SIAM J. CONTROL OPTIM.
Vol. 47, No. 6, pp. 2857–2885

UNIFORM BOUNDARY CONTROLLABILITY OF A SEMIDISCRETE 1-D WAVE EQUATION WITH VANISHING VISCOSITY*

SORIN MICU†

Abstract. This article deals with the approximation of the boundary control of the linear one-dimensional wave equation. It is known that the high frequency spurious oscillations that the classical methods of finite difference and finite element introduce lead to nonuniform controllability properties (see [J. A. Infante and E. Zuazua, *M2AN Math. Model. Numer. Anal.*, 33 (1999), pp. 407–438]. A space-discrete scheme with an added numerical vanishing viscous term is introduced and analyzed. The extra numerical damping filters out the high numerical frequencies and ensures the convergence of the sequence of discrete controls to a control of the continuous conservative wave equation when the mesh size tends to zero.

Key words. control, wave equation, semidiscrete approximation, numerical viscosity

AMS subject classifications. 93B40, 65M06, 93B05

DOI. 10.1137/070696933

1. Introduction.

The following boundary exact controllability property for the one-dimensional (1-D) linear wave equation is known to hold: given $T \geq 2$ and $(u^0, u^1) \in L^2(0,1) \times H^{-1}(0,1)$ there exists a control function $v \in L^2(0,T)$ such that the solution of the wave equation

$$\begin{cases} u'' - u_{xx} = 0 & \text{for } x \in (0,1), \ \ t > 0, \\ u(t,0) = 0, \quad u(t,1) = v(t) & \text{for } t > 0, \\ u(0,x) = u^0(x), \quad u'(0,x) = u^1(x) & \text{for } x \in (0,1) \end{cases} \tag{1.1}$$

satisfies

$$u(T,\cdot) = u'(T,\cdot) = 0. \tag{1.2}$$

By $'$ we denote the time derivative.

For the study of this controllability problem the moments theory has been successfully used (see, for instance, [1, 25]). Also, the Hilbert uniqueness method (HUM) (see [16]) has provided a different and general way to study this and similar multidimensional problems.

In past years there was an increasing interest in the numerical approximations of the controls. For instance, HUM was used in [8, 10, 11] to deduce numerical algorithms with finite differences in the context of the two dimensional (2-D) wave equation. In these references a bad behavior of the approximate controls was observed. Let us briefly explain this phenomenon.

Generally speaking, the control's approximation strategy consists in discretizing the continuous problem (1.1), finding a control for each discrete problem, and finally making the mesh size h tend to zero. What one normally expects is to get a control of (1.1). However, as mentioned before, this is not necessarily true. Indeed, negative

*Received by the editors July 11, 2007; accepted for publication (in revised form) July 9, 2008; published electronically November 19, 2008.
http://www.siam.org/journals/sicon/47-6/69693.html
†Facultatea de Matematica si Informatica, Universitatea din Craiova, Craiova DOLJ 200585, Romania (sd_micu@yahoo.com). This research was partially supported by grant MTM2005-00714 of MEC (Spain) and grant CEEX-05-D11-36/2005 (Romania).

numerical results may be obtained as a consequence of the spurious high frequency oscillations that do not exist at the continuous level but that are generated by any semidiscrete dynamics. Precisely the controls of the highest modes may have large norms which are not uniformly bounded in h. Moreover, a dispersion phenomenon appears, and the velocity of propagation of these high frequency numerical waves may converge to zero when the mesh size h tends to zero. Hence, neither is the control time uniformly bounded. All of these phenomena occur in the semidiscrete models obtained by finite differences or by the classical finite element method (see [13, 30] for a detailed analysis of the 1-D case and [29] for the 2-D case, in the context of the dual observability problem). In any of these models, the controllability property is *not uniform* as the discretization parameter h goes to zero. As a consequence there are initial data of the wave equation (even regular ones) for which the corresponding sequence of discrete controls will diverge in the L^2-norm.

Note that the spurious oscillations correspond to the high frequencies of the discrete model, and therefore they weakly converge to zero. Consequently, their existence is compatible with the convergence of the numerical scheme. However, when we are dealing with the exact controllability problem, the role of the high frequencies becomes much more important.

Since the main problem is the existence of the spurious high frequencies generated by the discretization process, the idea of eliminating or reducing them in one way or another arises naturally. All of the numerical experiments using a Tychonoff regularization technique [10, 11], a bigrid algorithm [8, 11] or a mixed finite element approximation [9] are based on this idea. How to do that in an optimal and general way and how to show mathematically the uniform controllability results are less clear. Nevertheless, in past years many theoretical results were obtained. In [18] the high frequency modes of the discrete initial data are filtered out in an appropriate manner to ensure the existence of a uniformly bounded sequence of discrete controls. In [2] a mixed finite element method is analyzed and an explicit sequence of discrete controls which tends to the HUM control of the limit wave equation (1.1) is constructed. The analysis of a bigrid method is presented in [21], where uniform results are also proved.

This paper considers a different method to achieve the uniform controllability. The idea is to introduce in the discrete equation a numerical viscous term which vanishes when the mesh size h tends to zero. The dissipation has the role to damp out the bad spurious high frequencies responsible for the large norm controls, eventually ensuring the uniform controllability of the system. The method has the advantage of being simple, general, and quite natural. Note that the amount of dissipation introduced in the discrete system is very important. If the dissipation is uniformly bounded in h (i.e., all of the eigenvalues belong to a vertical strip), it cannot compensate for the effect of the controls' growth, and the uniform result does not hold. On the other hand, the damping term should vanish in the limit and therefore cannot be enforced too much. We have treated a particular case in which we were able to prove the uniform controllability result. However, to give the optimal amount of dissipation capable of ensuring the best convergence rate and at the same time the uniformity needs further investigation (see Remark 1).

The proof of the uniform controllability result is based on the moments theory. More precisely, we construct a biorthogonal sequence in $L^2(-\frac{T}{2}, \frac{T}{2})$ to the family of complex exponentials $\Lambda = (e^{-\overline{\lambda}_n(h)t})_n$, where $\lambda_n(h)$ are the eigenvalues of the corresponding discrete operator. This is done via the Fourier transform of some entire functions of exponential type. A careful analysis of the behavior of these entire functions on the real axis gives estimates for the norm of the biorthogonal sequence.

These will allow us to show the uniform boundedness on h of the corresponding sequence of discrete controls for a large class of initial data.

Estimates for biorthogonal sequences to families of exponential functions may be found, for instance, in [6, 7] for the case of the heat equation, [5] for the case of a multidimensional wave equation, or [12] for the case of a dissipative beam equation. However, our family of exponentials has several different characteristics which make the study more difficult, such as the dependence on the discretization step h and the complexity of the exponents, which are contained neither in a sector of the real axis nor on a vertical strip of the imaginary axis as in the previous works. The Fourier transform of the biorthogonal elements has two qualitatively different behaviors on the real axis, depending on if the value of $|x|$ is smaller than $\frac{1}{h}$ or not. This gives interesting and new type estimates reflecting the behavior of the real parts of our exponents, which are of order h for the low frequencies but become of order $\frac{1}{h}$ for the highest ones.

In [28] the controllability of a hyperbolic-parabolic coupled system is studied. The corresponding spectrum is a union of two families $\{\lambda_l^l\}_{l\geq 1} \cup \{\lambda_k^h\}_{k\geq 1}$. The first one is purely real and behaves like $-l^2\pi^2$, and the second one is complex and looks like $-\frac{1}{\sqrt{2k\pi}} + k\pi i$. Taken separately, each family has good controllability properties, and the main problem is to join them. Although there are similarities with our case (the spectrum is contained neither in a sector of the real axis nor on a vertical strip of the imaginary axis), we cannot use the technique from [28] since the high part of our spectrum is far away from any known controllable family of eigenfunctions.

The numerical viscosity technique was used in many different contexts (see, for instance, [4, 17] in the case of hyperbolic conservation laws or [22, 23, 26, 27] in the case of the semidiscrete dissipative plate or wave equation). In [22, 26, 27] an additional vanishing viscosity term is introduced in the interior of the domain to make the initially dissipative system uniformly stable. Consequently, the uniform controllability property holds if a vanishing control is added in the interior of the domain. However, our aim is more ambitious and consists of showing that no additional control is needed to ensure the convergence of this scheme. From the numerical point of view this is an advantage since the extra storage and computation for an additional control are avoided.

In [3] a singular limit problem from a parabolic to a hyperbolic equation is studied from the controllability point of view. Although the context and the approach are different, there exists at least one similarity to our case: the parabolic character vanishes in the limit when a controlled hyperbolic system is obtained.

It is known that a structural damping of the form $-\varepsilon u_{txx}$ makes the continuous wave equation not even spectrally controllable (see [24]). This is due to the accumulation of the spectrum on $-\varepsilon$. In our semidiscrete case the viscosity parameter ε and the mesh size h are related and tend to zero simultaneously. The resulting system is consistent with the conservative wave equation, and therefore, unlike in [24], the uniform controllability property does hold.

The article is organized in the following way. The discrete control problem is presented in section 2, and an analysis of the energy decay rate is given in section 3. The control problem is transformed into an equivalent problem of moments in section 4. The main result is contained in section 5, where a biorthogonal sequence is constructed and evaluated. The technical but fundamental proof of the estimate on the real axis of the entire functions whose Fourier transforms give the biorthogonal sequence is given in the appendix at the end of the paper. In section 6 the existence of a bounded sequence of discrete controls is proved, and it is shown that any weak limit is a control of the continuous wave equation. Section 7 presents some numerical experiments to support the theoretical results obtained in the paper.

2. The discrete problem. In this paper we study a finite-difference space discretization of (1.1). In order to do this, let us consider $N \in \mathbb{N}^*$, a step $h = \frac{1}{N+1}$, and an equidistant mesh of the interval $(0,1)$, $0 = x_0 < x_1 < \cdots < x_N < x_{N+1} = 1$, with $x_j = jh$, $0 \le j \le N+1$. The central finite-difference approximation of the space derivatives leads to the following semidiscretization of (1.1):

$$\begin{cases} u_j''(t) - \frac{u_{j+1}(t)+u_{j-1}(t)-2u_j(t)}{h^2} = 0 & \text{for } 1 \le j \le N, \ \ t>0, \\ u_0(t) = 0, \ \ u_{N+1}(t) = v_h(t) & \text{for } t>0, \\ u_j(0) = u_j^0(x), \ \ u_j' = u_j^1(x) & \text{for } 1 \le j \le N. \end{cases} \tag{2.1}$$

System (2.1) consists of N linear differential equations with N unknowns $u_1, u_2, \ldots, u_N$. Roughly speaking, $u_j(t)$ approximates $u(t,x_j)$, the solution of (1.1), provided that $(u_j^0, u_j^1)_{0\le j\le N+1}$ is an approximation for the initial datum in (1.1). In fact, we shall choose

$$u_j^0 = u^0(jh), \quad u_j^1 = u^1(jh), \qquad 0 \le j \le N+1. \tag{2.2}$$

Our aim is to study the following controllability property for (2.1): *given $T > 2$ and $(u_j^0, u_j^1)_{1\le j\le N} \in \mathbb{C}^{2N}$, there exists a control function $v_h \in H^1(0,T)$ such that the corresponding solution $(u_j, u_j')_{1\le j\le N}$ of* (2.1) *satisfies*

$$u_j(T) = u_j'(T) = 0, \quad 1 \le j \le N. \tag{2.3}$$

If this holds for any $(u_j^0, u_j^1)_{1\le j\le N} \in \mathbb{C}^{2N}$, we say that (2.1) is *exactly controllable in time T*.

It is not difficult to see that the controllability problem we have just addressed has a positive answer and a sequence of discrete controls $(v_h)_{h>0}$ may be easily found. Considerably more difficult is to show that the sequence $(v_h)_{h>0}$ converges in some sense to a control v of the continuous wave equation (1.1), corresponding to the initial datum (u^0, u^1), which verifies (2.2). In fact, due to the spurious high frequencies introduced by the discretization, the numerical scheme (2.1) gives an unbounded sequence of controls $(v_h)_{h>0}$ even when very regular initial data (u^0, u^1) are considered (see [13, 18]).

In order to deal with the spurious high frequencies, we add a dissipative term, vanishing when the mesh size tends to zero. More precisely, we consider instead of (2.1) the following scheme:

$$\begin{cases} u_j''(t) - \frac{u_{j+1}(t)+u_{j-1}(t)-2u_j(t)}{h^2} - \varepsilon\frac{u_{j+1}'(t)+u_{j-1}'(t)-2u_j'(t)}{h^2} = 0, & t>0, \\ u_0(t) = 0, \ \ u_{N+1}(t) = v_h(t), & t>0, \\ u_j(0) = u_j^0(x), \ \ u_j' = u_j^1(x), & 1 \le j \le N, \end{cases} \tag{2.4}$$

and we address the same controllability problem as before.

The parameter ε, which multiplies the viscous term $\frac{u_{j+1}'(t)+u_{j-1}'(t)-2u_j'(t)}{h^2}$, depends on the step size h and tends to zero as $h \to 0$:

$$\lim_{h\to 0} \varepsilon(h) = 0. \tag{2.5}$$

Hence, in (2.4), the term $\varepsilon\frac{u'_{j+1}(t)+u'_{j-1}(t)-2u'_j(t)}{h^2}$ represents a vanishing numerical viscosity that will eventually ensure the boundedness of the sequence $(v_h)_{h>0}$.

Remark 1. A similar damping term, with $\varepsilon = h^2$, is used in [27] in order to ensure an exponentially uniform decay rate of the discrete energy, when a dissipative term is already present on the boundary. In this case, the extra damping constitutes a bounded perturbation sufficient to restore the uniform decay rate. The spectrum of the corresponding operator is shifted to the left, but it remains in a vertical strip. However, this is not sufficient to ensure the uniform boundary controllability property. Therefore, we shall reinforce the damping by considering $\varepsilon = h$. This selection verifies (2.5) and, at the same time, ensures the amount of dissipation needed for the uniform control of the high frequencies. Let us finally mention that we do not know the optimal choice for $\varepsilon(h)$, ensuring both the uniformity in h and the best convergence rate. For instance, any $\varepsilon(h) = h^\alpha$, with $\alpha \in (1, 2)$, could still produce uniform controllability results with a better convergence rate. The numerical experiments confirm this, but the theoretical study is more difficult and needs further investigation. In [23] such dissipative terms were used to achieve uniform stability of the approximating models.

Let us first write (2.4) in a vectorial form, which is easier to deal with. We define the following matrix from $\mathcal{M}_{N\times N}(\mathbb{R})$:

$$A_h = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix}.$$

If we denote the unknown of (2.4) by $U_h(t) = (u_1(t), u_2(t), \dots, u_N(t))^T$, system (2.4) may be written in the following equivalent vectorial form:

$$\begin{cases} U_h''(t) + A_h U_h(t) + \varepsilon A_h U_h'(t) = F_h(t), & \text{for } t > 0, \\ U_h(0) = U_h^0, \;\; U_h'(0) = U_h^1, \end{cases} \tag{2.6}$$

$U_h^0 = (u_j^0)_{1\le j\le N}$ and $U_h^1 = (u_j^1)_{1\le j\le N}$ being the initial data of (2.4). The vector F_h is given by

$$F_h(t) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \frac{1}{h^2}\left(v_h(t) + \varepsilon v_h'(t)\right) \end{pmatrix}.$$

In (2.6) we have taken into account that $u_{N+1}(t) = v_h(t)$ and $u_0(t) = 0$ for all $t > 0$.

Before studying the decay properties of the solutions of (2.6), let us define in $\mathbb{C}^N$ the canonical inner product

$$(f, g) = h\sum_{k=1}^{N} f_k \overline{g}_k, \tag{2.7}$$

where $f = (f_k)_{1\le k\le N}$ and $g = (g_k)_{1\le k\le N}$ belong to $\mathbb{C}^N$.

Also, we consider in $\mathbb{C}^{2N}$ the inner product defined by

$$(2.8)\quad (f,g)_1 = h\left[\sum_{k=1}^{N-1}\frac{f_{k+1}-f_k}{h}\frac{\overline{g}_{k+1}-\overline{g}_k}{h}+\frac{1}{h^2}(f_1\overline{g}_1+f_N\overline{g}_N)\right]+h\sum_{k=N+1}^{2N}f_k\overline{g}_k,$$

where $f=(f_k)_{1\le k\le 2N}$ and $g=(g_k)_{1\le k\le 2N}$ are two vectors from $\mathbb{C}^{2N}$. The corresponding norm will be denoted by $||\ \cdot\ ||_1$.

Remark 2. The following equivalent form of the inner product (2.8) justifies its definition and its usefulness for our problem:

$$(2.9)\qquad (f,g)_1 = \left(A_h f^1, g^1\right)+\left(f^2,g^2\right),$$

where $f^1=(f_k)_{1\le k\le N}$, $f^2=(f_k)_{N+1\le k\le 2N}$, $g^1=(g_k)_{1\le k\le N}$, $g^2=(g_k)_{N+1\le k\le 2N}$ and $f=(f_1,f_2)$, $g=(g_1,g_2)$.

The following discrete duality product will be needed in the study of the control problem:

$$(2.10)\qquad \left\langle \left(f^1,f^2\right),\left(g^1,g^2\right)\right\rangle_D = -\left(f^1,g^2\right)+\left(f^2+\varepsilon A_h f^1, g^1\right),$$

where $f=(f^1,f^2)$ and $g=(g^1,g^2)$ are two vectors from $\mathbb{C}^{2N}$ as in Remark 2.

3. Energy estimates. In this section we briefly study system (2.4) in the uncontrolled case, i.e., when $v_h=0$, and show its dissipative character. Here (2.4) is a homogeneous linear system of N differential equations of order two and has a unique solution $U\in\mathcal{C}^\omega\left([0,\infty),\mathbb{R}^N\right)$ (the set of analytic functions defined in $[0,\infty)$ and with values in $\mathbb{R}^N$). The energy of (2.4) is defined by

$$(3.1)\qquad E_h(t)=\frac{h}{2}\sum_{j=0}^{N}\left[|u_j'(t)|^2+\left|\frac{u_{j+1}(t)-u_j(t)}{h}\right|^2\right]$$

and represents a discretization of the continuous energy corresponding to (1.1)

$$(3.2)\qquad E(t)=\frac{1}{2}\int_0^1\left[|u'(t)|^2+|u_x(t)|^2\right]dx.$$

Remark 3. The energy may be expressed in terms of the inner product (2.7):

$$(3.3)\qquad E_h(t)=\frac{1}{2}\left[(U_h'(t),U_h'(t))+(U_h(t),A_hU_h(t))\right].$$

It is easy to show that system (2.4) is dissipative. More precisely, we have the following proposition.

PROPOSITION 3.1. *If* $v_h=0$ *in* (2.4) *and* $E_h(t)$ *is the energy function* (3.1), *then*

$$(3.4)\qquad \frac{dE_h}{dt}(t)=-\varepsilon(A_hU_h'(t),U_h'(t))=-\varepsilon h\sum_{j=0}^{N}\left|\frac{u_{j+1}'(t)-u_j'(t)}{h}\right|^2.$$

Proof. Equality (3.4) follows easily, multiplying (2.6) by U_h'. Indeed,

$$\begin{aligned}0&=(U_h''+A_hU_h+\varepsilon A_hU_h',U_h')\\&=\frac{1}{2}\left[(U_h',U_h')+(A_hU_h,U_h)\right]'+\varepsilon(A_hU_h',U_h')\\&=\frac{dE_h}{dt}(t)+\varepsilon(A_hU_h',U_h'),\end{aligned}$$

and the proof finishes. ∎

Proposition 3.1 indicates that the energy of (2.4) decreases with t. In the next theorem we shall give an estimate for the decay rate of the energy. Let us first consider the Fourier decomposition of the solutions of (2.4) by using the eigenvectors of the self-adjoint operator A_h. It is well known (see, for instance, [15]) that the eigenvalues of A_h are $(\nu_j^2)_{1\le j\le N}$, where

$$\nu_j = \frac{2}{h}\sin\left(\frac{j\pi h}{2}\right), \quad 1 \le j \le N, \tag{3.5}$$

and the corresponding eigenvectors are given by

$$\varphi^j = \sqrt{2}(\sin(j\pi hk))_{1\le k\le N} \in \mathbb{R}^N, \quad 1 \le j \le N. \tag{3.6}$$

The following property holds.

LEMMA 3.2. *The set of vectors* $(\varphi^j)_{1\le j\le N}$ *forms an orthonormal basis in* $\mathbb{C}^N$ *with respect to the inner product defined by* (2.7).

Proof. We have

$$\left(\varphi^l, \varphi^j\right) = 2h\sum_{k=1}^{N}\sin(j\pi hk)\sin(l\pi hk) = h\sum_{k=0}^{N}\left(\cos((l-j)\pi hk) - \cos((l+j)\pi hk)\right).$$

But, for $q \in \mathbb{Z}$,

$$\sum_{k=0}^{N}\cos(q\pi hk) = \begin{cases} N+1 & \text{if } q = 0,\\ \frac{1-(-1)^q}{2} & \text{if } q \in \mathbb{Z}^*. \end{cases}$$

It follows that $(\varphi^l, \varphi^j) = \delta_{lj}$, and the proof finishes. ◻

Let us expand the initial datum (U_h^0, U_h^1) of (2.6) as follows:

$$U_h^0 = \sum_{j=1}^{N} a_{jh}^0\varphi^j \text{ and } U_h^1 = \sum_{j=1}^{N} a_{jh}^1\varphi^j. \tag{3.7}$$

The corresponding solution $U_h(t)$ is given by

$$U_h(t) = \sum_{j=1}^{N} a_{jh}(t)\varphi^j, \tag{3.8}$$

where the coefficients $a_{jh}(t)$ can be computed explicitly. Indeed, we have the following lemma.

LEMMA 3.3. *If the initial datum* (U_h^0, U_h^1) *of system* (2.6) *are given by* (3.7), *then the corresponding solution of* (2.6), *with* $v_h = 0$, *is given by* (3.8), *where*

$$a_{jh}(t) = \frac{a_{jh}^1 - \lambda_j^- a_{jh}^0}{\lambda_j^+ - \lambda_j^-}e^{\lambda_j^+ t} - \frac{a_{jh}^1 - \lambda_j^+ a_{jh}^0}{\lambda_j^+ - \lambda_j^-}e^{\lambda_j^- t} \tag{3.9}$$

and

$$\lambda_j^{\pm} = \frac{1}{2}\left(-\varepsilon\nu_j^2 \pm \sqrt{\varepsilon^2\nu_j^4 - 4\nu_j^2}\right), \quad j = 1, 2, \ldots, N. \tag{3.10}$$

Proof. The proof is elementary, and we omit it. ◻

Remark 4. The values $(\lambda_j^{\pm})_{1\leq j\leq N}$ from (3.10) are the eigenvalues of the operator

$$\mathcal{A} = \begin{pmatrix} 0 & -I \\ A_h & \varepsilon A_h \end{pmatrix}$$

corresponding to (2.6). Note that they are complex numbers with a negative real part. We shall write

$$\lambda_j = \frac{1}{2}\left(-\varepsilon\nu_j^2 + \operatorname{sgn}(j)\sqrt{\varepsilon^2\nu_j^4 - 4\nu_j^2}\right), \quad 1 \leq |j| \leq N.$$

Let us remark that, in the case $\varepsilon = h$, λ_j has the simpler form

$$\lambda_j = i\frac{2}{h}\sin\left(\frac{j\pi h}{2}\right)\left(\cos\left(\frac{j\pi h}{2}\right) + i\sin\left(\frac{j\pi h}{2}\right)\right). \tag{3.11}$$

We pass now to evaluate the decay rate of the energy corresponding to (2.6).

From now on we shall suppose that $\varepsilon = h$. This is the case we shall analyze from the controllability point of view later on in the paper.

THEOREM 3.4. *Let $\varepsilon = h$ and $v_h = 0$ in (2.4). There exist two positive constants C and ω, not depending on h, such that*

$$E_h(t) \leq \frac{C}{h^2}e^{-\omega h t}E_h(0)\ \forall t > 0. \tag{3.12}$$

Proof. From (3.3) and Lemmas 3.2 and 3.3 we have that

$$E_h(t) = \frac{1}{2}\left[(U'(t), U'(t)) + (A_hU(t), U(t))\right] = \frac{1}{2}\left[\sum_{j=1}^{N}\left(|a'_{jh}(t)|^2 + \nu_j^2|a_{jh}(t)|^2\right)\right].$$

From (3.9) we obtain that, for $1 \leq j \leq N$,

$$\begin{aligned} |a'_{jh}(t)|^2 + \nu_j^2|a_{jh}(t)| &\leq 8e^{-\varepsilon\nu_j^2 t}\frac{|\lambda_j^+|^2 + \nu_j^2}{|\lambda_j^+ - \lambda_j^-|^2}\left[|a_{jh}^1|^2 + |\lambda_j^+|^2|a_{jh}^0|^2\right] \\ &= 16e^{-\varepsilon\nu_j^2 t}\frac{\nu_j^2}{|\lambda_j^+ - \lambda_j^-|^2}\left[|a_{jh}^1|^2 + \nu_j^2|a_{jh}^0|^2\right]. \end{aligned}$$

Hence,

$$E_h(t) \leq 8\sum_{j=1}^{N}\left[e^{-\varepsilon\nu_j^2 t}\frac{\nu_j^2}{|\lambda_j^+ - \lambda_j^-|^2}\left(|a_{jh}^1|^2 + \nu_j^2|a_{jh}^0|^2\right)\right]. \tag{3.13}$$

It follows that

$$E_h(t) \leq 8\max_{1\leq j\leq N}\left\{\frac{\nu_j^2}{|\lambda_j^+ - \lambda_j^-|^2}\right\}e^{-\omega h t}\sum_{j=1}^{N}\left[|a_{jh}^1|^2 + \nu_j^2|a_{jh}^0|^2\right] \leq \frac{1}{4h^2}e^{-\omega h t}E_h(0),$$

with

$$0 < \omega \leq 4 \leq \frac{4}{h^2}\sin^2\frac{\pi h}{2} = \frac{1}{h}\min_{1\leq j\leq N}\{\varepsilon\nu_j^2\}.$$

The proof is finished. ∎

Remark 5. Note that the decay rate of the energy is not uniform when h tends to zero. Nevertheless, from (3.13), it follows that

$$E_h(t) \leq \sum_{j=1}^{N} e^{-h\nu_j^2 t} \frac{2}{\cos\left(\frac{j\pi h}{2}\right)} \left(|a_{jh}^1|^2 + \nu_j^2 |a_{jh}^0|^2\right). \tag{3.14}$$

Since $h\nu_j^2$ increases with j, it follows that the high frequencies are sensibly more dissipated that the lower ones. This is precisely the mechanism we shall take advantage of in the control problem.

4. The problem of moments. We return now to the controllability problem for (2.6). Let us recall that (2.6) is exactly controllable in time T if, for any $(U_h^0, U_h^1) \in \mathbb{C}^{2N}$, there exists a control function $v_h \in H^1(0,T)$ such that the corresponding solution (U_h, U_h') of (2.6) satisfies $U_h(T) = U_h'(T) = 0$.

First, we deduce a variational characterization of the controllability property. Let (ϕ_h, ϕ_h') be the solution of the following backward homogeneous system:

$$\begin{cases} \phi_h''(t) + A_h\phi_h(t) - \varepsilon A_h \phi_h'(t) = 0 & \text{for } t \in (0,T), \\ \phi_h(T) = \phi_h^0, \ \ \phi_h'(T) = \phi_h^1, \end{cases} \tag{4.1}$$

where $(\phi^0, \phi^1) \in \mathbb{C}^{2N}$ are given.

Multiplying (4.1) by the solution U_h of (2.6) and integrating in time, we obtain

$$\begin{aligned} 0 &= \int_0^T (U_h, \phi_h'' + A_h\phi_h - \varepsilon A_h\phi_h')\, dt \\ &= [-(U_h' + \varepsilon A_h U_h, \phi_h) + (U_h, \phi_h')]\big|_0^T + \int_0^T (U_h'' + A_h U_h + \varepsilon A_h U_h', \phi_h)\, dt \\ &= [-(U_h' + \varepsilon A_h U_h, \phi_h) + (U_h, \phi_h')]\big|_0^T + \int_0^T (F_h, \phi_h) dt. \end{aligned}$$

Hence,

$$\langle (U_h(t), U_h'(t)), (\phi_h(t), \phi_h'(t))\rangle_D \Big|_0^T = \int_0^T \frac{1}{h} \left(v_h(t) + \varepsilon v_h'(t)\right) \overline{\phi_N(t)} dt. \tag{4.2}$$

For any $f_h \in L^2(0,T)$, let $v_h \in H^1(0,T)$ be a solution of

$$\varepsilon v_h' + v_h = f_h, \quad t \in (0,T). \tag{4.3}$$

From (4.2) we obtain the following variational characterization of the controllability property.

LEMMA 4.1. *Given $T > 0$, system* (2.4) *is exactly controllable in time T if and only if, for any $(U_h^0, U_h^1) \in \mathbb{C}^{2N}$, there exists $f_h \in L^2(0,T)$ such that, for any $(\phi_h^0, \phi_h^1) \in \mathbb{C}^{2N}$,*

$$\int_0^T f_h(t) \frac{\overline{\phi_N(t)}}{h} dt = -\left\langle \left(U_h^0, U_h^1\right), (\phi_h(0), \phi_h'(0))\right\rangle_D, \tag{4.4}$$

(ϕ_h, ϕ_h') being the corresponding solution of (4.1). *A control v_h for* (2.4) *is any solution of* (4.3).

In order to write (4.4) as an equivalent problem of moments, we use the Fourier expansion of the solutions of (4.1) as we have done for (2.6). Let us decompose the initial datum (ϕ^0, ϕ^1) of (4.1) as

$$\phi_h^0 = \sum_{j=1}^{N} b_{jh}^0 \varphi^j \text{ and } \phi_h^1 = \sum_{j=1}^{N} b_{jh}^1 \varphi^j. \tag{4.5}$$

The corresponding solution ϕ_h of (4.1) has the form

$$\phi_h(t) = \sum_{j=1}^{N} b_{jh}(t)\varphi^j, \tag{4.6}$$

where the coefficients $b_{jh}(t)$ are given by

$$b_{jh}(t) = \frac{b_{jh}^1 - \mu_j^- b_{jh}^0}{\mu_j^+ - \mu_j^-} e^{\mu_j^+(t-T)} + \frac{-b_{jh}^1 + \mu_j^+ b_{jh}^0}{\mu_j^+ - \mu_j^-} e^{\mu_j^-(t-T)}, \tag{4.7}$$

and $\mu_j^\pm = \frac{1}{2}\left(\varepsilon\nu_j^2 \pm \sqrt{\varepsilon^2\nu_j^4 - 4\nu_j^2}\right)$ are the roots of the characteristic equation

$$\mu^2 - \varepsilon\nu_j^2\mu + \nu_j^2 = 0, \quad 1 \le j \le N. \tag{4.8}$$

In the sequel we shall write

$$\mu_n = \frac{1}{2}\left(\varepsilon\nu_n^2 + \operatorname{sgn}(n)\sqrt{\varepsilon^2\nu_n^4 - 4\nu_n^2}\right) = \begin{cases} \mu_n^+ & \text{if } n > 0, \\ \mu_{-n}^- & \text{if } n < 0, \end{cases}$$

and these are the eigenvalues of the adjoint problem (4.1). We have the following new characterization of the controllability property in terms of a problem of moments.

THEOREM 4.2. *Let $T > 0$. System* (2.6) *is exactly controllable in time T if and only if, for any $(U_h^0, U_h^1) \in \mathbb{C}^{2N}$ of form* (3.7), *there exists $f_h \in L^2(0,T)$ such that*

$$\int_0^T f_h(t)e^{\mu_n t}dt = \frac{(-1)^n h}{\sqrt{2}\sin(|n|\pi h)}\left(\frac{\nu_n^2}{\mu_n}a_{|n|h}^0 + a_{|n|h}^1\right), \quad 1 \le |n| \le N. \tag{4.9}$$

Proof. The proof is a direct consequence of Lemma 4.1. Indeed, it is sufficient to verify (4.4) for $(\phi_h^0, \phi_h^1) = (\varphi^{|n|}, \mu_n\varphi^{|n|})$, $1 \le |n| \le N$. By taking into account (4.6)–(4.7), we deduce that in this case $\phi = e^{(t-T)\mu_n}\varphi^{|n|}$ and

$$\phi_N(t) = (-1)^{n+1}\sqrt{2}\sin(|n|\pi h)\, e^{(t-T)\mu_n}.$$

On the other hand,

$$\begin{aligned}
\left\langle \left(U_h^0, U_h^1\right), \left(\phi_h(0), \phi_h'(0)\right)\right\rangle_D &= \left\langle \left(U_h^0, U_h^1\right), e^{-\mu_n T}\left(\varphi^{|n|}, \mu_n\varphi^{|n|}\right)\right\rangle_D \\
&= e^{-\overline{\mu}_n T}\Bigg(-\overline{\mu}_n \sum_{1\le m\le N} a_{mh}^0 \left(\varphi^m, \varphi^{|n|}\right) \\
&\qquad + \sum_{1\le m\le N} \left(a_{mh}^1 + \varepsilon\nu_m^2 a_{mh}^0\right)\left(\varphi^m, \varphi^{|n|}\right)\Bigg) \\
&= e^{-\overline{\mu}_n T}\left(a_{|n|h}^0\left(-\overline{\mu}_n + \varepsilon\nu_n^2\right) + a_{|n|h}^1\right) \\
&= e^{-\overline{\mu}_n T}\left(\frac{\nu_n^2}{\overline{\mu}_n}a_{|n|h}^0 + a_{|n|h}^1\right).
\end{aligned}$$

From Lemma 4.1 it follows immediately that (4.9) holds. ∎

Remark 6. According to Theorem 4.2 a control v_h for (2.6) is obtained by solving the following problem of moments: find $g_h \in L^2(-\frac{T}{2}, \frac{T}{2})$ such that

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} g_h(t) e^{\mu_n t} dt = \beta_{nh} e^{-\mu_n \frac{T}{2}}, \quad 1 \le |n| \le N, \tag{4.10}$$

where $g_h(s - \frac{T}{2}) = f_h(s)$ and $\beta_{nh} = \frac{(-1)^{n+1} h}{\sqrt{2}\sin(|n|\pi h)}(\frac{\nu_n^2}{\mu_n} a^0_{|n|h} + a^1_{|n|h})$.

5. Biorthogonal sequences. Let us consider the sequence $(\mu_n)_{\substack{|n|\le N \\ n\neq 0}}$ of the eigenvalues of the matrix operator

$$\mathcal{A}^* = \begin{pmatrix} 0 & -I \\ A_h & -\varepsilon A_h \end{pmatrix}$$

corresponding to the adjoint problem (4.1). Recall that we have considered $\varepsilon = h$ and consequently

$$\mu_n = i\frac{2}{h} \sin\left(\frac{n\pi h}{2}\right)\left(\cos\left(\frac{n\pi h}{2}\right) - i \sin\left(\frac{n\pi h}{2}\right)\right). \tag{5.1}$$

To solve the problem of moments (4.10), we construct an explicit biorthogonal sequence $(\Theta_m)_{\substack{|m|\le N \\ m\neq 0}}$ to the family of complex exponentials $(e^{\mu_n t})_{\substack{|n|\le N \\ n\neq 0}}$ in $L^2(-\frac{T}{2}, \frac{T}{2})$ and we estimate the norm of the elements of this biorthogonal sequence.

We recall that $(\Theta_m)_{\substack{|m|\le N \\ m\neq 0}}$ is a biorthogonal sequence to $(e^{\mu_n t})_{\substack{|n|\le N \\ n\neq 0}}$ in $L^2(-\frac{T}{2}, \frac{T}{2})$ if (see [1] and [31])

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \Theta_m(t) e^{\mu_n t} dt = \delta_{mn} \ \forall m, n = \pm 1, \pm 2, \ldots, \pm N. \tag{5.2}$$

Remark 7. If $(\Theta_m)_{1\le|m|\le N}$ is a biorthogonal sequence in $L^2(-\frac{T}{2}, \frac{T}{2})$ to the family $(e^{\mu_n t})_{1\le|n|\le N}$, then

$$g_h(t) = \sum_{\substack{|n|\le N \\ n\neq 0}} \beta_{nh} e^{-\mu_n \frac{T}{2}} \Theta_n \tag{5.3}$$

is a solution of the problem of moments (4.10). Note that in (5.3) we have a finite sum, but the number of terms tends to infinity as h goes to zero. We also have

$$||g_h||_{L^2} \le \sum_{\substack{|n|\le N \\ n\neq 0}} |\beta_{nh}| e^{-\Re(\mu_n)\frac{T}{2}} ||\Theta_n||_{L^2}, \tag{5.4}$$

and an estimate for the norm of g_h is obtained from the estimate of the norm of the biorthogonal sequence. From (5.4) we see that the L^2-norm of g_h may be uniformly bounded in h even if $||\Theta_n||_{L^2}$ is large, provided that the negative exponentials $e^{-\Re(\mu_n)\frac{T}{2}}$ (due to the dissipation term we have introduced) are small enough. Our aim is to show that, for T sufficiently large, $||\Theta_n||_{L^2} e^{-\Re(\mu_n)\frac{T}{2}}$ is sufficiently small to ensure the uniform boundedness of the sum.

Since $(e^{\mu_n t})_{\substack{|n|\le N \\ n\neq 0}}$ is a finite family of exponential functions, it follows immediately that it has infinitely many biorthogonal families in $L^2(-\frac{T}{2}, \frac{T}{2})$. However, we are interested not only in the existence but also on the dependence of these biorthogonal families on N. Our aim is to construct an explicit biorthogonal and to evaluate the norm of its elements. We shall do that in several steps:

1. We construct an entire function of exponential type, the product Υ_m, with the property that $\Upsilon_m(-i\mu_n) = \delta_{mn}$ (Lemma 5.1).
2. We evaluate Υ_m on the real axis (Lemma 5.2).
3. We construct an entire function, the multiplier G, of exponential type such that $\Upsilon_m G$ is bounded on the real axis (Lemma 5.3).
4. The Fourier transform of the entire function $\Upsilon_m(z)G(z)(\frac{\sin(z+i\mu_m)}{z+i\mu_m})^2$ gives the element Θ_m of a biorthogonal sequence. Moreover, from the Plancherel theorem, an estimate for the norm of Θ_m is obtained too (Theorem 5.4).

This method was used in several controllability problems for the heat [6] or the wave equations [5]. However, here we deal with two parameters h and m and complex exponents μ_n. The most difficult part consists in the estimate at step 2. This estimate will have different forms, according to the relation existing between x and h.

5.1. The product. Let us first define, for each m such that $1 \leq | m | \leq N$, the following product function:

$$\Upsilon_m(z) = \prod_{\substack{1\leq|n|\leq N\\ n\neq m}} \left(\frac{1+\frac{z}{i\mu_n}}{1-\frac{\mu_m}{\mu_n}} \right) \prod_{1\leq|n|\leq N} \exp\left(-\frac{z+i\mu_m}{i\mu_n}\right). \tag{5.5}$$

Lemma 5.1. *The function Υ_m has the following properties:*
(i) $\Upsilon_m(-i\mu_n) = \delta_{nm}, \quad 1 \leq |n| \leq N$;
(ii) *There exists a constant B_1 independent of h and m such that Υ_m is an entire function of exponential type at most B_1; i.e., there exists a constant $0 < A_m < 1$ such that*

$$|\Upsilon_m(z)| \leq A_m \exp(B_1|z|) \quad \forall z \in \mathbb{C}. \tag{5.6}$$

Proof. The first property is evident. Let us pass directly to show (5.6). Since $\mu_{-n} = \overline{\mu}_n$, we have that, for any $z \in \mathbb{C}$,

$$\prod_{1\leq|n|\leq N} \exp\left(\frac{z}{\mu_n}\right) = \exp\left(z \sum_{1\leq|n|\leq N} \frac{1}{\mu_n}\right) = \exp\left(z \sum_{1\leq n\leq N} \frac{2\Re(\mu_n)}{|\mu_n|^2}\right) = \exp\left(Nh\, z\right)$$

and therefore

$$\left| \exp\left(-\frac{z}{i\mu_m}\right) \prod_{1\leq|n|\leq N} \exp\left(-\frac{\mu_m}{\mu_n}\right) \right| \leq \exp\left(-Nh\, \Re\left(\mu_m\right)\right) \exp\left(\frac{|z|}{2}\right). \tag{5.7}$$

On the other hand, for any $z \neq -i\mu_n$,

$$\left| \prod_{\substack{1\leq|n|\leq N\\ n\neq m}} \left(1+\frac{z}{i\mu_n}\right) \exp\left(-\frac{z}{i\mu_n}\right) \right| = \exp\left(\sum_{\substack{1\leq|n|\leq N\\ n\neq m}} \ln\left| \left(1+\frac{z}{i\mu_n}\right) \exp\left(-\frac{z}{i\mu_n}\right) \right| \right)$$

$$= \exp\left(\sum_{\substack{1\leq|n|\leq[|z|]\\ n\neq m}} \ln\left|1+\frac{z}{i\mu_n}\right| - \sum_{\substack{1\leq|n|\leq[|z|]\\ n\neq m}} \Re\left(\frac{z}{i\mu_n}\right) + \sum_{\substack{[|z|]+1\leq|n|\leq N\\ n\neq m}} \ln\left| \left(1+\frac{z}{i\mu_n}\right) \exp\left(-\frac{z}{i\mu_n}\right) \right| \right).$$

Since $|\mu_n| = \frac{2}{h}\sin(\frac{|n|\pi h}{2}) \geq 2|n|$ and $|\frac{z}{\mu_n}| < \frac{1}{2}$ if $[|z|]+1 \leq |\, n \,|$, we deduce that

$$\sum_{\substack{1\leq|n|\leq[|z|]\\ n\neq m}} \ln\left|1+\frac{z}{i\mu_n}\right| \leq 2\sum_{1\leq n\leq[|z|]} \ln\left(1+\frac{|z|}{2n}\right) \leq 2\int_0^{|z|} \ln\left(1+\frac{|z|}{2s}\right) ds$$

$$= |z|\,(3\ln 3 - 2\ln 2) < 2|z|,$$

$$\sum_{\substack{[|z|]+1\leq|n|\leq N\\ n\neq m}} \ln\left|\left(1+\frac{z}{i\mu_n}\right)\exp\left(-\frac{z}{i\mu_n}\right)\right| \leq \sum_{\substack{[|z|]+1\leq|n|\leq N\\ n\neq m}} \left|\frac{z}{i\mu_n}\right|^2 \leq \sum_{[|z|]+1\leq|n|\leq N} \frac{|z|^2}{4n^2}$$

$$\leq \frac{|z|}{2}, \exp\left(-\sum_{\substack{1\leq|n|\leq[|z|]\\ n\neq m}} \Re\left(\frac{z}{i\mu_n}\right)\right)$$

$$\leq \left|\exp\left(iz\sum_{1\leq|n|\leq[|z|]} \frac{2\Re(\mu_n)}{|\mu_n|^2}\right)\right| \exp\left(\frac{|z|}{|\mu_m|}\right)$$

$$\leq \exp\left(Nh|z| + \frac{|z|}{2}\right).$$

It follows that

$$\left|\prod_{\substack{1\leq|n|\leq N\\ n\neq m}} \left(1+\frac{z}{i\mu_n}\right)\,\exp\left(-\frac{z}{i\mu_n}\right)\right| \leq \exp\left(4|z|\right). \tag{5.8}$$

Finally, we evaluate

$$\prod_{\substack{1\leq|n|\leq N\\ n\neq m}} \left|\frac{\mu_n}{\mu_n-\mu_m}\right|^2 = \prod_{\substack{1\leq|n|\leq N\\ n\neq m}} \frac{\sin^2\left(\frac{n\pi h}{2}\right)}{\sin^2\left(\frac{(n-m)\pi h}{2}\right)}$$

$$= \frac{\prod_{k=N-|m|+1}^{N}\sin^2\left(\frac{k\pi h}{2}\right)}{\prod_{k=N+1}^{N+|m|}\sin^2\left(\frac{k\pi h}{2}\right)} = \frac{\prod_{k=1}^{|m|}\cos^2\left(\frac{k\pi h}{2}\right)}{\prod_{k=0}^{|m|-1}\cos^2\left(\frac{k\pi h}{2}\right)} = \cos^2\left(\frac{m\pi h}{2}\right).$$

Hence,

$$\prod_{\substack{1\leq|n|\leq N\\ n\neq m}} \left|\frac{\mu_n}{\mu_n-\mu_m}\right| = \cos\left(\frac{m\pi h}{2}\right). \tag{5.9}$$

From (5.7), (5.8), and (5.9) we obtain that

$$|\Upsilon_m(z)| \leq \cos\left(\frac{m\pi h}{2}\right)\exp\left(-Nh\,\Re\left(\mu_m\right)\right)\exp\left(\frac{9}{2}|z|\right),$$

and (5.6) follows if $B_1 > \frac{9}{2}$ and $A_m = \cos\left(\frac{m\pi h}{2}\right)\exp\left(-Nh\,\Re\left(\mu_m\right)\right)$. ∎

5.2. Estimate of the product on the real axis. A key point in the construction and evaluation of the biorthogonal sequence is the following result concerning the behavior of Υ_m on the real axis.

LEMMA 5.2. *The following estimate holds for the function Υ_m on the real axis:*

$$|\Upsilon_m(x)| \leq \begin{cases} C_1 \cos\left(\frac{m\pi h}{2}\right)\left|\frac{x-i\mu_m}{i\mu_m}\right| \exp\left(\omega_1\sqrt{\frac{|x|}{h}}\right), & |x| \geq \frac{1}{h}, \\ C_1 \cos\left(\frac{m\pi h}{2}\right)\left|\frac{x-i\mu_m}{i\mu_m}\right| \exp\left(\omega_1 h|x|^2\right), & |x| \leq \frac{1}{h}, \end{cases} \tag{5.10}$$

where ω_1 and C_1 are two positive constants independent of h.

Proof. The proof is technical, and it will be given in the appendix. □

Remark 8. An estimate of the product function on the real axis is always important in these type of problems. For instance, in [6], where the heat equation is considered and consequently only real exponents are used, the product has a behavior like $\exp(\omega\sqrt{x})$. On the other hand, if purely imaginary exponents are considered, as in the wave equation, it is easy to see that the corresponding product is bounded. Estimate (5.10) combines these two behaviors. Our product is like the wave equation if hx is small and like the heat equation if hx is sufficiently large.

5.3. The multiplier. The aim of this section is to construct an entire function with a sufficient decay on the real axis to compensate for the growth of the product Υ_m evaluated in Lemma 5.2. We adapt an idea of Ingham [14], used several times in the study of completeness problems for exponential functions.

LEMMA 5.3. *Let $\varepsilon > 0$ and $\varphi : \mathbb{R} \to \mathbb{R}$ be the function defined by*

$$\varphi(x) = \begin{cases} \varepsilon x^2, & |x| \leq \frac{1}{\varepsilon}, \\ \sqrt{\frac{|x|}{\varepsilon}}, & |x| > \frac{1}{\varepsilon}. \end{cases} \tag{5.11}$$

There exists an entire function G_ε of exponential type such that

$$|G_\varepsilon(x)| \leq C_2 \exp\left(-\varphi(x)\right) \quad \forall x \in \mathbb{R}, \tag{5.12}$$

$$|G_\varepsilon(-i\mu_m)| \geq \exp\left(-\omega_2 \Re(\mu_m)\right), \quad 1 \leq |m| \leq N, \tag{5.13}$$

where C_2 and ω_2 are two positive constants, independent of N and ε.

Proof. Let $(\rho_n)_{n\geq 1}$ be the nonincreasing sequence defined by

$$\rho_n = \begin{cases} e\varepsilon, & n \leq \frac{1}{\varepsilon}, \\ e\sqrt{\frac{1}{\varepsilon n^3}}, & n > \frac{1}{\varepsilon}. \end{cases} \tag{5.14}$$

Note that $\rho_n = e\frac{\varphi(n)}{n^2}$ and

$$\begin{aligned} \sum_{n\geq 1} \rho_n &= e\sum_{n=1}^{\left[\frac{1}{\varepsilon}\right]} \varepsilon + e\sum_{n=\left[\frac{1}{\varepsilon}\right]+1}^{\infty} \sqrt{\frac{1}{\varepsilon n^3}} \\ &\leq e + \int_{\left[\frac{1}{\varepsilon}\right]}^{\infty} \sqrt{\frac{1}{\varepsilon s^3}} ds = e + \frac{2e}{\sqrt{\varepsilon}}\frac{1}{\sqrt{\left[\frac{1}{\varepsilon}\right]}} \\ &\leq e + \frac{2e}{\sqrt{1-\varepsilon}} := l < \infty. \end{aligned}$$

Now, we define the function

$$H(z) = \prod_{n\geq 1} \frac{\sin(\rho_n z)}{\rho_n z}.$$

Since

$$\left|\frac{\sin(\rho_n z)}{\rho_n z}\right| = \left|\sum_{k\geq 0}(-1)^k \frac{(\rho_n z)^{2k}}{(2k+1)!}\right| \leq \sum_{k\geq 0} \frac{|\rho_n z|^{2k}}{(2k)!} \leq \exp(\rho_n |z|),$$

we have that

$$|H(z)| = \prod_{n\geq 1}\left|\frac{\sin(\rho_n z)}{\rho_n z}\right| \leq \exp\left(|z|\sum_{n\geq 1}\rho_n\right)$$

and H is an entire function of exponential type less than l.

We pass to evaluate $H(x)$ by considering the following two cases:

- $|x| > \frac{1}{\varepsilon}$. We consider $\nu = [\sqrt{\frac{|x|}{\varepsilon}}] \geq [\frac{1}{\varepsilon}]$, and we have that

$$|H(x)| \leq \prod_{n=1}^{\nu} \frac{|\sin(\rho_n x)|}{|\rho_n x|} \leq \prod_{n=1}^{\nu} \frac{1}{\rho_n |x|} \leq \left(\frac{1}{\rho_\nu |x|}\right)^{\nu}$$
$$= \left(\frac{\sqrt{\varepsilon \nu^3}}{e|x|}\right)^{\nu} \leq e^{-\nu} \leq e \,\exp\left(-\sqrt{\frac{|x|}{\varepsilon}}\right).$$

- $|x| \leq \frac{1}{\varepsilon}$. We consider $\nu = [\frac{1}{\varepsilon}]$, and we have that

$$|H(x)| \leq \prod_{n=1}^{\nu} \frac{|\sin(\rho_n x)|}{|\rho_n x|} = \left(\frac{|\sin(e\varepsilon x)|}{|e\varepsilon x|}\right)^{\nu}.$$

Since $e\varepsilon|x| \leq e$ and $\sin(t) \leq t - \frac{\sin(e)}{6e}t^3$ for all $t \in [0, e]$, it follows that

$$H(x) \leq \left(1 - \frac{\sin(e)}{6e}(e\varepsilon|x|)^2\right)^{\nu} = \exp\left(\nu \,\ln\left(1 - \frac{\sin(e)}{6e}(e\varepsilon|x|)^2\right)\right)$$
$$\leq \exp\left(-\nu\frac{\sin(e)}{6e}(e\varepsilon|x|)^2\right) \leq \exp\left(\frac{e\sin(e)}{6}\right)\exp\left(-\frac{1}{\varepsilon}\frac{\sin(e)}{6e}(e\varepsilon|x|)^2\right)$$
$$\leq e\exp\left(-\frac{e\sin(e)}{6}\varepsilon|x|^2\right) \leq e\exp\left(-\frac{1}{6}\varepsilon|x|^2\right).$$

It follows that the function $G_\varepsilon(z) = (H(z))^6$ verifies (5.12), with $C_2 = e^6$.

We prove that (5.13) holds too. We have that

$$|H(-i\mu_m)| = \prod_{n=1}^{\infty}\left|\frac{\sin(i\,\rho_n\mu_m)}{i\,\rho_n\mu_m}\right| = \prod_{\rho_n|\mu_m|\leq 1}\left|\frac{\sin(i\,\rho_n\mu_m)}{i\,\rho_n\mu_m}\right| \prod_{\rho_n|\mu_m|>1}\left|\frac{\sin(i\,\rho_n\mu_m)}{i\,\rho_n\mu_m}\right|.$$

If $\rho_n|\mu_m| \leq 1$,

$$|\sin(i\,\rho_n\mu_m)| \geq \sin(\rho_n|\mu_m|) \geq \rho_n|\mu_m| - \frac{(\rho_n|\mu_m|)^3}{6}$$

and consequently

$$\prod_{\rho_n|\mu_m|\leq 1}\left|\frac{\sin(i\,\rho_n\mu_m)}{i\,\rho_n\mu_m}\right| = \exp\left(\sum_{\rho_n|\mu_m|\leq 1}\ln\left|\frac{\sin(i\,\rho_n\mu_m)}{i\,\rho_n\mu_m}\right|\right)$$
$$\geq \exp\left(\sum_{\rho_n|\mu_m|\leq 1}\ln\left(1-\frac{(\rho_n|\mu_m|)^2}{6}\right)\right)$$
$$\geq \exp\left(-\frac{|\mu_m|^2}{6}\sum_{n\geq 1}\rho_n^2\right).$$

Since

$$\sum_{n\geq 1}\rho_n^2 = \sum_{n\leq\left[\frac{1}{\varepsilon}\right]} e^2\varepsilon^2 + \sum_{n>\left[\frac{1}{\varepsilon}\right]} e^2\frac{1}{\varepsilon n^3} \leq e^2\varepsilon + \frac{e^2}{\varepsilon}\int_{\left[\frac{1}{\varepsilon}\right]}^{\infty}\frac{ds}{s^3} \leq 4e^2\varepsilon,$$

it follows that

$$\prod_{\rho_n|\mu_m|\leq 1}\left|\frac{\sin(i\,\rho_n\mu_m)}{i\,\rho_n\mu_m}\right| \geq \exp\left(-\frac{2e^2}{3}\Re(\mu_m)\right). \tag{5.15}$$

If $\rho_n|\mu_m|>1$, we have that $|\mu_m|>\frac{1}{\rho_n}\geq\frac{1}{e\varepsilon}$ and

$$\Re(\rho_n\mu_m) = \rho_n\Re(\mu_m) = \rho_n\varepsilon|\mu_m|^2 \geq \varepsilon|\mu_m| \geq \frac{1}{e}.$$

It follows that

$$\prod_{\rho_n|\mu_m|>1}\left|\frac{\sin(i\,\rho_n\mu_m)}{i\,\rho_n\mu_m}\right| = \prod_{\rho_n|\mu_m|>1}\left|\frac{e^{\rho_n\mu_m}-e^{-\rho_n\mu_m}}{2\rho_n\mu_m}\right|$$
$$\geq \prod_{\rho_n|\mu_m|>1}\frac{e^{\rho_n\Re(\mu_m)}-e^{-\rho_n\Re(\mu_m)}}{2\rho_n|\mu_m|} \geq \prod_{\rho_n|\mu_m|>1}\frac{2\rho_n\Re(\mu_m)}{2\rho_n|\mu_m|}$$
$$= \prod_{\rho_n|\mu_m|>1}\varepsilon|\mu_m| \geq \prod_{\rho_n|\mu_m|>1}\frac{1}{e}$$
$$= \exp\left(-\left|\left\{n\geq 1\,:\,\rho_n\geq\frac{1}{|\mu_m|}\right\}\right|\right).$$

Since $|\mu_m|>\frac{1}{e\varepsilon}$ we have that

$$\left|\left\{n\geq 1\,:\,\rho_n\geq\frac{1}{|\mu_m|}\right\}\right| = \left[\sqrt[3]{\frac{e^2}{\varepsilon}|\mu_m|^2}\right] \leq e^2\varepsilon|\mu_m|^2 = e^2\Re(\mu_m)$$

and therefore

$$\prod_{\rho_n|\mu_m|>1}\left|\frac{\sin(i\,\rho_n\mu_m)}{i\,\rho_n\mu_m}\right| \geq \exp\left(-e^2\Re(\mu_m)\right). \tag{5.16}$$

From (5.15) and (5.16) it follows that inequality (5.13) holds with $\omega_2 = 10e^2$, and the proof ends. ∎

5.4. Biorthogonal function estimates. We are now ready to prove the desired result on the biorthogonal sequence.

THEOREM 5.4. *For any $T > 0$ sufficiently large but independent of h, there exists a sequence $(\Theta_m)_{\substack{|m|\leq N\\ m\neq 0}}$, biorthogonal in $L^2(-\frac{T}{2}, \frac{T}{2})$ to the family $(e^{\mu_n t})_{\substack{|n|\leq N\\ n\neq 0}}$, such that*

$$||\Theta_m||_{L^2(-\frac{T}{2},\frac{T}{2})} \leq M \cos\left(\frac{m\pi h}{2}\right) \exp\left(\omega\, \Re(\mu_m)\right), \quad 1 \leq |m| \leq N, \tag{5.17}$$

where M and ω are positive constants, independent of m and N.

Remark 9. Theorem 5.4 provides a biorthogonal set for any $T > 0$. However, for estimate (5.17) we need a time T sufficiently large (but independent of the discretized problem). The value $T = 2(B_1 + 74\pi^4 e + 2)$, for any $B_1 > \frac{9}{2}$, is obtained in the proof. We may improve it by finer estimates, but we are still far from $T = 2$, which is probably the optimal value.

Proof. We define

$$\Xi_m(z) = \Upsilon_m(z) \left[\frac{G_h(z)}{G_h(-i\mu_m)}\right]^{\omega_1} \left(\frac{\sin(z + i\mu_m)}{z + i\mu_m}\right)^2, \tag{5.18}$$

where Υ_m is given by (5.5) and G_h is the function constructed in Lemma 5.3, with $\varepsilon = h$. We have that

- $\Xi_m(-i\mu_n) = \delta_{nm}, \quad 1 \leq |n|, \ |m| \leq N$;
- Ξ_m is an entire function of exponential type $B = B_1 + 6l\omega_1 + 2$, independent of N;
- by using the properties of G_h from Lemma 5.3 and the estimates of Υ_m from Lemma 5.2, we obtain that

$$\begin{aligned}
\int_{-\infty}^{\infty} |\Xi_m(x)|^2 dx &\leq \frac{C_1^2 C_2^{2\omega_1} \cos^2\left(\frac{m\pi h}{2}\right)}{|G(-i\mu_m)|^{2\omega_1}} \int_{-\infty}^{\infty} \left|\frac{x - i\mu_m}{i\mu_m}\right|^2 \left|\frac{\sin(x + i\mu_m)}{x + i\mu_m}\right|^4 dx \\
&\leq \frac{2C_1^2 C_2^{2\omega_1} \cos^2\left(\frac{m\pi h}{2}\right)}{|G(-i\mu_m)|^{2\omega_1}} \left(\frac{e^{2\Re(\mu_m)}}{|\mu_m|^2} \int_{-\infty}^{\infty} \left|\frac{\sin(t + i\Re(\mu_m))}{t + i\Re(\mu_m)}\right|^2 dt\right. \\
&\quad \left. + 4\int_{-\infty}^{\infty} \left|\frac{\sin(t + i\Re(\mu_m))}{t + i\Re(\mu_m)}\right|^4 dt\right) \\
&\leq \frac{8C_1^2 C_2^{2\omega_1} \cos^2\left(\frac{m\pi h}{2}\right)}{|G(-i\mu_m)|^{2\omega_1}} e^{2(1+\pi)\Re(\mu_m)} \int_{-\infty}^{\infty} \left|\frac{\sin(t)}{t}\right|^2 dt \\
&\leq 8\pi C_1^2 C_2^{2\omega_1} \cos^2\left(\frac{m\pi h}{2}\right) e^{(2+2\pi+2\omega_1\omega_2)\Re(\mu_m)}.
\end{aligned}$$

We introduce now the Fourier transform of Ξ_m

$$\Theta_m(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Xi_m(x) e^{-xzi} dx, \tag{5.19}$$

and we note that $\{\Theta_m\}_{\substack{|m|\leq N\\ m\neq 0}}$ is the biorthogonal sequence we are looking for.

Indeed, from the properties of Ξ_m, by using the Paley–Wiener theorem, it follows that $\Theta_m(t)$ has compact support in $[-B, B]$, it belongs to $L^2(-B, B)$, and

$$\int_{-B}^{B} \Theta_m(t) e^{\mu_n t} dt = \Xi_m(-i\mu_n) = \delta_{nm}, \quad 1 \leq |n| \leq N.$$

It follows that $(\Theta_m)_{\substack{|m|\leq N\\ m\neq 0}}$ is a biorthogonal sequence to $\{e^{\mu_n t}\}_{\substack{|n|\leq N\\ n\neq 0}}$ in $L^2(-B,B)$.

Moreover, from Plancherel's theorem we have

$$\sqrt{2\pi}\parallel \Theta_m \parallel_{L^2(-B,B)}=\parallel \Xi_m \parallel_{L^2(-\infty,\infty)}\leq C_1 C_2^{\omega_1}\sqrt{8\pi}\cos\left(\frac{m\pi h}{2}\right)e^{(1+\pi+\omega_1\omega_2)\Re(\mu_m)},$$

and the proof is finished. □

6. Convergence results. The aim of this section is to show that a control for the continuous system (1.1) may be obtained as a limit of controls of the corresponding semidiscrete problems (2.6). The control time T will be considered sufficiently large (but independent of the discretized problem) such that estimate (5.17) from Theorem 5.4 holds.

Let $(u^0,u^1)\in L^2(0,1)\times H^{-1}(0,1)$ be the initial datum of (1.1). We consider that (u^0,u^1) has a Fourier decomposition

$$(u^0,u^1)=\sum_{n\geq 1}\left(a_n^0,a_n^1\right)\sqrt{2}\sin(n\pi x). \tag{6.1}$$

Since $\sqrt{2}\sin(n\pi x)$ is orthonormal in $L^2(0,1)$, it follows that $(u^0,u^1)\in L^2(0,1)\times H^{-1}(0,1)$ if and only if

$$\sum_{n\geq 1}\left(|a_n^0|^2+\frac{1}{|n|^2\pi^2}|a_n^1|^2\right)<\infty. \tag{6.2}$$

Now, let $(U_h^0,U_h^1)_{h>0}$ be a sequence of discretizations of (u^0,u^1) given by (3.7). Assume that $(a_{nh}^0,a_{nh}^1)_{n\in\mathbb{Z}^*}$, the Fourier coefficients of the discrete initial data, verify

$$\left(a_{nh}^0\right)_n \rightharpoonup \left(a_n^0\right)_n, \quad \left(\frac{a_{nh}^1}{\lambda_n}\right)_n \rightharpoonup \left(\frac{a_n^1}{n\pi i}\right)_n \text{ when } h\to 0 \text{ in } \ell^1. \tag{6.3}$$

Remark 10. The usual discretization by points

$$\left(U_h^0,U_h^1\right)=\left(\left(u^0(jh)\right)_{1\leq j\leq N},\left(u^1(jh)\right)_{1\leq j\leq N}\right) \tag{6.4}$$

leads to a convergence property of the Fourier coefficients sequence that depends on the regularity of (u^0,u^1). Indeed, it is not difficult to prove the following:

(i) If u^0 and u^1 are piecewise continuous functions in $[0,1]$, then

$$\left(a_{nh}^0\right)_n \rightharpoonup \left(a_n^0\right)_n, \quad \left(\frac{a_{nh}^1}{\lambda_n}\right)_n \rightharpoonup \left(\frac{a_n^1}{n\pi i}\right)_n \text{ when } h\to 0 \text{ in } \ell^1. \tag{6.5}$$

(ii) If u^0 and u^1 are one-time derivable with a continuous derivative in $[0,1]$, then

$$\left(a_{nh}^0\right)_n \to \left(a_n^0\right)_n, \quad \left(\frac{a_{nh}^1}{\lambda_n}\right)_n \to \left(\frac{a_n^1}{n\pi i}\right)_n \text{ when } h\to 0 \text{ in } \ell^1. \tag{6.6}$$

We prove now the existence of a bounded sequence of controls for the semidiscrete problem.

THEOREM 6.1. *Let us suppose that the initial data of* (1.1) *are such that*

$$\sum_{n\geq 1}\left(|a_n^0|+\frac{1}{n\pi}|a_n^1|\right)<\infty. \tag{6.7}$$

There exists a control v_h of the semidiscrete problem (2.4), *with $\varepsilon = h$, such that the sequence $(v_h)_{h>0}$ is bounded in $L^2(0,T)$.*

Proof. Let $(\Theta_m)_{\substack{|m|\leq N\\ m\neq 0}}$ be the biorthogonal sequence in $L^2(-\frac{T}{2},\frac{T}{2})$ to $\{e^{i\lambda_n t}\}_{\substack{|n|\leq N\\ n\neq 0}}$ constructed in Theorem 5.4. From Remarks 6 and 7, we obtain that

$$f_h(t) = \sum_{1\leq |n|\leq N} \frac{(-1)^{n+1}h}{\sqrt{2}\sin(|n|\pi h)}\left(\frac{\nu_n^2}{\mu_n}a^0_{|n|h} + a^1_{|n|h}\right)e^{-\mu_n\frac{T}{2}}\Theta_n\left(t-\frac{T}{2}\right) \tag{6.8}$$

is a solution of (4.10). Moreover,

$$\|f_h\|_{L^2(0,T)} \leq \sum_{1\leq |n|\leq N} \frac{h\exp\left(-\Re(\mu_n)\frac{T}{2}\right)}{\sqrt{2}|\sin(n\pi h)|}\left(|\nu_n||a^0_{|n|h}| + |a^1_{|n|h}|\right)\|\Theta_n\|_{L^2\left(-\frac{T}{2},\frac{T}{2}\right)}. \tag{6.9}$$

From the estimates for the norm of Θ_n given by Theorem 5.4, it follows that for any $T > 2\omega$,

$$\|f_h\|_{L^2(0,T)} \leq M\sum_{1\leq |n|\leq N}\left(a^0_{|n|h} + \frac{1}{|\nu_n|}a^1_{|n|h}\right).$$

It follows that any sequence of controls $(v_h)_{h>0}$ given by (4.3) is uniformly bounded in $L^2(0,T)$, and the proof ends. ▯

Remark 11. Theorem 6.1 shows that the initial data which verify (6.7) can be uniformly controlled with the scheme (2.4). The method we have used does not allow us to prove the optimal result for the initial data in $L^2\times H^{-1}$ which verifies (6.2). This is a general limitation of the biorthogonal technique already mentioned in [6].

Since the sequence of controls $(v_h)_h$ given by Theorem 6.1 is bounded in $L^2(0,T)$, there exists a subsequence, denoted in the same way, and $v\in L^2(0,T)$ such that $v_h \rightharpoonup v$ in $L^2(0,T)$ when $h\to 0$. In the next theorem we show that v is a control for the corresponding continuous problem.

THEOREM 6.2. *If $v \in L^2(0,T)$ is a weak limit of the bounded sequence $(v_h)_h$ given by Theorem* 6.1, *then v is a control for the continuous problem* (1.1).

Proof. Like in the case of the semidiscrete problem, it is easy to prove that $v\in L^2(0,T)$ is a control for (1.1) if and only if the equality

$$\int_0^T v(t)\overline{\varphi}_x(t,1)dt = \langle u^1,\overline{\varphi}(0)\rangle_{H^{-1},H_0^1} - \int_0^1 u^0\overline{\varphi}'(0) \tag{6.10}$$

holds for any $(\varphi^0,\varphi^1)\in H_0^1(0,1)\times L^2(0,1)$ and for φ the solution of the adjoint equation

$$\begin{cases}\varphi'' - \varphi_{xx} = 0 & \text{for } x\in(0,1),\ \ t>0,\\ \varphi(t,0)=\varphi(t,1)=0 & \text{for } t>0,\\ \varphi(T,x)=\varphi^0(x),\ \ \varphi'(T,x)=\varphi^1(x) & \text{for } x\in(0,1).\end{cases} \tag{6.11}$$

Now, the Fourier decomposition allows us to show that v is a control for (1.1) if and only if

$$\int_0^T v(t)e^{-in\pi t}dt = \frac{(-1)^n}{\sqrt{2}}\left(-a^0_{|n|}i + \frac{a^1_{|n|}}{n\pi}\right) \quad \forall n\neq 0. \tag{6.12}$$

Note that this is the moment problem for the continuous system (1.1), similar to (4.9) from Theorem 4.2. Let v be a weak limit in $L^2(0,T)$ of the sequence $(v_h)_h$. It follows that v is a weak limit of the sequence $(f_h)_h$ too.

Note that, for each $n \in \mathbb{Z}^*$,

$$e^{\mu_n t} \to e^{-in\pi t} \text{ in } L^2(0,T)$$

and

$$\frac{h}{\sin(|n|\pi h)}\left(\frac{\nu_n^2}{\mu_n}a^0_{|n|h} + a^1_{|n|h}\right) \to a^0_n i + \frac{a^1_n}{n\pi}$$

when h tends to zero.

By passing to the limit in (4.9), it follows that v satisfies (6.12). Hence, the limit v is a control for the problem (1.1), and the proof finishes. ∎

Remark 12. The weak convergence of the controlled discrete solutions to the controlled continuous solution of (1.1) may be proved too. Moreover, if the discrete initial data converge stronger to the continuous ones, the sequence of discrete HUM controls converges strongly to the continuous HUM control. Both proofs are rather technical and very similar to Theorems 3.2 and 3.3 from [2], and we omit them.

7. Numerical results. In this section we present a numerical experiment based on the scheme with an added viscosity term. More precisely, we approximate the HUM control for (1.1) (the control of minimal L^2-norm) denoted by $\widehat{v}$ by using (2.4) as the discretization of (1.1).

The algorithm we have used to compute the approximate controls is inspired by the one proposed by Glowinski, Li, and Lions [10] (see also [8, 11]), and it is based on a conjugate gradient implementation of the HUM method. To this end, we use the approximations (U_h^0, U_h^1) of the initial data by taking the values of u^0 and u^1 at the nodes. Then, we minimize the following functional

$$J\left(\phi_h^0, \phi_h^1\right) = \frac{1}{2}\int_0^T \left(\frac{\phi_N(t) - \varepsilon\phi_N'(t)}{h}\right)^2 dt + \left\langle \left(U_h^0, U_h^1\right), \left(\phi_h(0), \phi_h'(0)\right)\right\rangle_D \tag{7.1}$$

over all $(\phi^0, \phi^1) \in \mathbb{C}^{2N}$, ϕ_N being the last component of ϕ, the solution of the adjoint system (4.1).

The minimizer $(\widehat{\phi}_h^0, \widehat{\phi}_h^1)$ of J provides the control $\widehat{v}_h$ of the discrete system (2.4) with minimal L^2-norm. Since in Theorem 6.1 we have proved the existence of a bounded sequence of discrete controls, it follows that $(\widehat{v}_h)_{h>0}$ is bounded too. Its (unique) weak limit is the HUM control of (1.1) denoted by $\widehat{v}$.

In the algorithm several wave equations have to be solved. To do that, we also need a time discretization. Except for these numerical experiments, our article does not deal with the fully discrete problem, and the convergence remains to be done. However, the uniform results we have obtained for the semidiscrete case suggest that this is a good scheme to be discretized in time. The full discrete problem is analyzed, for instance, in [19, 20].

Let Δt be the time step and $l = \frac{\Delta t}{h}$ be the Courant number. We recall that the classical central difference scheme for the constant coefficients 1-D wave equation, with Courant number l equal to one, provides the exact solution at the nodes. Therefore, in this very particular situation, the 1-D version of the conjugate gradient algorithm described in [8, 11] gives a very accurate approximation of the control for any $h > 0$. We use this special situation to compute "exact" controls that allow us to compare the results obtained with our method. However, this scheme fails to converge if $l \neq 1$. Note that, in view of the possible generalizations to other types of equations

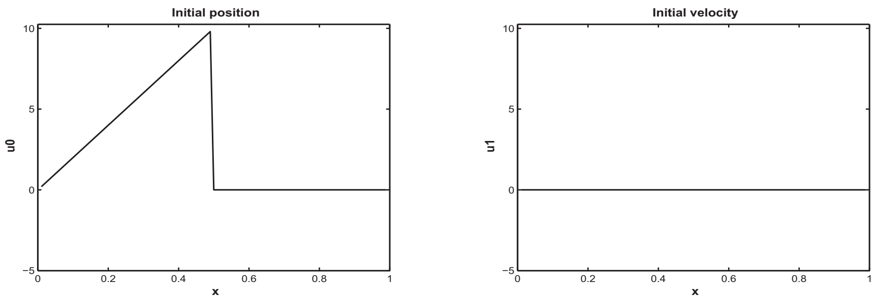

FIG. 1. *Initial data* (u^0, u^1) *to be controlled.*

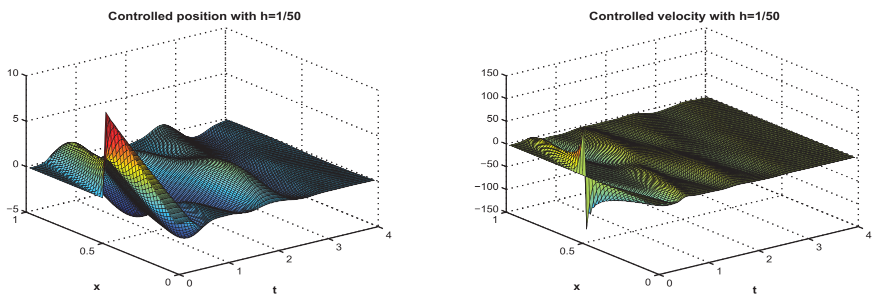

FIG. 2. *The controlled position and velocity with* $h = \frac{1}{50}$.

or multidimensional domains, it is of utmost importance to find robust algorithms which are not sensitive on the various discretization parameters. As the numerical results illustrate, the convergence of our method does not depend on the election of the Courant number $l = \Delta t/h$, which is related to the theoretical uniform result we have obtained.

Numerical example. In this example we consider a singular situation with discontinuous initial data (u^0, u^1). We take

$$u^0(x) = \begin{cases} 20x \text{ if } x \in (0, 1/2), \\ 0 \text{ if } x \in (1/2, 1), \end{cases} \qquad u^1(x) = 0.$$

In Figure 1 we present a picture of these data. Note that $u^0 \in L^2(0,1) \setminus H^1(0,1)$.

Since we are not looking for the optimal control time, we take $T = 4$. This is an arbitrary choice. The value of T for which we have proved the convergence is much larger, and it is given in Remark 9. It may be improved by finer estimates, but, in this case, our option was for simplicity.

The algorithm based on the finite difference scheme (2.1) gives, when $l := \Delta t/h < 1$, unsatisfactory approximations of the HUM control corresponding to (u^0, u^1). In the following experiments we have taken $l = 7/8$ and we have used (2.4). Figure 2 shows the controlled solution (position and velocity) when $h = 1/50$. Figure 3 illustrates the convergence of the discrete controls (dashed line) as h is decreased. The results are compared with those obtained by the finite difference scheme, with $\Delta t = h$ (solid line). We may conclude that, even in this singular situation, the viscosity method (2.4) provides satisfactory approximations of the HUM control.

The first line of results in Table 7.1 shows that the error in the L^2-norm decreases with h but at a slow rate when $\varepsilon = h$. It seems that this may be improved by choosing

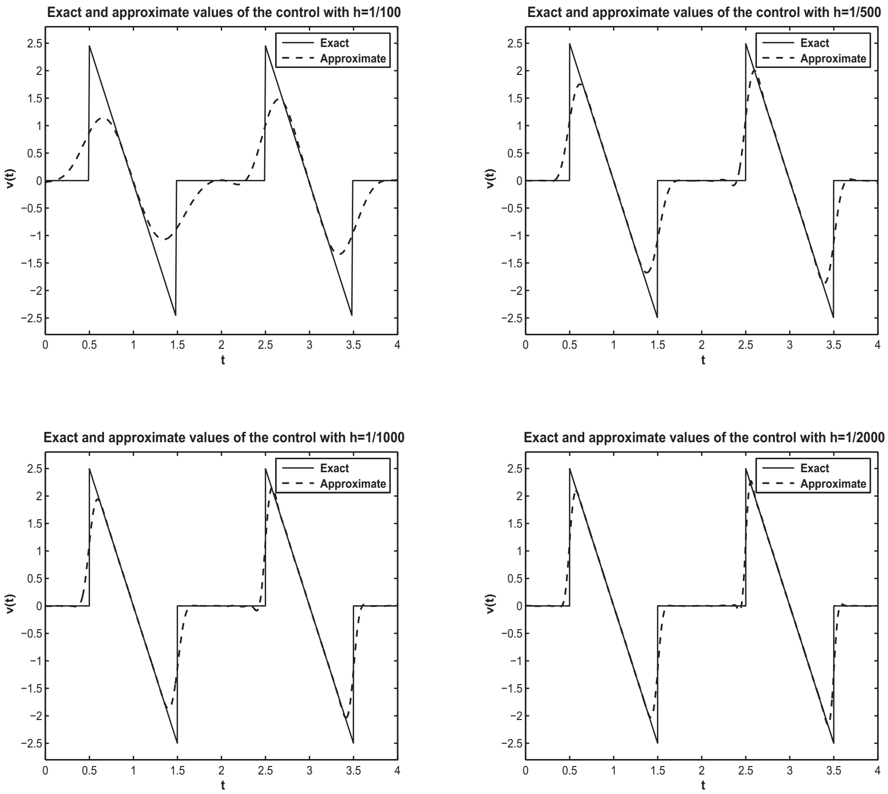

FIG. 3. *Approximations of the control with* $h = \frac{1}{100}, \frac{1}{500}, \frac{1}{1000}$, *and* $\frac{1}{2000}$.

TABLE 7.1
Numerical results for $\|v_h\|_{L^2}$ *obtained with* $\Delta t = 7/8h$ *and different values of the parameters* ε *and* h. *The exact result is* $||v||_{L^2} = 2.0106$.

h	1/100	1/500	1/1000
$\|v_h\|_{L^2}$ with $\varepsilon = h$	1.4656	1.8013	1.8750
$\|v_h\|_{L^2}$ with $\varepsilon = h^{1.5}$	1.8495	1.9877	2.0101
$\|v_h\|_{L^2}$ with $\varepsilon = h^{1.7}$	1.9117	2.0100	2.0242
$\|v_h\|_{L^2}$ with $\varepsilon = h^{1.9}$	1.9540	2.0225	2.0316

$\varepsilon = h^\alpha$, with $\alpha \in (1, 2)$ in (2.4). Numerical simulations with different values of the parameter ε are presented in Figure 4 and Table 7.1. We note better convergence rates with larger values of α, with an optimum close to 1.7. However, the theoretical analysis of this problem should be based on the error estimates for the control, which, to our knowledge, have not been yet obtained, regardless of the convergent discrete scheme.

8. Appendix. Proof of Lemma 5.2.

We have that

$$\Upsilon_m(x) = \left(1 + \frac{x}{i\mu_{-m}}\right) \prod_{\substack{1\leq|n|\leq N \\ n\neq m}} \frac{\mu_n}{\mu_m - \mu_n} \prod_{\substack{1\leq|n|\leq N \\ n\neq|m|}} \left(1 + \frac{x}{i\mu_n}\right) \prod_{1\leq|n|\leq N} \exp\left(-\frac{x + i\mu_m}{i\mu_n}\right).$$

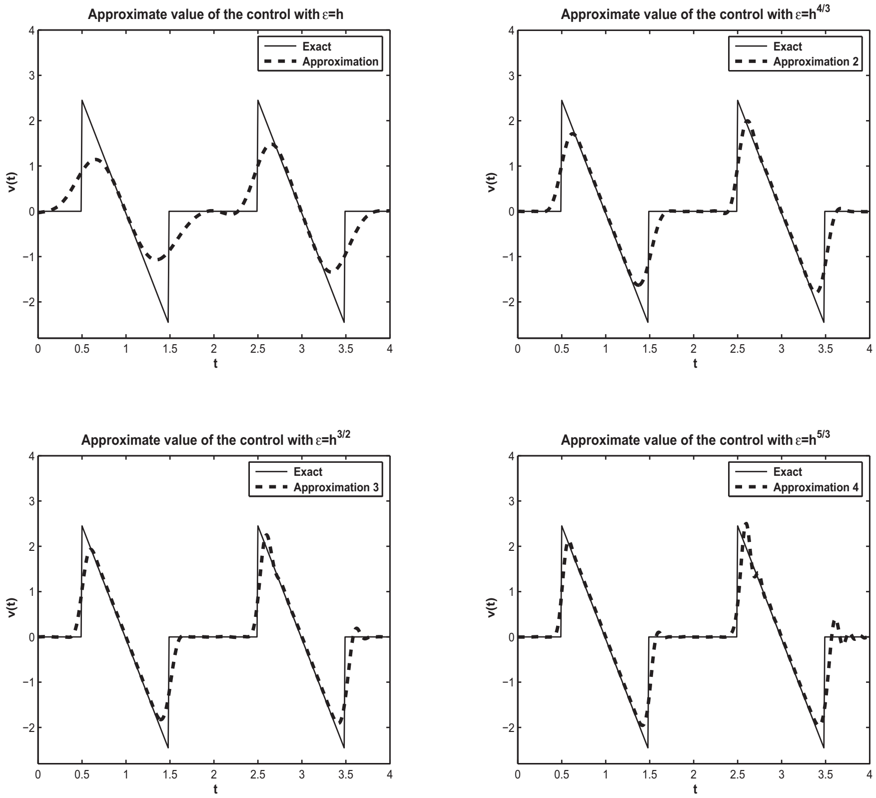

FIG. 4. *Approximations of the control with different values of the parameter* ε *when* $h = \frac{1}{100}$.

The first product in Υ_m is evaluated in (5.9), and we have that

$$\left| \prod_{\substack{1\leq |n|\leq N\\ n\neq m}} \frac{\mu_n}{\mu_m - \mu_n} \right| = \cos\left(\frac{m\pi h}{2}\right) \leq 1.$$

For the third product in Υ_m we have that

$$\left| \prod_{1\leq |n|\leq N} \exp\left(-\frac{x+i\mu_m}{i\mu_n}\right) \right| = \prod_{1\leq |n|\leq N} \left|\exp\left(\frac{ix}{\mu_n}\right)\right| \prod_{1\leq |n|\leq N} \left|\exp\left(-\frac{\mu_m}{\mu_n}\right)\right|$$
$$= \exp\left(-Nh\,\Re(\mu_m)\right) < 1.$$

We pass to evaluate the second product in Υ_m. We denote

$$P(x) = \prod_{\substack{1\leq |n|\leq N\\ n\neq |m|}} \left|1 + \frac{x}{i\mu_n}\right|^2 = \prod_{\substack{1\leq n\leq N\\ n\neq |m|}} \frac{\left(x^2 - \frac{4}{h^2}\sin^2\left(\frac{n\pi h}{2}\right)\right)^2 + \frac{16x^2}{h^2}\sin^4\left(\frac{n\pi h}{2}\right)}{\frac{16}{h^4}\sin^4\left(\frac{n\pi h}{2}\right)},$$

and we consider the cases $|x| \leq \pi$, $|x| > \frac{1}{2h}$, and $\pi < |x| \leq \frac{1}{2h}$.

Case 1 ($|x| \leq \pi$). In this case we have that

$$P(x) \leq \prod_{\substack{1\leq n\leq N\\ n\neq |m|}} \frac{\left(\pi^2+\frac{4}{h^2}\sin^2\left(\frac{n\pi h}{2}\right)\right)^2+\frac{16\pi^2}{h^2}\sin^4\left(\frac{n\pi h}{2}\right)}{\frac{16}{h^4}\sin^4\left(\frac{n\pi h}{2}\right)}$$

$$\leq \prod_{\substack{1\leq n\leq N\\ n\neq |m|}} \left(\left(1+\frac{\pi^2}{\frac{4}{h^2}\sin^2\left(\frac{n\pi h}{2}\right)}\right)^2+\pi^2h^2\right) \leq \prod_{\substack{1\leq n\leq N\\ n\neq |m|}} \left(\left(1+\frac{\pi^2}{4n^2}\right)^2+\pi^2h^2\right)$$

$$\leq \prod_{\substack{1\leq n\leq N\\ n\neq |m|}} \left(1+\pi h+\frac{\pi^2}{4n^2}\right)^2 \leq \exp\left(2\sum_{\substack{1\leq n\leq N\\ n\neq |m|}}\left(\pi h+\frac{\pi^2}{4n^2}\right)\right).$$

Hence,

$$P(x) \leq \exp\left(2\pi+\pi^2\right) \quad \forall |x|\leq \pi. \tag{8.1}$$

Case 2 ($|x| > \frac{1}{2h}$). We have that

$$P(x) \leq \prod_{\substack{1\leq n\leq N\\ n\neq |m|}} \frac{x^4+\frac{16}{h^4}\sin^4\left(\frac{n\pi h}{2}\right)+\frac{16x^2}{h^2}\sin^4\left(\frac{n\pi h}{2}\right)}{\frac{16}{h^4}\sin^4\left(\frac{n\pi h}{2}\right)} \leq \prod_{\substack{1\leq n\leq N\\ n\neq |m|}} \frac{x^4+\frac{16}{h^4}+\frac{16x^2}{h^2}}{\frac{16}{h^4}\sin^4\left(\frac{n\pi h}{2}\right)}$$

$$\leq \prod_{\substack{1\leq n\leq N\\ n\neq |m|}} \frac{x^4\left(1+256+64\right)}{\frac{16}{h^4}\sin^4\left(\frac{n\pi h}{2}\right)} \leq \prod_{\substack{1\leq n\leq N\\ n\neq |m|}} \frac{256x^4}{n^4} = \exp\left(4\sum_{\substack{1\leq n\leq N\\ n\neq |m|}} \ln\left(\frac{4|x|}{n}\right)\right)$$

$$\leq \exp\left(4\int_0^N \ln\left(\frac{4|x|}{s}\right)ds\right) = \exp\left(4N\ln\left(\frac{4|x|}{N}\right)+4N\right) \leq \exp\left(8N\sqrt{\frac{4|x|}{N}}\right),$$

where we have used that $\frac{4|x|}{N} > 2$ and $\ln(t)+1 \leq 2\sqrt{t}$ for any $t>1$. It follows that

$$P(x) \leq \exp\left(16\sqrt{\frac{|x|}{h}}\right), \quad \forall |x| > \frac{1}{2h}. \tag{8.2}$$

Case 3 ($\pi \leq |x| \leq \frac{1}{2h}$). The analysis of this case is much more difficult. First of all let us note that, in view of the particular form of $P(x)$, it is sufficient to consider $x>0$. Moreover, there exists a unique $x_h \in (1,N)$ such that

$$x = \frac{2}{h}\sin\left(\frac{x_h\pi h}{2}\right), \tag{8.3}$$

and let $p^* \in [1,N]\cap\mathbb{N}$ be the nearest integer to x_h. We have that

$$2x_h \leq x \leq \pi x_h, \tag{8.4}$$

$$p^*-\frac{1}{2} \leq x_h \leq p^*+\frac{1}{2}, \tag{8.5}$$

$$\frac{1}{2}x_h \leq p^* \leq x_h+\frac{1}{2} \leq \frac{x+1}{2} \leq \frac{1}{4h}+\frac{1}{2}. \tag{8.6}$$

We write $P(x) = P_1(x)P_2(x)P_3(x)$, where

$$P_1(x) = \frac{\left(x^2 - \frac{4}{h^2}\sin^2\left(\frac{p^*\pi h}{2}\right)\right)^2 + \frac{16x^2}{h^2}\sin^4\left(\frac{p^*\pi h}{2}\right)}{\frac{16}{h^4}\sin^4\left(\frac{p^*\pi h}{2}\right)},$$

$$P_2(x) = \prod_{\substack{1\le n\le N\\ n\neq p^*,|m|}} \frac{\left(x^2 - \frac{4}{h^2}\sin^2\left(\frac{n\pi h}{2}\right)\right)^2 + \frac{16x^2}{h^2}\sin^4\left(\frac{n\pi h}{2}\right)}{\left(x^2 - \frac{4}{h^2}\sin^2\left(\frac{n\pi h}{2}\right)\right)^2},$$

$$P_3(x) = \prod_{\substack{1\le n\le N\\ n\neq p^*,|m|}} \frac{\left(x^2 - \frac{4}{h^2}\sin^2\left(\frac{n\pi h}{2}\right)\right)^2}{\frac{16}{h^4}\sin^4\left(\frac{n\pi h}{2}\right)}.$$

We shall evaluate P_1, P_2, and P_3 successively. First of all note that

$$\left(x^2 - \frac{4}{h^2}\sin^2\left(\frac{n\pi h}{2}\right)\right)^2 = \frac{16}{h^4}\sin^2\left(\frac{(x_h - n)\pi h}{2}\right)\sin^2\left(\frac{(x_h + n)\pi h}{2}\right),$$

and since $(x_h + n)h \le \frac{3}{2}$ and $\sin^2\left(\frac{(x_h+n)\pi h}{2}\right) \ge \frac{2(x_h+n)^2h^2}{9}$ we obtain that

$$\frac{32(x_h + n)^2(x_h - n)^2}{9} \le \left(x^2 - \frac{4}{h^2}\sin^2\left(\frac{n\pi h}{2}\right)^2\right)^2 \le \pi^4(x_h + n)^2(x_h - n)^2. \tag{8.7}$$

We have that

$$P_1(x) \le \frac{\pi^4|x_h - p^*|^2|x_h + p^*|^2}{16(p^*)^4} + x^2h^2 \le \frac{9\pi^4}{16(p^*)^2}|x_h - p^*|^2 + x^2h^2$$

from where we deduce that

$$P_1(x) \le \frac{9\pi^6}{16x^2} + x^2h^2. \tag{8.8}$$

Now, we evaluate P_2:

$$P_2(x) \le \prod_{\substack{1\le n\le N\\ n\neq p^*,|m|}} \left(1 + \frac{\frac{16x^2}{h^2}\sin^4\left(\frac{n\pi h}{2}\right)}{\frac{32}{9}(x_h + n)^2(x_h - n)^2}\right) \le \prod_{\substack{1\le n\le N\\ n\neq p^*,|m|}} \left(1 + \frac{9\pi^4x^2h^2n^4}{32(x_h + n)^2(x_h - n)^2}\right)$$

$$\le \prod_{\substack{1\le n\le N\\ n\neq p^*,|m|}} \left(1 + \frac{9\pi^4x^2h^2n^2}{32(x_h - n)^2}\right) = \prod_{\substack{1\le n\le N\\ n\neq p^*,|m|}} \left(1 + \frac{r\,n^2}{(x_h - n)^2}\right),$$

where $r=\frac{9\pi^4x^2h^2}{32}$. We have that

$$\begin{aligned}
\prod_{\substack{1\le n\le N\\ n\ne p^*,|m|}}\left(1+\frac{r\,n^2}{(x_h-n)^2}\right) &\le \prod_{\substack{1\le n\le N\\ n\ne p^*}}\left(1+\frac{r\,n^2}{(x_h-n)^2}\right)\\
&= \prod_{\substack{1\le n\le N\\ n\notin\{p^*,p^*-1,p^*+1\}}}\left(1+\frac{r\,n^2}{(x_h-n)^2}\right)\left(1+\frac{r\,(p^*-1)^2}{(x_h-p^*+1)^2}\right)\\
&\quad\left(1+\frac{r\,(p^*+1)^2}{(x_h-p^*-1)^2}\right)\\
&\le \left(1+4r(p^*+1)^2\right)^2\prod_{\substack{1\le n\le N\\ n\notin\{p^*,p^*-1,p^*+1\}}}\left(1+\frac{r\,n^2}{(x_h-n)^2}\right)\\
&\le \exp\left(3\pi^2hx^2\right)\exp\left(\int_0^{N+1}\ln\left(1+\frac{r\,s^2}{(x_h-s)^2}\right)ds\right).
\end{aligned}$$

Now,

$$\begin{aligned}
\int_0^{N+1}\ln\left(1+\frac{r\,s^2}{(x_h-s)^2}\right)ds &= \left(\int_0^{x_h}+\int_{x_h}^{N+1}\right)\ln\left(1+\frac{r\,s^2}{(x_h-s)^2}\right)ds\\
&=(N+1-x_h)\ln\left(1+\frac{r\,(N+1)^2}{(N+1-x_h)^2}\right)\\
&\quad+\left(\int_0^{x_h}+\int_{x_h}^{N+1}\right)\frac{2rsx_h\,ds}{rs^2+(x_h-s)^2}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
(N+1-x_h)\ln\left(1+\frac{r\,(N+1)^2}{(N+1-x_h)^2}\right) &\le \frac{r\,(N+1)^2}{N+1-x_h}\le\frac{4r}{3}(N+1)\\
&=\frac{3\pi^4}{8}hx^2,\ \int_{\frac{x_h}{1+hx_h}}^{\frac{x_h}{1-hx_h}}\frac{2rsx_h\,ds}{rs^2+(s-x_h)^2}\\
&\le\int_{\frac{x_h}{1+hx_h}}^{\frac{x_h}{1-hx_h}}\frac{2rsx_h\,ds}{rs^2}=2x_h\ln\left(\frac{1+hx_h}{1-hx_h}\right)\\
&\le 4h(x_h)^2,\ \left(\int_0^{\frac{x_h}{1+hx_h}}+\int_{\frac{x_h}{1-hx_h}}^{N+1}\right)\frac{2rsx_h\,ds}{rs^2+(s-x_h)^2}\\
&\le\int_0^{\frac{x_h}{1+hx_h}}\frac{2r(x_h)^2\,ds}{(s-x_h)^2}\\
&\quad+\int_{\frac{x_h}{1-hx_h}}^{N+1}\left(\frac{2rx_h\,ds}{(s-x_h)}+\frac{2r(x_h)^2\,ds}{(s-x_h)^2}\right)\\
&=\frac{2r}{h}+4rx_h\ln\left(\frac{1-hx_h}{hx_h}\right)\\
&\quad+2r\frac{(1-hx_h)^2-h^2(x_h)^2}{h(1-hx_h)}\\
&\le\frac{2r}{h}+4\frac{r}{h}+2\frac{r}{h}\le\frac{9\pi^4}{4}hx^2.
\end{aligned}$$

It follows that

$$P_2(x) \leq \exp\left(\frac{3\pi^4}{8}hx^2 + \frac{4}{3}hx^2 + \frac{9\pi^4}{4}hx^2\right) \leq \exp\left(3\pi^4 hx^2\right). \tag{8.9}$$

Finally, we evaluate P_3. First, note that if $m \neq p^*$,

$$\left(\frac{\frac{4}{h^2}\sin^2\left(\frac{m\pi h}{2}\right)}{x^2 - \frac{4}{h^2}\sin^2\left(\frac{m\pi h}{2}\right)}\right)^2 \leq \frac{9m^4\pi^4}{32(x_h+m)^2(x_h-m)^2} \leq \frac{9\pi^4 m^2}{32(x_h-m)^2} \leq \frac{9\pi^4 x^2}{8}.$$

Therefore,

$$P_3(x) \leq \frac{9\pi^4 x^2}{8} \prod_{\substack{1\leq n\leq N\\ n\neq p^*}} \frac{\sin^2\left(\frac{(x_h-n)\pi h}{2}\right)\sin^2\left(\frac{(x_h+n)\pi h}{2}\right)}{\sin^4\left(\frac{n\pi h}{2}\right)}.$$

We have that $x_h = p^* + \alpha$, with $\alpha \in [-\frac{1}{2}, \frac{1}{2}]$, and we evaluate

$$\begin{aligned}
E(x) &= \prod_{\substack{1\leq n\leq N\\ n\neq p^*}} \frac{\sin^2\left(\frac{(p^*+\alpha-n)\pi h}{2}\right)\sin^2\left(\frac{(p^*+\alpha+n)\pi h}{2}\right)}{\sin^4\left(\frac{n\pi h}{2}\right)}\\
&= \frac{\prod_{\substack{1\leq k\leq 2p^*-1\\ k\neq p^*}}\sin^2\left(\frac{(k+\alpha)\pi h}{2}\right)\prod_{k=1}^{N-p^*}\sin^2\left(\frac{(k-\alpha)\pi h}{2}\right)\prod_{k=2p^*+1}^{N+p^*}\sin^2\left(\frac{(k+\alpha)\pi h}{2}\right)}{\prod_{\substack{1\leq k\leq N\\ k\neq p^*}}\sin^4\left(\frac{k\pi h}{2}\right)}\\
&= \frac{\sin^4\left(\frac{p^*\pi h}{2}\right)}{\sin^2\left(\frac{(2p^*+\alpha)\pi h}{2}\right)\sin^2\left(\frac{(p^*+\alpha)\pi h}{2}\right)}\\
&\quad\times\prod_{k=1}^{N}\frac{\sin^2\left(\frac{(k+\alpha)\pi h}{2}\right)\sin^2\left(\frac{(k-\alpha)\pi h}{2}\right)}{\sin^4\left(\frac{k\pi h}{2}\right)}\prod_{k=N-p^*+1}^{N}\frac{\sin^2\left(\frac{(k+p^*+\alpha)\pi h}{2}\right)}{\sin^2\left(\frac{(k-\alpha)\pi h}{2}\right)}.
\end{aligned}$$

By taking into account that $\sin(a)\sin(b) \leq \sin^2\left(\frac{a+b}{2}\right)$ and $2p^*+1 < \frac{1}{2h}$, we deduce that

$$E(x) \leq \frac{\sin^2\left(\frac{p^*\pi h}{2}\right)}{\sin^2\left(\frac{(p^*+\alpha)\pi h}{2}\right)}\prod_{k=N-p^*+1}^{N}\frac{\sin^2\left(\frac{(k+p^*+\alpha)\pi h}{2}\right)}{\sin^2\left(\frac{(k-\alpha)\pi h}{2}\right)}.$$

Since $0 \leq \frac{p^*+n}{2}\pi h \leq \pi$ and the function $h(x) = \frac{\sin x}{x}$ is decreasing on $[0,\pi]$, it follows that

$$\frac{\sin\left(\frac{(k+p^*+\alpha)\pi h}{2}\right)}{\sin\left(\frac{(k-\alpha)\pi h}{2}\right)} \leq \frac{k+p^*+\alpha}{k-\alpha}$$

and thus

$$
\begin{aligned}
E(x) &\le 2 \prod_{k=N-p^*+1}^{N} \left(\frac{k+p^*+\alpha}{k-\alpha}\right)^2 = 2\exp\left(2\sum_{k=N-p^*+1}^{N} \ln\left(1+\frac{p^*+2\alpha}{k-\alpha}\right)\right)\\
&\le 2\exp\left(2\int_{N-p^*}^{N} \ln\left(1+\frac{p^*+2\alpha}{s-\alpha}\right) ds\right) \le 2\exp\left(2(p^*+2\alpha)\int_{N-p^*}^{N} \frac{ds}{s-\alpha}\right)\\
&= 2\exp\left(2(p^*+2\alpha)\ln\left(\frac{N-\alpha}{N-p^*-\alpha}\right)\right) \le 2\exp\left(2(p^*+2\alpha)\frac{p^*}{N-p^*-\alpha}\right)\\
&\le 2\exp\left(4(p^*)^2\frac{1}{N-\frac{N+1}{4}-1}\right) \le 2\exp\left(16(p^*)^2\frac{1}{3N-5}\right).
\end{aligned}
$$

Hence,

$$
P_3(x) \le \frac{9\pi^4 x^2}{4}\exp\left(16hx^2\right). \tag{8.10}
$$

Finally from (8.8), (8.9), and (8.10) we obtain that

$$
P(x) \le \frac{9\pi^4 x^2}{4}\left(\frac{9\pi^6}{16x^2}+x^2h^2\right)\exp\left(3\pi^4hx^2+16hx^2\right) \le \frac{81\pi^{10}}{32}\exp\left(4\pi^4hx^2\right).
$$

The proof is complete, with $C_1 = \frac{81\pi^{10}}{32}$ and $\omega_1 = 4\pi^4$. ∎

Acknowledgments. The author is grateful to Enrique Zuazua for several discussions and suggestions related to this work and for encouraging him to study this problem. Also, the author thanks Carlos Castro for invaluable help concerning the program and numerical experiments.

REFERENCES

[1] S. A. Avdonin and S. A. Ivanov, *Families of Exponentials. The Method of Moments in Controllability Problems for Distributed Parameter Systems*, Cambridge University Press, New York, 1995.

[2] C. Castro and S. Micu, *Boundary controllability of a linear semi-discrete* 1*-D wave equation derived from a mixed finite element method*, Numer. Math., 102 (2006), pp. 413–462.

[3] J. M. Coron and S. Guerrero, *Singular optimal control: A linear* 1*-D parabolic-hyperbolic example*, Asymptot. Anal., 44 (2005), pp. 237–257.

[4] R. J. DiPerna, *Convergence of approximate solutions to conservation laws*, Arch. Ration. Mech. Anal., 82 (1983), pp. 27–70.

[5] H. O. Fattorini, *Estimates for sequences biorthogonal to certain complex exponentials and boundary control of the wave equation*, in New Trends in Systems Analysis, Lecture Notes in Control and Inform. Sci., Springer, Berlin, 1977, pp. 111–124.

[6] H. O. Fattorini and D. L. Russell, *Exact controllability theorems for linear parabolic equations in one space dimension*, Arch. Ration. Mech. Anal., 4 (1971), pp. 272–292.

[7] H. O. Fattorini and D. L. Russell, *Uniform bounds on biorthogonal functions for real exponentials with an application to the control theory of parabolic equations*, Q. J. Appl. Math., 32 (1974), pp. 45–69.

[8] R. Glowinski, *Ensuring well-posedness by analogy; Stokes problem and boundary control for the wave equation*, J. Comput. Phys., 103 (1992), pp. 189–221.

[9] R. Glowinski, W. Kinton, and M. F. Wheeler, *A mixed finite element formulation for the boundary controllability of the wave equation*, Internat. J. Numer. Methods Engrg., 27 (1989), pp. 623–636.

[10] R. Glowinski, C. H. Li, and J.-L. Lions, *A numerical approach to the exact boundary controllability of the wave equation (I). Dirichlet controls: Description of the numerical methods*, Japan J. Appl. Math., 7 (1990), pp. 1–76.

[11] R. Glowinski and J.-L. Lions, *Exact and approximate controllability for distributed parameter systems*, Acta Numer., (1996), pp. 159–333.

[12] S. Hansen, *Bounds on functions biorthogonal to sets of complex exponentials; control of damped elastic systems*, J. Math. Anal. Appl., 158 (1991), pp. 487–508.

[13] J. A. Infante and E. Zuazua, *Boundary observability for the space semi-discretization of the* 1-*D wave equation*, M2AN Math. Model. Numer. Anal., 33 (1999), pp. 407–438.

[14] A. E. Ingham, *A note on Fourier transform*, J. London Math. Soc., 9 (1934), pp. 29–32.

[15] E. Isaakson and H. B. Keller, *Analysis of Numerical Methods*, John Wiley, New York, 1996.

[16] J.-L. Lions, *Contrôlabilité exacte perturbations et stabilisation de systèmes distribués*, Tome 1, Masson, Paris, 1988.

[17] A. Majda and S. Osher, *Numerical viscosity and the entropy condition*, Comm. Pure Appl. Math., 32 (1979), pp. 797–838.

[18] S. Micu, *Uniform boundary controllability of a semi-discrete* 1-*D wave equation*, Numer. Math., 91 (2002), pp. 723–768.

[19] A. Münch, *A uniformly controllable and implicite scheme for the* 1-*D wave equation*, ESAIM M2AN Math. Model. Numer. Anal., 39 (2005), pp. 377–418.

[20] M. Negreanu and E. Zuazua, *Discrete Ingham inequalities and applications*, SIAM J. Numer. Anal., 44 (2006), pp. 412–448.

[21] M. Negreanu and E. Zuazua, *Uniform boundary controllability of a discrete* 1-*D wave equation*, Systems Control Lett., 48 (2003), pp. 261–280.

[22] K. Ramdani, T. Takahashi, and M. Tucsnak, *Internal stabilization of the plate equation in a square: The continuous and the semi-discretized problems*, J. Math. Pures Appl., 85 (2006), pp. 17–37.

[23] K. Ramdani, T. Takahashi, and M. Tucsnak, *Uniformly exponentially stable approximations for a class of second order evolution equations - Application to LQR problems*, ESAIM Control Optim. Calc. Var., 13 (2007), pp. 503–527.

[24] L. Rosier and P. Rouchon, *On the controllability of a wave equation with structural damping*, Int. J. Tomogr. Stat., 5 (2007), pp. 79–84.

[25] D. L. Russell, *Controllability and stabilizability theory for linear partial differential equations: Recent progress and open questions*, SIAM Rev., 20 (1978), pp. 639–739.

[26] L. R. Tcheugoué Tébou and E. Zuazua, *Uniform exponential long time decay for the space semi-discretization of a locally damped wave equation via an artificial numerical viscosity*, Numer. Math., 95 (2003), pp. 563–598.

[27] L. R. Tcheugoué Tébou and E. Zuazua, *Uniform boundary stabilization of the finite difference space discretization of the* 1 - *d wave equation*, Adv. Comput. Math., 26 (2007), pp. 337–365.

[28] X. Zhang and E. Zuazua, *Polynomial decay and control of a* 1-*d hyperbolic-parabolic coupled system*, J. Differential Equations, 204 (2004), pp. 380–438.

[29] E. Zuazua, *Boundary observability for the finite difference space semi-discretizations of the* 2-*D wave equation in the square*, J. Math. Pures Appl., 78 (1999), pp. 523–563.

[30] E. Zuazua, *Propagation, observation, and control of waves approximated by finite difference methods*, SIAM Rev., 47 (2005), pp. 197–243.

[31] R. M. Young, *An Introduction to Nonharmonic Fourier Series*, Academic Press, New York, 1980.