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# ON THE CONTROLLABILITY OF THE LINEARIZED BENJAMIN–BONA–MAHONY EQUATION*

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**Abstract.** We study the boundary controllability properties of the linearized Benjamin–Bona–Mahony equation

$$\begin{cases} u_t - u_{xxt} + u_x = 0, & x \in (0,1),\ t > 0, \\ u(t,0) = 0,\ u(1,t) = f(t), & t > 0. \end{cases}$$

We show that the equation is approximately controllable but not spectrally controllable (no finite linear combination of eigenfunctions, other than zero, is controllable). Next, we prove a finite controllability result and we estimate the norms of the controls needed in this case.

**Key words.** boundary control, moments, biorthogonals

**AMS subject classifications.** 93B05, 78M05

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## 1. Introduction.

The Benjamin–Bona–Mahony (BBM) equation

$$u_t + u_x + uu_x - u_{xxt} = 0, \tag{1.1}$$

like the Korteweg-de Vries (KdV) equation

$$u_t + u_x + uu_x + u_{xxx} = 0, \tag{1.2}$$

was originally derived as approximation for surface water waves in a uniform channel (see, for instance, [3], [4], and [5]).

Both (1.1) and (1.2) also cover cases of the following type: surface waves of long wavelength in liquids, acoustic-gravity waves in compressible fluids, hydromagnetic waves in cold plasma, acoustic waves in anharmonic crystals, etc. The wide applicability of these equations is the main reason why, during the last decades, they have attracted so much attention from mathematicians.

The main mathematical difference between KdV and BBM models can be most readily appreciated by comparing the dispersion relation for the respective linearized equations. It can be easily seen that these relations are comparable only for small wave numbers (i.e., long waves) and they generate drastically different responses to short waves (which are irrelevant to its role as a physical model). This is one of the reasons why, whereas existence and regularity theory for the KdV equation is difficult, the theory of the BBM equation is comparatively simple. The computing is also much easier for (1.1) than for (1.2).

The existence, uniqueness, and regularity of the BBM equation have been studied, for instance, in [7] and [18]. The large time behavior of the solutions of (1.1) was also intensively analyzed in the last decade (see, for instance, [1], [2], and [3]).

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Although it is generally considered that the BBM equation is easier to deal with than the KdV equation, it seems that, from the controllability point of view, (1.2) offers greater possibilities than (1.1). While important progress has been made in the last years for the KdV (see, for instance, [20], [24], and [21]), very little is known about the BBM. Some interior unique continuation properties for (1.1) and related problems (the linear case included) were studied in [9]. It is well known that, for the linear equation, by using the Hilbert uniqueness method due to J.-L. Lions (see [15]), the unique continuation property implies approximate controllability. Therefore, from [9], some approximate interior controllability results can be obtained for the linearized BBM equation. Nevertheless, the approximate controllability results for the nonlinear case do not seen to be entirely reducible to a unique continuation property and some estimates are needed on the dependence of the control function with respect to the perturbation introduced by the nonlinear term. In [23], (1.1) posed in $\mathbb{R}_+$ with boundary control is studied. It is proved that approximate controllability holds for the corresponding linear equation.

As far as we know there are no results for the controllability of the nonlinear BBM equation.

The present paper is concerned with the boundary controllability properties of the linearized BBM equation in finite domain. More precisely, given $T > 0$ and $u_0 \in H^{-1}(0,1)$ can we find a control function $f \in L^2(0,T)$ such that the solution $u$ of

$$\left\{\begin{array}{lll} u_t - u_{xxt} + u_x = 0, & t > 0, & x \in (0,1), \\ u(t,0) = 0,\ u(t,1) = f(t), & t > 0, & \\ u(0,x) = u_0(x), & x \in (0,1), & \end{array}\right. \tag{1.3}$$

satisfies

$$u(T,x) = 0, \quad x \in (0,1)? \tag{1.4}$$

We shall first show that (1.3) is not spectrally controllable. This means that no finite linear nontrivial combination of eigenvectors can be driven to zero in finite time by using a control $f \in L^2(0,T)$.

Nevertheless, (1.3) is approximately controllable, i.e., the set of reachable states

$$R(T,u_0) = \{u(T,x) \mid f \in L^2(0,T)\}$$

is dense in $L^2(0,1)$ for any $u_0 \in H^{-1}(0,1)$ and $T > 0$. Hence, given $T > 0$, $u_0 \in H^{-1}(0,1)$, $v_0 \in H^{-1}(0,1)$, and $\varepsilon > 0$, there exists a control function $f \in L^2(0,T)$ such that the solution $u$ of (1.3) satisfies $||u(T) - v_0||_{L^2(0,1)} < \varepsilon$.

These two results can be found at the beginning of the last section (Theorems 4.2 and 4.3).

We refer to [19] for similar negative results in the context of the exact controllability of the linear heat equation in a half-line.

Another interesting problem, with practical relevance, is the following finite controllability property: given $T > 0$, $N > 0$, and $u_0 \in H^{-1}(0,1)$, is there a control $f_N \in L^2(0,T)$ such that the projection of the solution of (1.3) over the finite dimensional space generated by the first $2N$ eigenvectors is equal to zero at $t = T$?

We give a positive answer to this question in Theorem 4.6. Moreover, by using some estimates for the corresponding biorthogonal sequences, we analyze how the norms of the controls change with $N$. We find an upper bound for the norms of the controls and we prove that this is, in some sense, sharp. More precisely, we prove that for any initial data $u_0 \in H^{-1}(0,1)$ there exists a control $f_N$ such that

$$\| f_N \|_{L^2(0,T)} \leq c_1 e^{\gamma_1 N \ln(N)} \| u_0 \|_{H^{-1}(0,1)}, \tag{1.5}$$

where $c_1$ and $\gamma_1$ do not depend on $N$. Moreover, there are initial data $u_0 \in H_0^1(0,1)$ for which any corresponding control $f_N$ satisfies

$$c_2 e^{\gamma_2 N \ln(N)} \| u_0 \|_{H_0^1(0,1)} \leq \| f_N \|_{L^2(0,T)}, \tag{1.6}$$

where $c_2$ and $\gamma_2$ do not depend on $N$.

We remark that the norms of the controls $f_N$ may increase very rapidly as $N$ goes to infinity. Hence, the cost needed to drive to zero the first $2N$ eigenmodes can be very high when $N$ is large.

The controllability of the KdV equation has been studied in [20], [21], and [24]. It has been proved that exact controllability holds for the linearized equation with different boundary conditions and number of controls. Hence, a Sobolev space of initial data can be controlled from the boundary. This implies that the linearized KdV equation is not only spectrally controllable but also N-partially controllable with uniformly bounded controls. Therefore the boundary controllability properties of the linearized KdV are much "nicer" than the corresponding ones for the linearized BBM (which is not spectrally controllable and not uniformly N-partially controllable). We also remark that, based on the linear case, local or global controllability results (depending on the number of controls) can be obtained for the nonlinear KdV equation.

The paper is organized in the following way. In the second section we study the differential operator $A$ corresponding to (1.3). We prove that $A$ has a sequence of purely imaginary eigenvalues $(i\lambda_n)_{n\in\mathbb{Z}^*}$ such that $\lim_{|n|\to\infty} \lambda_n = 0$.

In the third section we analyze the biorthogonal sequences to the exponentials family $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}^*}$ or to a subset of it. First, we prove that there is no full biorthogonal sequence. Next we concentrate our attention on the finite families $\{e^{i\lambda_n t}\}_{\substack{|n|\leq N \\ n\neq 0}}$. In this case various biorthogonal sequences can be constructed. We give an example and we analyze the behavior of the norms of the biorthogonals as $N$ goes to infinity. The techniques used in this section combine classical elements from the theory of analytic functions with constructions specific to our problem.

Finally, in the last section, we use the previous results to solve the controllability problems mentioned above.

## 2. Linearized BBM equation: Elementary properties.

Let us consider the following equation

$$\begin{cases} u_t - u_{xxt} + u_x = 0, & x \in (0,1), \quad t > 0, \\ u(t,0) = u(t,1) = 0, & t > 0, \\ u(0,x) = u_0(x), & x \in (0,1), \end{cases} \tag{2.1}$$

representing the linearized BBM equation.

In order to put (2.1) in an abstract Cauchy form, we apply the operator $(\mathcal{I}-\partial_x^2)^{-1}$. The following equivalent equation is obtained:

$$\begin{cases} u_t + Au = 0, & x \in (0,1), \quad t > 0, \\ u(0) = u_0, & x \in (0,1), \end{cases} \tag{2.2}$$

where $A : H_0^1(0,1) \to H_0^1(0,1)$ is given by

$$Au = (\mathcal{I} - \partial_x^2)^{-1} \partial_x u. \tag{2.3}$$

Here $\partial_x^2$ denote the Laplace operator

$$\partial_x^2 : H^2(0,1) \cap H_0^1(0,1) \to L^2(0,1), \quad \partial_x^2 u = u_{xx}.$$

The main properties of the operator $A$ are given in the following proposition.

PROPOSITION 2.1. *$A$ is a compact, skew-adjoint operator in $H_0^1(0,1)$.*

*Proof.* Due to the regularizing effect of the operator $(\mathcal{I} - \partial_x^2)^{-1}$ it follows immediately that $A$ takes values in $H^2(0,1) \cap H_0^1(0,1)$ which is compactly embedded in $H_0^1(0,1)$. Hence $A$ is compact.

Let us consider in $H_0^1(0,1)$ the inner product given by

$$(u, v) = \int_0^1 \partial_x u \partial_x v + \int_0^1 uv. \tag{2.4}$$

For any $u, v \in H^2(0,1) \cap H_0^1(0,1)$, we obtain

$$(Au, v) = \left( \left(\mathcal{I} - \partial_x^2\right)^{-1} \partial_x u, v \right) = \int_0^1 \partial_x \left[ \left(\mathcal{I} - \partial_x^2\right)^{-1} \partial_x u \right] \partial_x v + \int_0^1 \left[ \left(\mathcal{I} - \partial_x^2\right)^{-1} \partial_x u \right] v$$

$$= \int_0^1 \left(\mathcal{I} - \partial_x^2\right)^{-1} \left(\partial_x^2 u\right) \partial_x v - \int_0^1 \left[ \left(\mathcal{I} - \partial_x^2\right)^{-1} u \right] \partial_x v = - \int_0^1 u (\partial_x v)$$

$$= \int_0^1 (\partial_x u) v = - \int_0^1 \partial_x u \left(\mathcal{I} - \partial_x^2\right)^{-1} \left(\partial_x^2 v\right) + \int_0^1 \partial_x u \left[ \left(\mathcal{I} - \partial_x^2\right)^{-1} v \right]$$

$$= - \int_0^1 \partial_x u \partial_x \left[ \left(\mathcal{I} - \partial_x^2\right)^{-1} \partial_x v \right] - \int_0^1 u \left[ \left(\mathcal{I} - \partial_x^2\right)^{-1} \partial_x v \right] = - \left( u, \left(\mathcal{I} - \partial_x^2\right)^{-1} \partial_x v \right)$$

$$= -(u, Av).$$

By density we obtain that $(Au, v) = -(u, Av) \quad \forall u, v \in H_0^1(0,1)$ and therefore $A$ is skew-adjoint in $H_0^1(0,1)$. □

Since $A$ is compact, (2.2) can be treated like an ordinary differential equation in the Hilbert space $H_0^1(0,1)$. By using Cauchy–Lipschitz–Picard theorem the following properties concerning the solutions of (2.2) are immediate.

PROPOSITION 2.2. *Equation* (2.2) *has a unique solution $u \in C^1\left([0,\infty); H_0^1(0,1)\right)$ which satisfies*

$$\int_0^1 | \partial_x u(x,t) |^2 + \int_0^1 | u(x,t) |^2 = \int_0^1 | \partial_x u_0 |^2 + \int_0^1 | u_0 |^2 . \tag{2.5}$$

*Proof.* Since $A$ is a bounded linear operator the existence and uniqueness of solutions follow from Cauchy–Lipschitz–Picard theorem (see [8, p. 104]).

On the other hand, since $A$ is skew-adjoint, we have

$$\frac{1}{2} \frac{d}{dt} \| u \|_{H_0^1}^2 = Re(u, u_t) = Re(u, -Au) = 0.$$

Hence, the $H_0^1$ norm of the solution is conserved. □

Remark 2.1. *In fact much more can be said about the regularity of solutions of* (2.2). *Since* (2.2) *is linear and* $A$ *is a bounded operator we can easily deduce that* $u \in C^\omega\left([0,\infty);\, H_0^1(0,1)\right)$, *where* $C^\omega\left([0,\infty);\, H_0^1(0,1)\right)$ *represents the class of analytic functions defined in* $[0,\infty)$ *with values in* $H_0^1(0,1)$. *Indeed, for* $t_0 \in [0,\infty)$,

$$\left|\left|\sum_{n=0}^{\infty} u^{(n)}(t_0)\frac{(t-t_0)^n}{n!}\right|\right|_{H_0^1} \le \sum_{n=0}^{\infty}\frac{|t-t_0|^n}{n!}\left|\left|u^{(n)}(t_0)\right|\right|_{H_0^1}$$

$$\le \sum_{n=0}^{\infty}\frac{|t-t_0|^n}{n!}\,||A||^n\,||u(t_0)||_{H_0^1} < \infty.$$

*Hence the series* $\sum_{n=0}^{\infty} u^{(n)}(t_0)\frac{(t-t_0)^n}{n!}$ *is (absolutely) convergent and*

$$u(t) = \exp\left(-A(t-t_0)\right)u(t_0) = \sum_{n=0}^{\infty}(-1)^n\frac{(t-t_0)^n}{n!}A^n u(t_0) = \sum_{n=0}^{\infty}\frac{(t-t_0)^n}{n!}u^{(n)}(t_0).$$

Our next objective is to express the solution $u$ of (2.2) in Fourier series. To do so we need the spectral decomposition of the operator $A$.

Proposition 2.3. *$A$ has a sequence of purely imaginary eigenvalues* $(\mu_n)_{n\in\mathbb{Z}^*}$,

$$\mu_n = \operatorname{sgn}(n)\frac{i}{2\sqrt{1+n^2\pi^2}}, \quad n \in \mathbb{Z}^*. \tag{2.6}$$

*Moreover, to each eigenvalue* $\mu_n$ *corresponds an unique eigenfunction* $U_n$,

$$U_n(x) = \frac{1}{\sqrt{n^2\pi^2+1}}e^{-i\operatorname{sgn}(n)\sqrt{n^2\pi^2+1}x}\sin(n\pi x), \quad n \in \mathbb{Z}^*, \tag{2.7}$$

*such that* $\| U_n \|_{H_0^1} = 1$. *The family* $(U_n)_{n\in\mathbb{Z}^*}$ *forms an orthonormal basis in* $H_0^1(0,1)$.

*Proof.* We are looking for $\mu \in \mathbb{C}$ and $\tau \in H_0^1(0,1)$ such that $A\tau = \mu\tau$, which is equivalent to

$$\begin{cases} \mu\tau - \mu\tau_{xx} - \tau_x = 0, \\ \tau(0) = \tau(1) = 0. \end{cases} \tag{2.8}$$

Hence, $\tau(x) = c_1 e^{x\frac{-1-\sqrt{1+4\mu^2}}{2\mu}} + c_2 e^{x\frac{-1+\sqrt{1+4\mu^2}}{2\mu}}$.

From the boundary conditions we obtain, from one hand, that $c_1 = -c_2$ and from the other hand, that the eigenvalues of the operator are given by the equation

$$e^{\frac{\sqrt{1+4\mu^2}}{\mu}} = 1. \tag{2.9}$$

It results that the eigenvalues of the operator $A$ are

$$\mu_n = \operatorname{sgn}(n)\frac{i}{2\sqrt{1+n^2\pi^2}}, \quad n \in \mathbb{Z}^*.$$

To each $\mu_n$ corresponds an eigenfunction

$$U_n(x) = \frac{1}{\sqrt{n^2\pi^2+1}}e^{-i\operatorname{sgn}(n)\sqrt{1+n^2\pi^2}x}\sin(n\pi x)$$

with $||U_n||_{H_0^1} = 1$. ◻

REMARK 2.2. *Let us remark that, for each $n \in \mathbb{Z}^*$,*

$$(U_n)_x(1) = \frac{(-1)^n n\pi}{\sqrt{n^2\pi^2+1}} e^{-i\,\mathrm{sgn}(n)\sqrt{1+n^2\pi^2}} \neq 0$$

*and $|(U_n)_x(1)| \sim 1$ as $|n| \to \infty$.*

REMARK 2.3. *We have obtained that $\lim_{|n|\to\infty} \lambda_n = 0$. This is due to the compactness of the operator $A$ and will have some very important consequences for the controllability properties of the BBM equation.*

If we consider an initial data, $u_0 \in H_0^1(0,1)$, $u_0 = \sum_{n\in\mathbb{Z}^*} a_n U_n$, the solution of (2.2) corresponding to this initial data can be written as

$$u = \sum_{n\in\mathbb{Z}^*} a_n U_n e^{i\lambda_n t},$$

where $\lambda_n = \frac{\mathrm{sgn}(n)}{2\sqrt{1+n^2\pi^2}}$ and $\mu_n = i\lambda_n$ are the eigenvalues of the operator $A$ found in Proposition 2.3.

**3. Biorthogonal sequences.** Let $\lambda_n i$, $n \in \mathbb{Z}^*$, be the eigenvalues of the operator $A$. In this section we study the sequences biorthogonal to the family of exponentials $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}^*}$ or to some subset of it. All the results of this section will be used to study the boundary controllability properties of the BBM equation in the last section.

Let us first recall the following definition.

DEFINITION 3.1. *Let $(f_n)_{n\geq 1}$ be a sequence of vectors from a Hilbert space $H$. The sequence $(g_n)_{n\geq 1} \subset H$ is biorthogonal to $(f_n)_{n\geq 1}$ if and only if $(f_n, g_m) = \delta_{nm}$ $\forall n, m \geq 1$.*

We begin with the following negative result.

THEOREM 3.2. *Let $T > 0$ and $m \in \mathbb{Z}^*$. There is no function $\Theta_m \in L^2(-T,T)$ such that*

$$\int_{-T}^{T} \Theta_m(t) e^{i\lambda_n t} dt = \begin{cases} 0 & \text{if } n \in \mathbb{Z}^*,\ n \neq m, \\ 1 & \text{if } n = m. \end{cases} \tag{3.1}$$

*Proof.* Let us suppose that there exists a function $\Theta_m \in L^2(-T,T)$ such that (3.1) is satisfied.

We define $F : \mathbb{C} \to \mathbb{C}$ by

$$F(z) = \int_{-T}^{T} \Theta_m(t) e^{izt} dt. \tag{3.2}$$

From the Paley–Wiener theorem, $F$ is an entire function. Moreover, from (3.2) and (3.1) we obtain that

$$F(\lambda_n) = \delta_{nm} \quad \forall n \in \mathbb{Z}^*. \tag{3.3}$$

Since $\lim_{n\to\infty} \lambda_n = 0$, it follows that $F$ is zero on a set with a finite accumulation point. Therefore $F = 0$ which contradicts the fact that $F(\lambda_m) = 1$.

Hence, there is no function $\Theta_m \in L^2(-T,T)$ such that (3.1) is satisfied and the proof ends. ◻

REMARK 3.1. *From Theorem* 3.2 *the following result of nonobservability can be obtained: there is no sequence* $(\rho_n)_{n\in\mathbb{Z}^*}$ *of positive constants such that the following inequality*

$$\sum_{n\in\mathbb{Z}^*} \rho_n \mid a_n \mid^2 \leq \int_{-T}^{T} \left| \sum_{n\in\mathbb{Z}^*} a_n e^{i\lambda_n t} \right|^2 dt \tag{3.4}$$

*is true for any sequence* $(a_n)_{n\in\mathbb{Z}^*}$ *with a finite number of nonzero terms.*

*This is a direct consequence of Theorem* 3.2 *and the following result in moments theory (see* [22, p. 151]*).*

THEOREM A. *Let* $H$ *be a Hilbert space,* $(f_n)_n$ *a vector family from* $H$*, and* $(c_n)_n$ *a sequence of scalars. In order for a vector* $f \in H$ *to exist such that* $\| f \| \leq M$ *and* $(f, f_n) = c_n$ $\forall n$*, it is necessary and sufficient that*

$$\left| \sum_n a_n \bar{c}_n \right| \leq M \left\| \sum_n a_n f_n \right\|^2 \tag{3.5}$$

*for any finite number of scalars* $a_1, a_2, \ldots$.

*Let us suppose now that* (3.4) *is true. Then, for each* $m \in \mathbb{Z}^*$,

$$|a_m|^2 \leq \frac{1}{\rho_m} \int_{-T}^{T} \left| \sum_{n\in\mathbb{Z}^*} a_n e^{i\lambda_n t} \right|^2 dt$$

*for any sequence* $(a_n)_{n\in\mathbb{Z}^*}$ *with a finite number of nonzero terms.*

*From Theorem* A *it follows that there exists* $\Theta_m \in L^2(-T,T)$ *such that* $\int_{-T}^{T} \Theta_m(t) e^{i\lambda_n t} dt = \delta_{mn}$ *which contradicts Theorem* 3.2.

REMARK 3.2. *Theorem* 3.2 *proves that there is no sequence biorthogonal to* $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}^*}$ *in* $L^2(-T,T)$*. This is related to the fact that* $\lim_{|n|\to\infty} \lambda_n = 0$ *which affects the linear independency of the exponential family* $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}^*}$ *in* $L^2(-T,T)$.

*We shall use this result in the last section to prove that no eigenfunction of equation* (1.3) *can be driven to zero by using a control function in* $L^2(0,T)$ *(see Theorem* 4.2*).*

Let $N \in \mathbb{N}^*$. We shall pass now to prove the existence of a biorthogonal to the finite family of exponentials $\{e^{i\lambda_n t}\}_{\substack{|n|\leq N\\ n\neq 0}}$.

THEOREM 3.3. *Let* $T > 0$ *and* $N \in \mathbb{N}^*$. *There exists a biorthogonal sequence* $\{\Psi_m\}_{\substack{|n|\leq N\\ n\neq 0}}$ *to the family of exponentials* $\{e^{i\lambda_n t}\}_{\substack{|n|\leq N\\ n\neq 0}}$ *in* $L^2(-T,T)$ .

*Proof.* Let us first prove that there is a constant $C_1(N) > 0$ such that, for any scalars $(a_n)_{\substack{|n|\leq N\\ n\neq 0}}$,

$$C_1(N) \sum_{\substack{|n|\leq N\\ n\neq 0}} \mid a_n \mid^2 \leq \int_{-T}^{T} \left| \sum_{\substack{|n|\leq N\\ n\neq 0}} a_n e^{i\lambda_n t} \right|^2 dt. \tag{3.6}$$

We consider the space generated by $\{e^{i\lambda_n t}\}_{\substack{|n|\leq N\\ n\neq 0}}$,

$$X = \underset{L^2(-T,T)}{\mathrm{Span}} \left\{e^{i\lambda_n t}\right\}_{\substack{|n|\leq N\\ n\neq 0}}.$$

$X$ is a finite-dimensional space of dimension $2N$. Moreover, the application

$$\sum_{\substack{|n|\leq N\\ n\neq 0}} a_n e^{i\lambda_n t}\in X\longrightarrow \sqrt{\sum_{\substack{|n|\leq N\\ n\neq 0}} \mid a_n\mid^2}$$

is a norm in $X$. Since $X$ is finite dimensional this new norm and the one induced from $L^2(T,T)$ are equivalent. It follows that there is a constant $C_1(N)$ such that (3.6) is satisfied for any scalars $(a_n)$.

Now, for each $m$, $\mid m\mid\leq N$, $m\neq 0$, we can apply Theorem A from Remark 3.1 by taking $c_n=\delta_{nm}$, $f_n=e^{i\lambda_n t}$, and $H=L^2(-T,T)$. It follows that there exists a function $\Psi_m\in L^2(-T,T)$ such that $\int_{-T}^{T}\Psi_m(t)e^{i\lambda_n t}dt=\delta_{nm}\ \forall n$, $\mid n\mid\leq N$, $n\neq 0$.

Hence we get a biorthogonal sequence $\{\Psi_m\}_{\substack{|m|\leq N\\ m\neq 0}}\subset L^2(-T,T)$ to the family of exponentials $\{e^{i\lambda_n t}\}_{\substack{|n|\leq N\\ n\neq 0}}$ and the proof finishes. $\square$

Remark 3.3. *The following inequality is also true:*

$$\int_{-T}^{T}\left|\sum_{\substack{|n|\leq N\\ n\neq 0}} a_n e^{i\lambda_n t}\right|^2 dt\leq 4NT\sum_{\substack{|n|\leq N\\ n\neq 0}}\mid a_n\mid^2 . \tag{3.7}$$

*Indeed, from the Cauchy inequality*

$$\int_{-T}^{T}\left|\sum_{\substack{|n|\leq N\\ n\neq 0}} a_n e^{i\lambda_n t}\right|^2 dt\leq \int_{-T}^{T}\left(\sum_{\substack{|n|\leq N\\ n\neq 0}}\mid a_n\mid^2\right)\left(\sum_{\substack{|n|\leq N\\ n\neq 0}}\mid e^{i\lambda_n t}\mid^2\right)dt=4NT\sum_{\substack{|n|\leq N\\ n\neq 0}}\mid a_n\mid^2$$

*and* (3.7) *is proved.*

Remark 3.4. *The proof of Theorem* 3.3 *shows that there exists at least a biorthogonal sequence to any finite family of exponentials.*

Remark 3.5. *From Theorem* A *(Remark* 3.1*) we also obtain that the norm of the biorthogonal sequence* $\{\Psi_m\}_{\substack{|m|\leq N\\ m\neq 0}}$ *is bounded by* $C_1(N)$*. Since Theorem* 3.2 *proves that there is no biorthogonal sequence to* $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}^*}$ *we deduce again from Theorem* A *that* $C_1(N)$ *degenerates when* $N$ *goes to infinity. We shall analyze how this constant changes when* $N$ *increases.*

Theorem 3.4. *Let* $T>0$ *and* $N\in\mathbb{N}^*$*. There exists a biorthogonal sequence* $\{\Theta\}_{\substack{|n|\leq N\\ n\neq 0}}$ *to the family of exponentials* $\{e^{i\lambda_n t}\}_{\substack{|n|\leq N\\ n\neq 0}}$ *in* $L^2(-T,T)$ *such that*

$$\parallel\Theta_m\parallel^2_{L^2(-T,T)}\leq C_1 e^{\alpha N\ln(N)}, \tag{3.8}$$

*where* $C_1$ *and* $\alpha$ *are two constants which do not depend on* $N$*.*

*Proof.* Let us first define, for each $m$ such that $\mid m\mid\leq N$ and $m\neq 0$,

$$\xi_m(z)=\left(\prod_{\substack{|n|\leq N\\ n\neq 0,m}}\frac{z-\lambda_n}{\lambda_m-\lambda_n}\right)\left(\frac{\sin\frac{T(z-\lambda_m)}{2N}}{\frac{T(z-\lambda_m)}{2N}}\right)^{2N}. \tag{3.9}$$

Each function $\xi_m$ has the following properties:

- $\xi_m$ is an entire function,

- $\xi_m(\lambda_n) = \delta_{nm}$,
- $\xi_m(x) \in L^2(-\infty, \infty)$,
- $\xi_m$ is of the exponential type at most $T$, i.e., there exists a constant $A > 0$ such that $\forall \varepsilon > 0$, we have

$$|\xi_m(z)| \leq A e^{(T+\varepsilon)|z|} \quad \forall z \in \mathbb{C}.$$

Let us now define

$$\Theta_m(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \xi_m(x) e^{-ixt} dx, \tag{3.10}$$

and we shall show that $\{\Theta_m\}_{\substack{|n|\leq N\\ n\neq 0}}$ is the biorthogonal sequence we are looking for.

From the properties of $\xi_m$, by using Paley–Wiener theorem, it follows that $\Theta_m$ has compact support in $[-T, T]$, it belongs to $L^2(-T, T)$, and

$$\int_{-T}^{T} \Theta_m(t) e^{i\lambda_n t} dt = \xi_m(\lambda_n) = \delta_{nm}.$$

It follows that $(\Theta_m)_{\substack{|m|\leq N\\ m\neq 0}}$ is a biorthogonal sequence to $\{e^{i\lambda_n t}\}_{\substack{|n|\leq N\\ n\neq 0}}$.

Our next objective is to estimate the norm of $\Theta_m$ and to see that it satisfies (3.8).

From Plancherel's theorem we have that

$$\| \Theta_m \|_{L^2(-T,T)} = \| \xi_m \|_{L^2(-\infty,\infty)}. \tag{3.11}$$

Let us now estimate $\| \xi_m \|_{L^2(-\infty,\infty)}$.

$$\| \xi_m \|^2_{L^2(-\infty,\infty)} = \int_{-\infty}^{\infty} \left| \left( \prod_{\substack{|n|\leq N\\ n\neq 0,m}} \frac{x-\lambda_n}{\lambda_m - \lambda_n} \right) \left( \frac{\sin \frac{T(x-\lambda_m)}{2N}}{\frac{T(x-\lambda_m)}{2N}} \right)^{2N} \right|^2$$

$$= \left( \prod_{\substack{|n|\leq N\\ n\neq 0,m}} \frac{1}{|\lambda_n - \lambda_m|^2} \right) \int_{-\infty}^{\infty} \left| \left( \prod_{\substack{|n|\leq N\\ n\neq 0,m}} (x-\lambda_n) \right) \left( \frac{\sin \frac{T(x-\lambda_m)}{2N}}{\frac{T(x-\lambda_m)}{2N}} \right)^{2N} \right|^2 dx.$$

Let us first evaluate the constant

$$\gamma_1(N) = \prod_{\substack{|n|\leq N\\ n\neq 0,m}} \frac{1}{|\lambda_m - \lambda_n|^2}.$$

We have

$$\frac{1}{|\lambda_m - \lambda_n|} = \frac{1}{\left| \frac{\operatorname{sgn}(n)}{\sqrt{1+n^2\pi^2}} - \frac{\operatorname{sgn}(m)}{\sqrt{1+m^2\pi^2}} \right|} = \frac{2\sqrt{1+n^2\pi^2}\sqrt{1+m^2\pi^2}}{\left|\operatorname{sgn}(n)\sqrt{1+m^2\pi^2} - \operatorname{sgn}(m)\sqrt{1+n^2\pi^2}\right|}$$

$$\leq \frac{2\sqrt{1+n^2\pi^2}\sqrt{1+m^2\pi^2}}{\left|\sqrt{1+m^2\pi^2} - \sqrt{1+n^2\pi^2}\right|} = \frac{2\sqrt{1+n^2\pi^2}\sqrt{1+m^2\pi^2}\left(\sqrt{1+m^2\pi^2} + \sqrt{1+n^2\pi^2}\right)}{|m^2-n^2|\,\pi^2}$$

$$\leq \frac{2\sqrt{(1+n^2\pi^2)(1+m^2\pi^2)}\left(\frac{\pi^2}{2}m + \frac{\pi^2}{2}n\right)}{|m^2-n^2|\,\pi^2} \leq \sqrt{(1+n^2\pi^2)(1+m^2\pi^2)} \leq (2\pi N)^2.$$

It follows that

$$\gamma_1(N) = \prod_{\substack{|n|\leq N \\ n\neq 0,m}} \frac{1}{|\lambda_m - \lambda_n|^2} \leq (2\pi N)^{8N-4}. \tag{3.12}$$

Let us now evaluate the integral

$$\gamma_2(N) = \int_{-\infty}^{\infty} \left| \left( \prod_{\substack{|n|\leq N \\ n\neq 0,m}} (x-\lambda_n) \right) \left( \frac{\sin \frac{T(x-\lambda_m)}{2N}}{\frac{T(x-\lambda_m)}{2N}} \right)^{2N} \right|^2 dx.$$

We have

$$\gamma_2(N) = \int_{|x|\leq \frac{1}{2}} \left| \prod_{\substack{|n|\leq N \\ n\neq 0,m}} (x-\lambda_n) \right|^2 \left| \frac{\sin \frac{T(x-\lambda_m)}{2N}}{\frac{T(x-\lambda_m)}{N}} \right|^{4N} dx$$

$$+ \int_{|x|\geq \frac{1}{2}} \left| \prod_{\substack{|n|\leq N \\ n\neq 0,m}} (x-\lambda_n) \right|^2 \left| \frac{\sin \frac{T(x-\lambda_m)}{2N}}{\frac{T(x-\lambda_m)}{2N}} \right|^{4N} dx$$

$$\leq \int_{|x|\leq \frac{1}{2}} \left| \prod_{\substack{|n|\leq N \\ n\neq 0,m}} (x-\lambda_n) \right|^2 dx + \int_{|x|\geq \frac{1}{2}} \left| \prod_{\substack{|n|\leq N \\ n\neq 0,m}} (x-\lambda_n) \right|^2 \left( \frac{2N}{T(x-\lambda_m)} \right)^{4N} dx.$$

However, since $|\lambda_n| < \frac{1}{2}$,

$$\left| \prod_{\substack{|n|\leq N \\ n\neq 0,m}} (x-\lambda_n) \right| \leq \begin{cases} 1 & \text{if } |x| \leq \frac{1}{2}, \\ (2\,|\,x\,|)^{2N-1} & \text{if } |\,x\,| \geq \frac{1}{2}. \end{cases}$$

It follows that

$$\gamma_2(N) \leq 1 + \int_{|x|\geq \frac{1}{2}} \frac{2^{4N-2}}{|\,x\,|^2} \left| \frac{x}{x-\lambda_m} \right|^{4N} 2^{4N} \left( \frac{N}{T} \right)^{4N} dx$$

$$\leq 1 + 2^{12N-2} \left( \frac{N}{T} \right)^{4N} \int_{|x|\geq \frac{1}{2}} \frac{1}{|\,x\,|^2} = 1 + 2^{12N} \left( \frac{N}{T} \right)^{4N}.$$

Hence,

$$\gamma_2(N) \leq 1 + 2^{12N} \left( \frac{N}{T} \right)^{4N}. \tag{3.13}$$

From (3.12) and (3.13) it follows that, for $N$ large enough,

$$\|\, \xi_m \,\|_{L^2(-\infty,\infty)} \leq C_1 N^{\alpha N}, \tag{3.14}$$

where $C_1 > 0$ and $\alpha > 3$ are two constants which do not depend on $N$.

From (3.11) it follows that

$$\| \Theta_m \|_{L^2(-T,T)} \leq C_1 N^{\alpha N}$$

and (3.8) is obtained. ◻

REMARK 3.6. *In Theorem 3.4 we construct an explicit biorthogonal sequence which norm increases as* $\exp(\alpha N \ln(N))$ *as* $N \to \infty$. *Nevertheless, many other biorthogonals can be found. What can be said about the norms of these biorthogonals? We shall prove in the next theorem that the norm of any biorthogonal to* $\{e^{i\lambda_n t}\}_{\substack{|n|\leq N \\ n\neq 0}}$ *is bounded from below by a constant of the type* $\exp(\beta N \ln(N))$. *In this sense,* (3.8) *is sharp.*

THEOREM 3.5. *Let* $(\psi_n)_{\substack{|n|\leq N \\ n\neq 0}}$ *be biorthogonal to* $\{e^{i\lambda_n t}\}_{\substack{|n|\leq N \\ n\neq 0}}$ *in* $L^2(-T,T)$. *Then there exist two positive constants* $C_2$ *and* $\omega$, *not depending on* $N$, *such that*

$$\| \psi_m \|_{L^2(-T,T)} \geq C_2 e^{\omega N \ln(N)} \tag{3.15}$$

$\forall m \neq 0$ *such that* $| m | \leq N$.

*Proof.* In order to prove the theorem some arguments from [11] will be used. We shall give the proof in several steps.

*Step* 1. Let us define the following sequence of functions:

$$\tau_m(z) = \int_{-T}^{T} \psi_m(t) e^{itz} dt, \quad | m | \leq N, \quad m \neq 0. \tag{3.16}$$

From the Paley–Wiener theorem it follows that $\tau_m$ is an entire function of exponential type at most $T$. Moreover,

$$| \tau_m(x) | \leq \sqrt{2T} \| \psi_m \|_{L^2(-T,T)} \quad \forall x \in \mathbb{R}. \tag{3.17}$$

Since $\tau_m$ is a function of exponential type it follows from Hadamard's factorization theorem that

$$\tau_m(z) = a z^p e^{bz} \prod_{z_k \in E} \left(1 - \frac{z}{z_k}\right) e^{z/z_k}, \tag{3.18}$$

where $E$ is the set of the zeros $z_k$ of $\tau_m$ with $z_k \neq 0$, $E = \{z_k \in \mathbb{C} \mid \tau_m(z_k) = 0, \quad z_k \neq 0\}$.

From the definition of the function $\tau_m$ it follows that $\tau_m(\lambda_n) = \delta_{m,n}$. Therefore $\{\lambda_n : | n | \leq N,\ n \neq 0,\ n \neq m\} \subseteq E$. Let $E' = \{\lambda_n : | n | \leq N,\ n \neq 0,\ n \neq \pm m\}$ and define the polynomial function

$$P_m(z) = \prod_{\substack{|n|\leq N \\ n\neq 0, \pm m}} \frac{\lambda_n - z}{\lambda_n - \lambda_m}. \tag{3.19}$$

Let us now define function $\phi_m(z)$ by

$$\phi_m(z) = \frac{\tau_m(z)}{P_m(z)}. \tag{3.20}$$

The function $\phi_m$ has the following properties:

- it is an entire function of exponential type at most $T$,
- $\phi_m(\lambda_m) = 1$,
- $\tau_m(z) = P_m(z)\phi_m(z)$.

*Step* 2. In this step we shall give some estimates for $\mid P_m(z) \mid$.

$$\mid P_m(z) \mid = \left| \prod_{\substack{|n|\le N \\ n\neq 0,\pm m}} \frac{\lambda_n - z}{\lambda_n - \lambda_m} \right| = \left( \prod_{\substack{|n|\le N \\ n\neq 0,\pm m}} \mid \lambda_n - z \mid \right) \left[ \prod_{\substack{|n|\le N \\ n\neq 0,\pm m}} \mid \lambda_n - \lambda_m \mid \right]^{-1}.$$

By taking $z \in \mathbb{C}$ such that $\mid z \mid \geq 2$ we obtain that

$$\prod_{\substack{|n|\le N \\ n\neq 0,\pm m}} \mid \lambda_n - z \mid \geq (\mid z \mid -1)^{2N-2}. \tag{3.21}$$

On the other hand

$$\prod_{\substack{|n|\le N \\ n\neq 0,\pm m}} \mid \lambda_n - \lambda_m \mid \leq \prod_{\substack{|n|\le N \\ n\neq 0,\pm m}} \frac{1}{2}\left( \frac{1}{\sqrt{1+n^2\pi^2}} + \frac{1}{\sqrt{1+m^2\pi^2}} \right) \leq 1. \tag{3.22}$$

From (3.21) and (3.22) we deduce that

$$\mid P_m(z) \mid \geq (\mid z \mid -1)^{2N-2} \quad \forall z \in \mathbb{C},\ \mid z \mid \geq 2. \tag{3.23}$$

*Step* 3. From (3.17) and (3.23) we obtain that

$$\mid \phi_m(z) \mid = \frac{\mid \tau_m(z) \mid}{\mid P_m(z) \mid} \leq \frac{\sqrt{2T} e^{T \operatorname{Im} z} \parallel \psi_m \parallel_{L^2(-T,T)}}{(|z|-1)^{2N-2}} \quad \forall z \in \mathbb{C},\ |z| \geq 2. \tag{3.24}$$

It follows that

$$\mid \phi_m(x) \mid \leq \sqrt{2T} \parallel \psi_m \parallel_{L^2(-T,T)} \frac{1}{(\mid x \mid -1)^{2N-2}} \quad \forall x \in \mathbb{R},\ \mid x \mid \geq 2. \tag{3.25}$$

We shall show that (3.25) is not possible unless $\parallel \psi_m \parallel$ grows rapidly with $N$.

Let us first recall the following result (see [14, p. 21] and [6, p. 52]).

THEOREM B. *Let $f(z)$ be holomorphic in the circle $\mid z \mid \leq 2eR$ $(R > 0)$ with $f(0) = 1$ and let $\eta \in (0, \frac{3e}{2})$. Then inside the circle $\mid z \mid \leq R$, but outside of a family of excluded circles the sum of whose radii is not greater than $4\eta R$, we have*

$$\ln(\mid f(z) \mid) > -\left(2 + \ln\left(\frac{3e}{2\eta}\right)\right) \ln(M_f(2eR)), \tag{3.26}$$

*where* $M_f(2eR) = \max\limits_{|z|=2eR} \mid f(z) \mid$.

We apply this result to our case. Let us define $\varphi_m : \mathbb{C} \to \mathbb{C}$, $\quad \varphi_m(z) = \phi_m(\lambda_m - z)$.

Evidently, $\varphi_m$ is an entire function such that $\varphi_m(0) = 1$. Hence, $\varphi_m$ satisfies the hypothesis of Theorem B. It follows that, $\forall R > 0$ and $\eta \in (0, \frac{3e}{2})$,

$$\ln(\mid \varphi_m(z) \mid) > -2e\left(2 + \ln\left(\frac{3e}{2\eta}\right)\right) \ln(M_{\varphi_m}(2eR)) \quad \forall z \in \mathbb{C},\ \mid z \mid \leq R, \tag{3.27}$$

outside of a set of circles the sum of whose radii is not greater than $4\eta R$.

Let us denote $\delta = 2e(2+\ln(\frac{3e}{2\eta})) > 1$. Also, remark that, from (3.24),

$$M_{\varphi_m}(2eR) \leq e^{4eRT}||\psi_m||_{L^2(-T,T)}$$

if $2eR \geq 2$.

Hence, $\forall R > 0$ and $\eta \in (0, \frac{3e}{2})$ such that $2eR \geq 2$,

$$\ln(|\ \varphi_m(z)\ |) > -\delta \ln\left(e^{4eRT}||\psi_m||_{L^2(-T,T)}\right) \quad \forall z \in \mathbb{C},\ |\ z\ |\leq R, \tag{3.28}$$

outside of a set of circles the sum of whose radii is not greater than $4\eta R$.

Let us consider $R > 6$ and $\eta \in (0, \frac{1}{8})$.

It follows that there exists $x_0 \in [\frac{R}{2}, R]$ such that

$$\ln(|\ \varphi_m(x_0)\ |) > -\delta \ln\left(e^{4eRT}||\psi_m||_{L^2(-T,T)}\right). \tag{3.29}$$

On the other hand, from (3.25),

$$|\varphi_m(x_0)| = |\phi_m(\lambda_m - x_0)| \leq \sqrt{2T} \parallel \psi_m \parallel_{L^2(-T,T)} \frac{1}{(|\ \lambda_m - x_0\ | -1)^{2N-2}}. \tag{3.30}$$

From (3.30) and (3.29) the following estimate is obtained:

$$\ln\left(\sqrt{2T} \parallel \psi_m \parallel_{L^2(-T,T)} \frac{1}{(|\ \lambda_m - x_0\ | -1)^{2N-2}}\right) > -\delta \ln\left(e^{4eRT}||\psi_m||_{L^2(-T,T)}\right).$$

Hence

$$(1+\delta)\ln\left(||\psi_m||_{L^2(-T,T)}\right) > -4e\delta TR - \ln(\sqrt{2T}) + (2N-2)\ln(|x_0 - \lambda_m| - 1). \tag{3.31}$$

Let us now analyze the expression

$$G(N, x_0, R) = (2N-2)\ln(|\ \lambda_m - x_0\ | -1) - 4e\delta TR.$$

Remark that, for $R = N > 6$,

$$\begin{aligned} G(N, x_0, R) &\geq (2N-2)\ln(|\ x_0\ | - |\ \lambda_m\ | -1) - 4e\delta TN \\ &\geq (2N-2)\ln\left(\frac{N}{2} - 2\right) - 4e\delta TN \\ &= 2N\left(\frac{N-1}{N}\ln\left(\frac{N}{2}-2\right) - \underbrace{2e\delta T}_{\text{cte}}\right). \end{aligned}$$

It follows that there exists $\omega > 0$, not depending on $N$, such that

$$G(N, x_0, R) \geq \omega N \ln(N) \tag{3.32}$$

for any $N$ sufficiently large.

From (3.31) it follows that

$$\ln\left(||\psi_m||_{L^2(-T,T)}\right) > -\frac{\ln(\sqrt{2T})}{1+\delta} + \omega N \ln(N)$$

and the proof finishes. $\square$

**4. Controllability results.** In this section we study some boundary controllability properties of the BBM equation. We begin with the following exact controllability problem: given $T<0$ and an initial data $u_0 \in H^{-1}(0,1)$ find a control $f \in L^2(0,T)$ such that the solution $u$ of

$$\left\{\begin{array}{ll} u_t - u_{xxt} + u_x = 0, & x \in (0,1), \quad t>0, \\ u(t,0)=0, u(t,1)=f(t), & t>0, \\ u(0,x)=u_0(x), & x\in(0,1), \end{array}\right. \tag{4.1}$$

satisfies

$$u(T,x)=0, \quad x \in (0,1). \tag{4.2}$$

Remark 4.1. *Equation* (4.1) *has to be understood in a weak sense. For instance, the solution of* (4.1) *can be defined by transpositions (see* [16], [17]*). Let us briefly recall how can this be done.*

*Consider* $g \in L^1(0,T,L^2(0,1))$ *and* $v$ *the solution of the adjoint equation*

$$\left\{\begin{array}{ll} v_t - v_{xxt} + v_x = g, & x \in (0,1), \quad t>0, \\ v(t,0)=v(t,1)=0, & t>0, \\ v(T,x)=0, & x\in(0,1). \end{array}\right. \tag{4.3}$$

*By multiplying (formally)* (4.1) *by* $v$ *and integrating by parts we obtain*

$$\begin{aligned} 0 &= \int_0^T\int_0^1 (u_t - u_{xxt}+u_x)v = \int_0^1 \left(uv|_0^T - u_{xx}v|_0^T\right) + \int_0^T \left(u_x - uv_x + uv\right)|_0^1 \\ &- \int_0^T\int_0^1 u(v_t - v_{txx}+v_x) = \int_0^1 [-u_0v(0) + (u_0)_{xx}v(0)] - \int_0^T fv_{tx} - \int_0^T\int_0^1 ug. \end{aligned}$$

*Therefore we can say that* $u$ *is the solution of* (4.1) *if and only if*

$$\int_0^T\int_0^1 ug + \langle u_0, v(0)\rangle_{H^{-1},H_0^1} = -\int_0^T f(t)v_{tx}(t,1)dt \tag{4.4}$$

$\forall g \in L^1(0,T;L^2(0,1))$ *and* $v$ *the solution of* (4.3)*.* $\langle\cdot,\cdot\rangle$ *represents the duality product between* $H_0^1$ *and* $H^{-1}$*. As in* [16], [17] *it can be proved that* (4.4) *has a unique solution* $u \in C([0,T];L^2(0,1))$*. On the other hand we have just seen that a classical solution of* (4.1) *is the solution of* (4.4)*.*

Concerning the controllability of (4.1) let us begin with the following result which transforms the control problem into a moments problem.

Lemma 4.1.

(i) *The initial data* $u_0 \in H^{-1}(0,1)$ *is controllable to zero in time* $T>0$ *with a control* $f \in L^2(0,T)$ *if and only if*

$$\langle u_0, v(0)\rangle_{H^{-1},H_0^1} = -\int_0^T f(t)v_{tx}(t,1)dt \tag{4.5}$$

*for any solution* $v$ *of the equation*

$$\left\{\begin{array}{l} v_t - v_{txx} + v_x = 0, \\ v(t,0)=v(t,1)=0, \\ v(T,x)=v^T(x)\in H_0^1. \end{array}\right. \tag{4.6}$$

(ii) *The initial data $u_0 \in H^{-1}(0,1)$, $u_0(x) = \sum_{n\in\mathbb{Z}^*} a_n U_n(x)$, is controllable to zero in time $T > 0$ if and only if there exists $f \in L^2(0,T)$ such that*

$$\int_0^T f(t)e^{-i\lambda_n t}dt = \frac{i}{\lambda_n^2 (U_n)_x(1)} a_n \quad \forall n \in \mathbb{Z}^*. \tag{4.7}$$

*Proof.* (i) Let $u$ be the solution of (4.1) and $v$ the solution of (4.6). It follows that

$$0 = \int_0^T \int_0^1 (u_t - u_{xxt} + u_x)v = -\int_0^T \int_0^1 u(v_t - v_{txx} + v_x)$$

$$+ \int_0^1 (uv - u_{xx}v)\Big|_0^T + \int_0^T (u_x v_t - u v_{xt} + uv)\Big|_0^1 = -\int_0^1 (u_0 v(0) + (u_0)_x v_x)$$

$$+ \int_0^1 (u(T)v(T) + u_x(T)v_x(T)) - \int_0^T f(t)v_{xt}(t,1)dt.$$

We obtain that

$$\int_0^T f(t)v_{xt}(t,1)dt + \langle u_0, v(0)\rangle_{H^{-1},H_0^1} = \langle u(T), v^T\rangle_{H^{-1},H_0^1}$$

$\forall v^T \in H_0^1$.

Hence, $u_0$ is controllable to zero in time $T > 0$ if and only if (4.5) is satisfied.

(ii) For the second part let us put $v^T = \sum_{n\neq 0} b_n U_n$ and use (4.5). It follows that

$$\sum_{n\neq 0} \frac{1}{\lambda_n} a_n b_n e^{i\lambda_n T} = -\int_0^T f(t) \sum_{n\neq 0} i\lambda_n e^{i\lambda_n (T-t)} b_n (U_n)_x(1)dt$$

which is equivalent to

$$\sum_{n\neq 0} b_n e^{i\lambda_n T}\left[\int_0^T f(t) i\lambda_n e^{-i\lambda_n t}(U_n)_x(1)dt + \frac{1}{\lambda_n}a_n\right] = 0$$

for any $(b_n)_{n\neq 0} \in \ell^2$.

It follows that the control problem is equivalent to finding $f \in L^2(0,T)$ such that

$$\int_0^T f(t)e^{-i\lambda_n t}dt = \frac{i}{(\lambda_n)^2 (U_n)_x(1)} a_n \quad \forall n \in \mathbb{Z}^*. \qquad \square$$

By using Lemma 4.1 and Theorem 3.3 from section 3 the following negative result can be easily proved.

THEOREM 4.2. *No eigenfunction of the operator $A$ can be driven to zero in finite time.*

*Proof.* The controllability of an eigenfunction $U_m$ is equivalent, by Lemma 4.1, to finding $f \in L^2(0,T)$ such that

$$\int_0^T f(t)e^{-i\lambda_n t} = \begin{cases} 0 & \forall n \in \mathbb{Z}^*, \quad n \neq m, \\ \frac{i}{(\lambda_m)^2 (U_m)_x(1)}, & n = m. \end{cases}$$

Let us suppose that there exists $f \in L^2(0,T)$ with these properties. We define $g \in L^2(-\frac{T}{2}, \frac{T}{2})$ such that $g(t) = f(\frac{T}{2} - t)e^{-\frac{i\lambda_m T}{2}}$ almost everywhere in $(-\frac{T}{2}, \frac{T}{2})$. Then

$$\int_{\frac{T}{2}}^{\frac{T}{2}} g(t)e^{i\lambda_n t}dt = e^{\frac{iT}{2}(\lambda_n - \lambda_m)} \int_0^T f(t)e^{-i\lambda_n t}dt = \begin{cases} 0 & \forall n \in \mathbb{Z}^*, \quad n \neq m, \\ \frac{i}{(\lambda_m)^2 (U_m)_x(1)}, & n = m. \end{cases}$$

However, in Theorem 3.2, we have proved that this is not possible and the proof finishes. ∎

REMARK 4.2. *From Theorem* 4.2 *it follows that* (4.1) *is not spectrally controllable. This means that no finite linear nontrivial combination of eigenvectors can be driven to zero in finite time by using a control* $f \in L^2(0,T)$.

Let us now study the approximate controllability of (4.1). We recall that (4.1) is approximate controllable in time $T > 0$ if the set of reachable states

$$R(u_0, T) = \{u(T,x) \mid f \in L^2(0,T)\} \tag{4.8}$$

is dense in $L^2(0,1)$ for any $u_0 \in H^{-1}(0,1)$.

In other words, given $T > 0$, $u_0 \in H^{-1}(0,1)$, $v_0 \in L^2(0,1)$, and $\varepsilon > 0$ there exists a control function $f \in L^2(0,T)$ such that the solution $u$ of (4.1) satisfies $||u(T) - v_0||_{L^2(0,1)} < \varepsilon$.

THEOREM 4.3. *Equation* (4.1) *is approximate controllable in any time* $T > 0$ *with controls in* $L^2(0,T)$.

*Proof.* From the linearity of (4.1) it follows that it is sufficient to prove that the set $R(0,T)$ is dense in $H_0^1(0,1)$ for any $T > 0$. Therefore we shall consider only the case $u_0 = 0$. Let $u \in C([0,T], H_0^1(0,1))$ be the corresponding solution of (4.1).

Let also $v$ be the solution of the adjoint equation

$$\begin{cases} v_t - v_{txx} + v_x = 0, & x \in (0,1), \quad t < T, \\ v(t,0) = v(t,1) = 0, & t < T, \\ v(T,x) = v^T(x) \in H_0^1(\Omega). \end{cases} \tag{4.9}$$

It follows that

$$\int_0^T f(t) v_{xt}(t,1)dt = (u(T,x), v^T(x))_{H_0^1}. \tag{4.10}$$

Suppose that $R(0,T)$ is not dense in $H_0^1(0,1)$. Hence, there exists $v^T \in H_0^1(0,1)$, $v^T \neq 0$, such that

$$(u(T,x), v^T(x)) = 0 \quad \forall f \in L^2(0,T).$$

From (4.10) it follows that

$$\int_0^T f(t) v_{xt}(t,1) = 0 \quad \forall f \in L^2(0,T).$$

Therefore $v_{xt}(t,1) = 0 \quad \forall t \in (0,T)$. We show now that this contradicts the fact that $v^T \neq 0$. Hence, the problem is reduced to a unique continuation property.

Let us consider the Fourier decomposition of $v^T$:

$$v^T = \sum_{n \in \mathbb{Z}^*} a_n U_n,$$

where $(a_n)_{n \in \mathbb{Z}^*} \in \ell^2$ and the series converges in $H_0^1(0,1)$.

It follows that the corresponding solution of (4.9) is

$$v(t,x) = \sum_{n \in \mathbb{Z}^*} a_n e^{i\lambda_n (T-t)} U_n(x), \quad t \in (0,T).$$

From the equation $v$ verifies it follows that $v \in C^\omega\left([0,\infty);\, H_0^1(0,1)\right)$ (see Remark 2.1).

Hence, from the fact that $v_{xt}(t,1) = 0 \quad \forall t \in (0,T)$, we obtain that $v_{xt}(t,1) = 0 \quad \forall t \in \mathbb{R}$, i.e.,

$$\sum_{n\in\mathbb{Z}^*} a_n e^{i\lambda_n(T-t)}(U_n)_x(1)(-i\lambda_n) = 0 \quad \forall t \in \mathbb{R}.$$

For each $m \in \mathbb{Z}^*$,

$$\begin{aligned} 0 &= \lim_{S\to\infty} \frac{1}{2S}\int_{-S}^{S}\left[\sum_{n\in\mathbb{Z}^*} a_n e^{i\lambda_n(T-t)}(U_n)_x(1)(-i\lambda_n)\right] e^{i\lambda_m t}dt \\ &= a_m(U_m)_x(1)(-i\lambda_m)e^{i\lambda_m T}. \end{aligned}$$

From Remark 2.2 $(U_m)_x(1) \neq 0$. This implies that $a_m = 0 \quad \forall m \in \mathbb{Z}^*$ and therefore $v^T = 0$, which represents a contradiction. Hence, $R(0,T)$ is dense in $H_0^1(0,1)$ and the proof finishes. ∎

As we have seen in Theorem 4.2 no finite linear combination of eigenfunctions can be driven to zero. In this case the following question arises naturally: can we control to zero at least a part of the solution $u$ of (4.1)? And if we can do this, what is the cost we have to pay?

Therefore we shall now investigate the following special type of controllability.

Definition 4.4. *Equation* (4.1) *is $N$-partially controllable to zero in time $T > 0$ if, for any $u_0 \in H^{-1}(0,1)$, there exists a control $f \in L^2(0,T)$ such that the projection of the corresponding solution $u$ of* (4.1) *over the space generated by the eigenvectors $(U_n)_{\substack{|n|\le N\\ n\neq 0}}$ is zero at time $t = T$.*

Let $X_N = \operatorname{Span}\{U_n : \mid n \mid\le N,\, n \neq 0\}$ and let

$$\Pi_N : H^{-1}(0,1) \to X_N, \quad \Pi_N\left(\sum_{n\neq 0} a_n U_n\right) = \sum_{\substack{|n|\le N\\ n\neq 0}} a_n U_n,$$

be the projection operator.

Evidently, $u$ is $N$-partially controllable to zero if and only if

$$\Pi_N(u(T)) = 0. \tag{4.11}$$

By using the same argument as in Lemma 4.1, the following result can be obtained immediately.

Lemma 4.5. *The initial data $u_0 = \sum_{n\neq 0} a_n U_n$ is $N$-partially controlled to zero in time $T > 0$ if and only if there exists $f \in L^2(0,T)$ such that*

$$\int_0^T f(t)e^{i\lambda_n t}dt = \frac{i}{\lambda_n^2 (U_n)_x(1)} a_n \quad \forall \mid n \mid\le N, n \neq 0. \tag{4.12}$$

Now, the following theorem can be proved.

Theorem 4.6. *Any initial data $u_0 \in H^{-1}(0,1)$ can be $N$-partially controlled to zero in time $T > 0$ by using a control $f_N \in L^2(0,T)$ such that*

$$\| f_N \|^2_{L^2(0,T)} \le c_1 \| u_0 \|^2_{H^{-1}} e^{\alpha_1 N \ln(N)}, \tag{4.13}$$

*where $c_1$ and $\alpha_1$ are two constants which do not depend on $N$.*

*Moreover, there exists initial data $u_0 \in H_0^1(0,1)$ such that any control $f_N$ satisfies*

$$\| f_N \|^2_{L^2(0,T)} \geq c_2 \| u_0 \|^2_{H_0^1} e^{\omega_1 N \ln(N)}, \tag{4.14}$$

*where $c_2$ and $\omega_1$ are two constants which do not depend on $N$.*

*Proof.* Let us consider the initial data $u_0 = \sum_{n\neq 0} a_n U_n$ from $H^{-1}(0,1)$. We prove that there exists a function $f_N \in L^2(0,T)$ such that (4.12) is satisfied. This will be the control we are looking for.

Let $(\Theta_n)_{\substack{n\leq N\\ n\neq 0}}$ be the biorthogonal sequence to $(e^{i\lambda_n t})_{\substack{n\leq N\\ n\neq 0}}$ in $L^2(-\frac{T}{2},\frac{T}{2})$ constructed in Theorem 3.4.

Then we can define

$$f_N(t) = \sum_{\substack{n\neq 0\\ |n|\leq N}} \frac{ia_n}{\lambda_n^2 (U_n)_x(1)} \Theta_n \left(\frac{T}{2} - t\right) e^{\frac{i\lambda_n T}{2}}.$$

Evidently, $f_N \in L^2(0,T)$ and $\int_0^T f_N(t) e^{-i\lambda_n t} dt = \frac{ia_n}{\lambda_n^2 (U_n)_x(1)}$ $\forall \mid n \mid\leq N, n \neq 0$.

From Lemma 4.5 it follows that $f_N$ is the control we are looking for.

By using inequality (3.8) from Theorem 3.4 it follows that

$$\| f_N \|^2_{L^2(0,T)} \leq \sum_{\substack{|n|\leq N\\ n\neq 0}} \frac{\mid a_n \mid^2}{\mid \lambda_n \mid^2 \mid (U_n)_x(1) \mid^2} \sum_{\substack{|n|\leq N\\ n\neq 0}} \| \Theta_n \|^2$$

$$\leq \sum_{\substack{|n|\leq N\\ n\neq 0}} \frac{\mid a_n \mid^2}{\mid \lambda_n \mid^2 \mid (U_n)_x(1) \mid^2} \sum_{\substack{|n|\leq N\\ n\neq 0}} c_1 e^{\alpha N \ln(N)} \leq c_1 \| u_0 \|^2_{H^{-1}} e^{\alpha_1 N \ln(N)}$$

for any $\alpha_1 > \alpha$.

On the other hand let us consider $u_0 = U_m$, $\mid m \mid\leq N$, $m \neq 0$. From Lemma 4.5 $u_0$ is $N$-partially controllable to zero in time $T > 0$ if and only if there exists a control $f_N^m \in L^2(0,T)$ such that

$$\int_0^T f_N^m(t) e^{-i\lambda_n t} dt = \begin{cases} 0, & n \neq m, \\ \frac{i}{(\lambda_m)^2 (U_m)_x(1)}, & n = m. \end{cases}$$

We define $g_N^m \in L^2(-\frac{T}{2},\frac{T}{2})$ such that $g_N^m(t) = f_N^m(\frac{T}{2} - t) e^{-\frac{i\lambda_m T}{2}}$ almost everywhere in $(-\frac{T}{2},\frac{T}{2})$. Then

$$\int_{\frac{T}{2}}^{\frac{T}{2}} g_N^m(t) e^{i\lambda_n t} dt = e^{\frac{iT}{2}(\lambda_n - \lambda_m)} \int_0^T f_N^m(t) e^{-i\lambda_n t} dt = \begin{cases} 0 & \forall n \in \mathbb{Z}^*, \quad n \neq m, \\ \frac{i}{(\lambda_m)^2 (U_m)_x(1)}, & n = m. \end{cases}$$

Now, by using Theorem 3.5, it follows that

$$\| f_N^m \|^2_{L^2(0,T)} = \| g_N^m \|^2_{L^2(-\frac{T}{2},\frac{T}{2})} \geq C_2^2 \mid \lambda_m \mid^4 \mid (U_m)_x(1) \mid^2 e^{2\omega N \ln(N)}.$$

It follows that (4.14) is true for any $\omega_1 < 2\omega$ and $c_2 = C_2^2$ and the proof finishes. ◻

REMARK 4.3. *Theorem* 4.6 *proves that the cost (the norm of the control functions) needed to drive to zero the projection of the solutions of* (4.1) *over the space generated by the first* $2N$ *eigenfunctions may increase very rapidly when* $N$ *goes to infinity. Theorem* 4.6 *gives an upper bound for these norms (essentially,* $e^{\alpha N \ln(N)}$*) and shows that there exists a lower bound of the same order.*

**5. Comments.** As we have mentioned in the introduction, based on the linear case, local or global controllability results (depending on the number of controls) have been obtained for the nonlinear KdV equation in [20], [21], and [24].

The same cannot be said for the nonlinear equation (1.1). In fact, to our knowledge, no result for the controllability of the BBM equation is available. The controllability properties of the nonlinear systems are usually studied by linearizing the problem at an equilibrium state, by proving exact controllability results for this linear problem and by applying next the implicit function theorem. This method was first used in [13] for the ordinary differential equations and next generalized for the nonlinear wave equation (see, for instance, [12]). In [10] and [25] exact and local controllability results were given by using Schauder's fixed point theorem instead of the implicit function theorem. All approaches use the exact controllability result for the linearized equation. Taking into account the negative results (like nonspectral controllability) obtained in this paper for the linearized BBM equation it is not possible to study the controllability properties of (1.1) by using one of the classical techniques mentioned above. Probably, the controllability results for (1.1) are not better than the ones for the corresponding linear case but this is still to be proved.

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