, \tag{4.44}$$

and, for all $v \in V$,

$$\begin{aligned}\left(D_2^2 j_\rho(\bar{u}, \bar{u})w, v\right)_V = &\int_{\Gamma_3} \partial_2 g\left(x, \sqrt{(\gamma\bar{u}(x))^2 + \rho^2} - \rho\right) \frac{(\gamma\bar{u}(x))^2}{(\gamma\bar{u}(x))^2 + \rho^2}\gamma w(x)\gamma v(x)\mathrm{d}\Gamma \\ &+ \int_{\Gamma_3} g(x, \sqrt{(\gamma\bar{u}(x))^2 + \rho^2} - \rho)\frac{\rho^2}{\left[(\gamma\bar{u}(x))^2 + \rho^2\right]^{3/2}}\gamma w(x)\gamma v(x)\mathrm{d}\Gamma,\end{aligned}$$

$\bar{u} = u(\bar{f}_2)$ *being the solution of (PS2) with* $f_2 = \bar{f}_2$.

Proof. In order to obtain an optimality condition we will apply Theorem 2.5.

Using the definition of the functional (3.24), for every $(u, f_2) \in V \times L^2(\Gamma_2)$, we get

$$\partial_1 L(f_2, u) : L^2(\Gamma_2) \to \mathbb{R}; \quad \partial_1 L(f_2, u)\zeta = \beta(f_2, \zeta)_{L^2(\Gamma_2)} \quad \forall \zeta \in L^2(\Gamma_2),$$

and

$$\partial_2 L(f_2, u) : V \to \mathbb{R}; \quad \partial_2 L(f_2, u)v = \alpha\,(u - u_d, v)_V \quad \forall v \in V.$$

Now, let us define $F : L^2(\Gamma_2) \times V \to V$,

$$F(f_2, u) = Au + \nabla_2 j_\rho(u, u) - z - y(f_2), \quad \forall f_2 \in L^2(\Gamma_2),\ u \in V, \tag{4.45}$$

where z is given by (4.38) and $y(f_2)$ by (4.39).

We shall apply Theorem 2.5 with F given by (4.45) and J from (4.41).

For every $(u, f_2) \in V \times L^2(\Gamma_2)$, we have

$$\partial_1 F(f_2, u) : L^2(\Gamma_2) \to V; \quad \partial_1 F(f_2, u)\zeta = -y(\zeta) \quad \forall \zeta \in L^2(\Gamma_2),$$

and

$$\partial_2 F(f_2, u): V \to V; \quad \partial_2 F(f_2, u)q = Aq + D_2^2 j_\rho(u, u)q \quad \forall q \in V,$$

where, for all $w \in V$,

$$\left(D_2^2 j_\rho(u, u)q, w\right)_V = \int_{\Gamma_3} \partial_2 g\left(x, \sqrt{(\gamma u(x))^2 + \rho^2} - \rho\right) \frac{(\gamma u(x))^2}{(\gamma u(x))^2 + \rho^2} \gamma q(x) \gamma w(x) \mathrm{d}\Gamma$$
$$+ \int_{\Gamma_3} g\left(x, \sqrt{(\gamma u(x))^2 + \rho^2} - \rho\right) \frac{\rho^2}{\left[(\gamma u(x))^2 + \rho^2\right]^{3/2}} \gamma q(x) \gamma w(x) \mathrm{d}\Gamma.$$

Let $f_2 \in L^2(\Gamma_2)$ and $u(f_2) \in V$ be the corresponding solution of the regularized state problem (PS2). Then, $\partial_2 F(f_2, u(f_2)) : V \to V$ is an isomorphism.

Indeed, $\partial_2 F(f_2, u(f_2)) : V \to V$ is an isomorphism from V to V if and only if for every $h \in V$, there exists an unique element $v^* \in V$ such that

$$\left(Av^*, w\right)_V + \left(D_2^2 j_\rho\left(u(f_2), u(f_2)\right) v^*, w\right)_V = (h, w)_V. \tag{4.46}$$

Let us denote by $a_{f_2,u(f_2)}$ the following bilinear functional

$$a_{f_2,u(f_2)} : V \times V \to \mathbb{R}; \quad a_{f_2,u(f_2)}(v, w) = (Av, w)_V + \left(D_2^2 j_\rho\left(u(f_2), u(f_2)\right) v, w\right)_V.$$

Taking into account the properties of the operator A and the functional j_ρ, together with the hypotheses (3.16) and (4.42), we deduce that the bilinear form $a_{f_2,u(f_2)}(\cdot, \cdot)$ is continuous and V-elliptic. Thus, the existence of a unique element $v^* \in V$ such that (4.46) holds is obtained from Lax–Milgram's lemma.

We note that the adjoint of the operator $\partial_2 F(f_2, u(f_2))$ is also an isomorphism. Let us denote by $p(f_2) \in V$ the unique solution of the equation

$$\left[\partial_2 F\left(f_2, u(f_2)\right)\right]^* p(f_2) = \partial_2 L\left(f_2, u(f_2)\right).$$

Since

$$\partial_2 L\left(f_2, u(f_2)\right) w = \left(\left[\partial_2 F\left(f_2, u(f_2), \right)\right]^* p(f_2), w\right)_V = \left(p(f_2), \partial_2 F\left(f_2, u(f_2)\right) w\right)_V,$$

$p(f_2) \in V$ is in fact the unique solution of the following variational equation,

$$\alpha\left(u(f_2) - u_d, w\right)_V = \left(p(f_2), Aw + D_2^2 j_\rho\left(u(f_2), u(f_2)\right) w\right)_V. \tag{4.47}$$

Now, from Theorem 2.5, we deduce that for all $q \in L^2(\Gamma_2)$,

$$\frac{\mathrm{d}J}{\mathrm{d}f_2}(f_2)q = \partial_1 L\left(f_2, u(f_2)\right) q - \left(p(f_2), \partial_1 F\left(f_2, u(f_2)\right) q\right)_V$$
$$= \beta(f_2, q)_{L^2(\Gamma_2)} + \left(p(f_2), y(q)\right)_V,$$

where $y(q)$ is defined by (4.39). Since $\bar{f}_2$ is a minimizer of J, we obtain that

$$\frac{\mathrm{d}J}{\mathrm{d}f_2}(\bar{f}_2)q = 0$$

and thus, we get the following optimality condition

$$\beta\left(\bar{f}_2, q\right)_{L^2(\Gamma_2)} + \left(p(\bar{f}_2), y(q)\right)_V = 0 \quad \forall q \in L^2(\Gamma_2), \tag{4.48}$$

where $p(\bar{f}_2)$ is the unique solution of (4.47) with $f_2 = \bar{f}_2$.

By taking into account (4.39), relation (4.48) is equivalent to (4.43) that concludes the proof. □

Remark 4.15. The replacement of the functional j from (PS1) by the Gâteaux differentiable function j_ρ in (PS2) has enabled us to deduce the optimality condition (4.43)–(4.44). How to deduce an optimality condition for (PS1) is an open question.

5. Convergence results

Our investigation in this section is made under the hypotheses (3.12), (3.13), (a) and (c) of (3.14), (3.16) and (4.42). In the first part of this section, we prove that the unique solution of the problem (4.36) converges to the unique solution of the problem (3.22). More precisely, the following theorem takes place.

Theorem 5.16. *Let $\rho > 0$, $f_0 \in L^2(\Omega)$ and $f_2 \in L^2(\Gamma_2)$ be given. If $u^\rho, u \in V$ are the solutions of problems (PS2) and (PS1), respectively, then,*

$$u^\rho \to u \text{ in } V \quad \text{as } \rho \to 0. \tag{5.49}$$

Proof. Let us take $v = u^\rho$ in (3.22) and $v = u$ in (4.36). By adding, we get

$$\begin{aligned}(Au - Au^\rho, u - u^\rho)_V &\le j(u, u^\rho) - j(u, u) + j_\rho(u^\rho, u) - j_\rho(u^\rho, u^\rho)\\ &\le j(u, u^\rho) - j(u, u) + j(u^\rho, u) - j(u^\rho, u^\rho)\\ &\quad + \left[j_\rho(u^\rho, u) - j(u^\rho, u)\right] - \left[j_\rho(u^\rho, u^\rho) - j(u^\rho, u^\rho)\right]\\ &\le L_g c_0^2 \|u - u^\rho\|_V^2 + |j_\rho(u^\rho, u) - j(u^\rho, u)| + |j_\rho(u^\rho, u^\rho) - j(u^\rho, u^\rho)|.\end{aligned}$$

Thus, by the properties of the operator A we have,

$$\left(\mu^* - L_g c_0^2\right) \|u^\rho - u\|_V^2 \le |j_\rho(u^\rho, u) - j(u^\rho, u)| + |j_\rho(u^\rho, u^\rho) - j(u^\rho, u^\rho)|.$$

Taking into account the definitions of the functionals j and j_ρ, we obtain

$$|j_\rho(u^\rho, u) - j(u^\rho, u)| \to 0 \quad \text{as } \rho \to 0.$$

Using in addition (4.37), we get

$$|j_\rho(u^\rho, u^\rho) - j(u^\rho, u^\rho)| \to 0 \quad \text{as } \rho \to 0.$$

By (3.16) we deduce now (5.49). □

Next, we prove a convergence result involving the solutions of the problems (POC2) and (POC1).

Theorem 5.17. *Let $[\bar{u}^\rho, \bar{f_2}^\rho]$ be a solution of the problem (POC2). Then, there exists a solution of the problem (POC1), $[u^*, f_2^*]$, such that*

$$\begin{aligned}&\bar{u}^\rho \to u^* \text{ in } V \quad \text{as } \rho \to 0,\\ &\bar{f_2}^\rho \rightharpoonup f_2^* \text{ in } L^2(\Gamma_2) \quad \text{as } \rho \to 0.\end{aligned} \tag{5.50}$$

Proof. Let u_0^ρ be the unique solution of the problem (PS2) with $f_2 = 0_{L^2(\Gamma_2)}$.

$$L\left(0_{L^2(\Gamma_2)}, u_0^\rho\right) = \frac{\alpha}{2}\|u_0^\rho - u_d\|_V^2 \le \alpha\left(\|u_0^\rho\|_V^2 + \|u_d\|_V^2\right).$$

Since

$$\|u_0^\rho\|_V \le \frac{1}{\mu^*}\|f_0\|_{L^2(\Omega)}$$

then, there exists $c^* > 0$ such that

$$L\left(\bar{f_2}^\rho, \bar{u}^\rho\right) \le L\left(0_{L^2(\Gamma_2)}, u_0^\rho\right) \le c^*\left(\|f_0\|_{L^2(\Omega)}^2 + \|u_d\|_V^2\right).$$

Thus, we deduce that $(\bar{u}^\rho, \bar{f_2}^\rho)_\rho$ is a bounded sequence in $V \times L^2(\Gamma_2)$. Consequently, there exists $[u^*, f_2^*] \in V \times L^2(\Gamma_2)$ such that

$$\bar{u}^\rho \rightharpoonup u^* \text{ in } V \quad \text{as } \rho \to 0$$

and

$$\bar{f_2} \rightharpoonup f_2^* \text{ in } L^2(\Gamma_2) \quad \text{as } \rho \to 0.$$

In fact,

$$\bar{u}^\rho \to u^* \text{ in } V \quad \text{as } \rho \to 0. \tag{5.51}$$

Indeed, since the operator A is strongly monotone, by (4.35), we have

$$\begin{aligned}\mu^*\|\bar{u}^\rho - u^*\|_V^2 &\le \left(Au^*, u^* - \bar{u}^\rho\right)_V + \left(A\bar{u}^\rho, \bar{u}^\rho - u^*\right)_V\\ &\le \left(Au^*, u^* - \bar{u}^\rho\right)_V + j_\rho\left(\bar{u}^\rho, u^*\right) - j_\rho(\bar{u}^\rho, \bar{u}^\rho) + \left(f_0, u^* - \bar{u}^\rho\right)_{L^2(\Omega)} + \left(\bar{f}_2^\rho, \gamma u^* - \gamma\bar{u}^\rho\right)_{L^2(\Gamma_2)}.\end{aligned} \tag{5.52}$$

Using the compactness of the trace operator and the estimate

$$|j_\rho\left(\bar{u}^\rho, u^*\right) - j_\rho\left(\bar{u}^\rho, \bar{u}^\rho\right)| \le M \int_{\Gamma_3} |\gamma \bar{u}^\rho(x) - \gamma u^*(x)| \mathrm{d}\Gamma,$$

we deduce that

$$\lim_{\rho\to 0} \left(j_\rho\left(\bar{u}^\rho, u^*\right) - j_\rho\left(\bar{u}^\rho, \bar{u}^\rho\right) + \left(f_0, u^* - \bar{u}^\rho\right)_{L^2(\Omega)} + \left(\bar{f}_2^\rho, \gamma u^* - \gamma \bar{u}^\rho\right)_{L^2(\Gamma_2)}\right) = 0.$$

Combining this last relation with (5.52) and passing to the limit as $\rho \to 0$, we get (5.51). Moreover, it can be proved that

$$\left[u^*, f_2^*\right] \in \mathcal{V}_{ad}. \tag{5.53}$$

Indeed, since $\bar{u}^\rho \to u^*$ in V as $\rho \to 0$ and $\bar{f}_2^{\,\rho} \rightharpoonup f_2^*$ in $L^2(\Gamma_2)$ as $\rho \to 0$ we have, for every $v \in V$,

$$(A\bar{u}^\rho, v - \bar{u}^\rho)_V \to \left(Au^*, v - u^*\right)_V \quad \text{as } \rho \to 0, \tag{5.54}$$

$$\left(\bar{f}_2^\rho, \gamma v - \gamma \bar{u}^\rho\right)_{L^2(\Gamma_2)} \to \left(f^*, \gamma v - \gamma u^*\right)_{L^2(\Gamma_2)} \quad \text{as } \rho \to 0, \tag{5.55}$$

$$(f_0, v - \bar{u}^\rho)_{L^2(\Omega)} \to \left(f_0, v - u^*\right)_{L^2(\Omega)} \quad \text{as } \rho \to 0. \tag{5.56}$$

Taking into account the definitions of j_ρ and j, it follows that, for every $v \in V$,

$$j_\rho\left(\bar{u}^\rho, v\right) - j_\rho\left(\bar{u}^\rho, \bar{u}^\rho\right) \to j\left(u^*, v\right) - j\left(u^*, u^*\right) \text{ as } \rho \to 0. \tag{5.57}$$

Therefore, we deduce that (u^*, f_2^*) verifies (3.22). Thus, (5.53) holds.

Let $(\bar{u}, \bar{f}_2)$ be a solution of (POC1) and let us consider the sequence $(u^\rho)_\rho$ such that, for each $\rho > 0$, u^ρ is the unique solution of the problem (PS2) with the data $f_0 \in L^2(\Omega)$ and $\bar{f}_2 \in L^2(\Gamma_2)$. Obviously, for every $\rho > 0$, $(u^\rho, \bar{f}_2) \in \mathcal{V}_{ad}^\rho$. Using Theorem 5.16 we deduce that

$$\left(u^\rho, \bar{f}_2\right) \to \left(\bar{u}, \bar{f}_2\right) \text{ in } V \times L^2(\Gamma_2) \quad \text{as } \rho \to 0. \tag{5.58}$$

Since the functional L is convex and continuous, we have

$$L\left(f_2^*, u^*\right) \le \liminf_{\rho\to 0} L\left(\bar{f}_2^\rho, \bar{u}^\rho\right). \tag{5.59}$$

Moreover, since $[\bar{u}^\rho, \bar{f}_2^{\,\rho}]$ is a solution of (POC2) we can write

$$\limsup_{\rho\to 0} L\left(\bar{f}_2^\rho, \bar{u}^\rho\right) \le \limsup_{\rho\to 0} L\left(\bar{f}_2, u^\rho\right). \tag{5.60}$$

Taking into account (5.58) we have

$$\limsup_{\rho\to 0} L\left(\bar{f}_2, u^\rho\right) = L\left(\bar{f}_2, \bar{u}\right) \tag{5.61}$$

and, due to the fact that $[\bar{u}, \bar{f}_2]$ is a solution of (POC1),

$$L\left(\bar{f}_2, \bar{u}\right) \le L\left(f_2^*, u^*\right). \tag{5.62}$$

Thus, from (5.59)–(5.62), we deduce that

$$L\left(\bar{f}_2, \bar{u}\right) = L\left(f_2^*, u^*\right)$$

and the proof ends. □

Remark 5.18. Theorem 5.17 shows that the regularized problem (POC2), for which we dispose of the optimality condition (4.43)–(4.44), may be used to approximate a solution of (POC1).

Acknowledgements

We would like to thank Professor Dan Tiba for interesting suggestions and comments. The work of the first author was supported by CNCSIS Grant PNII-420. The second author is partially supported by Grant MTM2008-03541 funded by MICINN (Spain) and Grant PNII-420 of CNCSIS (Romania).

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