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Boundary controllability of a linear semi-discrete 1-D wave equation derived from a mixed finite element method

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Abstract In this article one discusses the controllability of a semi-discrete system obtained by discretizing in space the linear 1-D wave equation with a boundary control at one extremity. It is known that the semi-discrete models obtained with finite difference or the classical finite element method are not uniformly controllable as the discretization parameter h goes to zero (see [8]).

Here we introduce a new semi-discrete model based on a mixed finite element method with two different basis functions for the position and velocity. We show that the controls obtained with these semi-discrete systems can be chosen uniformly bounded in $L^2(0, T)$ and in such a way that they converge to the HUM control of the continuous wave equation, i.e. the minimal L^2-norm control. We illustrate the mathematical results with several numerical experiments.

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1 Introduction

The following boundary controllability property for the 1-D linear wave equation is known to hold: given $T > 2$ and $(u^0, u^1) \in L^2(0,1) \times H^{-1}(0,1)$ there exists a control function $v \in L^2(0,T)$ such that the solution of the system of equations

$$\begin{cases} u'' - u_{xx} = 0, & \text{for } x \in (0,1), \quad t > 0, \\ u(t,0) = 0, & \text{for } t > 0, \\ u(t,1) = v(t), & \text{for } t > 0, \\ u(0,x) = u^0(x), & \text{for } x \in (0,1), \\ u'(0,x) = u^1(x), & \text{for } x \in (0,1), \end{cases} \tag{1}$$

satisfies

$$u(T,\cdot) = u'(T,\cdot) = 0. \tag{2}$$

By $'$ we denote the time derivative.

This controllability problem has been studied and solved some decades ago and several approaches are now known. The moment theory is one of the oldest and most successful (see, for instance, [1] and [17]). More recently, The Hilbert Uniqueness Method (HUM) provided a different and very general way to solve this and other multi-dimensional similar problems (see [11]).

In the last years many works have dealt with the numerical approximations of the control problem (1)–(2) using the HUM approach. For instance, in references [4], [6] and [7] numerical algorithms based on both the finite differences and finite element approximations of the two dimensional wave equation were described. In these references a bad behavior of the approximate controls was observed and various numerical cures were provided in order to eliminate the spurious oscillations. More precisely, a Tychonoff regularization technique was successfully implemented in [6] and [7], and a bi-grid algorithm in [4] and [7]. A mixed finite element approximation was also proposed in [5] with good results.

In general, any semi-discrete dynamics generates spurious high-frequency oscillations that do not exist at the continuous level. Moreover, a dispersion phenomenon appears and the velocity of propagation of these high frequency numerical waves may converge to zero when the mesh size tends to zero. Note that these spurious oscillations correspond to the high frequencies of the discrete model and therefore, they weakly converge to zero when the discretization parameter h does. Consequently, their existence is compatible with the convergence of the numerical scheme. However, when we are dealing with the exact controllability problem, an uniform time for the control of *all numerical waves* is needed. Since the velocity of propagation of some high frequency numerical waves may tend to zero with the mesh size, the uniform controllability properties of the semi-discrete model may eventually disappear for a fixed time $T > 0$. This is the case when the semi-discrete model is obtained by finite differences or the classical finite element method (see [8],[19] for a detailed analysis of the 1-D case and [18] for the 2-D case, in the context of the dual observability problem).

The conclusion was that, with any of the above semi-discrete models, the controllability property is not uniform as the discretization parameter h goes to zero and, consequently, there are initial data of the wave equation (even very regular ones) for which the corresponding controls of the semi-discrete model will diverge in the L^2-norm as $h \to 0$.

In [8] it was shown that, after filtering the high frequency modes, an uniform observability inequality for the adjoint system holds. This is equivalent with the uniform controllability of the projection of the solutions over the space generated by the remaining eigenmodes. Observe that the dimension of this space tends to infinity as the step size h goes to zero and that, in the limit, we would obtain the control of the continuous system. However, in practice these projection methods are not very efficient.

In [12] the problem with finite difference approximations was considered again. It was proved that, if the high frequency modes of the *discrete initial data* are filtered out in an appropriate manner (or if the initial data are sufficiently regular), there are controls of the semi-discrete model which converge to a control of (1). This is one of the ways of taking care of the spurious high frequency oscillations that the numerical method introduces. Note that in this case the uniform controllability of the entire discrete solutions is ensured and not only that of the projections, as in [8]. Moreover, it was also shown that the norm of the discrete HUM controls may increase exponentially with the number of points in the mesh if no filtering is applied.

As we have said, from a numerical point of view, several techniques have been proposed as possible cures of the high frequency spurious oscillations. For example, in [6] a Tychonoff regularization procedure was successfully implemented. Roughly speaking, this method introduces an additional control, tending to zero with the mesh size, but acting on the interior of the domain. Other proposed numerical techniques are multi-grid (see [4] and [7]) and mixed finite element methods (see [5]).

This paper considers a different method to obtain the uniform controllability as the discretization parameter h goes to zero. It consists in a different space discretization scheme of the equation (1) derived from a *mixed finite element method*, which is based on different discretizations for the position and velocity. More precisely, while the classical first order splines are used for the former, discontinuous elements approximate the latter. This method is different to the one used in [5] where u and ∇u are approximated in different finite dimensional spaces.

With this method, we explicitly construct a sequence of discrete controls which tends to the HUM control of the limit wave equation (1). Recall that the controls are not unique and that the HUM characterizes the one with minimal L^2-norm. The controls of our semi-discrete system have also this minimal L^2-norm property.

Let us briefly explain why the method that we introduce here leads to a uniformly controllable semi-discrete system. As we have mentioned before the main problem from the controllability point of view was pointed out in [8] and it is due to the fact that the velocity of propagation of some high frequency numerical waves may converge to zero when the discretization step, h tends to zero. As a consequence, in order to obtain a uniform controllability result we should consider a controllability time T that tends to infinity as h tends to zero. In our discrete model this phenomenon does not appear. In fact, we show that the speed of propagation of the high frequency oscillations is larger than the one corresponding to the continuous solutions. For the controllability property, this does not present any problem.

To our knowledge, this mixed finite element approach was considered for the first time in the context of the wave equation in [2], in order to obtain an uniform decay rate for the semi-discrete wave equation with boundary dissipation. Later on, in [10], a rigorous analysis of the convergence and error estimates were given. It is generally known that uniform stability implies controllability. Nevertheless, it is not clear how to establish such principle when passing to the limit from the finite dimensional discrete systems to the infinite dimensional continuous one. Moreover, here we offer an explicit way to construct the discrete controls which tend to the HUM control of the limit wave equation (1).

We observe also that there exist some full discretization schemes associated to system (1) which provide uniformly controllable discrete systems as both, the space and time steps, converge to zero (see [14]). This is due to the fact that, for some numerical schemes discretizing the wave equation, the dispersion phenomenon introduced by the space semi-discretization is corrected by the time-discretization. Indeed, these very particular fully-discrete systems provide the exact solution of the wave equation at the nodes. For example, this is the case for the classical central finite difference scheme with equal space and time steps. Of course, this unbeatable situation is only possible for the one-dimensional wave equation. The main advantage of our scheme is that it can be generalized to higher dimensions. This will be done elsewhere.

The rest of the paper is organized in the following way. In the second section briefly recall some controllability results for the wave equation (1). In the third section the semi-discrete model under consideration is derived and the main results of existence, characterization and convergence of the discrete controls and solutions are presented. The fourth section discusses the corresponding adjoint homogeneous equation. The main result is an uniform observability inequality which will be fundamental for our study. In the fifth section the controllability problem for the discrete equation is addressed and an uniformly bounded sequence of controls is obtained. Next, the convergence of the discrete controls to the HUM control of the continuous equation (1) is proved. The convergence of the corresponding solutions is also discussed. The weak and strong convergence properties are given in terms of the behavior of the Fourier coefficients of the discrete initial data. Finally, section 6 is devoted to the numerical results and some comments.

2 The continuous problem: results and notation

In this section we recall the controllability property of the wave equation (1) and we introduce some notation that will be used in the article. The results are given without proofs. Full details may be found, for instance, in [11].

Theorem 2.1 *Given any $T > 2$ and $(u^0, u^1) \in L^2(0, 1) \times H^{-1}(0, 1)$ there exists a control function $v \in L^2(0, T)$ such that the solution (u, u') of (1) verifies (2).*

In fact, following HUM, the control of minimal L^2-norm may be obtained by minimizing the functional

$$\mathcal{J} : H_0^1(0, 1) \times L^2(0, 1) \to \mathbb{R},$$

$$\mathcal{J}(w^0, w^1) = \frac{1}{2}\int_0^T (w_x)^2(t, 1)dt + \int_\Omega u^0(x)w'(0, x)dx - \langle u^1, w(0, \cdot)\rangle_{H^{-1},H_0^1}, \tag{3}$$

where (w, w') is the solution of the adjoint backward homogeneous equation

$$\begin{cases} w'' - w_{xx} = 0, & \text{for } x \in (0, 1), \quad t > 0 \\ w(t, 0) = w(t, 1) = 0, & \text{for } t > 0, \\ w(T, x) = w^0(x), \;\; w'(T, x) = w^1(x), & \text{for } x \in (0, 1). \end{cases} \tag{4}$$

In (3) and in the rest of this paper, $< \cdot, \cdot >_{H^{-1},H_0^1}$ denotes the duality pairing between $H^{-1}(0, 1)$ and $H_0^1(0, 1)$.

Theorem 2.2 *Given any $T > 2$ and $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ the functional $\mathcal{J}$ has an unique minimizer $(\widehat{w}^0, \widehat{w}^1) \in H_0^1(\Omega) \times L^2(\Omega)$. If $(\widehat{w}, \widehat{w}')$ is the corresponding solution of (4) with initial data $(\widehat{w}^0, \widehat{w}^1)$ then $v(t) = \widehat{w}_x(t, 1)$ is the control of (1) with minimal L^2-norm.*

Remark 2.1 The control v from Theorem 2.2 is usually called the HUM control. It may be characterized by the following two properties

(i) v is a control for (1), or equivalently,

$$\int_0^T v(t)w_x(t, 1)dt =< u^1, w(0) >_{H^{-1},H_0^1} - \int_0^1 u^0(x)w'(0, x)dx, \tag{5}$$

for any $(w^0, w^1) \in H_0^1(0, 1) \times L^2(0, 1)$, w being the solution of the adjoint equation (4).

(ii) There exists $(\widehat{w}^0, \widehat{w}^1) \in H_0^1(0, 1) \times L^2(0, 1)$ such that $v(t) = \widehat{w}_x(t, 1)$, where $(\widehat{w}, \widehat{w}')$ is the solution of the adjoint equation (4) with initial data $(\widehat{w}^0, \widehat{w}^1)$.

□

Associated to system (1) we introduce the following eigenvalue problem,

$$\begin{pmatrix} 0 & -1 \\ -\partial_x^2 & 0 \end{pmatrix} \begin{pmatrix} \Phi^1 \\ \Phi^2 \end{pmatrix} = \lambda \begin{pmatrix} \Phi^1 \\ \Phi^2 \end{pmatrix}, \tag{6}$$

where ∂_x^2 is the Laplace operator with homogeneous Dirichlet boundary conditions.

The eigenvalues of (6) are given by

$$\lambda^n = n\pi i, \quad n \in \mathbb{Z}^*, \tag{7}$$

and the corresponding eigenfunctions are

$$\Phi^n = \begin{pmatrix} \Phi^{n,1} \\ \Phi^{n,2} \end{pmatrix} = \begin{pmatrix} \frac{1}{in\pi} \sin(n\pi x) \\ -\sin(n\pi x) \end{pmatrix}. \tag{8}$$

Thus, any initial data (u^0, u^1) of (1) may be expanded in Fourier series as follows

$$(u^0, u^1) = \sum_{n \in \mathbb{Z}^*} a_0^n \Phi^n. \tag{9}$$

3 The semi-discrete model

Let us consider $N \in \mathbb{N}^*$, $h = \frac{1}{N+1}$ and an uniform grid of the interval $(0, 1)$ given by $0 = x_0 < x_1 < \cdots < x_N < x_{N+1} = 1$, with $x_j = jh$, $0 \le j \le N + 1$.

We introduce the following semi-discrete system

$$\begin{cases} \frac{h}{4}\left[2u_j''(t) + u_{j+1}''(t) + u_{j-1}''(t)\right] \\ -\frac{1}{h}\left[u_{j+1}(t) + u_{j-1}(t) - 2u_j(t)\right] = 0, & \text{for } 1 \le j \le N, \ t > 0, \\ u_0(t) = 0, \ u_{N+1}(t) = v_h(t), & \text{for } t > 0, \\ u_j(0) = u_j^0, \ u_j'(0) = u_j^1, & \text{for } 1 \le j \le N. \end{cases} \tag{10}$$

System (10) consists of N linear differential equations with N unknowns $u_1, u_2, \ldots, u_N$. The function $u_j(t)$ is an approximation of the solution u of (1) in (t, x_j), provided that $(u_j^0, u_j^1)_{1 \le j \le N}$ approximates the initial datum (u^0, u^1).

Our aim is *to study the controllability properties of (10) and to show the convergence of the controls of (10) to a control of (1).*

As we have mentioned in the introduction, the space discretization is the most delicate step in the approximation of the controls of (1). Recall that the lack of the convergence of the controls when finite difference or finite element methods are used may be seen in the semi-discrete models. Generally, the time discretization does not introduce more difficulties, except the ones related to the usual stability condition.

In the rest of this section we deduce the above semi-discrete model and state the main results of the paper.

3.1 Derivation of the model

Let us introduce a variational formulation associated to (1): find $(u, z) = (u, z)(t, x)$ such that $u(t) \in H_t = \{\varphi \in H^1(0,1) : \varphi(0) = 0, \varphi(1) = v(t)\}$ and $z(t) \in L^2(0,1)$, for any $t \in (0,T)$, and the following holds

$$\begin{cases} \frac{d}{dt}\int_0^1 u(t,x)\psi(x)dx = \int_0^1 z(t,x)\psi(x)dx, & \text{for all } \psi \in L^2(0,1), \\ \frac{d}{dt} < z(t,\cdot),\varphi >_{H^{-1},H_0^1} = \int_0^1 u_x(t,x)\varphi_x(x)dx, & \text{for all } \varphi \in H_0^1(0,1), \\ u(0,x) = u^0(x),\ z(0,x) = u^1(x), & \text{for } x \in (0,1). \end{cases} \tag{11}$$

In order to discretize (11), we consider $N \in \mathbb{N}^*$, $h = \frac{1}{N+1}$ and $x_j = jh$, with $0 \le j \le N+1$.

For each $1 \le j \le N$, we introduce the functions $\varphi_j, \psi_j : [0,1] \to \mathbb{R}$ defined by

$$\varphi_j(x) = \begin{cases} \frac{x - x_{j-1}}{h} & \text{if } x \in (x_{j-1}, x_j) \\ \frac{x_{j+1} - x}{h} & \text{if } x \in [x_j, x_{j+1}] \\ 0 & \text{otherwise,} \end{cases} \qquad \psi_j(x) = \begin{cases} \frac{1}{2} & \text{if } x \in (x_{j-1}, x_{j+1}) \\ 0 & \text{otherwise,} \end{cases}$$

and $\varphi_{N+1} : [0,1] \to \mathbb{R}$ defined by

$$\varphi_{N+1}(x) = \begin{cases} \frac{x - x_N}{h} & \text{if } x \in (x_N, x_{N+1}] \\ 0 & \text{otherwise.} \end{cases}$$

System (11) is discretized in the following way: find

$$u_h(t,x) = \sum_{i=1}^N u_i(t)\varphi_i(x) + v_h(t)\varphi_{N+1}(x),$$
$$z_h(t,x) = \sum_{i=1}^N b_i(t)\psi_i(x), \tag{12}$$

such that

$$\begin{cases} \frac{d}{dt}\int_0^1 u_h(t,x)\psi_j(x)dx = \int_0^1 z_h(t,x)\psi_j(x)dx, & \text{for all } 1 \le j \le N, \\ \frac{d}{dt} < z_h(t,\cdot),\varphi_j >_{H^{-1},H_0^1} = \int_0^1 (u_h)_x(t,x)(\varphi_j)_x(x)dx, & \text{for all } 1 \le j \le N, \\ u_h(0,x) = u_h^0(x),\ z_h(0,x) = u_h^1(x), & \text{for } x \in (0,1). \end{cases} \tag{13}$$

Observe that in (12), whereas the classical linear splines approximate the position u, discontinuous piecewise constant functions are used for the velocity z. This type of methods, with different basis functions for approximating the two components of the solutions, are usually called mixed finite element methods (see, for instance, [3] or [15]).

By taking into account that, for any $1 \le i, j \le N$,

$$< \psi_i, \varphi_j >_{H^{-1},H_0^1} = \int_0^1 \psi_i(x)\varphi_j(x)dx = \int_0^1 \psi_i(x)\psi_j(x)dx = \begin{cases} \frac{h}{2} & \text{if } i = j \\ \frac{h}{4} & \text{if } |i-j| = 1 \\ 0 & \text{othewise,} \end{cases}$$

and

$$\int_0^1 (\varphi_i)_x(x)(\varphi_j)_x(x)dx = \begin{cases} \frac{2}{h} & \text{if } i = j \\ -\frac{1}{h} & \text{if } |i-j| = 1 \\ 0 & \text{othewise,} \end{cases}$$

we obtain that (13) is equivalent with system (10).

3.2 Main results

We first write (10) in an equivalent vectorial form and introduce some notation.

Let us define the matrices $K_h, M_h \in \mathcal{M}_{N\times N}(\mathbb{R})$ as follows:

$$K_h = \frac{1}{h}\begin{pmatrix} 2 & -1 & 0 & 0 & ..... & 0 & 0 \\ -1 & 2 & -1 & 0 & ..... & 0 & 0 \\ 0 & -1 & 2 & -1 & ..... & 0 & 0 \\ ... & ... & ... & ... & ..... & ... & ... \\ 0 & 0 & 0 & 0 & ..... & 2 & -1 \\ 0 & 0 & 0 & 0 & ..... & -1 & 2 \end{pmatrix}, \quad M_h = \frac{h}{4}\begin{pmatrix} 2 & 1 & 0 & 0 & ..... & 0 & 0 \\ 1 & 2 & 1 & 0 & ..... & 0 & 0 \\ 0 & 1 & 2 & 1 & ..... & 0 & 0 \\ ... & ... & ... & ... & ..... & ... & ... \\ 0 & 0 & 0 & 0 & ..... & 2 & 1 \\ 0 & 0 & 0 & 0 & ..... & 1 & 2 \end{pmatrix}.$$

If we denote the unknown by $U_h(t) = (u_1(t), u_2(t), \dots, u_N(t))^T$, equation (10) may be written as

$$\begin{cases} M_h U_h''(t) + K_h U_h(t) = F_h(t), & \text{for } t > 0 \\ U_h(0) = U_h^0, \ U_h'(0) = U_h^1, \end{cases} \tag{14}$$

where $U_h^0 = (u_j^0)_{1\le j\le N}$, $U_h^1 = (u_j^1)_{1\le j\le N}$ and the vector F_h is given by

$$F_h(t) = \begin{pmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 0 \\ -\frac{h}{4}v_h''(t) + \frac{1}{h}v_h(t) \end{pmatrix}.$$

In (14) we have taken into account that $u_{N+1}(t) = v_h(t)$ and $u_0(t) = 0$ for all $t > 0$.

Our aim is to study the following controllability property for (14): *given $T > 2$, and $(U_h^0, U_h^1) \in \mathbb{R}^{2N}$, there exists a control function $v_h \in H^2(0,T)$ such that the solution (U_h, U_h') of (14) satisfies*

$$U_h(T) = U_h'(T) = 0. \tag{15}$$

If this holds for any $(U_h^0, U_h^1) \in \mathbb{R}^{2N}$ we say that (14) is *exactly controllable*.

It is not difficult to see that the controllability problem we have just addressed has a positive answer and a sequence of discrete controls $(v_h)_{h>0}$ may be found. Let us now describe how the sequence $(v_h)_{h>0}$ may be constructed such that the convergence to the HUM control of the continuous problem (1) is ensured.

Now we introduce the Fourier expansion of the initial data in (14). In order to do this we consider the eigenvalue problem associated to system (14),

$$\begin{pmatrix} 0 & -I \\ M_h^{-1}K_h & 0 \end{pmatrix} \begin{pmatrix} \varphi^1 \\ \varphi^2 \end{pmatrix} = \lambda \begin{pmatrix} \varphi^1 \\ \varphi^2 \end{pmatrix}. \tag{16}$$

We observe that the eigenvalues of system (16) are given by

$$\lambda_h^n = i\,\frac{2}{h}\tan\left(\frac{n\pi h}{2}\right), \quad 1 \le |n| \le N, \tag{17}$$

while the corresponding eigenfunctions are given by

$$\varphi_h^n = \begin{pmatrix} \varphi_h^{n,1} \\ \varphi_h^{n,2} \end{pmatrix} = \frac{1}{\cos\left(\frac{n\pi h}{2}\right)} \begin{pmatrix} \frac{1}{\lambda_h^n}\theta_h^n \\ -\theta_h^n \end{pmatrix}, \tag{18}$$

where $\theta_h^n = (\sin(n\pi jh))_{1\le j\le N}$.

Remark 3.1 In Figure 1 we illustrate the different behavior of the eigenvalues of the continuous problem (6), the ones of the semi-discrete problem with mixed finite element (16) and the ones corresponding to the finite difference scheme studied in [8]. Note that in the continuous case the gap between two consecutive eigenvalues, $\gamma = \lambda^{n+1} - \lambda^n$, is constantly equal to π whereas in the finite difference scheme this is tending to zero with the mesh size. The mixed finite element scheme produces a spectrum with a constantly increasing gap. This fact has very important consequences for the controllability problem. It indicates that it is more difficult to control the high frequencies of the finite difference model than those obtained from the mixed finite element one.

Any initial data (U_h^0, U_h^1) of (14) may be expanded in Fourier series as follows

$$(U_h^0, U_h^1) = \sum_{1\le|n|\le N} a_{0h}^n \varphi_h^n. \tag{19}$$

In the sequel, $(a_{0h}^n)_{n\in\mathbb{Z}^*}$ (or $(a_{0h}^n)_n$ to abbreviate) will denote the sequence of the Fourier coefficients extended by zero, i.e. we assume $a_{0h}^n = 0$, when $|n| > N$.

We consider in $\mathbb{C}^{2N}$ the inner product defined by

$$\begin{aligned} < f, g >_0 = h\left[\sum_{k=1}^{N-1} \frac{f_{k+1}-f_k}{h}\frac{\overline{g}_{k+1}-\overline{g}_k}{h} + \frac{1}{h^2}(f_1\overline{g}_1 + f_N\overline{g}_N)\right] \\ +h\left[\sum_{k=N+1}^{2N-1} \frac{f_{k+1}+f_k}{2}\frac{\overline{g}_{k+1}+\overline{g}_k}{2} + \frac{1}{4}(f_{N+1}\overline{g}_{N+1} + f_{2N}\overline{g}_{2N})\right]. \end{aligned} \tag{20}$$

where $f = (f_k)_{1\le k\le 2N}$ and $g = (g_k)_{1\le k\le 2N}$ are two vectors from $\mathbb{C}^{2N}$.

The corresponding norm will be denoted by $||\ \cdot\ ||_0$.

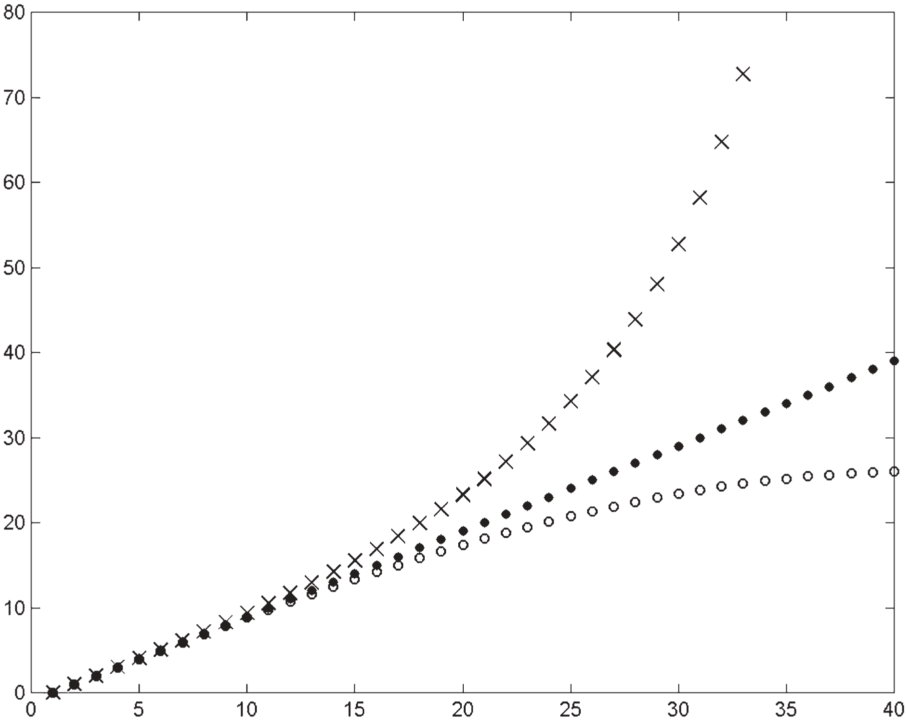

Fig. 1 Spectra of the eigenvalue problem (6) (dots), the classical central finite difference discretization of (6) (circles) and the eigenvalue problem (16), associated to the mixed finite elements scheme, (crosses), when $N = 40$

Remark 3.2 The following equivalent form of the inner product (20) justifies its definition and its usefulness for our problem:

$$< f, g >_0 = < K_h f^1, g^1 > + < M_h f^2, g^2 >, \tag{21}$$

where $f^1 = (f_k)_{1\leq k\leq N}$, $f^2 = (f_k)_{N+1\leq k\leq 2N}$, $g^1 = (g_k)_{1\leq k\leq N}$ and $g^2 = (g_k)_{N+1\leq k\leq 2N}$ and $< \cdot, \cdot >$ denotes the canonical inner product in $\mathbb{C}^N$. □

Given $T > 2$, let us also consider a cut-off function $\rho \in C^\infty[0, T]$ with the property that there exists a positive number $\varepsilon > 0$ such that $T - 2\varepsilon > 2$ and

$$\begin{aligned} &\text{(i) } \operatorname{supp}(\rho) \subset (\varepsilon/2, T - \varepsilon/2),\\ &\text{(ii) } 0 \leq \rho(t) \leq 1 \text{ for all } t \in [0, T],\\ &\text{(iii) } \rho(t) \geq 1/2 \text{ for all } t \in [\varepsilon, T - \varepsilon]. \end{aligned} \tag{22}$$

Finally, let $J : \mathbb{R}^{2N} \to \mathbb{R}$ be the semi-discrete version of the functional $\mathcal{J}$ in (3), defined by

$$\begin{aligned} J((W_h^0, W_h^1)) = \frac{1}{8}\int_0^T \rho(t)(w_N')^2(t)dt + \frac{1}{2h^2}\int_0^T (w_N)^2(t)dt -\\ - < (-(K_h^{-1})M_h U_h^1, U_h^0), (W_h(0), W_h'(0)) >_0, \end{aligned} \tag{23}$$

where (W_h, W_h') is the solution of the following adjoint homogeneous system:

$$\begin{cases} M_h W_h''(t) + K_h W_h(t) = 0, & \text{for } t \in (0, T), \\ W_h(T) = W_h^0, \ W_h'(T) = W_h^1. \end{cases} \tag{24}$$

The unknown of (24) is the vector-valued function $W_h(t) = (w_1(t), w_2(t), \dots, w_N(t))^T$. Note that we do not make explicit the dependence in h of the components $w_j(t)$, to simplify the notation.

The following result is a discrete version of Theorem 2.2.

Theorem 3.1 *Given any $T > 2$ and $(U_h^0, U_h^1) \in \mathbb{R}^{2N}$, the functional J defined by (23) has a unique minimizer $(\widehat{W}_h^0, \widehat{W}_h^1) \in \mathbb{R}^{2N}$. Let $v_h \in C^\infty[0, T]$ be the unique solution of the differential equation*

$$\begin{cases} -\frac{h}{4} v_h'' + \frac{1}{h} v_h = -\frac{1}{4}\left[\rho(t)\widehat{w}_N'(t)\right]' + \frac{1}{h^2}\widehat{w}_N(t), & t \in (0, T) \\ v_h'(0) = v_h'(T) = 0, \end{cases} \tag{25}$$

where $(\widehat{W}_h, \widehat{W}_h')$ is the solution of (24) with initial data $(\widehat{W}_h^0, \widehat{W}_h^1)$. Then v_h is a control for (14).

The proof of Theorem 3.1 will be given in Section 5.1.

The main results of this paper are the following two convergence theorems.

Theorem 3.2 *Assume that $T > 2$. Let $(U_h^0, U_h^1)_{h>0}$ be a sequence of discretizations of the continuous initial data (u^0, u^1). Assume that $(a_{0h}^n)_{n \in \mathbb{Z}^*}$, the Fourier coefficients of the discrete initial data $(U_h^0, U_h^1)_{h>0}$, verify*

$$\left(\frac{a_{0h}^n}{\lambda_h^n}\right)_n \rightharpoonup \left(\frac{a_0^n}{n\pi i}\right)_n \quad \textit{when } h \to 0 \textit{ in } \ell^2, \tag{26}$$

where $(a_0^n)_{n \in \mathbb{Z}^*}$ are the Fourier coefficients of the continuous initial data (u^0, u^1).

Let $(v_h)_{h>0}$ be the sequence of controls given by Theorem 3.1. Then $(v_h)_{h>0}$ and $(hv_h')_{h>0}$ are uniformly bounded in $L^2(0, T)$, $(h^2 v_h')_{h>0}$ is uniformly bounded in $L^\infty(0, T)$ and there exists a subsequence (denoted in the same way) and $v \in L^2(0, T)$ such that

$$\begin{aligned} v_h &\rightharpoonup v \textit{ in } L^2(0, T), \\ hv_h' &\rightharpoonup 0 \textit{ in } L^2(0, T), \\ h^2 v_h' &\rightharpoonup 0 \textit{ in } L^\infty(0, T). \end{aligned} \tag{27}$$

Moreover, the limit v is the HUM control of the continuous equation (1).

If the convergence in (26) is strong in ℓ^2 then $(v_h)_{h>0}$, $(hv_h')_{h>0}$ and $(h^2 v_h')_{h>0}$ converge strongly too.

Theorem 3.3 *Assume that $T > 2$. Let $(U_h^0, U_h^1)_h$ be a sequence of discretizations of the initial data (u^0, u^1). Assume that the Fourier coefficients of the discrete and continuous initial data, $(U_h^0, U_h^1)_{h>0}$ and (u^0, u^1) respectively, verify (26). Let*

$\left(a_h^n(t)\right)_n$ and $(a^n(t))_n$ be the Fourier coefficients of the HUM-controlled solutions of the semi-discretized and continuous problem respectively. Then

$$\left(\frac{a_h^n(t)}{\lambda_h^n}\right)_n \rightharpoonup \left(\frac{a^n(t)}{\lambda^n}\right)_n \text{ weakly in } C([0,T];\ell^2) \text{ as } h \to 0. \tag{28}$$

If the convergence in (26) is strong in ℓ^2, then the convergence in (28) holds strongly too.

The proofs of Theorems 3.2 and 3.3 will be given in Sections 5.3 and 5.4 respectively.

Remark 3.3 In Theorems 3.2 and 3.3 we have assumed the convergence of the sequence of Fourier coefficients of the discrete initial data to the sequence of Fourier coefficients of the continuous ones. The usual discretization by points

$$(U_h^0, U_h^1) = ((u^0(jh))_{1\le j\le N}, (u^1(jh))_{1\le j\le N}), \tag{29}$$

leads to a convergence property of the Fourier coefficients sequence that depends on the regularity of the initial data (u^0, u^1) of the continuous problem. Indeed, it is not difficult to prove the following:

(i) If u^0 and u^1 are piecewise continuous functions in $[0, 1]$ then

$$\left(\frac{a_{0h}^n}{\lambda_h^n}\right)_n \rightharpoonup \left(\frac{a_0^n}{n\pi i}\right)_n \text{ when } h \to 0 \text{ in } \ell^2. \tag{30}$$

(ii) If u^0 and u^1 are one time derivable with continuous derivative in $[0, 1]$ then

$$\left(\frac{a_{0h}^n}{\lambda_h^n}\right)_n \to \left(\frac{a_0^n}{n\pi i}\right)_n \text{ when } h \to 0 \text{ in } \ell^2. \tag{31}$$

Remark 3.4 Let $(a_h^n(t))_{n\in\mathbb{Z}^*}$, $(a^n(t))_{n\in\mathbb{Z}^*} \in L^\infty((0,T);\ell^2)$ be the sequence of Fourier coefficients from Theorem 3.3. We define

$$(\overline{W}_h^1(t,x), \overline{W}_h^2(t,x)) = \sum_{n\in\mathbb{Z}^*} a_h^n(t)\Phi^n(x),$$

where Φ^n are the eigenfunctions of the wave operator given by (8). We also consider

$$(w(t,x), w'(t,x)) = \sum_{n\in\mathbb{Z}^*} a^n(t)\Phi^n(x),$$

the controlled solution of equation (1). We have that

(i) If the weak convergence in (28) holds then $((\overline{W}^1_h, \overline{W}^2_h))_{h>0}$ converges weakly to (w, w') in $C([0,T]; L^2(0,1) \times H^{-1}(0,1))$ when h tends to zero.
(ii) If the convergence in (28) holds strongly in $L^\infty((0,T), \ell^2)$ then $((\overline{W}^1_h, \overline{W}^2_h))_{h>0}$ converges strongly to (w, w') in $C([0,T]; L^2(0,1) \times H^{-1}(0,1))$ when h tends to zero.

□

4 The homogeneous adjoint problem

As it is well known, the controllability property stated in Theorem 2.1 is equivalent with a certain observability inequality for the homogeneous adjoint system (4). In this section we prove that the semi-discrete version of such inequality holds for the homogeneous adjoint system (32) below, independently of the discretization step h. Let us consider the system

$$\begin{cases} M_h W_h''(t) + K_h W_h(t) = 0, & \text{for } t \in (0,T), \\ W_h(T) = W_h^0, \ \ W_h'(T) = W_h^1, \end{cases} \tag{32}$$

where $(W_h^0, W_h^1) \in \mathbb{R}^{2N}$.

System (32) stands for the homogeneous adjoint of (14). Its unknown is the vector-valued function $W_h(t) = (w_1(t), w_2(t), \ldots, w_N(t))^T$.

We may write (32) in the following equivalent form

$$\begin{cases} \begin{pmatrix} W_h \\ W_h' \end{pmatrix}' + \begin{pmatrix} 0 & -I \\ M_h^{-1} K_h & 0 \end{pmatrix} \begin{pmatrix} W_h \\ W_h' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & \text{for } t \in (0,T), \\ W_h(T) = W_h^0, \ \ W_h'(T) = W_h^1. \end{cases} \tag{33}$$

It follows that (33) is a system of $2N$ linear differential equations of order one and has an unique solution $(W_h, W_h') \in \mathcal{C}^\omega\left([0,\infty), \mathbb{R}^{2N}\right)$, the class of the analytic functions in $[0,\infty)$ with values in $\mathbb{R}^{2N}$.

The energy of (32) is defined by

$$E_h(t) = \frac{h}{2} \sum_{j=0}^{N} \left[\left| \frac{w_{j+1}' + w_j'(t)}{2} \right|^2 + \left| \frac{w_{j+1}(t) - w_j(t)}{h} \right|^2 \right]. \tag{34}$$

The following proposition indicates that the energy of (32) is conserved in time and the system is conservative.

Proposition 4.1 *If $E_h(t)$ is the energy function corresponding to (32) defined by (34) then*

$$\frac{dE_h}{dt}(t) = 0, \qquad \forall t > 0. \tag{35}$$

Proof Multiplying the first equation of (32) by w_j' and adding the relations we obtain

$$0=\sum_{j=1}^{N}\frac{h}{4}\left[w''_{j+1}(t)+w''_{j-1}(t)+2w''_{j}(t)\right]w'_{j}$$
$$-\sum_{j=1}^{N}\frac{w_{j+1}(t)+w_{j-1}(t)-2w_{j}(t)}{h}w'_{j}(t). \qquad (36)$$

The first term in this expression can be simplified as follows

$$\sum_{j=1}^{N}\frac{h}{4}\left[w''_{j+1}+w''_{j-1}+2w''_{j}\right]w'_{j}=\frac{h}{4}\sum_{j=1}^{N}\left((w''_{j+1}+w''_{j})w'_{j}+(w''_{j}+w''_{j-1})w'_{j}\right)$$
$$=\frac{h}{4}\left(\sum_{j=0}^{N}(w''_{j+1}+w''_{j})(w'_{j}+w'_{j+1})\right)-(w''_{1}+w''_{0})w'_{0}-(w''_{N+1}+w''_{N})w'_{N+1}$$
$$=\frac{h}{4}\sum_{j=0}^{N}(w''_{j+1}+w''_{j})(w'_{j}+w'_{j+1}). \qquad (37)$$

Concerning the second term in (36) we have

$$\sum_{j=1}^{N}\frac{1}{h}\left(w_{j+1}+w_{j-1}-2w_{j}\right)w'_{j}=\frac{1}{h}\sum_{j=1}^{N}\left((w_{j+1}-w_{j})w'_{j}-(w_{j}-w_{j-1})w'_{j}\right)$$
$$=\frac{1}{h}\left(\sum_{j=0}^{N-1}(w_{j+1}-w_{j})(w'_{j}-w'_{j+1})\right)-(w_{1}-w_{0})w'_{0}+(w_{N+1}-w_{N})w'_{N+1}$$
$$=\frac{1}{h}\sum_{j=0}^{N}(w_{j+1}-w_{j})(w'_{j}-w'_{j+1}).$$

Combining formulas (36)–(38) it follows that

$$0=\frac{h}{4}\sum_{j=0}^{N}(w''_{j+1}+w''_{j})(w'_{j}+w'_{j+1})+\frac{1}{h}\sum_{j=0}^{N}(w_{j+1}-w_{j})(w'_{j+1}-w'_{j}).$$

Therefore, we obtain

$$\frac{dE_h}{dt}(t)=\frac{h}{2}\frac{d}{dt}\left[\sum_{j=0}^{N}\left|\frac{w'_{j}(t)+w'_{j+1}(t)}{2}\right|^{2}+\sum_{j=0}^{N}\left|\frac{w_{j+1}(t)-w_{j}(t)}{h}\right|^{2}\right]$$
$$=\frac{h}{4}\sum_{j=0}^{N}(w'_{j+1}(t)+w'_{j}(t))(w'_{j}(t)+w'_{j+1}(t))'$$
$$+\frac{1}{h}\sum_{j=0}^{N}(w_{j+1}(t)-w_{j}(t))(w_{j+1}(t)-w_{j}(t))'=0.$$

□

We divide the rest of this section in three parts where we give two different proofs of the observability inequality for the solutions of (32) and some convergence results.

4.1 Uniform observability inequality

In this section we prove the following.

Theorem 4.1 *Let $E_h(t)$ be the energy corresponding to (32) defined by (34) and $T > 2$. There exists a positive constant $C > 0$, not depending on N, such that*

$$E_h(0) \le C \left[\int_0^T \left(\frac{w_N}{h}\right)^2 (t)dt + \frac{h^2}{4} \int_0^T \left(\frac{w_N'}{h}\right)^2 (t)dt \right]. \tag{38}$$

Proof Multiplying the first equation of (32) by the multiplier $j\frac{w_{j+1}-w_{j-1}}{2}$, integrating in time and adding the relations we obtain

$$\begin{aligned}
0 &= \frac{h}{4}\sum_{j=1}^{N}\int_0^T \left[w_{j+1}'' + w_{j-1}'' + 2w_j''\right] j\frac{w_{j+1}-w_{j-1}}{2}dt \\
&\quad -\frac{1}{h}\sum_{j=1}^{N}\int_0^T \left[w_{j+1} + w_{j-1} - 2w_j\right] j\frac{w_{j+1}-w_{j-1}}{2}dt \\
&= X_h(t)|_0^T - \frac{h}{4}\sum_{j=1}^{N}\int_0^T \left[w_{j+1}' + w_{j-1}' + 2w_j'\right] j\frac{w_{j+1}'-w_{j-1}'}{2}dt \\
&\quad -\frac{1}{2h}\sum_{j=1}^{N}\int_0^T \left[w_{j+1} + w_{j-1} - 2w_j\right] j\frac{w_{j+1}-w_{j-1}}{2}dt,
\end{aligned} \tag{39}$$

where

$$X_h(t) = \frac{h}{4}\sum_{j=1}^{N}\left[w_{j+1}'(t) + w_{j-1}'(t) + 2w_j'(t)\right] j\frac{w_{j+1}(t)-w_{j-1}(t)}{2}. \tag{40}$$

We simplify the last two terms in (39). We have

$$\begin{aligned}
&\sum_{j=1}^{N}\left[w_{j+1} + w_{j-1} - 2w_j\right] j(w_{j+1}-w_{j-1}) \\
&\quad = \sum_{j=1}^{N}\left[(w_{j+1}-w_j)jw_{j+1} - (w_j - w_{j-1})jw_{j+1}\right. \\
&\qquad \left. -(w_{j+1}-w_j)jw_{j-1} + (w_j - w_{j-1})jw_{j-1}\right]
\end{aligned}$$

$$= \sum_{j=1}^{N} \big[(w_{j+1} - w_j) j (w_{j+1} - w_j) + (w_j - w_{j-1}) j (w_{j-1} - w_j)$$
$$-(w_j - w_{j-1}) j w_{j+1} - (w_{j+1} - w_j) j w_{j-1}$$
$$+(w_{j+1} - w_j) j w_j + (w_j - w_{j-1}) j w_j\big]$$
$$= \sum_{j=1}^{N} j (w_{j+1} - w_j)^2 - \sum_{j=0}^{N-1} (j+1)(w_{j+1} - w_j)^2$$
$$= -\sum_{j=0}^{N} (w_{j+1} - w_j)^2 + (N+1)|w_N|^2. \tag{41}$$

On the other hand

$$\sum_{j=1}^{N} \big[w'_{j+1} + w'_{j-1} + 2w'_j\big] j (w'_{j+1} - w'_{j-1}) = \sum_{j=1}^{N} \big[(w'_{j+1} + w'_j) j w'_{j+1}$$
$$+(w'_j + w'_{j-1}) j w'_{j+1} - (w'_{j+1} + w'_j) j w'_{j-1} - (w'_j + w'_{j-1}) j w'_{j-1}\big]$$
$$= \sum_{j=1}^{N} \big[(w'_{j+1} + w'_j) j (w'_{j+1} + w'_j) - (w'_j(t) + w'_{j-1}) j (w'_{j-1} + w'_j)$$
$$+(w'_j + w'_{j-1}) j w'_{j+1} - (w'_{j+1} + w'_j) j w'_{j-1}$$
$$-(w'_{j+1} + w'_j) j w'_j + (w'_j + w'_{j-1}) j w'_j\big]$$
$$= \sum_{j=1}^{N} j (w'_{j+1} + w'_j)^2 - \sum_{j=0}^{N-1} (j+1)(w'_{j+1} + w'_j)^2$$
$$= -\sum_{j=0}^{N} (w'_{j+1} + w'_j)^2 + (N+1)|w'_N|^2. \tag{42}$$

Hence, from (39)–(42) we have

$$0 = X(t)\big|_0^T + \frac{h}{8} \int_0^T \left(\sum_{j=0}^{N} (w'_{j+1} + w'_j)^2 - (N+1)|w'_N|^2 \right) dt$$
$$+ \frac{1}{2h} \int_0^T \left(\sum_{j=0}^{N} (w_{j+1} - w_j)^2 - (N+1)|w_N|^2 \right) dt. \tag{43}$$

Now, observe that, using Young's inequality in (40), we obtain, for any $a > 0$,

$$|X_h(t)| \le \frac{h}{8} \left[\sum_{j=1}^{N} \frac{1}{a} \left(w'_{j+1}(t) + w'_{j-1}(t) + 2w'_j(t) \right)^2 \right.$$
$$\left. + \sum_{j=1}^{N} a j^2 \left(\frac{w_{j+1}(t) - w_{j-1}(t)}{2} \right)^2 \right]$$

$$
\begin{aligned}
&\leq \frac{h}{4a}\sum_{j=1}^{N}\left[\left(w'_{j+1}(t)+w'_j(t)\right)^2+\left(w'_{j-1}(t)+w'_j(t)\right)^2\right]\\
&\quad+\frac{ah}{16}\sum_{j=1}^{N} j^2h^2\left[\left(\frac{w_{j+1}(t)-w_j(t)}{h}\right)^2+\left(\frac{w_j(t)-w_{j-1}(t)}{h}\right)^2\right]\\
&\leq \frac{h}{2a}\sum_{j=0}^{N}\left(w'_{j+1}(t)+w'_j(t)\right)^2+\frac{ah}{8}\sum_{j=0}^{N}\left(\frac{w_{j+1}(t)-w_j(t)}{h}\right)^2\\
&=\frac{2h}{a}\sum_{j=0}^{N}\left(\frac{w'_{j+1}(t)+w'_j(t)}{2}\right)^2+\frac{ah}{8}\sum_{j=0}^{N}\left(\frac{w_{j+1}(t)-w_j(t)}{h}\right)^2.
\end{aligned}
$$

If we take in particular $a=4$, we obtain that

$$
|X_h(t)|\leq E_h(t). \tag{44}
$$

From (43) and (44), it follows that

$$
\begin{aligned}
0&=X(t)|_0^T+\int_0^T E_h dt-\frac{1}{8}\int_0^T (w'_N)^2dt-\frac{1}{2}\int_0^T\left(\frac{w_N}{h}\right)^2dt\\
&\geq -2E_h(0)+TE_h(0)-\frac{1}{8}\int_0^T (w'_N)^2dt-\frac{1}{2}\int_0^T\left(\frac{w_N}{h}\right)^2dt.
\end{aligned}
$$

Inequality (38) follows now by taking $C=\frac{1}{2(T-2)}$. □

Remark 4.1 Let us consider ρ, a cut-off function with the properties (i)–(iii) in (22). The same proof and the conservation of the energy of (32) allow to deduce that there exists a positive constant $C=\frac{1}{2(T-2\varepsilon-2)}>0$, which does not depend on h, such that

$$
\begin{aligned}
E_h(0)&\leq C\left[\int_\varepsilon^{T-\varepsilon}\left(\frac{w_N}{h}\right)^2(t)dt+\frac{h^2}{4}\int_\varepsilon^{T-\varepsilon}\left(\frac{w'_N}{h}\right)^2(t)dt\right]\\
&\leq 2C\left[\int_\varepsilon^{T-\varepsilon}\rho(t)\left(\frac{w_N}{h}\right)^2(t)dt+\frac{h^2}{4}\int_\varepsilon^{T-\varepsilon}\rho(t)\left(\frac{w'_N}{h}\right)^2(t)dt\right].
\end{aligned}
$$

Therefore,

$$
E_h(0)\leq 2C\left[\int_0^T\left(\frac{w_N}{h}\right)^2(t)dt+\frac{h^2}{4}\int_0^T\rho(t)\left(\frac{w'_N}{h}\right)^2(t)dt\right]. \tag{45}
$$

This is the observability inequality we shall use in our control problem. □

The following direct inequality for the solutions of (32) establishes the optimality of (38).

Theorem 4.2 *Let $E_h(t)$ be the energy corresponding to (32) defined by (34) and $T > 0$. There exists a positive constant $C' > 0$, not depending on N, such that*

$$\left[\int_0^T \left(\frac{w_N}{h}\right)^2 (t)dt + \frac{h^2}{4}\int_0^T \left(\frac{w_N'}{h}\right)^2 (t)dt\right] \leq C' E_h(0). \tag{46}$$

Proof From (43) we obtain that

$$-X(t)|_0^T + \int_0^T \left(\frac{w_N}{h}\right)^2 (t)dt + \frac{1}{4}\int_0^T (w_N')^2(t)dt = 2\int_0^T E_h(t)dt = 2TE_h(0). \tag{47}$$

Combining this inequality with (44) we easily obtain

$$\int_0^T \left(\frac{w_N(t)}{h}\right)^2 dt + \frac{1}{4}\int_0^T (w_N'(t))^2 dt$$
$$= X(t)|_0^T + 2TE_h(0) \leq 2E_h(0) + 2E_h(T) + 2TE_h(0) = 2(T+2)E_h(0).$$

Inequality (46) follows by taking $C' = 2(T+2)$. □

4.2 A new proof of the observability inequality

In this section we give a different proof of the observability inequality (38) in Theorem 4.1 by using the following result on nonharmonic Fourier series due to Ingham [9].

Theorem 4.3 *Let $(\lambda_n)_{n\in\mathbb{Z}^*}$ be a sequence of real numbers and $\gamma > 0$ be such that*

$$\lambda_n - \lambda_{n-1} \geq \gamma > 0, \quad \forall n \in \mathbb{Z}^*. \tag{48}$$

For any $T > \frac{2\pi}{\gamma}$ there exists a positive constant $C = C(T,\gamma) > 0$ such that, for any finite sequence $(a_n)_{n\in\mathbb{Z}^*}$,

$$C\sum_{n\in\mathbb{Z}^*} | a_n |^2 \leq \int_0^T \left|\sum_{n\in\mathbb{Z}^*} a_n e^{i\lambda_n t}\right|^2 dt. \tag{49}$$

Let us first prove the following result on the eigenfunctions of (33) given by (18).

Proposition 4.2 *The eigenfunctions $(\varphi_h^n)_{1\leq|n|\leq N}$ form an orthonormal basis in $\mathbb{C}^{2N}$ with respect to the inner product defined by (20).*

Proof We have that

$$\cos\left(\frac{n\pi h}{2}\right)\cos\left(\frac{m\pi h}{2}\right) < \varphi_h^n, \varphi_h^m >_0 = \frac{1}{\lambda_h^n \overline{\lambda_m^h}} < K_h\theta_h^n, \theta_h^m > + < M_h\theta_h^n, \theta_h^m >$$
$$= \frac{1}{\lambda_h^n\overline{\lambda_h^m}}\frac{4}{h}\sin^2\left(\frac{n\pi h}{2}\right) < \theta_h^n, \theta_h^m > + h\cos^2\left(\frac{n\pi h}{2}\right) < \theta_h^n, \theta_h^m > .$$

Observe that

$$< \theta_h^n, \theta_h^m >= \sum_{j=1}^{N} \sin(j\pi hn)\sin(j\pi hm) = \frac{N+1}{2}\delta_{nm},$$

where $\delta_{nm} = 1$ if $n = m$ and 0 otherwise.

Hence,

$$\cos\left(\frac{n\pi h}{2}\right)\cos\left(\frac{m\pi h}{2}\right) < \varphi_h^n, \varphi_h^m >_0 = \frac{1}{\lambda_h^n \overline{\lambda}_h^m} < K_h\theta_h^n, \theta_h^m > + < M_h\theta_h^n, \theta_h^m >$$
$$= \left[\frac{1}{|\lambda_h^n|^2}\frac{4}{h}\sin^2\left(\frac{n\pi h}{2}\right) + h\cos^2\left(\frac{n\pi h}{2}\right)\right]\frac{1}{2h}\delta_{nm} = \cos^2\left(\frac{n\pi h}{2}\right)\delta_{nm}.$$

□

If we consider an initial data of (33) of the form

$$(W_h^0, W_h^1) = \sum_{1\le|n|\le N} b_{0h}^n \varphi_h^n, \tag{50}$$

we obtain that the corresponding solution is given by

$$(W_h, W_h')(t) = \sum_{1\le|n|\le N} b_{0h}^n e^{\lambda_h^n(T-t)}\varphi_h^n. \tag{51}$$

Remark 4.2 The energy corresponding to (32) may be written as

$$E_h(t) = \frac{1}{2} < (W_h(t), W_h'(t)), (W_h(t), W_h'(t)) >_0 . \tag{52}$$

□

Theorem 4.4 *There exists a positive constant $C > 0$, not depending on N, such that, for any N sufficiently large and for any initial data of the form (50), the solution (51) satisfies*

$$\sum_{1\le|n|\le N} |b_{0h}^n|^2 \le C\int_0^T \left[\left|\frac{w_N}{h}\right|^2(t) + \frac{h^2}{4}\left|\frac{w_N'}{h}\right|^2(t)\right]dt. \tag{53}$$

Proof Observe that

$$\frac{w_N}{h}(t) = \sum_{1\le|n|\le N} \frac{\sin(n\pi h)}{h\lambda_h^n\cos\left(\frac{n\pi h}{2}\right)} b_{0h}^n e^{\lambda_h^n(T-t)}.$$

In order to apply Theorem 4.3, note that, given $\varepsilon > 0$, there exists $h_\varepsilon > 0$ such that

$$|\lambda_h^n - \lambda_h^{n-1}| = \frac{2}{h}\left|\frac{\sin\left(\frac{\pi h}{2}\right)}{\cos\left(\frac{n\pi h}{2}\right)\cos\left(\frac{(n-1)\pi h}{2}\right)}\right| \ge \frac{2}{h}\sin\left(\frac{\pi h}{2}\right) \ge \pi - \varepsilon,$$

for any $h < h_\varepsilon$. Hence, for any $\varepsilon > 0$ and h sufficiently small,

$$|\lambda_h^n - \lambda_h^{n-1}| \geq \pi - \varepsilon. \tag{54}$$

By Theorem 4.3, for any $T > 2$ and h sufficiently small, the following inequality holds

$$\sum_{1\leq|n|\leq N} \cos^2\left(\frac{n\pi h}{2}\right)|b_{0h}^n|^2 = \sum_{1\leq|n|\leq N}\left|\frac{\sin(n\pi h)}{h\lambda_h^n \cos\left(\frac{n\pi h}{2}\right)}\right|^2 |b_{0h}^n|^2 \leq C\int_0^T \left|\frac{w_N}{h}(t)\right|^2 dt,$$

where C is a positive constant independent of N.

Now, remark that

$$\frac{w_N'}{h}(t) = \sum_{1\leq|n|\leq N} -\frac{\sin(n\pi h)}{h\cos\left(\frac{n\pi h}{2}\right)} b_{0h}^n e^{\lambda_h^n(T-t)}.$$

By Theorem 4.3, for any $T > 2$ and h sufficiently small, we have that

$$\frac{4}{h^2}\sum_{1\leq|n|\leq N} \sin^2\left(\frac{n\pi h}{2}\right)|b_{0h}^n|^2 = \sum_{1\leq|n|\leq N}\left|\frac{\sin(n\pi h)}{h\cos\left(\frac{n\pi h}{2}\right)}\right|^2 |b_{0h}^n|^2 \leq C\int_0^T \left|\frac{w_N'}{h}(t)\right|^2 dt.$$

Consequently, (53) holds. □

Remark 4.3 Inequality (53) is the observability inequality proved in Theorem 4.1. The key point in the proof presented above is the existence of an uniform (in h) gap (54). This property is a characteristic of the mixed finite element method we have used. If the finite difference or the classical finite elements are used instead, no such uniform gap exists and the uniform observability inequality (53) does not hold (see Figure 1). □

4.3 Some convergence results

In this paragraph we prove some convergence results for the homogeneous discrete equation (32) that will be used later.

Assume that the initial data of the adjoint continuous homogeneous problem (4) has the following Fourier decomposition

$$(w^0, w^1) = \sum_{n\neq 0} b_0^n \Phi^n, \tag{55}$$

and let (w, w') be the corresponding solution of (4) given by

$$(w, w')(t) = \sum_{n\neq 0} b_0^n e^{in\pi(T-t)}\Phi^n. \tag{56}$$

Lemma 4.1 *Let (W_h^0, W_h^1) and (w^0, w^1) be given by (50) and (55) respectively. Assume that $\left(b_{0h}^n\right)_n \rightharpoonup \left(b_0^n\right)_n$ in ℓ^2 when $h \to 0$. Then the sequence $\left(\frac{w_N}{h}\right)_{h>0}$ converges weakly to $w_x(\cdot, 1)$ in $L^2(0, T)$ when h tends to zero.*

Proof By taking into account formulas (51) and (56), the problem consists of proving that the following weak convergence holds in $L^2(0,T)$

$$\sum_{1\le|n|\le N}(-1)^{n+1} i b_{0h}^n \cos\left(\frac{n\pi h}{2}\right) e^{\lambda_h^n(T-t)} \rightharpoonup \sum_{n\neq 0}(-1)^{n+1} i b_0^n e^{n\pi i(T-t)}. \quad (57)$$

Indeed, it suffices to see that for any test function $\varphi \in \mathcal{D}(0,T)$,

$$\int_0^T \sum_n (-1)^{n+1} b_{0h}^n \cos\left(\frac{n\pi h}{2}\right) e^{\lambda_h^n(T-t)} \varphi(t)dt \\ \to \int_0^T \sum_n (-1)^{n+1} b_0^n e^{n\pi i(T-t)} \varphi(t)dt,$$

or, equivalently,

$$\int_0^T \sum_n (-1)^{n+1} \frac{b_{0h}^n}{(\lambda_n^h)^2} \cos\left(\frac{n\pi h}{2}\right) e^{\lambda_h^n(T-t)} \varphi''(t)dt \\ \to \int_0^T \sum_n (-1)^{n+1} \frac{b_0^n}{(n\pi i)^2} e^{n\pi i(T-t)} \varphi''(t)dt.$$

The last convergence holds since $(b_{0h}^n/\lambda_h^n)_n$ tends strongly to $(b_0^n/\lambda^n)_n$ in ℓ^2 and the proof finishes. □

5 The controllability problem

In this section we prove the main results of this article, i.e. Theorems 3.1, 3.2 and 3.3.

We divide the section in four subsections. In the first one we construct an explicit sequence of discrete controls v_h. In the second subsection we prove that this sequence of controls is uniformly bounded in $L^2(0,T)$ as h goes to zero. The convergence of the controls and solutions are addressed in the last two subsections respectively.

5.1 Existence of the discrete controls. Proof of Theorem 3.1

Firstly, we deduce a variational characterization of the controllability property for the system (14). Let (W_h, W_h') be the solution of the adjoint backward homogeneous system (32). Multiplying system (32) by the solution U_h of system (14) and integrating in time we obtain

$$\begin{aligned} 0 &= \int_0^T \left[< U_h, M_h W_h'' > + < U_h, K_h W_h > \right] dt \\ &= [< U_h, M_h W_h' > - < U_h', M_h W_h >]\Big|_0^T \\ &\quad + \int_0^T \left[< M_h U_h'', W_h > + < K_h U_h, W_h > \right] dt \\ &= [< U_h, M_h W_h' > - < U_h', M_h W_h >]\Big|_0^T + \int_0^T < F_h, W_h > dt. \end{aligned}$$

Hence,

$$\begin{aligned} &< U_h^0, M_h W_h'(0) > - < U_h^1, M_h W_h(0) > - < U_h(T), M_h W_h^1 > \\ &\quad + < U_h'(T), M_h W_h^0 > \\ &= \int_0^T \left(-\frac{h}{4} v_h''(t) + \frac{1}{h} v_h(t) \right) w_N(t) dt. \end{aligned} \tag{58}$$

If we assume that

$$v_h'(0) = v_h'(T) = 0, \tag{59}$$

we deduce that

$$-\frac{h}{4} \int_0^T v_h''(t) w_N(t) dt = \frac{h}{4} \int_0^T v_h'(t) w_N'(t) dt. \tag{60}$$

Moreover, we have that

$$\begin{aligned} &< U_h^0, M_h W_h'(0) > - < U_h^1, M_h W_h(0) > \\ &=< U_h^0, M_h W_h'(0) > - < K_h^{-1} M_h U_h^1, K_h W_h(0) > \\ &=< (-K_h^{-1} M_h U_h^1, U_h^0), (W_h(0), W_h'(0)) >_0 . \end{aligned} \tag{61}$$

From (58)–(61) we obtain the following variational characterization of the controllability property.

Lemma 5.1 *Given $T > 0$, system (14) is controllable if, for any $(U_h^0, U_h^1) \in \mathbb{R}^{2N}$, there exists $v_h \in H^2(0, T)$ which satisfies (59) and*

$$\begin{aligned} &\int_0^T \left(\frac{h}{4} v_h'(t) w_N'(t) + \frac{1}{h} v_h(t) w_N(t) \right) dt \\ &\qquad - < (-K_h^{-1} M_h U_h^1, U_h^0), (W_h(0), W_h'(0)) >_0 = 0, \end{aligned} \tag{62}$$

for any $(W_h^0, W_h^1) \in \mathbb{R}^{2N}$, (W_h, W_h') being the corresponding solution of (32).

For any $(f_1, f_2), (g_1, g_2) \in \mathbb{C}^{2N}$ we introduce the "duality product":

$$\langle (f_1, f_2), (g_1, g_2) \rangle_D =< f_1, M_h g_2 > - < f_2, M_h g_1 > . \tag{63}$$

The following holds:

Lemma 5.2 *If* $(\varphi_h^n)_{1\le|n|\le N}$ *are the eigenvectors of (16), given by (18), then*

$$\left\langle \varphi_h^n, \varphi_h^m \right\rangle_D = -\frac{1}{\lambda_h^n}\delta_{nm}, \quad \forall 1 \le |n|, |m| \le N. \tag{64}$$

Proof We have that

$$\begin{aligned}
\left\langle \varphi_h^n, \varphi_h^m \right\rangle_D &= \frac{1}{\cos\left(\frac{n\pi h}{2}\right)\cos\left(\frac{m\pi h}{2}\right)}\left[-\frac{1}{\lambda_h^n} < \theta_h^n, M_h\theta_h^m > + \frac{1}{\overline{\lambda_h^m}} < \theta_h^n, M_h\theta_h^m >\right] \\
&= \frac{1}{\cos\left(\frac{n\pi h}{2}\right)\cos\left(\frac{m\pi h}{2}\right)}\left[-\frac{1}{\lambda_h^n} < \theta_h^n, M_h\theta_h^m > + \frac{1}{\overline{\lambda_h^m}} < M_h\theta_h^n, \theta_h^m >\right] \\
&= \frac{ih^2}{2}\left[\frac{\cos\left(\frac{m\pi h}{2}\right)}{\sin\left(\frac{n\pi h}{2}\right)} + \frac{\cos\left(\frac{n\pi h}{2}\right)}{\sin\left(\frac{m\pi h}{2}\right)}\right] < \theta_h^n, \theta_h^m > \\
&= \frac{ih}{4}\left[\frac{\cos\left(\frac{n\pi h}{2}\right)}{\sin\left(\frac{n\pi h}{2}\right)} + \frac{\cos\left(\frac{n\pi h}{2}\right)}{\sin\left(\frac{n\pi h}{2}\right)}\right]\delta_{mn} = \frac{ih}{2}\text{ctan}\left(\frac{n\pi h}{2}\right)\delta_{mn}.
\end{aligned}$$

□

Remark 5.1 The definition of $< \cdot, \cdot >_D$ is justified by the following relation

$$< (-K_h^{-1}M_hU_h^1, U_h^0), (W_h(0), W_h'(0)) >_0 = < (U_h^0, U_h^1), (W_h(0), W_h'(0)) >_D . \tag{65}$$

Thus, (62) may be written in an equivalent form as

$$\int_0^T \left(\frac{h}{4}v_h'(t)w_N'(t) + \frac{1}{h}v_h(t)w_N(t)\right)dt = < (U_h^0, U_h^1), (W_h(0), W_h'(0)) >_D . \tag{66}$$

□

With the above notation the functional $J : \mathbb{R}^{2N} \to \mathbb{R}$ introduced in (23) reads

$$\begin{aligned}
J((W_h^0, W_h^1)) = \frac{1}{8}\int_0^T \rho(t)(w_N')^2(t)dt + \frac{1}{2h^2}\int_0^T (w_N)^2(t)dt \\
- < (U_h^0, U_h^1), (W_h(0), W_h'(0)) >_D,
\end{aligned} \tag{67}$$

where (W_h, W_h') is the solution of (32) with initial data (W_h^0, W_h^1) and ρ is a cut-off function with properties (i)–(iii) in (22).

Proof of Theorem 3.1. Assume that J has a minimum which is attained at $(\widehat{W}_h^0, \widehat{W}_h^1)$. It follows that, for any $\lambda > 0$ and (W_h^0, W_h^1),

$$\begin{aligned}
0 &\le \frac{1}{\lambda}\left[J((\widehat{W}_h^0, \widehat{W}_h^1) + \lambda(W_h^0, W_h^1)) - J((\widehat{W}_h^0, \widehat{W}_h^1))\right] \\
&= \lambda\left(\frac{1}{8}\int_0^T \rho(t)(w_N')^2(t)dt + \frac{1}{2h^2}\int_0^T (w_N)^2(t)dt\right) \\
&\quad + \int_0^T \left(\frac{1}{4}\rho(t)\widehat{w}_N'(t)w_N'(t) + \frac{1}{h^2}\widehat{w}_N(t)w_N(t)\right)dt \\
&\quad - < (U_h^0, U_h^1), (W_h(0), W_h'(0)) >_D .
\end{aligned}$$

By making λ tending to zero we obtain that

$$0 \le \int_0^T \left(\frac{\rho(t)}{4} \widehat{w}'_N(t) w'_N(t) + \frac{1}{h^2} \widehat{w}_N(t) w_N(t) \right) dt \\ - < (U_h^0, U_h^1), (W_h(0), W'_h(0)) >_D .$$

In the same way, taking $\lambda < 0$,

$$0 \ge \int_0^T \left(\frac{\rho(t)}{4} \widehat{w}'_N(t) w'_N(t) + \frac{1}{h^2} \widehat{w}_N(t) w_N(t) \right) dt \\ - < (U_h^0, U_h^1), (W_h(0), W'_h(0)) >_D .$$

Since $\rho \widehat{w}'_N$ has compact support in $(0, T)$, it follows that, for any (W_h^0, W_h^1),

$$\int_0^T \left(-\frac{1}{4} \left(\rho(t) \widehat{w}'_N(t) \right)' + \frac{1}{h^2} \widehat{w}_N(t) \right) w_N(t) dt \\ - < (U_h^0, U_h^1), (W_h(0), W'_h(0)) >_D = 0. \tag{68}$$

Since $\widehat{w}_N \in C^\infty[0, T]$, equation (25) has a unique solution $v_h \in C^\infty[0, T]$ which verifies

$$\int_0^T \left(\frac{h}{4} v'_h(t) w'_N(t) + \frac{1}{h} v_h(t) w_N(t) \right) dt \\ = \int_0^T \left(-\frac{1}{4} \left(\rho(t) \widehat{w}'_N(t) \right)' + \frac{1}{h^2} \widehat{w}_N(t) \right) w_N(t) dt. \tag{69}$$

Relations (68) and (69) show that v_h is a control for (14).

Hence, the proof of the Theorem 3.1 is complete if we show that there exists a unique minimizer of the functional J. This is stated in the following Lemma. □

Lemma 5.3 *The functional J defined by (67) has a unique minimizer $(\widehat{W}_h^0, \widehat{W}_h^1)$.*

Proof Let us first observe that J is continuous and strictly convex, implying uniqueness if there is existence of a minimizer. In order to prove that J has a minimizer it is sufficient to show that it is coercive, i.e.

$$\lim_{||(W_h^0, W_h^1)||_0 \to \infty} J((W_h^0, W_h^1)) = +\infty. \tag{70}$$

Note that (70) follows from the observability inequality (45). Indeed,

$$\begin{aligned} J((W_h^0, W_h^1)) &= \frac{1}{8} \int_0^T \rho(t) (w'_N)^2(t) dt + \frac{1}{2h^2} \int_0^T (w_N)^2(t) dt \\ &\quad - < (U_h^0, U_h^1), (W_h(0), W'_h(0)) >_D \\ &= \frac{1}{8} \int_0^T \rho(t) (w'_N)^2(t) dt + \frac{1}{2h^2} \int_0^T (w_N)^2(t) dt \\ &\quad - < (-K_h^{-1} M_h U_h^1, U_h^0), (W_h, W'_h)(0) >_0 \\ &\ge \frac{1}{4C} ||(W_h^0, W_h^1)||_0^2 - ||(-K_h^{-1} M_h U_h^1, U_h^0)||_0 \; ||(W_h, W'_h)(0)||_0 \\ &= \frac{1}{4C} ||(W_h^0, W_h^1)||_0^2 - ||(W_h^0, W_h^1)||_0 \; ||(-K_h^{-1} M_h U_h^1, U_h^0)||_0. \end{aligned}$$

The last expression tends to infinity when $||(W_h^0, W_h^1)||_0$ does and the proof of the lemma is complete. □

This concludes the proof of Theorem 3.1.

5.2 Boundedness of the discrete controls

One of the most important properties of the controls v_h given by Theorem 3.1 is the following uniform boundedness result.

Theorem 5.1 *The control v_h given by Theorem 3.1 satisfies*

$$\int_0^T (v_h)^2(t)dt + \frac{h^2}{4}\int_0^T (v_h')^2(t)dt \leq 16C||(-K_h^{-1}M_hU_h^1, U_h^0)||_0^2, \tag{71}$$

where C is the constant from (45) and does not depend of h.

Moreover, there exists another constant $C' > 0$, independent of h, such that

$$\left\|h^2 v_h'\right\|_{L^\infty(0,T)} \leq C'||(-K_h^{-1}M_hU_h^1, U_h^0)||_0. \tag{72}$$

Proof We multiply (25) by hv_h and integrate in time. It follows that

$$\begin{aligned}
&\frac{h^2}{4}\int_0^T (v_h')^2(t)dt + \int_0^T (v_h)^2(t)dt \\
&\quad = \int_0^T \left(\frac{h}{4}\rho(t)\widehat{w}_N'(t)v_h'(t) + \frac{1}{h}\widehat{w}_N(t)v_h(t)\right)dt \\
&\quad \leq \left[\int_0^T \left(\frac{h^2}{4}\rho(t)(v_h')^2(t) + (v_h)^2(t)\right)dt\right]^{1/2} \\
&\qquad \times \left[\int_0^T \left(\frac{1}{4}\rho(t)(\widehat{w}_N')^2(t) + \frac{1}{h^2}(\widehat{w}_N)^2(t)\right)dt\right]^{1/2} \\
&\quad \leq \left[\int_0^T \left(\frac{h^2}{4}(v_h')^2(t) + (v_h)^2(t)\right)dt\right]^{1/2} \\
&\qquad \times \left[\int_0^T \left(\frac{1}{4}\rho(t)(\widehat{w}_N')^2(t) + \frac{1}{h^2}(\widehat{w}_N)^2(t)\right)dt\right]^{1/2}.
\end{aligned}$$

Hence,

$$\frac{h^2}{4}\int_0^T (v_h')^2(t)dt + \int_0^T (v_h)^2(t)dt \leq \int_0^T \left(\frac{1}{4}\rho(t)(\widehat{w}_N')^2(t) + \frac{1}{h^2}(\widehat{w}_N)^2(t)\right)dt. \tag{73}$$

Since $(\widehat{W}_h^0, \widehat{W}_h^1)$ is a minimizer for J,

$$J((\widehat{W}_h^0, \widehat{W}_h^1)) \leq J((0,0)) = 0.$$

Consequently,

$$\int_0^T \left(\frac{1}{4}\rho(t)(\widehat{w}'_N)^2(t) + \frac{1}{h^2}(\widehat{w}_N)^2(t)\right) dt$$
$$\leq 2 < (-K_h^{-1}M_hU_h^1, U_h^0), (\widehat{W}_h(0), \widehat{W}'_h(0)) >_0$$
$$\leq 2||(\widehat{W}_h^0, \widehat{W}_h^1)||_0 \; ||(-K_h^{-1}M_hU_h^1, U_h^0)||_0 = 2\sqrt{2E_h(T)}\, ||(-K_h^{-1}M_hU_h^1, U_h^0)||_0$$
$$\leq 2\sqrt{4C}\sqrt{\frac{1}{4}\int_0^T \rho(t)(\widehat{w}'_N)^2(t)dt + \frac{1}{h^2}\int_0^T (\widehat{w}_N)^2(t)dt}\; ||(-K_h^{-1}M_hU_h^1, U_h^0)||_0.$$

Hence,

$$\int_0^T \left(\frac{1}{4}\rho(t)(\widehat{w}'_N)^2(t) + \frac{1}{h^2}(\widehat{w}_N)^2(t)\right) dt \leq 16C||(-K_h^{-1}M_hU_h^1, U_h^0)||_0^2. \tag{74}$$

Inequality (71) follows by taking into account (73) and (74).
We now prove (72). Integrating in (25) and multiplying by h we obtain

$$-\frac{h^2}{4}v'_h(t) + \int_0^t v_h(s)\, ds = -\frac{h}{4}\rho(t)\widehat{w}'_N(t) + \frac{1}{h}\int_0^t \widehat{w}_N(s)\, ds. \tag{75}$$

Then,

$$h^2|v'_h(t)| \leq 4\left|\int_0^t v_h(s)\, ds\right| + h\rho(t)\left|\widehat{w}'_N(t)\right| + \frac{4}{h}\left|\int_0^t \widehat{w}_N(s)\, ds\right|$$
$$\leq 4\sqrt{T}\, \|v_h\|_{L^2(0,T)} + 4\sqrt{T}\left\|\frac{\widehat{w}_N}{h}\right\|_{L^2(0,T)} + h\left|\widehat{w}'_N(t)\right|.$$

The first two terms can be bounded by (71) and (74). Concerning the last one we observe that it is part of the energy of the solution (w, w'). Indeed,

$$h^2\left|\widehat{w}'_N(t)\right|^2 \leq 2h^2\sum_{j=0}^N\left[\left|\frac{\widehat{w}'_{j+1}(t) + \widehat{w}'_j(t)}{2}\right|^2 + \left|\frac{\widehat{w}_{j+1}(t) - \widehat{w}_j(t)}{h}\right|^2\right]$$
$$\leq 4hE(t) \leq 8hC\left(\frac{1}{4}\int_0^T \rho(t)(\widehat{w}'_N)^2(t)dt + \frac{1}{h^2}\int_0^T (\widehat{w}_N)^2(t)dt\right)$$
$$\leq 144hC^2||(-K_h^{-1}M_hU_h^1, U_h^0)||_0^2.$$

The proof of the theorem is now complete. □

For any $(f^1, f^2), (g^1, g^2) \in \mathbb{C}^{2N}$ we introduce the notations

$$< (f^1, f^2), (g^1, g^2) >_{-1} = < (-K_h^{-1}M_hf^2, f^1), (-K_h^{-1}M_hg^2, g^1) >_0,$$
$$||(f^1, f^2)||_{-1} = ||(-K_h^{-1}M_hf^2, f^1)||_0. \tag{76}$$

Observe that $< \cdot, \cdot >_{-1}$ is an inner product and $|| \cdot ||_{-1}$ is a norm on $\mathbb{C}^{2N}$. The norm $|| \cdot ||_{-1}$ is the discrete equivalent of the norm in $L^2(0, 1) \times H^{-1}(0, 1)$.
The following property of the eigenvectors will be useful.

Lemma 5.4 *If* $(\varphi_h^n)_{1\le|n|\le N}$ *are the eigenvectors of (16) given by (18) then*

$$< \varphi_h^n, \varphi_h^m >_{-1} = \frac{1}{|\lambda_h^n|^2}\delta_{nm}, \quad \forall 1 \le |n|, |m| \le N. \tag{77}$$

Proof We have that

$$\begin{aligned} < \varphi_h^n, \varphi_h^m >_{-1} &= \frac{1}{\cos\left(\frac{n\pi h}{2}\right)\cos\left(\frac{m\pi h}{2}\right)} \\ &\quad < (K_h^{-1}M_h\theta_h^n, \frac{1}{\lambda_h^n}\theta_h^n), (K_h^{-1}M_h\theta_h^m, \frac{1}{\lambda_h^m}\theta_h^m)) >_{-1} \\ &= \frac{1}{\cos\left(\frac{n\pi h}{2}\right)\cos\left(\frac{m\pi h}{2}\right)} \\ &\quad \times \left[< M_h\theta_h^n, K_h^{-1}M_h\theta_h^m > + \frac{1}{\lambda_h^n\overline{\lambda}_h^m} < M_h\theta_h^n, \theta_h^m > \right]. \end{aligned}$$

Since

$$M_h\theta_h^n = h\cos^2\left(\frac{n\pi h}{2}\right)\theta_h^n, \; K_h\theta_h^n = \frac{4}{h}\sin^2\left(\frac{n\pi h}{2}\right)\theta_h^n,$$

we obtain that

$$\begin{aligned} < \varphi_h^n, \varphi_h^m >_{-1} &= \frac{h^3}{\cos\left(\frac{n\pi h}{2}\right)\cos\left(\frac{m\pi h}{2}\right)} \\ &\quad \times \left[\frac{\cos^2\left(\frac{n\pi h}{2}\right)\cos^2\left(\frac{m\pi h}{2}\right)}{4\sin^2\left(\frac{m\pi h}{2}\right)} + \frac{\cos^3\left(\frac{n\pi h}{2}\right)\cos\left(\frac{m\pi h}{2}\right)}{4\sin\left(\frac{m\pi h}{2}\right)\sin\left(\frac{n\pi h}{2}\right)} \right] < \theta_h^n, \theta_h^m > \\ &= \frac{h^2}{8}\left[\frac{\cos^2\left(\frac{n\pi h}{2}\right)}{\sin^2\left(\frac{n\pi h}{2}\right)} + \frac{\cos^2\left(\frac{n\pi h}{2}\right)}{\sin^2\left(\frac{n\pi h}{2}\right)} \right]\delta_{mn} = \frac{h^2}{4\tan^2\left(\frac{n\pi h}{2}\right)}\delta_{mn}, \end{aligned}$$

and relation (77) follows. □

Remark 5.2 With the new norm, the result from Theorem 5.1 reads as follows

$$\int_0^T (v_h)^2(t)dt + \frac{h^2}{4}\int_0^T (v_h')^2(t)dt \le 16C||(U_h^0, U_h^1)||_{-1}^2, \tag{78}$$

where C is the constant from (45). □

Remark 5.3 From the observability inequality (45) and (71) we deduce that

$$\begin{aligned} ||(\widehat{W}_h^0, \widehat{W}_h^1)||_0^2 &\le 2E_h(T) \\ &\le 2C\int_0^T \left(\frac{1}{4}\rho(t)(\widehat{w}_N')^2(t) + \frac{1}{h^2}(\widehat{w}_N)^2(t)\right)dt \le 32C^2||(U_h^0, U_h^1)||_{-1}^2. \end{aligned}$$

Consequently, the sequence of the minimizers $((\widehat{W}_h^0, \widehat{W}_h^1))_{h>0}$ is bounded in the $||\;||_0-$ norm if the same property holds for $((U_h^0, U_h^1))_{h>0}$ in the $||\;||_{-1}-$ norm. □

5.3 Convergence of the discrete controls. Proof of Theorem 3.2

Let us now show the convergence of the controls v_h of the discrete equation (14) to the HUM control of the continuous problem (1). In order to do that we assume that the discrete and the continuous initial data have the following Fourier decomposition

$$(U_h^0, U_h^1) = \sum_{1\leq|n|\leq N} a_{0h}^n \varphi_h^n,$$
$$(u^0, u^1) = \sum_{n\neq 0} a_0^n \Phi^n. \tag{79}$$

The solutions of (1) and (14) respectively are given by

$$(U_h, U_h')(t) = \sum_{1\leq|n|\leq N} a_h^n(t)\varphi_h^n,$$
$$(u, u')(t) = \sum_{n\neq 0} a^n(t)\Phi^n. \tag{80}$$

In the rest of this section we prove the Theorem 3.2 and we discuss the hypothesis on the convergence of the Fourier coefficients of the initial data in the statement of Theorem 3.2.

Proof of Theorem 3.2. We shall consider the following steps:

1. Proof of the weak convergence
2. Identification of the limit
3. Proof of the strong convergence

1. *Proof of the weak convergence:* For every $h > 0$, let v_h be the control given by Theorem 3.1 and $(\widehat{W}_h^0, \widehat{W}_h^1)$ be the minimizer of the functional J.

From (26) it follows that $(a_{0h}^n/\lambda_h^n)_n$ is uniformly bounded in ℓ^2 and, consequently, the sequence $(U_h^0, U_h^1)_{h>0}$ is also uniformly bounded in the $||\cdot||_{-1}$ norm.

Theorem 5.1 and the boundedness of the sequence $(U_h^0, U_h^1)_{h>0}$ implies that the sequences $(v_h)_{h>0}$ and $(hv_h')_{h>0}$ are uniformly bounded in $L^2(0, T)$. Hence, there exists a subsequence (denoted in the same way) and $v, \tilde{v} \in L^2(0, T)$ such that

$$\begin{aligned} v_h &\rightharpoonup v \text{ in } L^2(0, T), \\ hv_h' &\rightharpoonup \tilde{v} \text{ in } L^2(0, T). \end{aligned} \tag{81}$$

To show that $\tilde{v} = 0$ note that the sequence $(hv_h)_{h>0}$ is bounded in $H^1(0, T)$. Hence there exists $\widehat{v} \in H^1(0, T)$ such that

$$\begin{aligned} hv_h &\to \widehat{v} \text{ in } L^2(0, T), \\ hv_h' &\rightharpoonup \widehat{v}' \text{ in } L^2(0, T). \end{aligned}$$

Since $v_h \rightharpoonup v$ in $L^2(0, T)$ it follows that $\widehat{v} = 0$ and therefore

$$hv_h' \rightharpoonup 0 \text{ in } L^2(0, T). \tag{82}$$

The last convergence in (27) is a consequence of the bound of $h^2 v_h'$ in $L^\infty(0,T)$ stated in (72) and the fact that its weak limit in $L^2(0,T)$ is 0.

2. *Identification of the limit:* We show that v is the HUM control of the continuous equation (1). In order to do that, it is necessary and sufficient to verify the assumptions from Remark 2.1.

To prove (i) let us first remark that it is sufficient to show that (5) is verified by the eigenfunctions of the wave operator, $(w^0, w^1) = \Phi^n$ for all $n \in \mathbb{Z}^*$.

Indeed, from the continuity of the linear form $\Lambda : H_0^1(0,1) \times L^2(0,1) \to \mathbb{C}$,

$$\Lambda(w^0, w^1) = \int_0^T v(t) w_x(t,1) dt - < u^1, w(0) >_{H^{-1}, H_0^1} + \int_0^1 u^0(x) w'(0,x),$$

it follows that (5) holds for any $(w^0, w^1) \in H_0^1(0,1) \times L^2(0,1)$ iff it is verified on a basis of the space $H_0^1(0,1) \times L^2(0,1)$. But $(\Phi^n)_{n \neq 0}$ is exactly such a basis.

Thus, by considering $(w^0, w^1) = \Phi^n$, we obtain that v is a control for (1) iff

$$\int_0^T v(t) e^{in\pi t} dt = \frac{(-1)^{n+1}}{n\pi} a_0^n, \quad \forall n \neq 0. \tag{83}$$

Now, since v_h is a control for the discrete problem (14), by taking $(W_h^0, W_h^1) = \varphi_h^n$ in Lemma 5.1, we obtain that

$$\int_0^T e^{\lambda_h^n (t-T)} \left[\frac{h}{4} v_h'(t)(-1)^{n+1} 2 \sin\left(\frac{n\pi h}{2}\right) + \frac{1}{h} v_h(t)(-1)^n ih \cos\left(\frac{n\pi h}{2}\right)\right] dt$$
$$- < (U_h^0, U_h^1), e^{\lambda_h^n T} \varphi_h^n >_D . \tag{84}$$

But, from Lemma 5.2,

$$< (U_h^0, U_h^1), e^{\lambda_h^n T} \varphi_h^n >_D = -\frac{1}{\lambda_h^n} e^{-\lambda_h^n T} a_{0h}^n.$$

Hence,

$$(-1)^n i \cos\left(\frac{n\pi h}{2}\right) \int_0^T e^{\lambda_h^n t} \left[\frac{ih}{2} \tan\left(\frac{n\pi h}{2}\right) v_h'(t) + v_h(t)\right] dt = -a_{0h}^n \frac{1}{\lambda_h^n}. \tag{85}$$

Taking into account that, for each $n \in \mathbb{Z}^*$,

$$e^{\lambda_h^n t} \to e^{in\pi t}, \quad v_h \rightharpoonup v, \quad h v_h' \rightharpoonup 0 \quad \text{in } L^2(0,T),$$
$$\frac{1}{\lambda_h^n} a_{0h}^n \to \frac{1}{in\pi} a_0^n,$$

when h tends to zero, by passing to limit in (84) we obtain that

$$\int_0^T v(t) e^{in\pi t} dt = \frac{(-1)^{n+1}}{n\pi} a_0^n, \quad \forall n \neq 0.$$

It follows that the limit v satisfies (83). Therefore v is a control for (1) and (i) is proved.

Let us now prove (ii). Let $(\widehat{W}_h^0, \widehat{W}_h^1)$ be the minimizer of the functional J that we expand in Fourier series as follows

$$(\widehat{W}_h^0, \widehat{W}_h^1) = \sum_{1\leq|n|\leq N} b_{0h}^n \varphi_h^n. \tag{86}$$

From Remark 5.3 and the boundedness of $(U_h^0, U_h^1)_{h>0}$ we deduce that the sequence $(b_{0h}^n)_n$ is uniformly bounded in ℓ^2. Therefore, there exists a subsequence, denoted in the same way, which converges weakly to $(b_0^n)_n$ in ℓ^2 when h tends to zero. Let

$$(\widehat{w}^0, \widehat{w}^1) = \sum_n b_0^n \Phi^n \in H_0^1(0,1) \times L^2(0,1). \tag{87}$$

By Lemma 4.1 the sequence $\left(\frac{\widehat{w}_N}{h}\right)_{h>0}$ tends weakly to $\widehat{w}_x(\cdot, 1)$ in $L^2(0,T)$ where $(\widehat{w}, \widehat{w}')$ is the solution of (4) with initial data $(\widehat{w}^0, \widehat{w}^1)$.

Let ψ be a test function from $H^1(0,T)$, we have that

$$\begin{aligned}\int_0^T v(t)\psi(t)dt &= \lim_{h\to 0}\left(\int_0^T \frac{h^2}{4} v_h'(t)\psi'(t)dt + \int_0^T v_h(t)\psi(t)dt\right)\\ &= \lim_{h\to 0}\left(\int_0^T \left[\frac{h}{4}\rho(t)\widehat{w}_N'(t)\psi'(t)dt + \frac{\widehat{w}_N(t)}{h}\psi(t)\right]dt\right).\end{aligned}$$

In order to pass to the limit in the last term remark that, from (74) it follows that

$$\int_0^T \frac{h^2}{4}\rho^2(t)(\widehat{w}_N')^2(t)dt \leq \int_0^T \frac{h^2}{4}\rho(t)(\widehat{w}_N')^2(t)dt \leq 8h^2 C||(U_h^0, U_h^1)||_{-1}^2. \tag{88}$$

and consequently $\lim_{h\to 0} h\rho\widehat{w}_N' = 0$ in $L^2(0,T)$.

We obtain that

$$\int_0^T v(t)\psi(t)dt = \int_0^T \widehat{w}_x(t,1)\psi(t)dt$$

which implies that $v(t) = \widehat{w}_x(t,1)$, and the proof of (ii) is complete.

3. *Proof of the strong convergence:* We have already proved that $(v_h)_{h>0}$ converges weakly to $v = \widehat{w}_x(\cdot, 1)$ in $L^2(0,T)$ when h tends to zero where $(\widehat{w}, \widehat{w}')$ is the solution of (4) with initial data $(\widehat{w}^0, \widehat{w}^1)$, the minimizer of $\mathcal{J}$ from Theorem 2.2.

By considering in (66) test function $(\widehat{W}_h^0, \widehat{W}_h^1)$ it follows that

$$\int_0^T \left(\frac{1}{4}\rho(t)(\widehat{w}_N')^2(t) + \left(\frac{\widehat{w}_N}{h}\right)^2(t)\right)dt =< (U_h^0, U_h^1), (\widehat{W}_h(0), \widehat{W}_h'(0)) >_D .$$

From Lemma 5.2, the strong convergence of the sequence $\left(\frac{a_{0h}^n}{\lambda_h^n}\right)_n$ and the weak convergence of the sequence $\left(b_{0h}^n\right)_n$ we obtain that

$$\begin{aligned}\lim_{h\to 0} &< (U_h^0, U_h^1), (\widehat{W}_h(0), \widehat{W}_h'(0)) >_D = \lim_{h\to 0} \sum_{n\in\mathbb{Z}^*} a_{0h}^n b_{0h}^n \left(-\frac{e^{T\lambda_h^n}}{\lambda_h^n}\right) \\ &= \sum_{n\in\mathbb{Z}^*} a_0^n b_0^n \left(-\frac{e^{T\lambda^n}}{\lambda^n}\right) =< u^1, \widehat{w}(0) >_{H^{-1},H_0^1} - \int_0^1 u^0(x)\widehat{w}'(0,x)dx.\end{aligned}$$

It follows that

$$\lim_{h\to 0}\int_0^T \left(\frac{1}{4}\rho(t)(\widehat{w}_N')^2(t) + \left(\frac{\widehat{w}_N}{h}\right)^2(t)\right)dt = \int_0^T (\widehat{w}_x)^2(t,1)dt. \tag{89}$$

Now, note that

$$\begin{aligned}\int_0^T (\widehat{w}_x)^2(t,1)dt &\le \liminf_{h\to 0}\int_0^T \left(\frac{\widehat{w}_N}{h}\right)^2(t)dt \le \limsup_{h\to 0}\int_0^T \left(\frac{\widehat{w}_N}{h}\right)^2(t)dt \\ &\le \limsup_{h\to 0}\int_0^T \left(\frac{1}{4}\rho(t)(\widehat{w}_N')^2(t) + \left(\frac{\widehat{w}_N}{h}\right)^2(t)\right)dt \\ &= \int_0^T (\widehat{w}_x)^2(t,1)dt,\end{aligned}$$

where we have take into account (89) and the weak convergence of $\left(\frac{\widehat{w}_N}{h}\right)$ to $\widehat{w}_x(\,\cdot\,,1)$ proved in Lemma 4.1.

It follows that,

$$\lim_{h\to 0}\int_0^T \left(\frac{\widehat{w}_N}{h}\right)^2(t)dt = \int_0^T (\widehat{w}_x)^2(t,1)dt, \tag{90}$$

and $\widehat{w}_N/h$ converges strongly in $L^2(0,T)$ to $\widehat{w}_x(\,\cdot\,,1)$ when h tends to zero.

Thus, from (89) we deduce that

$$\lim_{h\to 0}\int_0^T \rho^2(t)(\widehat{w}_N')^2(t) = \lim_{h\to 0}\int_0^T \rho(t)(\widehat{w}_N')^2(t) = 0. \tag{91}$$

Now, since

$$\frac{1}{4}\int_0^T (hv_h'(t))^2dt + \int_0^T (v_h(t))^2dt = \frac{h}{4}\int_0^T \left(\rho(t)\widehat{w}_N'(t)v_h' + \frac{\widehat{w}_N(t)}{h}v_h\right)dt,$$

and if we take into account the strong convergence results (89) and (90) and the weak convergence of the sequences $(v_h)_{h>0}$ and $(hv_h')_{h>0}$ it follows that

$$\lim_{h\to 0}\left(\frac{1}{4}\int_0^T (hv_h')^2(t)dt + \int_0^T (v_h)^2(t)dt\right) = \int_0^T v^2(t)dt. \tag{92}$$

On the other hand, we have that

$$\int_0^T (v_h(t)-v(t))^2dt \le \int_0^T (v_h(t)-v(t))^2dt + \frac{1}{4}\int_0^T (hv_h'(t))^2dt$$
$$= \left(\frac{1}{4}\int_0^T (hv_h'(t))^2dt + \int_0^T (v_h(t))^2dt\right) + \int_0^T (v(t))^2dt - 2\int_0^T v_h(t)v(t)dt.$$

By passing to the limit in the last equality, it follows that $(v_h)_{h>0}$ converges strongly in $L^2(0,T)$ to $v(t)=\widehat{w}_x(t,1)$. Next, from (92) we deduce that $(hv_h')_{h>0}$ converges strongly in $L^2(0,T)$ to zero.

Let un now prove that the last convergence in (27) holds strongly in $L^\infty(0,T)$. Integrating in (25) and multiplying by h we easily obtain

$$|h^2v_h'(t)| = \left|h\rho(t)\widehat{w}_N'(t) - 4\int_0^t \frac{\widehat{w}_N(s)}{h}ds + 4\int_0^t v_h(s)\,ds\right|$$
$$\le \left|h\widehat{w}_N'(t)\right| \max_{t\in[0,T]} \rho(t) + 4\sqrt{T}\left(\left\|\frac{\widehat{w}_N}{h}-v\right\|_{L^2(0,T)} + \|v_h - v\|_{L^2(0,T)}\right).$$

These two terms converge to zero uniformly in $t\in[0,T]$ by the strong convergence of both $\widehat{w}_N/h$ and v_h to the limit control v and the bound (76).

The proof of Theorem 3.2 is complete. □

5.4 Convergence of the solutions. Proof of Theorem 3.3

In this section we study the convergence of the controlled solutions of the discrete equation (14) to the controlled solution of the continuous problem (1). In order to prove the main result we introduce a characterization of the Fourier coefficients for both the continuous and the discrete solutions. This is given in the following proposition.

Proposition 5.1 *Let $(U_h^0, U_h^1)_h$ be a sequence of discretizations of the initial data (u^0,u^1) and let $\left(a_{0h}^n\right)_n$, $\left(a_0^n\right)_n$ be their Fourier coefficients, respectively. Then $\left(a_h^n(t)\right)_n$ is the sequence of Fourier coefficients of the solution of the discrete system (14) with initial data $(U_h^0, U_h^1)_h$ if and only if*

$$\int_0^T \sum_{1\le|n|\le N} g_h^n(t)\frac{a_h^n(t)}{\lambda_h^n}dt = \frac{1}{4}\sum_{1\le|n|\le N} \varphi_{h,N}^{n,2}\int_0^T hv_h'(t)\int_t^T g_h^n(s)e^{\lambda_h^n(t-s)}ds\,dt$$
$$-\frac{1}{4}\sum_{1\le|n|\le N}\varphi_{h,N}^{n,1}\int_0^T hv_h'(t)g_h^n(t)\,dt$$
$$+\sum_{1\le|n|\le N}\frac{\varphi_{h,N}^{n,1}}{h}\int_0^T v_h(t)\int_t^T g_h^n(s)e^{\lambda_h^n(t-s)}ds\,dt$$
$$+\sum_{1\le|n|\le N}\frac{a_{0h}^n}{\lambda_h^n}\int_0^T g_h^n(s)\,e^{-\lambda_h^n s}ds,$$
$$\forall\left(g_h^n(s)\right)_n \in L^1(0,T;\mathbb{R}^{2N}). \qquad (93)$$

Here $\varphi_h^{n,1}$ and $\varphi_h^{n,2}$ are the first and second components of the eigenfunction φ_h^n respectively (see (18)). The values $\varphi_{h,N}^{n,1}$ and $\varphi_{h,N}^{n,2}$ are the N component of the vectors $\varphi_h^{n,1}$ and $\varphi_h^{n,2}$ respectively.

Analogously, $\left(a_h^n(t)\right)_n$ is the sequence of Fourier coefficients of the solution of the continuous system (1) with initial data (u^0, u^1) if and only if

$$\int_0^T \sum_{n\in\mathbb{Z}^*} \frac{a^n(t)}{\lambda^n} g^n(t)\, dt = -\sum_{n\in\mathbb{Z}^*} \Phi_x^{n,1}(1) \int_0^T v(t) \int_t^T g^n(s) e^{\lambda^n(t-s)} ds\, dt + \sum_{n\in\mathbb{Z}^*} \frac{a_0^n}{\lambda^n} \int_0^T e^{-\lambda^n s} g^n(s) ds, \quad \forall (g^n)_n \in L^1(0, T; \ell^2), \tag{94}$$

where $\Phi^{n,1}$ is the first component of the eigenfunction Φ^n (see (8)).

Proof We only consider the semi-discrete system, the continuous one being analogous. We first need a variational formulation for (14).

Note that (14) may be written as follows:

$$\begin{cases} \begin{pmatrix} U_h \\ U_h' \end{pmatrix}' + \begin{pmatrix} 0 & -I \\ M_h{}^{-1} K_h & 0 \end{pmatrix} \begin{pmatrix} U_h \\ U_h' \end{pmatrix} = \begin{pmatrix} 0 \\ M_h{}^{-1} F_h \end{pmatrix}, & \text{for } t \in (0, T), \\ U_h(0) = U_h^0, \ U_h'(0) = U_h^1. \end{cases} \tag{95}$$

We now introduce a new characterization of the solutions of this system in the sense of transposition. For any $(G_1, G_2) \in L^1((0, T), \mathbb{R}^{2N})$, let $(W_h^1, W_h^2) \in C([0, T], \mathbb{R}^{2N})$ be the solution of the following nonhomogeneous system

$$\begin{cases} \begin{pmatrix} W_h^1 \\ W_h^2 \end{pmatrix}' + \begin{pmatrix} 0 & -I \\ M_h^{-1} K_h & 0 \end{pmatrix} \begin{pmatrix} W_h^1 \\ W_h^2 \end{pmatrix} = \begin{pmatrix} G^1 \\ G^2 \end{pmatrix}, & \text{for } t \in (0, T), \\ W_h^1(T) = W_h^2(T) = 0. \end{cases} \tag{96}$$

Multiplying system (95) by the solution (W_h^1, W_h^2) of (96) with the duality product introduced in (63) and integrating we obtain,

$$\int_0^T \left\langle (U_h, U_h')', (W_h^1, W_h^2) \right\rangle_D dt + \int_0^T \left\langle (-U_h', M_h^{-1} K_h U_h), (W_h^1, W_h^2) \right\rangle_D dt = \int_0^T - < M_h^{-1} F_h, M_h W_h^1 > dt. \tag{97}$$

Integrating by parts in the first term of this expression we obtain

$$\begin{aligned} \int_0^T \left\langle (U_h, U_h')', (W_h^1, W_h^2) \right\rangle_D dt &= -\int_0^T \left\langle (U_h, U_h'), (W_h^1, W_h^2)' \right\rangle_D dt \\ &\quad + \left\langle (U_h, U_h'), (W_h^1, W_h^2) \right\rangle_D \Big|_0^T \\ &= -\int_0^T \left\langle (U_h, U_h'), (W_h^1, W_h^2)' \right\rangle_D dt - \left\langle (U_h^0, U_h^1), (W_h^1(0), W_h^2(0)) \right\rangle_D \end{aligned} \tag{98}$$

Concerning the second term in (97) we have

$$\int_0^T \left\langle (-U_h', M_h^{-1}K_hU_h), (W_h^1, W_h^2)\right\rangle_D dt$$
$$= \int_0^T \left\langle (U_h', U_h), (M_h^{-1}K_hW_h^1, -W_h^2)\right\rangle_D dt$$
$$= -\int_0^T \left\langle (U_h, U_h'), (-W_h^2, M_h^{-1}K_hW_h^1)\right\rangle_D dt. \tag{99}$$

The third term in (97) reads as follows:

$$\int_0^T - < M_h^{-1}F_h, M_hW_h^1 > dt = -\int_0^T \left(-\frac{h}{4}v_h'' + \frac{1}{h}v_h\right) w_N^1 dt$$
$$= -\int_0^T \left(\frac{h}{4}v_h'(w_N^1)' + \frac{1}{h}v_hw_N^1\right) dt, \tag{100}$$

where w_N^1 is the N-component of the vector W_h^1.

Taking into account that (W_h^1, W_h^2) is the solution of (96) and substituting (98)–(100) in (97) we easily deduce that, for any $(G^1, G^2) \in L^1((0,T), \mathbb{R}^{2N})$,

$$\int_0^T \left\langle (U_h, U_h')(t), (G_1, G_2)(t)\right\rangle_D dt$$
$$= \int_0^T \left(\frac{h}{4}v_h'(t)(w_N^1)'(t) + \frac{1}{h}v_h(t)w_N^1(t)\right) dt$$
$$-\left\langle (U_h^0, U_h^1), (W_h^1(0), W_h^2(0))\right\rangle_D . \tag{101}$$

This identity provides a characterization of the solutions of (95).

Now we write (101) in terms of the Fourier coefficients of both the solutions of (95) and (96). The Fourier representation of the solutions of (14) was introduced in (80). For the solutions of (96) we use the following lemma: □

Lemma 5.5 *Assume that $(G^1, G^2) \in L^1((0,T), \mathbb{R}^{2N})$. Then, the solution of system (96) is given by*

$$(W_h^1, W_h^2) = -\sum_{1\le|n|\le N} \varphi_h^n \int_t^T g_h^n(s)\, e^{\lambda_n(t-s)} ds. \tag{102}$$

where $(g_h^n(t))_n$ are the Fourier coefficients of the nonhomogeneous data, i.e.

$$(G^1, G^2) = \sum_{1\le|n|\le N} g_h^n(t)\varphi_h^n. \tag{103}$$

Moreover, there exists a constant $C > 0$, independent of h, such that

$$\int_0^T \left[|w_N^2|^2 + \left|\frac{w_N^1}{h}\right|^2\right] dt \le C\left\|g_h^n(t)\right\|^2_{L^1(0,T;\ell^2)} . \tag{104}$$

Proof Formula (102) can be checked directly. Concerning estimate (104) we observe that, by the Duhamel's principle, W_h can be written as

$$W_h^1(t) = -\int_t^T Z_h(t-s,s)ds, \qquad W_h^2(t) = -\int_t^T Z_h'(t-s,s)ds, \tag{105}$$

where, for each $s \in (t,T)$, $(Z_h(t,s), Z_h'(t,s))$ is the solution of

$$\begin{cases} \begin{pmatrix} Z_h \\ Z_h' \end{pmatrix}' + \begin{pmatrix} 0 & -I \\ M_h^{-1}K_h & 0 \end{pmatrix} \begin{pmatrix} Z_h \\ Z_h' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & \text{for } t \in (0,T), \\ Z_h(0,s) = G^1(s), \qquad Z_h'(0,s) = G^2(s). \end{cases}$$

By the regularity inequality proved in Theorem 4.2, we deduce

$$\begin{aligned} \int_0^T \left[|(z_N)'|^2 + \left| \frac{z_N}{h} \right|^2 \right] dt &\le C||(G^1(s), G^2(s))||_0^2 \\ &= C \left\| g_h^n(s) \right\|_{\ell^2}^2, \quad \forall s \in (0,T), \end{aligned} \tag{106}$$

where z_N is the N-component of the vector Z_h. We now use (105) to write the left hand side of (106) in terms of w_N^1:

$$\begin{aligned} \left\| \frac{w_N^1}{h} \right\|_{L^2(0,T)} &= \left\| \int_t^T \frac{z_N(t-s,s)}{h} ds \right\|_{L^2(0,T)} \le \left\| \int_0^T \chi_{(t,T)}(s) \frac{z_N(t-s,s)}{h} ds \right\|_{L^2(0,T)} \\ &\le \int_0^T \left\| \chi_{(0,s)}(t) \frac{z_N(t-s,s)}{h} \right\|_{L^2(0,T)} ds \le \int_0^T \left\| \frac{z_N(\cdot,s)}{h} \right\|_{L^2(0,T)} ds. \end{aligned} \tag{107}$$

Analogously, we deduce that

$$\left\| w_N^2 \right\|_{L^2(0,T)} \le \int_0^T \left\| z_N'(\cdot,s) \right\|_{L^2(0,T)} ds. \tag{108}$$

From (107) and (108) we easily obtain (104). □

Now we write the Fourier representation of (101). Given any $(G^1, G^2) \in L^1((0,T), \mathbb{R}^{2N})$, we consider the Fourier expansion of (G^1, G^2) given by (103).

By Lemma 5.5, the solution (W_h, W_h') of (96) is given by (102). Then,

$$\frac{w_N^1}{h} = -\frac{1}{h} \sum_{1 \le |n| \le N} \varphi_{h,N}^{n,1} \int_t^T g_h^n(s)\, e^{\lambda_n^h(t-s)} ds,$$

$$(w_N^1)' = w_N^2 + G_N^1 = -\sum_{1 \le |n| \le N} \varphi_{h,N}^{n,2} \int_t^T g_h^n(s)\, e^{\lambda_n^h(t-s)} ds + \sum_{1 \le |n| \le N} \varphi_{h,N}^{n,1} g_h^n(t).$$

Substituting in (101) the above expressions and the Fourier representations (80) and (102), and taking into account Lemma 5.4, we easily obtain formula (93). □

Proof of Theorem 3.3. We shall consider the following steps:

1. Boundedness
2. Weak convergence
3. Strong convergence

The weak and strong convergence are proved for the $L^\infty(0,T;\ell^2)$ topology. The fact that the solutions of the semi-discrete problem are regular in time and a density argument provide the convergence results in $C([0,T];\ell^2)$, as stated in Theorem 3.3.

1. *Boundedness:* It suffices to estimate the four terms in the right hand side of (93). The last term in (93) can be bounded as follows:

$$\left| \sum_{1\le|n|\le N} \frac{a_{0h}^n}{\lambda_h^n} \int_0^T g_h^n(s)\, e^{-\lambda_h^n s} ds \right| \le \int_0^T \sum_{1\le|n|\le N} \left| \frac{a_{0h}^n}{\lambda_h^n} \right| \left| g_h^n(s) \right| ds$$
$$\le \left\| \left(\frac{a_{0h}^n}{\lambda_h^n} \right) \right\|_{\ell^2} \int_0^T \left\| (g_h^n(s)) \right\|_{\ell^2} ds = \left\| \left(\frac{a_{0h}^n}{\lambda_h^n} \right) \right\|_{\ell^2} \left\| (g_h^n) \right\|_{L^1(0,T;\ell^2)} . \quad (109)$$

Concerning the first and third term in the right hand side of (93), we estimate them by formula (104) in Lemma 5.5. Indeed,

$$\left| \frac{1}{4} \sum_{1\le|n|\le N} \varphi_{h,N}^{n,2} \int_0^T h v_h'(t) \int_t^T g_h^n(s) e^{\lambda_h^n(t-s)} ds\, dt \right.$$
$$\left. + \sum_{1\le|n|\le N} \varphi_{h,N}^{n,1} \int_0^T v_h(t) \int_t^T g_h^n(s) e^{\lambda_h^n(t-s)}\, ds\, dt \right|$$
$$= \left| \int_0^T \left(\frac{h}{4} v_h' w_N^2 + \frac{1}{h} v_h w_N^1 \right) dt \right|$$
$$\le C \left(\|v_h\|_{L^2(0,T)} + \left\| h v_h' \right\|_{L^2(0,T)} \right) \left\| g_h^n \right\|_{L^1(0,T;\ell^2)} . \quad (110)$$

Finally, for estimating the second term in (93) we first observe that

$$\left| \varphi_{h,N}^{n,1} h v_h'(t) \right| = \left| \frac{1}{\lambda_h^n \cos(\frac{n\pi h}{2})} \sin(n\pi N h) h v_h'(t) \right|$$
$$= \left| \cos(\frac{n\pi h}{2}) h^2 v_h'(t) \right| \le \left| h^2 v_h'(t) \right| .$$

Thus, we deduce

$$\left| \frac{1}{4} \sum_{1\le|n|\le N} \varphi_{h,N}^{n,1} \int_0^T h v_h'(t) g_h^n(t)\, dt \right| \le \frac{1}{4} \sum_{1\le|n|\le N} \int_0^T \left| h^2 v_h'(t) \right| \left| g_h^n(t) \right| dt$$
$$\le C(T) \left\| h^2 v_h'(t) \right\|_{L^\infty(0,T)} \left\| g_h^n \right\|_{L^1(0,T;\ell^2)} . \quad (111)$$

From the above estimates (109)–(111) we easily estimate the right hand side of (93). Thus,

$$\left|\int_0^T \sum_{1\le|n|\le N} g_h^n(t)\frac{a_h^n(t)}{\lambda_h^n}dt\right|$$
$$\le C\left(\left\|\left(\frac{a_{0h}^n}{\lambda_h^n}\right)\right\|_{\ell^2} + \|v_h\|_{L^2(0,T)} + \|hv_h'\|_{L^2(0,T)} + \|h^2 v_h'(t)\|_{L^\infty(0,T)}\right)$$
$$\times \left\| g_h^n(s)\right\|_{L^1(0,T;\ell^2)},$$

and therefore

$$\left\|\left(\frac{a_h^n(t)}{\lambda_h^n}\right)_n\right\|_{L^\infty(0,T;\ell^2)}$$
$$\le C\left(\left\|\left(\frac{a_{0h}^n}{\lambda_h^n}\right)_n\right\|_{\ell^2} + \|v_h\|_{L^2(0,T)} + \|hv_h'\|_{L^2(0,T)} + \|h^2 v_h'(t)\|_{L^\infty(0,T)}\right).$$

Now observe that the right hand side of this estimate is uniformly bounded as $h\to 0$, due to the weak convergence of the sequence $\left(\frac{a_{0h}^n}{\lambda_h^n}\right)_n$ and the uniform bounds for the controls stated in Theorem 5.1.

2. *Weak convergence:* By the boundedness of the sequence $\left(\frac{a_h^n(t)}{\lambda_h^n}\right)_n$ in $L^\infty(0,T;\ell^2)$ we deduce that there exists a subsequence, still denoted by the parameter h , such that

$$\left(\frac{a_h^n(t)}{\lambda_h^n}\right)_n \rightharpoonup (b^n(t))_n \text{ weakly* in } L^\infty(0,T;\ell^2).$$

We prove that

$$b^n(t) = \frac{a^n(t)}{\lambda_h^n}, \qquad \forall n\in\mathbb{Z}^*, \tag{112}$$

where $(a^n(t))$ is the sequence of Fourier coefficients of the solution of the continuous limit problem. When considering (93) for a particular sequence $(g_h^n(t))_n$ with an unique nonzero term, i.e.

$$g_h^j(t) = \begin{cases} 0 \text{ if } j\neq n,\\ g(t) \text{ if } j=n,\end{cases}$$

we obtain the following characterization of the Fourier coefficient $a_h^n(t)$

$$\int_0^T g(t)\frac{a_h^n(t)}{\lambda_h^n}dt$$
$$= \frac{\varphi_{h,N}^{n,2}}{4}\int_0^T hv_h'(t)\int_t^T g(s)e^{\lambda_h^n(t-s)}ds\,dt$$
$$-\frac{1}{4}\varphi_{h,N}^{n,1}\int_0^T hv_h'(t)g(t)\,dt + \frac{\varphi_{h,N}^{n,1}}{h}\int_0^T v_h(t)\int_t^T g(s)e^{\lambda_h^n(t-s)}ds\,dt$$
$$+\frac{a_{0h}^n}{\lambda_h^n}\int_0^T g(s)\,e^{-\lambda_h^n s}ds, \qquad \forall g\in C_0^\infty(0,T). \tag{113}$$

Analogously, from (94) we easily obtain the following characterization for the Fourier coefficient $a^n(t)$ of the limit system

$$\int_0^T \frac{a^n(t)}{\lambda_h^n} g(t)\, dt = -\Phi_x^{n,1}(1) \int_0^T v(t) \int_t^T g(s) e^{\lambda_n (t-s)} ds\, dt + \frac{a_0^n}{\lambda_h^n} \int_0^T e^{-\lambda_h^n s} g(s) ds, \qquad \forall g \in C_0^\infty(0,T). \tag{114}$$

Once we have characterized the Fourier coefficients of the limit problem we pass to the limit in (113). The convergence of the eigenpairs $(\lambda_h^n, \varphi_h^n)$ to those of the continuous system (λ^n, Φ^n), and in particular

$$\varphi_{h,N}^{n,2} = -\frac{\sin(n\pi Nh)}{\cos\left(\frac{n\pi h}{2}\right)} = (-1)^n 2\left(\frac{n\pi h}{2}\right) \to 0, \text{ as } h \to 0,$$

$$-\frac{\varphi_{h,N}^{n,1}}{h} = -h\frac{\sin(n\pi Nh)}{\lambda_h^n \cos\left(\frac{n\pi h}{2}\right)} = (-1)^n \cos\left(\frac{n\pi h}{2}\right) \to (-1)^n = \Phi_x^{n,1}(1), \text{ as } h \to 0,$$

for all $n \in \mathbb{Z}^*$, the weak convergence of the controls stated in Theorem 3.2 and the hypothesis on the weak convergence of the Fourier coefficients of the initial data, give the following:

$$\int_0^T b^n(t) g(t)\, dt = -\Phi_x^{n,1}(1) \int_0^T v(t) \int_t^T g(s) e^{\lambda_n (t-s)} ds\, dt + \frac{a_0^n}{\lambda_h^n} \int_0^T e^{-\lambda_n s} g(s) ds, \tag{115}$$
$$\forall g \in C_0^\infty(0,T), \forall n \in \mathbb{Z}^*.$$

This identity combined with (114) gives (112) for each $n \in \mathbb{Z}^*$.

3. *Strong convergence:* It is enough to see that, for any sequence $(g_h^n(t))_n$ such that

$$\left(g_h^n(t)\right)_n \rightharpoonup \left(g^n(t)\right)_n \text{ weakly in } L^1(0,T;\ell^2), \tag{116}$$

the following holds

$$\sum_{1\le|n|\le N} \int_0^T g_h^n(t) \frac{a_h^n(t)}{\lambda_h^n} dt \to \sum_{n\in\mathbb{Z}^*} \int_0^T g^n(t) \frac{a^n(t)}{\lambda^n} dt. \tag{117}$$

Note that the two terms here are the left hand sides of (93) and (94) respectively. Thus, it is enough to see that the right hand side of (93) converges to the right hand side of (94). In order to do that we need the following lemma:

Lemma 5.6 *Let $\left(g_h^n(t)\right)_n$ be a sequence satisfying*

$$\left(g_h^n(t)\right)_n \rightharpoonup \left(g^n(t)\right)_n \textit{ weakly in } L^1(0,T;\ell^2).$$

Then,

$$\left(\int_t^T g_h^n(s)e^{\lambda_h^n(t-s)}ds\right)_n \rightharpoonup \left(\int_t^T g^n(s)e^{\lambda^n(t-s)}ds\right)_n \text{ weakly in } L^1(0,T;\ell^2),$$
$$\left(\int_0^T g_h^n(s)e^{-\lambda_h^n s}ds\right)_n \rightharpoonup \left(\int_0^T g^n(s)e^{-\lambda^n s}ds\right)_n \text{ weakly in } \ell^2.$$

Proof We have to see that, for all $(\xi^n(t))_n \in L^\infty(0,T;\ell^2)$,

$$\sum_{1\le|n|\le N}\int_0^T \xi^n(t)\int_t^T g_h^n(s)e^{\lambda_h^n(t-s)}ds\,dt \to \sum_{n\in\mathbb{Z}^*}\int_0^T \xi^n(t)\int_t^T g^n(s)e^{\lambda^n(t-s)}ds\,dt.$$

By Fubini's theorem we easily obtain

$$\int_0^T \xi^n(t)\int_t^T g_h^n(s)e^{\lambda_h^n(t-s)}ds\,dt = \int_0^T g_h^n(s)\int_0^s \xi^n(t)e^{\lambda_h^n(t-s)}dt\,ds,$$
$$\int_0^T \xi^n(t)\int_t^T g^n(s)e^{\lambda^n(t-s)}ds\,dt = \int_0^T g^n(s)\int_0^s \xi^n(t)e^{\lambda^n(t-s)}dt\,ds.$$

So, it is enough to see that

$$\left(\int_0^s \xi^n(t)e^{\lambda_h^n(t-s)}dt\right)_n \to \left(\int_0^s \xi^n(t)e^{\lambda^n(t-s)}dt\right)_n,$$

strongly in $L^\infty(0,T;\ell^2)$ for all $(\xi^n(t))_n \in L^\infty(0,T;\ell^2)$.

By Minkowski inequality we have,

$$\left\|\left(\int_0^s \xi^n(t)e^{\lambda_h^n(t-s)}dt - \int_0^s \xi^n(t)e^{\lambda^n(t-s)}dt\right)_n\right\|_{\ell^2}$$
$$\le \int_0^s \left\|\left(\xi^n(t)\left(e^{\lambda_h^n(t-s)} - e^{\lambda^n(t-s)}\right)\right)_n\right\|_{\ell^2} dt.$$

We prove that this converges to zero as $h \to 0$. Given $\epsilon > 0$ we choose n_0 such that

$$\int_0^s \left(\sum_{|n|>n_0}\left|\xi^n(t)\right|^2\right)^{1/2} dt < \frac{\epsilon}{4}.$$

This is always posible because $(\xi^n(t))_n \in L^\infty(0,T;\ell^2)$. Then, we chose h small enough such that

$$\sup_{t\in[0,T]}\sup_{n\le n_0}\left(e^{\lambda_h^n t} - e^{\lambda^n t}\right) \le \frac{\epsilon}{2\left\|(\xi^n(t))_n\right\|_{L^1(0,T;\ell^2)}}.$$

Thus we obtain

$$\begin{aligned}
&\int_0^s \left\|\left(\xi^n(t)\left(e^{\lambda_h^n(t-s)}-e^{\lambda^n(t-s)}\right)\right)_n\right\|_{\ell^2} dt \\
&\le \int_0^s \left(\sum_{1\le|n|\le n_0}\left|\xi^n(t)\left(e^{\lambda_h^n(t-s)}-e^{\lambda^n(t-s)}\right)\right|^2\right)^{1/2} dt \\
&\quad+\int_0^s \left(\sum_{|n|>n_0}\left|\xi^n(t)\left(e^{\lambda_h^n(t-s)}-e^{\lambda^n(t-s)}\right)\right|^2\right)^{1/2} dt \\
&\le \left\|\left(\xi^n(t)\right)_n\right\|_{L^1(0,T;\ell^2)} \sup_{t\in[0,T]}\sup_{n\le n_0}\left(e^{\lambda_h^n t}-e^{\lambda^n t}\right) \\
&\quad+2\int_0^s \left(\sum_{|n|>n_0}\left|\xi^n(t)\right|^2\right)^{1/2} \le \frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.
\end{aligned}$$

This concludes the proof of the lemma. □

By hypothesis (116), Lemma 5.6, the convergence of the eigenfunctions stated in (115) and the strong convergence of the controls stated in Theorem 3.2, we deduce that the right hand side in (93) converges to the right hand side of (94). Thus, we deduce that (117) holds.

This concludes the proof of the theorem. □

6 Numerical results

In this section we present some numerical results which validate the approximation of the boundary control of the wave equation with the mixed finite elements scheme (MFES) discussed in this article. We compare also these results with those obtained by classical finite differences (FDS) approximations.

The algorithm that we describe here approximates the control $v(t)$, at the extremity $x = 1$, that drives the system to a desired final state, i.e. we consider the following control problem: *Given $T > 2$ and (u^0, u^1), find a control $v(t)$ such that the solution of the system*

$$\begin{cases} u'' - u_{xx} = 0, & \text{for } x \in (0,1), \quad t > 0 \\ u(t,0) = 0, & \text{for } t > 0, \\ u(t,1) = v(t), & \text{for } t > 0, \\ u(0,x) = u'(0,x) = 0, & \text{for } x \in (0,1), \end{cases} \tag{118}$$

satisfies

$$u(T,\cdot) = u^0(x), \qquad u'(T,\cdot) = u^1(x). \tag{119}$$

Of course, the above control problem is equivalent to (1)–(2), due to the time reversibility of the wave equation and the linearity of the problem.

We remark that, as we have said in the introduction, the classical central difference scheme for (118) with time step Δt satisfying $\Delta t = h$ provides the exact solution at the nodes. Therefore, in this very particular situation, the one-dimensional version of the conjugate gradient algorithm described in [4] and [7] provides a very accurate approximation of the control for any $h > 0$. We use this very special situation to compute 'exact' controls that allow us to compare the results obtained with the MFES.

In [7], several cures are proposed to obtain the control when using finite difference or finite elements (with $\Delta t < h$) to approximate (118), as for example a bi-grid scheme. However, a complete analysis of the convergence of the controls with these techniques remains to be done (see [13] for some preliminary results in this context). Since we are mainly interested on the efficiency of the mixed finite element method, in our experiments we have not implemented any of these cures.

The rest of this section is divided in three subsections. In the first one we describe the numerical algorithm. In the second one we make some remarks on the time discretization of the wave equations. The last subsection is devoted to the numerical experiments.

6.1 The algorithm

The algorithm that we describe here is inspired by the one proposed by Glowinski, Li and Lions [6] (see also [4] and [7]) and is based on a conjugate gradient implementation of the HUM method. To this end we approximate the initial data by taking the values of u^0 and u^1 at the nodes. Then, we minimize the following functional:

$$J(W_h^0, W_h^1) = \frac{1}{8}\int_0^T \rho(t)(w_{0,N}')^2(t)dt + \frac{1}{2h^2}\int_0^T (w_{0,N})^2(t)dt + < (U_h^0, U_h^1), (W_h^0, W_h^1) >_D, \tag{120}$$

over all $(W_h^0, W_h^1) \in \mathbb{R}^{2N}$, where $w_{0,N}$ is the last component of the vector W_h, solution of the adjoint system (122) below.

The minimizer of J, $(\widehat{W}_h^0, \widehat{W}_h^1)$, provides a control v_h of the discrete system associated to (118) and the sequence $(v_h)_{h>0}$ converges to the HUM control of (118).

Since the initial data in (118) are identically zero, it seems natural to us to consider controls that satisfy $v_h(0) = 0$. We will impose this condition in our algorithm. As we will see, this condition provides a discontinuous control since $v_h(t)$ is nonzero immediately after $t = 0$, in general. This drawback can be avoided by considering compact support controls. This amounts to consider a modified functional in (120) where the second term is replaced by

$$\frac{1}{2h^2}\int_0^T \rho(t)(w_{0,N})^2(t)dt.$$

The minimizer of J,$(\widehat{W}_h^0, \widehat{W}_h^1)$, is characterized by the optimality condition $J'(\widehat{W}_h^0, \widehat{W}_h^1) = 0$, where $J'(\widehat{W}_h^0, \widehat{W}_h^1)$ denotes the gradient of J at $(\widehat{W}_h^0, \widehat{W}_h^1)$.

Moreover, it is not difficult to see that this optimality condition is equivalent to the following linear problem

$$\mathcal{A}_h(\widehat{W}_h^0, \widehat{W}_h^1) = (U_h^0, U_h^1), \tag{121}$$

where the linear operator $\mathcal{A}_h$ is defined as follows: $\mathcal{A}_h(W_h^0, W_h^1) = (U(T), U'(T))$, with $U(T)$ the solution of the coupled system

$$\begin{cases} M_h W_0''(t) + K_h W_0(t) = 0, & 0 < t < T, \\ W_0(T) = W_h^0, \;\; W_0'(T) = W_h^1, \end{cases} \tag{122}$$

$$\begin{cases} M_h U''(t) + K_h U(t) = \left(0, \ldots, 0, -\frac{1}{4}(\rho w_{0,N}')'(t) + \frac{1}{h^2} w_{0,N}(t)\right)^T, & t > 0, \\ U(0) = 0, \;\; U'(0) = 0, \end{cases} \tag{123}$$

where $w_{0,N}$ is the $N-$th component of the solution W_0 of (122).

We solve the linear system (121) by a conjugate gradient algorithm. An important fact to be taken into account is the choice of the scalar product used to compute de gradients. We consider the natural one $< \cdot, \cdot >_0$, defined in (21). Note that the operator $\mathcal{A}_h$ is uniformly (in h) elliptic with respect to the norm associated to this scalar product, due to the inverse inequality (38). More precisely, the inverse inequality (38) is equivalent to the ellipticity condition

$$< \mathcal{A}_h(W_h^0, W_h^1), (W_h^0, W_h^1) >_D \geq C \|(W_h^0, W_h^1)\|_0^2, \tag{124}$$

where $\| \cdot \|_0$ denotes the scalar product associated to $< \cdot, \cdot >_0$ and $< \cdot, \cdot >_D$ the duality product introduced in (63).

Let us briefly describe the main steps of the algorithm we have implemented.

Step 1: Initialization.

(1) Consider $\left(W_0^0, W_0^1\right) \in \mathbb{R}^{2N}$ any arbitrarily chosen initial data ((0, 0) for example).
(2) Compute $(R_0^0, R_0^1) = \mathcal{A}_h\left(W_0^0, W_0^1\right) - (U_h^0, U_h^1)$ solving the linear systems (122)–(123).
(3) Compute (G_0^0, G_0^1), by solving

$$K_h G_0^0 = -M_h R_0^1, \quad G_0^1 = R_0^0.$$

(4) Take $(D_0^0, D_0^1) = \left(G_0^0, G_0^1\right)$.

Take $j \geq 0$. Assume that (W_j^0, W_j^1), (R_j^0, R_j^1), (G_j^0, G_j^1) and (D_j^0, D_j^1) are known. Assume also that (G_j^0, G_j^1) and (D_j^0, D_j^1) are nonzero. We compute (W_{j+1}^0, W_{j+1}^1), (R_{j+1}^0, R_{j+1}^1), (G_{j+1}^0, G_{j+1}^1) and (D_{j+1}^0, D_{j+1}^1) as follows:

Step 2: Descent.

(5) $(\widetilde{R}_j^0, \widetilde{R}_j^1) = \mathcal{A}_h(D_j^0, D_j^1)$,
(6) Compute $(\widetilde{G}_j^0, \widetilde{G}_j^1)$, by solving

$$K_h \widetilde{G}_j^0 = -M_h \widetilde{R}_j^1, \quad \widetilde{G}_j^1 = \widetilde{R}_j^0.$$

(7)

$$\rho_j = \frac{< K_h G_j^0, G_j^0 > + < M_h G_j^1, G_j^1 >}{< K_h \widetilde{G}_j^0, D_j^0 > + < M_h \widetilde{G}_j^1, D_j^1 >},$$

(8) $(W_{j+1}^0, W_{j+1}^1) = (W_j^0, W_j^1) - \rho_j (D_j^0, D_j^1)$,
(9) $(R_{j+1}^0, R_{j+1}^1) = (R_j^0, R_j^1) - \rho_j (\widetilde{R}_j^0, \widetilde{R}_j^1)$,
(10) $(G_{j+1}^0, G_{j+1}^1) = (G_j^0, G_j^1) - \rho_j (\widetilde{G}_j^0, \widetilde{G}_j^1)$,

Step 3: Test of convergence and construction of the new direction.

(11)

If $\dfrac{< K_h G_{j+1}^0, G_{j+1}^0 > + < M_h G_{j+1}^1, G_{j+1}^1 >}{< K_h G_0^0, G_0^0 > + < M_h G_0^1, G_0^1 >} \leq \varepsilon^2$ (ε given)

then $(\widehat{W}^0, \widehat{W}^1) = (W_{j+1}^0, W_{j+1}^1)$,

else

(12)

$$\gamma_j = \frac{< K_h G_{j+1}^0, G_{j+1}^0 > + < M_h G_{j+1}^1, G_{j+1}^1 >}{< K_h G_j^0, G_j^0 > + < M_h G_j^1, G_j^1 >},$$

(13) $(D_{j+1}^0, D_{j+1}^1) = (G_{j+1}^0, G_{j+1}^1) + \gamma_j (D_j^0, D_j^1)$,
(14) Do $j = j + 1$ and return to Step 2.

6.2 Comments on the time discretization of the wave equation

Observe that in the algorithm we have presented above several semi-discretized wave equations must be solved.

When the semi-discrete system is obtained using a classical finite difference approximation in space (FDS), an explicit central difference scheme in time is usually implemented. This leads to the stability condition $\Delta t \leq h$, coming from the classical Courant-Friederichs-Levy criterium (CFL). This condition is not altered if a multigrid scheme is considered, as in [4] and [7].

When considering the same central difference method for the time discretization of the semi-discrete systems obtained with the MFES we obtain the following CFL condition

$$\Delta t \leq h^2. \tag{125}$$

This means that we have to consider a very small time step and therefore a very costly scheme. This is due to the fact that the eigenvalues corresponding to our model, λ_h^n, have a quadratic growth rate for large values of n (see Figure 1). Consequently, the velocity of propagation of the numerical waves associated to the high frequencies is large and this produces a bad CFL condition.

Therefore we have used the well-known Newmark scheme with parameters $\beta = 1/2$ and $\gamma = 1/4$ which is inconditially stable (see [16]). Of course, for this scheme there is no need to introduce a condition between Δt and h to achieve stability. In most of our experiments we have considered $\Delta t/h = 0.9$. However, for irregular data, a smaller time step has been used in order to obtain a more accurate approximation of the normal derivatives for the adjoint system. Thus, for a discontinuous data we have chosen $\Delta t/h = 0.1$.

6.3 Numerical experiments

Example 1 We first show an example where the one-dimensional version of the scheme described in [7], i.e. the FDS with the central difference discretization in time, provides a control that grows exponentially as $h \to 0$, when considering $\Delta t < h$. It is worth noting that the ill-conditioning of this scheme was already pointed out in [7], where the authors observed that the algorithm based on this scheme does not converge. In [12] it is observed that, in fact, at the semi-discrete level, there would exist some initial data for which the controls will blow-up as $h \to 0$. The novelty of this example in this context is that it exhibits this phenomenon numerically.

Let us consider as final data

$$u^0(x) = \begin{cases} 16x^3 \text{ if } x \in (0, 1/2) \\ 16(1-x)^3 \text{ if } x \in (1/2, 1) \end{cases}, \qquad u^1(x) = 0.$$

In Figure 2 we present a picture of these data.

When we compute the boundary control for these data with the algorithm in [7], in time $T = 2.4$, we obtain an unbounded sequence of controls. In figure 3 we have presented, in a log-scaled diagram, the L^2-norm of the controls when $\Delta t/h = 0.9,\ 0.7,\ 0.5$ and 0.3, for different values of $h = 1/(N+1)$. We observe that as the ratio $l = \Delta t/h$ becomes smaller, the norm of the control increases more rapidly.

When considering the MFES the sequence of controls remain uniformly bounded as $h \to 0$.

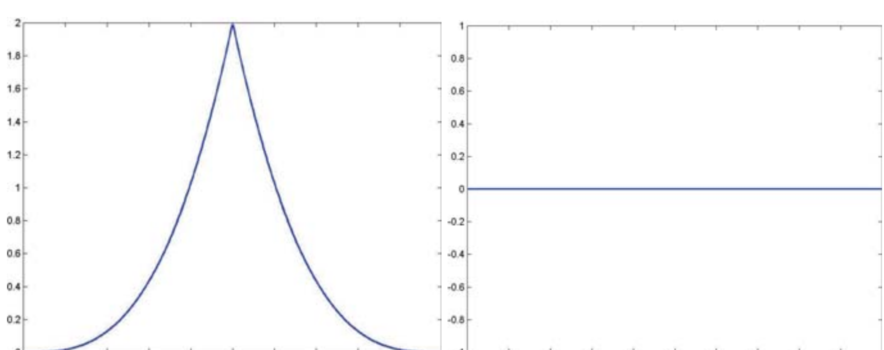

Fig. 2 (Example 1) Final position $u^0(x)$ (left) and final velocity $u^1(x)$

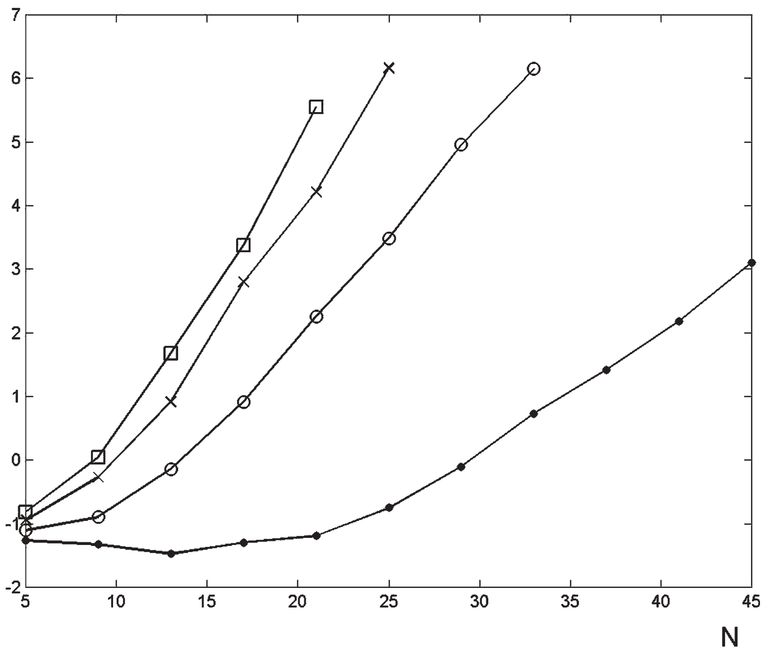

Fig. 3 (Example 1) Number of nodes N versus the log. of the L^2-norm of the controls computed with the classical FDS and the central difference discretization in time for different values of $\Delta t/h = 0.9$ (dots), 0.7 (circles), 0.5 (+) and 0.3 (squares)

In Figure 4 we have represented the control obtained with the MFES (left). We observe that at $t = 2$ there are some small oscillations which are due to the discontinuity of the control at $t = 0$. Recall that we have assumed that the control is zero at $t = 0$ but this is not the case immediately after. Then, this discontinuity propagates along the domain and it is controlled at $t = 2$. The presence of some oscillations in the control before $t = 2$ is natural due to the fact that the numerical high frequencies travel faster. We mention that this fact can be avoided by regularizing the control in our algorithm. This can be done multiplying the control by a given compact support function at each iteration. This produces a more regular control (see Figure 4, right).

In the numerical results presented in Figure 4 we have considered $\Delta t/h = 0.9$. Other choices of this ratio $0 < \Delta t/h < 1$ provide very similar results.

In Table 1 we compare the L^2-norms of the controls $\tilde{v}_h$ obtained with the FDS and $\Delta t/h = 1$ (special case in which we have convergence with FDS) and the L^2-norms of the controls v_h obtained with the MFES and $\Delta t/h = 0.9$.

Example 2 In this example we consider the most singular situation with a discontinuous final data. As in the previous example, if we take $T = 2.4$, the algorithm based on the FDS provides an unbounded sequence of controls as $h \to 0$, when $\Delta t/h < 1$. We show that, even in this very singular situation, the MFES provides an approximation of the controls. For this example we took a smaller ratio $\Delta h/t = 0.1$ to obtain a more accurate approximation. The results are compared with those obtained by the FDS with $\Delta t = h$.

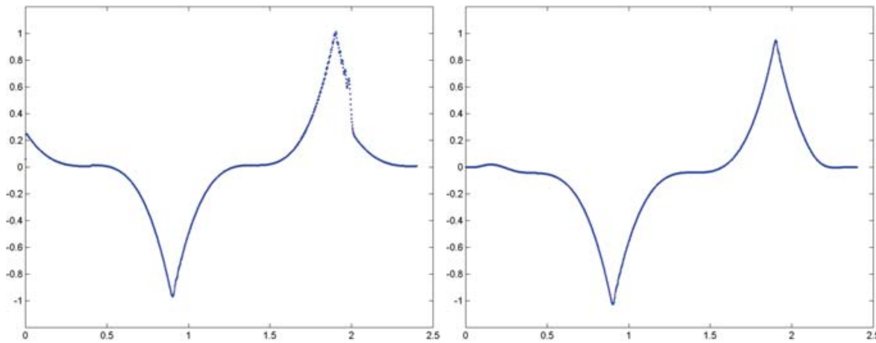

Fig. 4 (Example 1) Control obtained with the MFES with $\Delta t/h = 0.9$ and $h = 1/512$ (left), and the same when we introduce a regularizing function on the algorithm in order to obtain a compact support control (right)

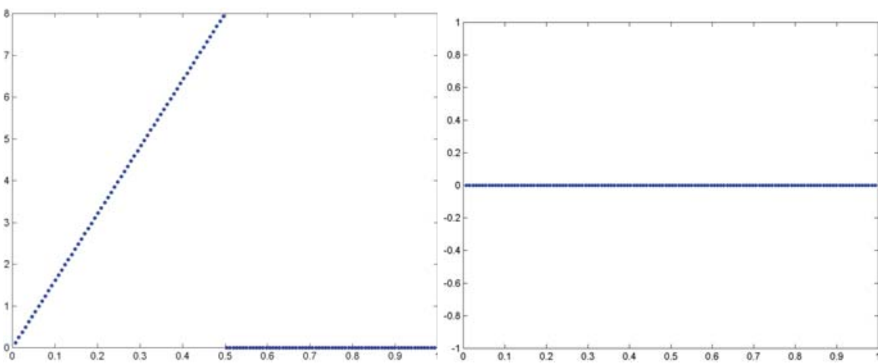

Fig. 5 (Example 2) Final position $u^0(x)$ (left) and final velocity $u^1(x)$

We take

$$u^0(x) = \begin{cases} 16x \text{ if } x \in (0, 1/2) \\ 0 \text{ if } x \in (1/2, 1). \end{cases}, \qquad u^1(x) = 0.$$

In Figure 5 we present a picture of these data.

In Figure 6 we show the convergence of the control (with the MFES) as h is decreased. We observe a number of oscillations that become smaller as h goes to zero.

Example 3 Finally we consider an example where the HUM-control can be computed explicitly. The analogous to this example in 2-D was introduced in [7] to

Table 1 (Example 1) The L^2-norms of the controls v_h obtained with the MFES (with $\Delta t/h = 0.9$) compared to the L^2-norms of the controls $\tilde{v}_h$ obtained with the FDS (with $\Delta t/h = 1$)

h	1/64	1/128	1/256	1/512
CG iter.	3	3	3	3
$\|v_h\|_{L^2}$	0.53682	0.53733	0.53751	0.53757
$\|\tilde{v}_h\|_{L^2}$	0.53908	0.53875	0.53801	0.53760

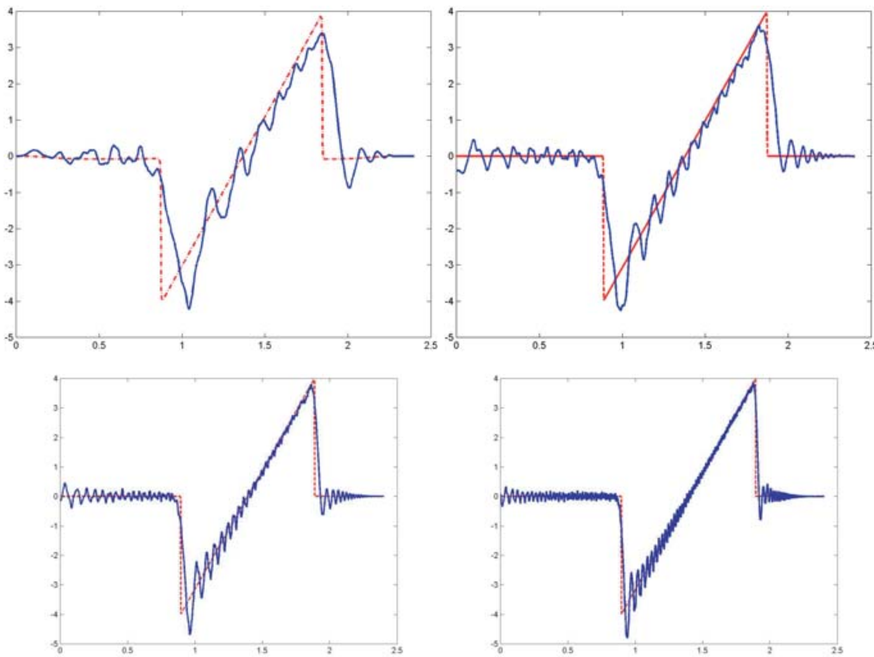

Fig. 6 (Example 2) Control of the semi-discrete system with MFES (solid) and with FDS with $\Delta t = h$ (dash) when $h = 1/32$ (top left),$h = 1/64$ (top right), $h = 1/128$ (bottom left), $h = 1/256$ (bottom right)

test the proposed cures to the ill-posedness of the discrete boundary controllability problem. We have observed that in the 1-D case these examples produce a converging sequence of controls, even for the classical FDS with any ratio $\Delta t/h < 1$, and without any of the cures proposed in [7]. We have included it to illustrate that these examples do not exhibit the ill-conditioning of the FDS for the one-dimensional case.

Let $w(t,x)$ be the following solution of the backwards adjoint system (4) with $T = 2 + 3/4$,

$$w(t,x) = \sqrt{2}\cos\left[\pi(T - t - 1/4)\right]\sin(\pi x). \tag{126}$$

Then, the normal derivative at $x = 1$ is given by

$$v(t) = w_x(t,1) = -\sqrt{2}\pi\cos\left[\pi(T - t - 1/4)\right]. \tag{127}$$

The solution of system (118) with the above nonhomogeneous data $v(t)$ is given by $u = u_1 + u_2$ where

$$u_1(t,x) = -\sqrt{2}\pi\cos\left[\pi(T - t - 1/4)\right]\sin(\pi x/2),$$

$$u_2(t,x) = 8\sqrt{2}\sum_{n=1}^{\infty}\frac{\sin(n\pi x)\sin(n\pi t)}{4n^2-1}$$
$$-6\sqrt{2}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin(n\pi x)}{4n^2-1}\left[\frac{n}{n^2-1}\cos(\pi(T-t-1/4))\right.$$
$$\left.-\frac{1}{2(n+1)}\cos(\pi(T+nt-1/4))-\frac{1}{2(n-1)}\cos(\pi(T-nt-1/4))\right]. \quad (128)$$

In Figure 7 we give a picture of the final position $u(T,x)$ and velocity $u'(T,x)$. We observe that the final position is continuous but not C^1 while the final velocity has a discontinuity. Thus, these data are more irregular than those considered in the Example 1 above. By construction, when considering these final data, the HUM control is given by (127) (see Figure 8).

The results that we obtain with the FDS when $\Delta t/h = 7/8$ are given in Table 2. We observe that the algorithm converges in a few iterations with very

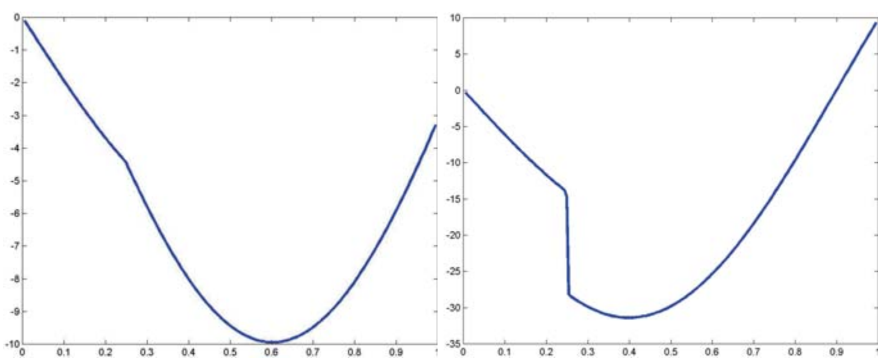

Fig. 7 (Example 3) Final position (left) and velocity (right)

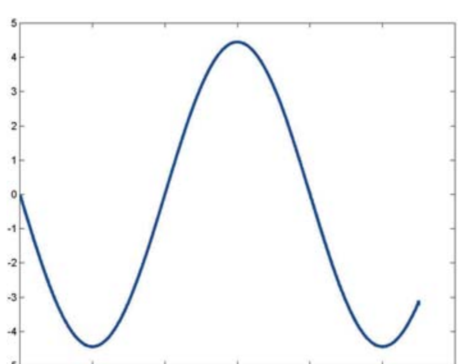

Fig. 8 (Example 3) Control obtained with the FDS when $h = 1/512$ and $\Delta t/h = 7/8$

Table 2 (Example 3) Numerical results obtained with the FDS, when $\Delta t/h = 7/8$.

h	1/64	1/128	1/256	1/512
CG iter.	3	3	3	3
$\|\hat{W}_h^0 - w(T,x)\|_{L^2}$	1.45×10^{-2}	6.38×10^{-3}	2.34×10^{-3}	1.72×10^{-3}
$\|\hat{W}_h^0 - w(T,x)\|_{H_0^1}$	5.24×10^{-2}	2.46×10^{-2}	1.18×10^{-2}	7.88×10^{-3}
$\|\hat{W}_h^1 - w'(T,x)\|_{L^2}$	2.12×10^{-2}	1.27×10^{-2}	8.29×10^{-3}	5.15×10^{-3}
$\|v_h - w_x(t,1)\|_{L^2}$	8.14×10^{-2}	4.01×10^{-2}	2.14×10^{-2}	1.31×10^{-2}
$\|v_h\|_{L^2}$	5.4056	5.3769	5.3625	5.3640

accurate approximations. Other choices of the ratio $\Delta t/h$ provide similar results. The approximations obtained with the MFES are also very similar.

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