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ON THE CONTROLLABILITY OF A COUPLED SYSTEM OF TWO KORTEWEG–DE VRIES EQUATIONS

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This paper proves the local exact boundary controllability property of a nonlinear system of two coupled Korteweg–de Vries equations which models the interactions of weakly nonlinear gravity waves (see [10]). Following the method in [24], which combines the analysis of the linearized system and the Banach's fixed point theorem, the controllability problem is reduced to prove a nonstandard unique continuation property of the eigenfunctions of the corresponding differential operator.

Keywords: Korteweg–de Vries equation and system; boundary control; observation; local controllability.

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1. Introduction

In this work, we analyze the boundary controllability property of a nonlinear system of two coupled Korteweg–de Vries equations. More precisely, given $T > 0$, and functions u^0, v^0, u^1 and v^1, we study the existence of four control functions h_1, h_2, g_1 and g_2, such that the solution of the system

$$\begin{cases} u_t + uu_x + u_{xxx} + a_3 v_{xxx} + a_1 vv_x + a_2(uv)_x = 0, & \text{in } (0,T)\times(0,L) \\ b_1 v_t + rv_x + vv_x + b_2 a_3 u_{xxx} + v_{xxx} + b_2 a_2 uu_x & \\ \qquad + b_2 a_1 (uv)_x = 0, & \text{in } (0,T)\times(0,L) \\ u(0,x) = u^0(x), \quad v(0,x) = v^0(x), & \text{on } (0,L), \end{cases} \tag{1}$$

$$\begin{cases} u(t,0)=0, \quad u(t,L)=h_1(t), \quad u_x(t,L)=h_2(t), \quad \text{on } (0,T), \\ v(t,0)=0, \quad v(t,L)=g_1(t), \quad v_x(t,L)=g_2(t), \quad \text{on } (0,T), \end{cases} \tag{2}$$

satisfies

$$u(T,\cdot)=u^1, \quad v(T,\cdot)=v^1. \tag{3}$$

The spaces to which the initial and final data as well as the controls where these belong to will be given latter.

In (1), a_i, b_i and r are real constants such that $b_i>0$ for $i=1,2$ and $a_3^2 b_2 < 1$.

System (1) was derived by Gear and Grimshaw in [10] as a model to describe strong interactions of two long internal gravity waves in a stratified fluid, where the two waves are assumed to correspond to different modes of the linearized equations of motion. It has the structure of a pair of KdV equations with both linear and nonlinear coupling terms. This somewhat complicated system has been object of intensive research in recent years (see, for instance [1, 2, 15, 17–19]).

The controllability problem we address here has been intensively investigated in the context of the wave or heat equations but there are fewer results for the KdV type equation under the boundary condition as in (1). Rosier in [24] proved that the underlying scalar linear equation,

$$\begin{cases} u_t+u_{xxx}+u_x=0, & \text{in } (0,T)\times(0,L) \\ u(t,0)=u(t,L)=0, \quad u_x(t,L)=h(t), & \text{on } (0,T) \\ u(0,x)=u^0(x), & \text{on } (0,L) \end{cases} \tag{4}$$

is exactly controllable by means of a single boundary control $h\in L^2(0,T)$, except when L lies in a countable set of critical lengths, of the form

$$\Lambda=\left\{\frac{2\pi}{\sqrt{3}}\sqrt{k^2+kl+l^2}, k \text{ and } l \text{ are positive natural numbers}\right\}. \tag{5}$$

This was done by combining the Lions' HUM method and a nonstandard unique continuation principle for the eigenfunctions of the differential operator. The critical lengths in (5) are such that there are eigenfunctions of the linear scalar problem for which the observability inequality fails. By a linearization argument a local controllability result for the semilinear scalar equation,

$$\begin{cases} u_t+u_{xxx}+u_x+uu_x=0, & \text{in } (0,T)\times(0,L) \\ u(t,0)=u(t,L)=0, \quad u_x(t,L)=h(t), & \text{on } (0,T) \\ u(0,x)=u^0(x), & \text{on } (0,L), \end{cases} \tag{6}$$

was also proved.

Later on, Zhang in [28] showed that using three controls, acting on all the boundary conditions, controllability holds for all values of L. More recently, in [7] Crépeau and Coron proved that the presence of the nonlinear term uu_x in the Korteweg–de Vries equation (6) gives the controllability around the origin in the case $L=2k\pi$, $k\in\mathbb{N}^*$. Using the same approach their result was subsequently improved by Cerpa in [6] and Cerpa and Crépeau [8]. Finally, Guerrero and Glass [11] established a result of null controllability for KdV via the left Dirichlet boundary condition and of exact controllability via both Dirichlet boundary conditions. As a consequence, they obtain some local exact controllability results for (6).

Concerning the controllability properties for system (1), we prove that the corresponding linear system

$$\begin{cases} u_t + u_{xxx} + av_{xxx} = 0, & \text{in } (0,T)\times(0,L) \\ cv_t + rv_x + abu_{xxx} + v_{xxx} = 0, & \text{in } (0,T)\times(0,L) \\ u(0,x) = u^0, \quad v(0,x) = v^0, & \text{on } (0,L), \end{cases} \tag{7}$$

under boundary conditions (2) is exactly controllable in $L^2(0,L)$, with control functions $h_1, g_1 \in H_0^1(0,T)$ and $h_2, g_2 \in L^2(0,T)$. The arguments of the proof follow closely the ones developed in [24] for the analysis of the scalar KdV. Therefore, the exact controllability property is reduced to a unique continuation result for the eigenfunctions of the adjoint differential operator corresponding to (7).

The result for the linear system allows to prove the local controllability property of the nonlinear system (1) by means of a fixed point argument. Our main result reads as follows:

Theorem 1.1. *Let $L > 0$ and $T > 0$. Then there exists a constant $\delta > 0$ such that for any initial and final data $u^0, v^0, u^1, v^1 \in L^2(0,L)$ verifying*

$$\|(u^0,v^0)\|_{(L^2(0,L))^2} \le \delta, \quad \textit{and} \quad \|(u^1,v^1)\|_{(L^2(0,L))^2} \le \delta,$$

there exist four control functions $h_1, g_1 \in H_0^1(0,T)$ and $h_2, g_2 \in L^2(0,T)$, such that the solution

$$(u,v) \in C([0,T];(L^2(0,L))^2) \cap L^2(0,T;(H^1(0,L))^2) \cap H^1(0,T;(H^{-2}(0,L))^2)$$

of (1)–(2) *verifies* (3).

In [20], a linear problem similar to (7), but with periodic boundary conditions and $r = 0$, was analyzed. The main characteristic of that system was the symmetry in the unknowns u and v which allowed to uncouple the system and to reduce the problem to the controllability of two independent KdV equations. Remark that the uncoupling is not possible in (7) unless $r = 0$. Moreover, if we consider periodic boundary conditions for (7), we cannot obtain a control result for the nonlinear system, since this problem does not have a regularizing effect.

Let us also remark that, as for the scalar KdV equation, the controllability of (1) is very sensitive to changes of the boundary conditions and controls. For instance, the use of a smaller number of controls is likely to impose some restrictions on the length L of the interval, as it is the case for the scalar KdV equation. However, due to the complexity of the system, it would be almost impossible to characterized the set of the uncontrolled lengths.

The rest of the article is organized in the following way: In Sec. 2, we give some fundamental results on the existence, uniqueness and regularity of solutions of the corresponding linear systems. The controllability results are given in Sec. 3 for the linearized system and the fully nonlinear one in Sec. 4. Some comments are given at the end of the paper.

2. Preliminaries

2.1. *The linear homogeneous system*

In this section, we study the existence of solutions of the linear system corresponding to (1):

$$\begin{cases} u_t + u_{xxx} + av_{xxx} = 0, & \text{in } (0,T)\times(0,L) \\ cv_t + rv_x + bau_{xxx} + v_{xxx} = 0, & \text{in } (0,T)\times(0,L) \\ u(t,0) = u(t,L) = u_x(t,L) = 0, & \text{on } (0,T), \\ v(t,0) = v(t,L) = v_x(t,L) = 0, & \text{on } (0,T), \\ u(0,x) = u^0(x), \quad v(0,x) = v^0(x), & \text{on } (0,L). \end{cases} \tag{8}$$

In (8) a, b, r and c are positive constants with

$$1 - a^2 b > 0.$$

Let $X = (L^2(0,L))^2$ endowed with the inner product

$$\langle (u,v),(\varphi,\psi)\rangle = \frac{b}{c}\int_0^L u\varphi dx + \int_0^L v\psi dx, \tag{9}$$

and consider the operator

$$A : D(A) \subset X \to X$$

where

$$D(A) = \{(u,v) \in (H^3(0,L))^2 \,:\, u(0) = v(0) = u(L) = v(L) = u_x(L) = v_x(L) = 0\}$$

and

$$A(u,v) = \begin{pmatrix} -u_{xxx} - av_{xxx} \\ -\frac{r}{c}v_x - \frac{1}{c}v_{xxx} - \frac{ab}{c}u_{xxx} \end{pmatrix}, \quad \forall\,(u,v) \in D(A). \tag{10}$$

With the notation introduced above, system (8) can be now written as an abstract Cauchy problem in X:

$$\begin{cases} (u,v)_t = A(u,v) \\ (u,v)(0) = (u^0,v^0). \end{cases} \tag{11}$$

On the other hand, it is easy to see that the adjoint of the operator A is the operator A^* defined by

$$A^*(\varphi,\psi) = \begin{pmatrix} \varphi_{xxx} + a\psi_{xxx} \\ \frac{r}{c}\psi_x + \frac{1}{c}\psi_{xxx} + \frac{ab}{c}\varphi_{xxx} \end{pmatrix} \tag{12}$$

where

$$A^* : D(A^*) \subset X \to X$$

and

$$D(A^*) = \{(\varphi, \psi) \in (H^3(0,L))^2 : \varphi(0) = \psi(0) = \varphi(L) = \psi(L) = \varphi_x(0) = \psi_x(0) = 0\}.$$

We are interested in the following property of these two operators:

Proposition 2.1. *The operator A and its adjoint A^* are dissipative in X.*

Proof. Let $u, v \in D(A)$. By multiplying the first equation of (8) by u and by integrating by parts in $(0, L)$ we obtain that

$$\int_0^L (-u_{xxx} - av_{xxx})u = -\frac{1}{2}u_x^2(0) + \int_0^L av_{xx}u_x. \tag{13}$$

On the other hand, by multiplying the second equation of (8) by v and by integrating by parts in $(0, L)$, we obtain that

$$\int_0^L \left(-\frac{r}{c}v_x - \frac{1}{c}v_{xxx} - \frac{ab}{c}u_{xxx}\right) v = -\frac{1}{2c}v_x^2(0) + \frac{ab}{c}\int_0^L u_{xx}v_x. \tag{14}$$

Thus, from (13) and (14), we get

$$\begin{aligned}\langle A(u,v),(u,v)\rangle_{(L^2(0,L))^2} &= -\frac{b}{2c}u_x^2(0) + \frac{ab}{c}\int_0^L v_{xx}u_x - \frac{1}{2c}v_x^2(0) + \frac{ab}{c}\int_0^L u_{xx}v_x \\ &= -\frac{b}{2c}u_x^2(0) - \frac{1}{2c}v_x^2(0) + \frac{ab}{c}\int_0^L (u_xv_x)_x \\ &= -\frac{1}{2c}(bu_x^2(0) + v_x^2(0) + 2abu_x(0)v_x(0)) \\ &= -\frac{1}{2c}([\sqrt{b}u_x(0) + \sqrt{a^2b}v_x(0)]^2 + (1-a^2b)v_x^2(0)) \\ &\le 0. \end{aligned} \tag{15}$$

Hence, A is a dissipative operator in $(L^2(0,L))^2$. Analogously, we can deduce that

$$\begin{aligned}&\langle(\varphi,\psi), A^*(\varphi,\psi)\rangle_{(L^2(0,L))^2} \\ &\quad = -\frac{1}{2c}([\sqrt{b}\varphi_x(L) + \sqrt{a^2b}\psi_x(L)]^2 + (1-a^2b)\psi_x^2(L)) \le 0\end{aligned} \tag{16}$$

and therefore A^* is also dissipative in $(L^2(0,L))^2$. □

Since A and A^* are both dissipatives, A is a closed operator and the respective domains $D(A)$ and $D(A^*)$ are dense and compactly embedded in X we conclude that A generates a C^0 semigroup of contractions on $(L^2(0,L))^2$ (see [22]) which will be denoted by $(S(t))_{t\geq 0}$. Classical existence results then give us the global well-posedness for (8):

Theorem 2.1. *Let* $(u^0, v^0) \in L^2(0,L)$. *There exists a unique weak solution* $(u,v) = S(\cdot)(u^0,v^0)$ *of* (8) *such that*

$$(u,v) \in C([0,T];(L^2(0,L))^2) \cap H^1(0,T;(H^{-2}(0,L))^2). \tag{17}$$

Moreover, if $(u_0, v_0) \in D(A)$, *then* (8) *has a unique* (*classical*) *solution* (u,v) *such that*

$$(u,v) \in C([0,T];D(A)) \cap C^1(0,T;(L^2(0,L))^2). \tag{18}$$

Additional regularity results for the weak solutions on (8) are proven in the next theorem.

Theorem 2.2. *Let* $(u^0, v^0) \in (L^2(0,L))^2$ *and* $(u,v) = S(\cdot)(u^0,v^0)$ *the weak solution of* (8). *Then,* $(u,v) \in L^2(0,T;(H^1(0,L))^2)$ *and there exists a positive constant* c_0 *such that*

$$\|(u,v)\|_{L^2(0,T;(H^1(0,L))^2)} \leq c_0 \|(u^0,v^0)\|_{(L^2(0,L))^2}. \tag{19}$$

Moreover, there exist two positive constants c_1 *and* c_2 *such that*

$$\|(u_x(\cdot,0), v_x(\cdot,0))\|^2_{(L^2(0,T))^2} \leq c_1 \|(u^0,v^0)\|^2_{(L^2(0,L))^2} \tag{20}$$

and

$$\begin{aligned}\|(u^0,v^0)\|^2_{(L^2(0,L))^2} \leq{}& \frac{1}{T}\|(u,v)\|^2_{L^2(0,T;(L^2(0,L))^2)} \\ &+ c_2 \|(u_x(\cdot,0), v_x(\cdot,0))\|^2_{(L^2(0,T))^2}.\end{aligned} \tag{21}$$

Proof. A density argument allows us to consider only the case $(u^0,v^0) \in D(A)$.

Let us first remark that, if $q \in C^\infty((0,T)\times(0,L))$, then the following identities hold

$$\begin{aligned}0 &= \int_0^T\int_0^L qu(u_t + u_{xxx} + av_{xxx}) \\ &= -\int_0^T\int_0^L u(qu)_t + \int_0^L qu^2|_0^T - \int_0^T\int_0^L (u_{xx} + av_{xx})(qu)_x,\end{aligned} \tag{22}$$

and

$$\begin{aligned}0 &= \int_0^T\int_0^L qv(cv_t + rv_x + v_{xxx} + abu_{xxx}) \\ &= -\int_0^T\int_0^L cv(qv)_t + \int_0^L cqv^2|_0^T - \int_0^T\int_0^L (rv)(qv)_x \\ &\quad - \int_0^T\int_0^L (v_{xx} + abu_{xx})(qv)_x.\end{aligned} \tag{23}$$

Now, if we consider $q = x$ we have that

$$\begin{aligned} 0 &= \int_0^T \int_0^L xu(u_t + u_{xxx} + av_{xxx}) \\ &= \frac{1}{2}\int_0^L xu^2(t,x)|_0^T - \int_0^L \int_0^T (xu_x)(u+av)_{xx} + \int_0^T \int_0^L u_x(u+av)_x \end{aligned} \tag{24}$$

and

$$\begin{aligned} 0 &= \int_0^T \int_0^L xv(cv_t + rv_x + v_{xxx} + abu_{xxx}) \\ &= \frac{c}{2}\int_0^L xv^2(t,x)|_0^T - \frac{r}{2}\int_0^T \int_0^L v^2(t,x) - \int_0^L \int_0^T (xv_x)(abu+v)_{xx} \\ &\quad + \int_0^T \int_0^L v_x(abu+v)_x. \end{aligned} \tag{25}$$

By multiplying (24) by b and by adding to (25) we obtain

$$\begin{aligned} 0 &= \frac{1}{2}\int_0^L x(bu^2(t,x) + cv^2(t,x))|_0^T + \frac{r}{2}\int_0^T \int_0^L v^2 + \frac{3}{2}\int_0^T \int_0^L (bu_x^2 + v_x^2) \\ &\quad + 3ab\int_0^T \int_0^L u_x v_x - \frac{1}{2}\int_0^T x(bu_x^2 + 2abu_xv_x + v_x^2)|_0^L. \end{aligned} \tag{26}$$

Since $u_x(L) = v_x(L) = 0$, the following holds

$$\begin{aligned} \frac{3}{2}\int_0^T \int_0^L (bu_x^2 + 2abu_xv_x + v_x^2) &= \frac{r}{2}\int_0^T \int_0^L v^2 - \frac{1}{2}\int_0^L x(bu^2(t,x) + cv^2(t,x))|_0^T \\ &\le \frac{r}{2}\int_0^T \int_0^L v^2 + \frac{1}{2}\int_0^L x(b(u^0)^2(x) + c(v^0)^2(x)) \\ &\le \frac{r}{2}\int_0^T \int_0^L v^2 + \frac{L}{2}\int_0^L (b(u^0)^2(x) + c(v^0)^2(x)). \end{aligned} \tag{27}$$

Choosing $\varepsilon > 0$ such that $\sqrt{a^2b} < \varepsilon < 1$, we obtain that

$$bu_x^2 + 2abu_xv_x + v_x^2 \ge b(1-\varepsilon^2)u_x^2 + v_x^2\left(1 - \frac{a^2b}{\varepsilon^2}\right) \tag{28}$$

and, consequently,

$$\int_0^T \int_0^L b(1-\varepsilon^2)u_x^2 + v_x^2\left(1 - \frac{a^2b}{\varepsilon^2}\right) \le \frac{r}{3}\int_0^T \int_0^L v^2 + \frac{L}{3}\int_0^L (b(u^0)^2 + c(v^0)^2). \tag{29}$$

Thus, there exits a positive constant $c_0 > 0$ such that

$$\|(u_x, v_x)\|^2_{L^2((0,T)\times(0,L))^2} \le c_0(\|v\|^2_{L^2((0,T)\times(0,L))^2} + \|(u^0, v^0)\|^2_{(L^2(0,L))^2}). \tag{30}$$

Since the semigroup is continuous, we conclude that there exists a constant $c_0 > 0$ satisfying

$$\|(u_x, v_x)\|^2_{L^2((0,T)\times(0,L))^2} \leq c_0 \|(u^0, v^0)\|^2_{L^2(0,L)^2} \tag{31}$$

and (19) is proven.

By taking $q = 1$, we obtain from (22) and (23) that

$$0 = \frac{1}{2}\int_0^L u^2(T,x) - \frac{1}{2}\int_0^L u^2(0,x) - \int_0^T\int_0^L (u_{xx} + av_{xx})u_x, \tag{32}$$

and

$$0 = \frac{1}{2}\int_0^L cv^2(T,x) - \frac{1}{2}\int_0^L cv^2(0,x) - \int_0^T\int_0^L (v_{xx} + abu_{xx})v_x. \tag{33}$$

By multiplying (32) by b and by adding to (33), we have that

$$\begin{aligned} 0 &= \frac{b}{2}\int_0^L u^2(T,x) - \frac{b}{2}\int_0^L u^2(0,x) - \int_0^T\int_0^L (bu_{xx} + abv_{xx})u_x \\ &\quad + \frac{1}{2}\int_0^L cv^2(T,x) - \frac{1}{2}\int_0^L cv^2(0,x) - \int_0^T\int_0^L (v_{xx} + abu_{xx})v_x \\ &= \frac{b}{2}\int_0^L u^2(T,x) - \frac{b}{2}\int_0^L u^2(0,x) + \frac{1}{2}\int_0^L cv^2(T,x) - \frac{1}{2}\int_0^L cv^2(0,x) \\ &\quad + \int_0^T \left[\frac{b}{2}u_x^2(t,0) + abv_x(t,0)u_x(t,0) + \frac{1}{2}v_x^2(t,0)\right], \end{aligned} \tag{34}$$

that is

$$\begin{aligned} &\frac{b}{c}\int_0^L u^2(0,x) - \frac{b}{c}\int_0^L u^2(T,x) + \int_0^L v^2(0,x) - \int_0^L v^2(T,x) \\ &\quad = \int_0^T \left[\frac{b}{c}u_x^2(t,0) + 2\frac{ab}{c}v_x(t,0)u_x(t,0) + \frac{1}{c}v_x^2(t,0)\right]. \end{aligned} \tag{35}$$

Hence,

$$\begin{aligned} \|(u^0, v^0)\|^2_{(L^2(0,L))^2} &= \frac{b}{c}\int_0^L u^2(0,x) + \int_0^L v^2(0,x) \\ &= \int_0^T \left[\frac{b}{c}u_x^2(t,0) + 2\frac{ab}{c}v_x(t,0)u_x(t,0) + \frac{1}{c}v_x^2(t,0)\right]. \end{aligned} \tag{36}$$

If ε is a constant such that $1 > \varepsilon > \sqrt{a^2 b}$, then

$$\begin{aligned} 2abv_x(t,0)u_x(t,0) &= 2\left(\frac{1}{\varepsilon}\sqrt{a^2 b}v_x(t,0)\right)(\varepsilon\sqrt{b}u_x(t,0)) \\ &\geq -\varepsilon^2 bu_x^2(t,0) - \frac{a^2 b}{\varepsilon^2}v_x^2(t,0). \end{aligned} \tag{37}$$

Consequently, we can guarantee the existence of a constant $c_1 > 0$ such that

$$\begin{aligned}\|(u^0, v^0)\|^2_{(L^2(0,L))^2} &= \frac{b}{c}\int_0^L u^2(0,x) + \int_0^L v^2(0,x)\\ &\geq \int_0^T \left[\frac{b}{c}u_x^2(t,0) + 2\frac{ab}{c}v_x(t,0)u_x(t,0) + \frac{1}{c}v_x^2(t,0)\right]\\ &\geq \int_0^T \left[\frac{b}{c}u_x^2(0,t)(1-\varepsilon^2) + \frac{1}{c}v_x^2(t,0)\left(1 - \frac{a^2b}{\varepsilon^2}\right)\right]\\ &\geq c\|(u_x(\cdot,0), v_x^2(\cdot,0))\|^2_{(L^2(0,T))^2} \end{aligned} \tag{38}$$

and (20) is proved.

In order to obtain (21) we use a similar argument with $q = (T-t)$ to obtain

$$\begin{aligned} 0 &= \frac{1}{T}\int_0^T\int_0^L (bu^2 + cv^2) - \int_0^L (b(u^0)^2 + c(v^0)^2)\\ &\quad + \frac{1}{T}\int_0^T (T-t)(bu_x^2(t,0) + v_x^2(t,0) + 2abv_x(t,0)u_x(t,0)), \end{aligned} \tag{39}$$

which implies that

$$\|(u^0, v^0)\|^2_{(L^2(0,L))^2} \leq \frac{1}{T}\|(u,v)\|^2_{L^2((0,T)\times(0,L))^2} + c_2\|(u_x(\cdot,0), v_x(\cdot,0))\|^2_{(L^2(0,T))^2}, \tag{40}$$

for a suitable constant $c_2 > 0$. Now the proof is complete. □

Remark 2.1. If $\mu = \sqrt{\left(\frac{1}{c} - 1\right)^2 + 4\frac{a^2b}{c}}$, the change of variable

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2a & 2a \\ \left(\frac{1}{c} - 1\right) + \mu & \left(\frac{1}{c} - 1\right) - \mu \end{pmatrix}\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}$$

transforms the linear system (8) into

$$\begin{cases} \tilde{u}_t + \left(\frac{1}{c} + 1 + \mu\right)\tilde{u}_{xxx} - \frac{ra}{c}\left(\frac{1}{c} - 1 + \mu\right)\tilde{u}_x \\ \qquad + \frac{ra}{c}\left(\frac{1}{c} - 1 - \mu\right)\tilde{v}_x = 0, & \text{in } (0,T)\times(0,L)\\ \tilde{v}_t + \left(\frac{1}{c} + 1 - \mu\right)\tilde{v}_{xxx} + \frac{ra}{c}\left(\frac{1}{c} - 1 + \mu\right)\tilde{v}_x \\ \qquad - \frac{ra}{c}\left(\frac{1}{c} - 1 - \mu\right)\tilde{u}_x = 0, & \text{in } (0,T)\times(0,L)\\ \tilde{u}(t,0) = \tilde{u}(t,L) = \tilde{u}_x(t,L) = 0, & \text{on } (0,T),\\ \tilde{v}(t,0) = \tilde{v}(t,L) = \tilde{v}_x(t,L) = 0, & \text{on } (0,T),\\ \tilde{u}(0,x) = \tilde{u}^0(x), \quad \tilde{v}(0,x) = \tilde{v}^0(x), & \text{on } (0,L). \end{cases} \tag{41}$$

This change of variables was used in [20] where we have analyzed the case $r = 0$ and where the resulting system (41) consists of two uncoupled KdV equations. If $r \neq 0$, system (41) is still coupled. However, it is simpler than (8), in the sense that the coupling is realized only through the lower terms $\tilde{u}_x$ and $\tilde{v}_x$. This is an alternative equivalent approach for the study of our system.

2.2. *The nonhomogeneous system*

Now, we study the nonhomogeneous system corresponding to (8):

$$\begin{cases} u_t + u_{xxx} + av_{xxx} = 0, & \text{in } (0,T)\times(0,L) \\ cv_t + rv_x + abu_{xxx} + v_{xxx} = 0, & \text{in } (0,T)\times(0,L) \\ u(t,0) = 0, \quad u(t,L) = h_1, \quad u_x(t,L) = h_2, & \text{on } (0,T), \\ v(t,0) = 0, \quad v(t,L) = g_1, \quad v_x(t,L) = g_2, & \text{on } (0,T), \\ u(0,x) = u^0, \quad v(0,x) = v^0, & \text{on } (0,L) \end{cases} \tag{42}$$

where $h_1, g_1 \in H_0^1(0,T)$ and $h_2, g_2 \in L^2(0,T)$.

The following result ensures the existence of classical solutions of (42).

Theorem 2.3. *Let* $(u^0, v^0) \in D(A)$ *and*

$$h_i, g_i \in C_0^2[0,T] = \{z \in C^2[0,T] : z(0) = z(T) = 0\}, \quad i = 1, 2.$$

Then, system (42) *has a unique* (*classical*) *solution*

$$(u,v) \in C([0,T];(H^3(0,L))^2) \cap C^1([0,T];(L^2(0,L))^2) \tag{43}$$

and there exists a positive constant C *such that*

$$\|(u,v)\|^2_{C([0,T];(L^2(0,L))^2)} + \|(u,v)\|^2_{L^2(0,T;(H^1(0,L))^2)} \\ \leq C\left[\|(u^0,v^0)\|^2_{(L^2(0,L))^2} + \sum_{i=1}^{2}(\|h_i\|^2_{H^1(0,T)} + \|g_i\|^2_{H^1(0,T)})\right]. \tag{44}$$

Proof. As in [24], let $\phi_i, \eta_i \in C^\infty([0,L])$ be functions such that

$$\phi_1(0) = \phi_1'(L) = \eta_1(0) = \eta_1'(L) = 0 \quad \text{and} \quad \phi_1(L) = \eta_1(L) = 1 \tag{45}$$

$$\phi_2(0) = \phi_2(L) = \eta_2(0) = \eta_2(L) = 0 \quad \text{and} \quad \phi_2'(L) = \eta_2'(L) = 1. \tag{46}$$

The solution (u,v) of (42) can be written as

$$(u,v) = S(t)(u^0,v^0) + (\varphi,\psi), \tag{47}$$

where $(S(t))_{t\geq 0}$ is the semigroup associated to the homogeneous problem (8),

$$\widehat{\varphi}(x,t) = \sum_{i=1}^{2}\phi_i(x)g_i(t), \quad \widehat{\psi}(x,t) = \sum_{i=1}^{2}\eta_i(x)h_i(t)$$

and (φ, ψ) is the solution of the problem

$$\begin{cases} \varphi_t + \varphi_{xxx} + a\psi_{xxx} = F, & \text{in } (0,T)\times(0,L) \\ c\psi_t + r\psi_x + ab\varphi_{xxx} + \psi_{xxx} = G, & \text{in } (0,T)\times(0,L) \\ \varphi(t,0) = \varphi(t,L) = \varphi_x(t,L) = 0, & \text{on } (0,T), \\ \psi(t,0) = \psi(t,L) = \psi_x(t,L) = 0, & \text{on } (0,T), \\ \varphi(0,x) = \psi(0,x) = 0, & \text{on } (0,L), \end{cases} \tag{48}$$

where

$$\begin{aligned} F(x,t) &= \widehat{\varphi}_t + \widehat{\varphi}_{xxx} + a\widehat{\psi}_{xxx} \\ G(x,t) &= c\widehat{\psi}_t + r\widehat{\psi}_x + \widehat{\psi}_{xxx} + ab\widehat{\varphi}_{xxx}. \end{aligned} \tag{49}$$

Since $F, G \in C^1([0,T]; (L^2(0,L))^2)$, it follows that (48) has a unique solution

$$(\varphi, \psi) \in C([0,T]; D(A)) \cap C^1([0,T], (L^2(0,L))^2).$$

Thus, for every $(u^0, v^0) \in D(A)$ and $h_i, g_i \in C_0^2[0,T]$, we have a solution of (42) with the property (43). The uniqueness follows from the corresponding property of the homogeneous linear system.

For estimate (44) we note that

$$\begin{aligned} &\|(u,v)\|^2_{C([0,T];(L^2(0,L))^2)} \\ &\quad \le 2\|S(t)(u^0,v^0)\|^2_{C([0,T];(L^2(0,L))^2)} + 2\|(\varphi,\psi)\|^2_{C([0,T];(L^2(0,L))^2)} \\ &\quad \le C[\|(u^0,v^0)\|^2_{(L^2(0,L))^2} + \|(F,G)\|^2_{L^2([0,T];(L^2(0,L))^2)}]. \end{aligned}$$

Since

$$\|(F,G)\|^2_{L^2([0,T];(L^2(0,L))^2)} \le C\sum_{i=1}^{2}(\|h_i\|^2_{H^1(0,T)} + \|g_i\|^2_{H^1(0,T)}), \tag{50}$$

we deduce that

$$\begin{aligned} &\|(u,v)\|^2_{C([0,T];(L^2(0,L))^2)} \\ &\quad \le C\left[\|(u^0,v^0)\|^2_{(L^2(0,L))^2} + \sum_{i=1}^{2}(\|h_i\|^2_{H^1(0,T)} + \|g_i\|^2_{H^1(0,T)})\right]. \end{aligned} \tag{51}$$

On the other hand, from (19) we deduce that

$$\begin{aligned} &\|(u,v)\|^2_{L^2(0,T;(H^1(0,L))^2)} \\ &\quad \le 2\|S(t)(u^0,v^0)\|^2_{L^2(0,T;(H^1(0,L))^2)} + 2\|(\varphi,\psi)\|^2_{L^2(0,T;(H^1(0,L))^2)} \\ &\quad \le C[\|(u^0,v^0)\|^2_{(L^2(0,L))^2} + \|(\varphi,\psi)\|^2_{L^2(0,T;(H^1(0,L))^2)}]. \end{aligned}$$

As in the proof of Theorem 2.2, we obtain that the solution (φ, ψ) of (48) verifies

$$\begin{aligned} \int_0^T\int_0^L x(bF\varphi + G\psi) &= \frac{1}{2}\int_0^L x(b\varphi^2(t,x) + c\psi^2(t,x))\Big|_0^T \\ &\quad + \frac{r}{2}\int_0^T\int_0^L \psi^2 + \frac{3}{2}\int_0^T\int_0^L (b\varphi_x^2 + 2ab\varphi_x\psi_x + \psi_x^2). \end{aligned}$$

It follows that there exists a positive constant such that

$$\begin{aligned}\|(\varphi,\psi)\|_{L^2(0,T;(H^1(0,L))^2)} &\le C[\|(\varphi(T),\psi(T))\|^2_{(L^2(0,L))^2} \\ &\quad + \|(\varphi,\psi)\|^2_{L^2(0,T;(L^2(0,L))^2)} + \|(F,G)\|^2_{L^2(0,T;(L^2(0,L))^2)}] \\ &\le C\|(F,G)\|^2_{L^2(0,T;(L^2(0,L))^2)} \\ &\le C\sum_{i=1}^{2}(\|h_i\|^2_{H^1(0,T)} + \|g_i\|^2_{H^1(0,T)}).\end{aligned}$$

Hence,

$$\begin{aligned}&\|(u,v)\|^2_{L^2(0,T;(H^1(0,L))^2)} \\ &\quad \le C\left[\|(u^0,v^0)\|^2_{(L^2(0,L))^2} + \sum_{i=1}^{2}(\|h_i\|^2_{H^1(0,T)} + \|g_i\|^2_{H^1(0,T)})\right]. \qquad (52)\end{aligned}$$

From (51) and (52), it follows that (44) holds and the proof ends. □

Now, we shall prove two results of existence and regularity of weak solutions of (42).

Theorem 2.4. *There exists a unique linear and continuous map*

$$\begin{aligned}\Psi : (L^2(0,L))^2 \times (H^1_0(0,T))^2 \times (L^2(0,T))^2 \\ \to C([0,T];(L^2(0,L))^2) \cap L^2(0,T;(H^1(0,L))^2)\end{aligned}$$

such that, for any $(u^0,v^0) \in D(A)$ *and* $h_1, g_1, h_2, g_2 \in C^2_0[0,T]$,

$$\Psi((u^0,v^0),(h_1,g_1,h_2,g_2)) = (u,v)$$

where (u,v) *is the unique classical solution of* (42).

Remark 2.2. If $(u^0,v^0) \in (L^2(0,L))^2$, (u,v) will be the weak solution of (42).

Proof. We decompose the solution of (42) as $(u,v) = (u_1,v_1) + (u_2,v_2)$, where (u_1,v_1) is the solution of (42) with $h_1 = g_1 = 0$ and (u_2,v_2) is the solution of (42) with null data (u^0,v^0) and $h_2 = g_2 = 0$.

We first consider the case $h_1 = g_1 = 0$. The proof will done using the multipliers method. Let $q \in C^\infty([0,L]\times[0,T])$, $(u^0,v^0) \in D(A)$ and $h_2, g_2 \in C^2_0[0,T]$. From Theorem 2.3, it follows that there exists a unique solution $(u,v) \in C([0,T];(H^3(0,1))^2) \cap C^1([0,T];(L^2(0,1))^2)$ of (42). If $s \in [0,T]$ we have

$$\begin{aligned}0 = &-\frac{1}{2}\int_0^s\int_0^L q_t(bu^2+cv^2) + \frac{1}{2}\int_0^L q(bu^2+cv^2)|_0^s - \frac{r}{2}\int_0^s\int_0^L q_x v^2 \\ &+\frac{r}{2}\int_0^s qv^2|_0^L - \int_0^s\int_0^L (qu)_x(bu+abv)_{xx} + \int_0^s (qu)(bu+abv)_{xx}|_0^L \\ &-\int_0^s\int_0^L (qv)_x(abu+v)_{xx} + \int_0^s (qv)(abu+v)_{xx}|_0^L. \qquad (53)\end{aligned}$$

Consequently, if $q = 1$ the following holds

$$\begin{aligned} 0 &= \frac{1}{2}\int_0^L (bu^2 + cv^2)|_0^s - \int_0^s\int_0^L u_x(bu + abv)_{xx} - \int_0^s\int_0^L v_x(abu + v)_{xx} \\ &= \frac{1}{2}\int_0^L (bu^2 + cv^2)|_0^s - \frac{1}{2}\int_0^s (bu_x^2 + v_x^2 + 2abu_xv_x)|_0^L, \end{aligned} \tag{54}$$

i.e.

$$\begin{aligned} &\frac{1}{2}\int_0^L (bu^2(s,x) + cv^2(s,x)) + \frac{1}{2}\int_0^s (bu_x^2(t,0) + v_x^2(t,0) + 2abu_x(t,0)v_x(t,0)) \\ &\quad = \frac{1}{2}\int_0^L (b(u^0)^2(x) + c(v^0)^2(x)) + \frac{1}{2}\int_0^s (bh_2^2(t) + g_2^2(t) + 2abh_2(t)g_2(t)). \end{aligned} \tag{55}$$

If we consider $\varepsilon > 0$ satisfying $\sqrt{a^2b} < \varepsilon < 1$, as in (28), we obtain a constant $M > 0$ such that

$$\begin{aligned} &\|(u(s,\cdot), v(s,\cdot))\|^2_{(L^2(0,L))^2} + \|(u_x(\cdot,0), v_x(\cdot,0))\|^2_{(L^2(s,0))^2} \\ &\quad \leq M(\|(u^0,v^0)\|^2_{(L^2(0,L))^2} + \|(h_2,g_2)\|^2_{(L^2(s,0))^2}) \end{aligned} \tag{56}$$

which give us that

$$\|(u,v)\|_{C([0,T];(L^2(0,L))^2)} \leq M(\|(u^0,v^0)\|^2_{L^2(0,L)^2} + \|(h_2,g_2)\|^2_{L^2(0,T)^2}) \tag{57}$$

and

$$\|(u_x(0,\cdot), v_x(0,\cdot))\|^2_{L^2(0,T)^2} \leq M(\|(u^0,v^0)\|^2_{L^2(0,L)^2} + \|(h_2,g_2)\|^2_{L^2(0,T)^2}). \tag{58}$$

Now, letting $q(x,t) = x$ and performing as in the previous computation we get

$$\begin{aligned} 0 &= \frac{1}{2}\int_0^L x(bu^2 + cv^2)|_0^T - \frac{r}{2}\int_0^T\int_0^L v^2 \\ &\quad + \frac{3}{2}\int_0^T\int_0^L (bu_x^2 + v_x^2 + 2abu_xv_x) - \frac{1}{2}\int_0^T L(bh_2^2 + g_2^2 + 2abh_2g_2), \end{aligned} \tag{59}$$

or

$$\begin{aligned} &\int_0^L x(bu^2(T,x) + cv^2(T,x)) + 3\int_0^T\int_0^L (bu_x^2 + v_x^2 + 2abu_xv_x) \\ &\quad = \int_0^L x(b(u^0)^2 + c(v^0)^2) + r\int_0^T\int_0^L v^2 + \int_0^T L(bh_2^2 + g_2^2 + 2abh_2g_2). \end{aligned} \tag{60}$$

On the other hand, from the last relation, we obtain

$$\int_0^T\int_0^L (u_x^2 + v_x^2) \leq C\left[\int_0^L ((u^0)^2 + (v^0)^2) + r\int_0^T\int_0^L v^2 + \int_0^T (h_2^2 + g_2^2)\right]. \tag{61}$$

Finally, by taking (57) into account we conclude that there exists a constant $C > 0$ such that

$$\|(u,v)\|^2_{L^2(0,T;H^1(0,L))^2} \leq C(\|(u^0,v^0)\|^2_{L^2(0,L)^2} + \|(h_2,g_2)\|^2_{L^2(0,T)^2}). \tag{62}$$

From the density of $D(A)$ in $(L^2(0,L))^2$ and from the density of $C_0^2[0,T]$ in $(L^2(0,T))^2$, we obtain that the linear map

$$\Psi_1 : D(A) \times (C_0^2[0,T])^2 \to C([0,T];(L^2(0,L))^2) \cap L^2(0,T;(H^1(0,L))^2)$$
$$\Psi_1((u^0,v^0),(h_2,g_2)) = (u_1,v_1)$$

can be uniquely extended by continuity to a linear and continuous map

$$\Psi_1 : (L^2(0,L))^2 \times (L^2(0,T))^2 \to C([0,T];(L^2(0,L))^2) \cap L^2(0,T;(H^1(0,L))^2).$$

In order to estimate the second component, i.e. the solution of (42) when $h_2 = g_2 = 0$ and $u^0 = v^0 = 0$, we use (44). For $(u^0,v^0) \in D(A)$ and $h_1, g_1 \in C_0^2[0,T]$ it follows that there exists a unique solution $(u_2,v_2) \in C([0,T];(H^3(0,1))^2) \cap C^1([0,T];(L^2(0,1))^2)$ of (42). Moreover, (44) implies that

$$\Psi_2 : D(A) \times (C_0^2[0,T])^2 \to C([0,T];(L^2(0,L))^2) \cap L^2(0,T;(H^1(0,L))^2)$$
$$\Psi_2((u^0,v^0),(h_1,g_1)) = (u_2,v_2)$$

can be uniquely extended by continuity to a linear and continuous map

$$\Psi_2 : (L^2(0,L))^2 \cap (H_0^1(0,T))^2 \to C([0,T];(L^2(0,L))^2) \cap L^2(0,T;(H^1(0,L))^2).$$

Then, defining $\Psi = \Psi_1 + \Psi_2$, the proof is complete. □

Remark 2.3. For the nonhomogeneous Korteweg–de Vries equation

$$\begin{cases} u_t + u_{xxx} + u_x = 0, & \text{in } (0,T)\times(0,L) \\ u(t,0) = 0, \quad u(t,L) = h(t), \quad u_x(t,L) = g(t), & \text{on } (0,T) \\ u(0,x) = u^0(x), & \text{on } (0,L) \end{cases} \tag{63}$$

the well-posedness in $L^2(0,L)$ is obtained by taking $h \in H^{\frac{1}{3}}(0,T)$ and $g \in L^2(0,T)$ (see, for instance, [3, 4, 9, 12]). Moreover, additional regularity for the solution may be obtained in the case $h = g = 0$. The same properties should be true for our system but we shall not need them in the controllability study.

2.3. *The adjoint system*

This section is devoted to study the properties of the adjoint system of (8):

$$\begin{cases} \varphi_t - \varphi_{xxx} - a\psi_{xxx} = 0, & \text{in } (0,T)\times(0,L) \\ c\psi_t - r\psi_x - ba\varphi_{xxx} - \psi_{xxx} = 0, & \text{in } (0,T)\times(0,L) \\ \varphi(t,0) = \varphi(t,L) = \varphi_x(t,0) = 0, & \text{on } (0,T), \\ \psi(t,0) = \psi(t,L) = \psi_x(t,0) = 0, & \text{on } (0,T), \\ \varphi(0,x) = \varphi^0(x), \quad \psi(0,x) = \psi^0(x), & \text{on } (0,L) \end{cases} \tag{64}$$

which is equivalent to

$$\begin{cases}(\varphi,\psi)_t = A^*(\varphi,\psi)\\ (\varphi,\psi)(0) = (\varphi^0,\psi^0),\end{cases}$$

where A^* is given by (12).

Remark that the change of variable $x = L - x$ reduces system (64) to (8). Therefore, the properties of the solutions of (64) are similar to the ones deduced in Theorems 2.1 and 2.2 for the linear homogeneous system (8). More precisely, we have

Theorem 2.5. *Let* $\varphi^0, \psi^0 \in L^2(0,L)$. *There exists a unique solution* (φ,ψ) *of* (64) *such that*

$$(\varphi,\psi) \in C([0,T];(L^2(0,L))^2) \cap L^2(0,T;(H^1(0,L))^2) \tag{65}$$

and the following estimate holds

$$\|(\varphi,\psi)\|_{L^2(0,T;(H^1(0,L))^2)} \le c_0\|(\varphi^0,\psi^0)\|_{(L^2(0,L))^2}. \tag{66}$$

Moreover, there exist $c_1, c_2 > 0$ *such that following inequalities hold*

$$\|(\varphi_x(\cdot,L),\psi_x(\cdot,L))\|^2_{(L^2(0,T))^2} \le c_1\|(\varphi^0,\psi^0)\|^2_{(L^2(0,L))^2} \tag{67}$$

and

$$\begin{aligned}\|(\varphi^0,\psi^0)\|^2_{(L^2(0,L))^2} \le{}& \frac{1}{T}\|(\varphi,\psi)\|^2_{(L^2((0,T)\times(0,L)))^2}\\ &+ c_2\|(\varphi_x(\cdot,L),\psi_x(\cdot,L))\|^2_{(L^2(0,T))^2}.\end{aligned} \tag{68}$$

Moreover, we have the following estimate of the second derivative of the solution on the boundary.

Theorem 2.6. *Let* $\varphi^0, \psi^0 \in L^2(0,L)$. *The corresponding solution* (φ,ψ) *of* (64) *verifies*

$$\|(\varphi_{xx}(\cdot,L),\psi_{xx}(\cdot,L))\|^2_{(H^{-1}(0,T))^2} \le C\|(\varphi^0,\psi^0)\|^2_{(L^2(0,L))^2}. \tag{69}$$

Proof. Let $(\varphi^0,\psi^0) \in D(A^*)$. Then $(\varphi,\psi) \in C([0,T];D(A^*))$ and $\varphi_{xx}(\cdot,L), \psi_{xx}(\cdot,L) \in C[0,T]$. Let now $h_1, g_1 \in C_0^2[0,T]$ and (u,v) be the (classical) solution of

$$\begin{cases} u_t + u_{xxx} + av_{xxx} = 0, & \text{in } (0,T)\times(0,L)\\ cv_t + rv_x + abu_{xxx} + v_{xxx} = 0, & \text{in } (0,T)\times(0,L)\\ u(t,0) = 0, \quad u(t,L) = h_1, \quad u_x(t,L) = 0, & \text{on } (0,T),\\ v(t,0) = 0, \quad v(t,L) = g_1, \quad v_x(t,L) = 0, & \text{on } (0,T),\\ u(0,x) = 0, \quad v(0,x) = 0, & \text{on } (0,L).\end{cases} \tag{70}$$

Multiplying the first equation of (70) by φ, the second one by ψ and integrating by parts we obtain that

$$\begin{cases} \int_0^L u\varphi|_0^T + \int_0^T u\varphi_{xx}|_0^L + a\int_0^T v\varphi_{xx}|_0^L = \int_0^T\int_0^L (\varphi_t u + \varphi_{xxx}u + a\varphi_{xxx}v) \\ c\int_0^L v\psi|_0^T + ab\int_0^T u\psi_{xx}|_0^L + \int_0^T v\psi_{xx}|_0^L \\ \quad = \int_0^T\int_0^L (c\psi_t v + r\psi_x v + ab\psi_{xxx}u + \psi_{xxx}v). \end{cases} \tag{71}$$

Multiplying the first equation of (70) by b and adding it to the second one, we obtain that

$$\int_0^L (bu(T)\varphi(T) + cv(T)\psi(T)) = -\int_0^T [\varphi_{xx}(t,L)(bu(t,L) + abv(t,L)) + \psi_{xx}(t,L)(abu(t,L) + v(t,L))]. \tag{72}$$

On the other hand, by (44), we have that

$$\left|\int_0^L (bu(T)\varphi(T) + cv(T)\psi(T))\right| \leq C\|(\varphi^0,\psi^0)\|^2_{(L^2(0,L))^2}(\|h_1\|_{H^1(0,T)} + \|g_1\|_{H^1(0,T)})$$

where C does depend on h_1 and g_1. then, it follows that the map

$$\Upsilon : (H^1(0,T))^2 \to \mathbb{R},$$
$$\Upsilon(\tilde{h}_1,\tilde{g}_1) = \int_0^T \varphi_{xx}(t,L)\tilde{h}_1 + \int_0^T \psi_{xx}(t,L)\tilde{g}_1$$

verifies

$$|\Upsilon(\tilde{h}_1,\tilde{g}_1)| \leq C\|(\varphi^0,\psi^0)\|^2_{(L^2(0,L))^2}(\|\tilde{h}_1\|^2_{H^1(0,T)} + \|\tilde{g}_1\|^2_{H^1(0,T)})$$

from which (69) follows and the proof is completed. □

Let us also remark that the following backward system

$$\begin{cases} \varphi_t + \varphi_{xxx} + a\psi_{xxx} = 0, & \text{in } (0,T)\times(0,L) \\ c\psi_t + r\psi_x + ba\varphi_{xxx} + \psi_{xxx} = 0, & \text{in } (0,T)\times(0,L) \\ \varphi(t,0) = \varphi(t,L) = \varphi_x(t,0) = 0, & \text{on } (0,T), \\ \psi(t,0) = \psi(t,L) = \psi_x(t,0) = 0, & \text{on } (0,T), \\ \varphi(T,x) = \varphi^1(x), \quad \psi(T,x) = \psi^1(x), & \text{on } (0,L) \end{cases} \tag{73}$$

is well-posed for any $(\varphi^1,\psi^1) \in (L^2(0,L))^2$ and the simple change of variable $t = T - t$ transforms (73) in (64). Therefore, all the results from Theorems 2.5 and 2.6 hold, with minor changes, for system (73) too.

3. Exact Boundary Controllability Results: The Linear System

This section is devoted to the analysis of the exact controllability property for the linear system corresponding to (1) with boundary controls (2). More precisely, given $T > 0$ and (u^0, v^0), $(u^1, v^1) \in (L^2(0, L))^2$, we study the existence of four control functions $h_1, g_1 \in H_0^1(0, T)$ and $h_2, g_2 \in L^2(0, T)$, such that the solution (u, v) of the system

$$\begin{cases} u_t + u_{xxx} + av_{xxx} = 0, & \text{in } (0, T) \times (0, L) \\ cv_t + rv_x + bau_{xxx} + v_{xxx} = 0, & \text{in } (0, T) \times (0, L) \\ u(0, x) = u^0(x), \quad v(0, x) = v^0(x), & \text{on } (0, L) \end{cases} \tag{74}$$

with boundary conditions

$$\begin{cases} u(t, 0) = 0, \quad u(t, L) = h_1(t), \quad u_x(t, L) = h_2(t), \quad \text{on } (0, T), \\ v(t, 0) = 0, \quad v(t, L) = g_1(t), \quad v_x(t, L) = g_2(t), \quad \text{on } (0, T), \end{cases} \tag{75}$$

satisfies

$$u(T, \cdot) = u^1, \quad v(T, \cdot) = v^1 \quad \text{in } L^2(0, L). \tag{76}$$

Definition 3.1. Let $T > 0$. System (74) is *exactly controllable in time* T if for any initial an final data $(u^0, v^0), (u^1, v^1) \in (L^2(0, L))^2$ there exist control functions $h_1, g_1 \in H_0^1(0, T)$ and $h_2, g_2 \in L^2(0, T)$ such that the solution (u, v) of (74)–(75) satisfies (76).

Remark 3.1. Without any lost of generality, we may study only the exact controllability property for the case $u^0 = v^0 = 0$. Indeed, let $(u^0, v^0), (u^1, v^1)$ be arbitrarily in $(L^2(0, L))^2$ and let $h_1, g_1 \in H_0^1(0, T)$, $h_2, g_2 \in L^2(0, T)$ be controls which lead the solution (u, v) of (74) from the zero initial data to the final state $(u^1, v^1) - S(T)(u^0, v^0)$ (we recall that $(S(t))_{t \geq 0}$ is the semigroup generated by the differential operator A corresponding to (74)). It follows immediately that these controls also lead to the solution $(u, v) + S(\cdot)(u^0, v^0)$ of (74) from (u^0, v^0) to the final state (u^1, v^1).

From now on, we shall consider only the case $u^0 = v^0 = 0$. Firstly, we give an equivalent condition for the exact controllability property.

Lemma 3.1. *Let $(u^1, v^1) \in (L^2(0, T))^2$. Then, there exist controls $h_1, g_1 \in H_0^1(0, T)$ and $h_2, g_2 \in L^2(0, T)$, such that the solution (u, v) of* (74)–(75), *satisfies* (76) *if and only if*

$$\begin{aligned} &\int_0^L (b\varphi^1(x)u^1(x) + c\psi^1(x)v^1(x))dx \\ &\qquad = \int_0^T bh_2(t)(\varphi_x(t, L) + a\psi_x(t, L))dt \\ &\qquad\quad + \int_0^T g_2(t)(ab\varphi_x(t, L) + \psi_x(t, L))dt \end{aligned}$$

$$- \langle bh_1(t), \varphi_{xx}(t,L) + a\psi_{xx}(t,L)\rangle_{H_0^1\times H^{-1}}$$
$$- \langle g_1(t), ab\varphi_{xx}(t,L) + \psi_{xx}(t,L)\rangle_{H_0^1\times H^{-1}} \tag{77}$$

for any $(\varphi^1, \psi^1) \in (L^2(0,L))^2$, (φ, ψ) *being the solution of the backward system* (73).

Proof. Relation (77) is obtained multiplying (74) by the solution (φ, ψ) of (73) and integrating by parts. □

For the study of the controllability property, a fundamental role will be played by the following observability result

Theorem 3.1. *For any* $L > 0$ *and* $T > 0$ *there exists a positive constant* $C = C(L,T) > 0$ *such that the inequality*

$$\begin{aligned} &C\|(w^1, z^1)\|^2_{(L^2(0,L))^2} \\ &\quad\le \|w_x(\cdot,L) + az_x(\cdot,L)\|^2_{L^2} + \|abw_x(\cdot,L) + z_x(t,L)\|^2_{L^2} \\ &\qquad + \|w_{xx}(\cdot,L) + az_{xx}(\cdot,L)\|^2_{H^{-1}} + \|abw_{xx}(\cdot,L) + z_{xx}(\cdot,L)\|^2_{H^{-1}} \end{aligned} \tag{78}$$

holds for any $w^1, z^1 \in L^2(0,L)$, *where* (w,z) *is the solution of* (73) *with initial data* (w^1, z^1).

Proof. The change of variable $t = T - t$ transforms (73) into (64). Hence, inequality (78) is equivalent to

$$\begin{aligned} &C\|(w^0, z^0)\|^2_{(L^2(0,L))^2} \\ &\quad\le \|w_x(\cdot,L) + az_x(\cdot,L)\|^2_{L^2} + \|abw_x(\cdot,L) + z_x(t,L)\|^2_{L^2} \\ &\qquad + \|w_{xx}(\cdot,L) + az_{xx}(\cdot,L)\|^2_{H^{-1}} + \|abw_{xx}(\cdot,L) + z_{xx}(\cdot,L)\|^2_{H^{-1}}, \end{aligned} \tag{79}$$

for any $w^0, z^0 \in L^2(0,L)$, where (w,z) is the solution of (64) with initial data (w^0, z^0).

Let us suppose that (79) does not hold. In this case, it follows that there exists a sequence $(w_n^0, z_n^0)_{n\ge1} \subset (L^2(0,L))^2$ such that

$$\|(w_n^0, z_n^0)\|_{(L^2(0,L))^2} = 1, \quad \forall\, n \ge 1 \tag{80}$$

and

$$\begin{aligned} &\lim_{n\to\infty} (\|(w_n)_x(\cdot,L) + a(z_n)_x(\cdot,L)\|^2_{L^2} + \|ab(w_n)_x(\cdot,L) + (z_n)_x(\cdot,L)\|^2_{L^2}) = 0, \\ &\lim_{n\to\infty} (\|(w_n)_{xx}(\cdot,L) + a(z_n)_{xx}(\cdot,L)\|^2_{H^{-1}} \\ &\qquad + \|ab(w_n)_{xx}(\cdot,L) + (z_n)_{xx}(\cdot,L)\|^2_{H^{-1}}) = 0, \end{aligned} \tag{81}$$

where (w_n, z_n) is the solution of (64) with initial data (w_n^0, z_n^0).

Since $(1 - a^2 b) > 0$, from (81) we deduce that

$$\begin{aligned} &\lim_{n\to\infty} ((w_n)_x(\cdot,L)) = 0 \quad \text{and} \quad \lim_{n\to\infty} ((z_n)_x(\cdot,L)) = 0 \quad \text{in } L^2(0,T) \\ &\lim_{n\to\infty} ((w_n)_{xx}(\cdot,L)) = 0 \quad \text{and} \quad \lim_{n\to\infty} ((z_n)_{xx}(\cdot,L)) = 0 \quad \text{in } H^{-1}(0,T). \end{aligned} \tag{82}$$

From Theorem 2.5, we obtain that $(w_n, z_n)_{n\geq 1}$ is bounded in $(L^2(0,T; H^1(0,L)))^2$ and from (64), we have that $((w_n)_t, (z_n)_t)_{n\geq 1}$ is bounded in $(L^2(0,T; H^{-2}(0,L)))^2$.

Since $H^1(0,L) \subset L^2(0,L) \subset H^{-2}(0,L)$ and the first embedding is compact, it follows that $(w_n, z_n)_{n\geq 1}$ is relatively compact in $L^2(0,T;(L^2(0,L))^2)$. Therefore, there exists a subsequence, still denoted by the same index, which converges to (w,z) in $L^2(0,T;(L^2(0,L))^2)$.

Now, from (68) and the above convergences, we deduce that $(w_n^0, z_n^0)_{n\geq 1}$ is a Cauchy sequence in $(L^2(0,L))^2$. Hence, it converges to $(w^0, z^0) \in (L^2(0,L))^2$. Moreover, $(w,z) \in C([0,T];(L^2(0,L))^2)$ is a weak solution of (64) and

$$w(0,\cdot) = w^0 \quad \text{and} \quad z(0,\cdot) = z^0. \tag{83}$$

Indeed, if $(\widehat{w}, \widehat{z}) \in C([0,T];(L^2(0,L))^2)$ is the solution of (64) with data $(w^0, z^0) \in (L^2(0,L))^2$, then

$$\begin{aligned}
&\|(\widehat{w}, \widehat{z}) - (w,z)\|_{L^2(0,T;(L^2(0,L))^2)} \\
&\quad \leq \|(\widehat{w}, \widehat{z}) - (w_n,z_n)\|_{L^2(0,T;(L^2(0,L))^2)} + \|(w_n,z_n) - (w,z)\|_{L^2(0,T;(L^2(0,L))^2)} \\
&\quad \leq C\|(w^0, z^0) - (w_n^0, z_n^0)\|_{(L^2(0,L))^2} + \|(w_n,z_n) - (w,z)\|_{L^2(0,T;(L^2(0,L))^2)}.
\end{aligned}$$

The last expression tends to zero when n goes to infinity and (83) follows. On the other hand, since $\|(w_n^0, z_n^0)\|_{(L^2(0,L))^2} = 1$, for all $n \geq 1$, it follows that

$$\|(w^0, z^0)\|_{(L^2(0,L))^2} = 1. \tag{84}$$

From (67), and (69), we deduce that

$$\begin{aligned}
(w_n)_x(\cdot,0) \to w_x(\cdot,L) \quad &\text{and} \quad (z_n)_x(\cdot,L) \to z_x(\cdot,L) \\
&\text{in } L^2(0,T), \quad \text{as } n \to \infty, \\
(w_n)_{xx}(\cdot,L) \to w_{xx}(\cdot,L) \quad &\text{and} \quad (z_n)_{xx}(\cdot,L) \to z_{xx}(\cdot,L) \\
&\text{in } H^{-1}(0,T), \quad \text{as } n \to \infty
\end{aligned}$$

and from (82) we conclude that $w_x(\cdot,L) = z_x(\cdot,L) = w_{xx}(\cdot,L) = z_{xx}(\cdot,L) = 0$.

Hence, (w,z) is a solution of

$$\begin{cases}
w_t - w_{xxx} - az_{xxx} = 0, & \text{in } (0,L)\times(0,T) \\
cz_t - rz_x - baw_{xxx} - z_{xxx} = 0, & \text{in } (0,L)\times(0,T) \\
w(0,t) = w(L,t) = w_x(0,t) = 0, & \text{on } (0,T), \\
z(0,t) = z(L,t) = z_x(0,t) = 0, & \text{on } (0,T), \\
w(x,0) = w^0, \quad z(x,0) = z^0, & \text{on } (0,L)
\end{cases} \tag{85}$$

and satisfies

$$w_x(\cdot,L) = z_x(\cdot,L) = w_{xx}(\cdot,L) = z_{xx}(\cdot,L) = 0. \tag{86}$$

Remark that (84) implies that the solutions of (85)–(86) cannot be identically zero. Therefore, the proof of the theorem will be complete if we prove the following unique continuation result:

Lemma 3.2. *Let $w^0, z^0 \in L^2(0,L)$. Then if*

$$\left.\begin{array}{l}(w,z)\ \textit{is a solution of}\ (85)\\ (w,z)\ \textit{satisfies}\ (86)\end{array}\right\} \Rightarrow w^0 = z^0 = 0. \tag{87}$$

Proof. The additional regularity of the solutions given by Theorem 2.5 allows to reduce (87) to a unique continuation principle for the eigenfunctions of the system. This argument was used before in [16, 24].

Let N_T be the space of initial data $(w^0, z^0) \in (L^2(0,L))^2$ such that the corresponding solution of (85) satisfies $w_x(\cdot, L) = z_x(\cdot, L) = 0$ in $L^2(0,T)$ and $w_{xx}(\cdot, L) = z_{xx}(\cdot, L) = 0$ in $H^{-1}(0,T)$. The space N_T has the following properties:

(1) $\dim N_T < \infty$;
(2) $N_T \subset D(A^*)$;
(3) If $N_T \neq \{0\}$, then $A^* : \overline{N_T} \to \overline{N_T}$ has at least one (complex) eigenvalue ($\overline{N_T}$ is the complexification of N_T).

Let us prove these properties of the space N_T.

(1) As above, we deduce that any bounded set in N_T is relatively compact. Then, it follows that the unit ball is relatively compact in N_T and therefore N_T has finite dimension.
(2) First of all let us remark that $T_1 < T$ implies $N_T \subset N_{T_1}$. On the other hand, since $\dim(N_T) < \infty$, there exists $T_1 > 0$ and $\varepsilon > 0$ such that

$$N_{T_1} = N_t, \quad \forall\, t \in [T_1, T_1 + \varepsilon].$$

We shall prove that $N_{T_1} \subset D(A^*)$. Let $y \in N_{T_1}$ and let $\{S^*(t)\}_{t\geq 0}$ be the semigroup generated by the operator A^*. We have that

$$y \in D(A^*) \Leftrightarrow \lim_{t\to 0} \frac{S^*(t)y - y}{h} \text{ exists in } (L^2(0,L))^2.$$

We obtain immediately that, for t sufficiently small,

$$\frac{S^*(t)y - y}{h} \in N_{T_1}.$$

On the other hand, from the extrapolation theory (see [5, p. 27]), we deduce that there exists a Banach space Y such that $(L^2(0,L))^2 \subset Y$ and

$$S^*(\cdot)(y) \in C^1([0,T], Y).$$

Hence,

$$\lim_{t\to 0}\frac{S^*(t)y-y}{h}$$

exists in Y. But, $N_{T_1}\subset (L^2(0,L))^2\subset Y$ and, since $\dim(N_{T_1})<\infty$, the norms induced from $(L^2(0,L))^2$ and Y on N_{T_1} are equivalent. Consequently, it follows that

$$\lim_{t\to 0}\frac{S^*(t)y-y}{h}$$

exists in $(L^2(0,L))^2$ and therefore $y\in D(A^*)$.

(3) Since A^* is a linear operator defined in a finite dimensional space the property follows immediately.

Remark that the unique continuation principle (87) does not hold if and only if $N_T\neq\{0\}$ or, equivalently,

$$\text{there exists } \lambda\in\mathbb{C} \text{ and } (w,z)\in N_T \text{ such that } (w,z)\neq 0 \text{ and } A^*(w,z)=\lambda(w,z). \tag{88}$$

Note that (88) means that there exists a nontrivial solution (w,z) of the system

$$\begin{cases} \lambda w - w_{xxx} - a z_{xxx} = 0, & \text{in } (0,L)\\ c\lambda z - r z_x - ba w_{xxx} - z_{xxx} = 0, & \text{in } (0,L)\\ w(0)=w(L)=w_x(0)=w_x(L)=w_{xx}(L)=0, & \\ z(0)=z(L)=z_x(0)=z_x(L)=z_{xx}(L)=0. & \end{cases} \tag{89}$$

□

To conclude the proof of the Lemma 3.2, the following result is needed.

Lemma 3.3. *If* $(\lambda,(w,z))$ *is a solution of* (89), *then*

$$w=z=0. \tag{90}$$

Proof. Let us remark that $\lambda=0$ is not a solution of (88). Indeed, $\lambda=0$ implies that

$$\begin{cases} w_{xxx} + a z_{xxx} = 0, & \text{in } (0,L)\\ r z_x + ba w_{xxx} + z_{xxx} = 0, & \text{in } (0,L)\\ w(0)=w(L)=w_x(0)=w_{xx}(0)=0, & \\ z(0)=z(L)=z_x(0)=z_{xx}(0)=0. & \end{cases} \tag{91}$$

By noting $z_x=\zeta$, we obtain that ζ satisfies

$$\begin{cases} (a^2b-1)\zeta_{xx} - r\zeta = 0, & \text{in } (0,L)\\ \zeta(0)=\zeta_x(0)=0. & \end{cases}$$

The unique solution of the last equation is $\zeta = 0$ (and consequently $w = z = 0$), hence $\lambda = 0$ is not a solution of (88).

Now, we introduce the Fourier transforms of w and z,

$$\widehat{w}(\xi) = \int_0^L w(x)e^{ix\xi}dx, \quad \widehat{z}(\xi) = \int_0^L z(x)e^{ix\xi}dx.$$

Thus $\widehat{w}$ and $\widehat{z}$ are entire functions.

By multiplying the first and second equations in (89) $e^{ix\xi}$ and integrating by parts, we obtain

$$\begin{cases} (\lambda - i\xi^3)\widehat{w}(\xi) - i\xi^3 a\widehat{z}(\xi) = \alpha_1' + a\beta_1' \\ (\lambda c - i\xi^3 + ir\xi)\widehat{z}(\xi) - i\xi^3 ab\widehat{w}(\xi) = ab\alpha_1' + \beta_1' \end{cases} \tag{92}$$

where $\alpha_1' = -w_{xx}(0)$ and $\beta_1' = -z_{xx}(0)$ are complex values. Remark that α_1' and β_1' determine uniquely $\widehat{w}$ and $\widehat{z}$ and, consequently, w and z.

Also we have that

$$w = z = 0 \Leftrightarrow \alpha_1' = \beta_1' = 0.$$

By eliminating $\widehat{w}$ from (92), we obtain the identity

$$\begin{aligned} &[(\lambda - i\xi^3)(\lambda c - i\xi^3 + ir\xi) + a^2b\xi^6]\widehat{z} \\ &\quad = [(iab\xi^3)(\alpha_1' + a\beta_1') + (\lambda - i\xi^3)(ab\alpha_1' + \beta_1')]. \end{aligned} \tag{93}$$

Let us denote

$$\begin{cases} P_\lambda(\xi) = (a^2b - 1)\xi^6 + r\xi^4 - i\lambda(1+c)\xi^3 + ir\lambda\xi + \lambda^2 c, \\ Q_\lambda(\xi, L) = [(iab\xi^3)(\alpha_1' + a\beta_1') + (\lambda - i\xi^3)(ab\alpha_1' + \beta_1')], \end{cases}$$

from (93) we obtain that,

$$P_\lambda(\xi)\widehat{z}(\xi) = Q_\lambda(\xi, L). \tag{94}$$

Since $\widehat{z}$ is an entire function, it follows that all the roots of P_λ are also roots of Q_λ. Furthermore, the following properties of P_λ are evident:

(1) The polynomial P_λ has degree 6 (since $(1 - a^2b) > 0$).
(2) There is no root of P_λ with multiplicity 6 (since there is no term containing ξ^5).

On the other hand, the polynomial Q_λ has the properties:

(1) The polynomial Q_λ has degree 3.
(2) There is no root of Q_λ with multiplicity 3 (since there is no term containing ξ^2).

Since the degree of Q_λ is smaller than the degree of P_λ, we conclude that $\hat{z}$ is an entire function if $Q_\lambda = 0$ or equivalently $\alpha_1' = \beta_1' = 0$.

Therefore we have that $w = z = 0$, which completes the proofs of the lemmas and of Theorem 3.1. □

The following theorem solves the control problem (74)–(75):

Theorem 3.2. *Let $T > 0$ and $L > 0$. Then system* (74)–(75) *is exactly controllable in time T.*

Proof. Let us define the following functional

$$J(w^1, z^1) = \frac{1}{2}(\|b(w_{xx}(\cdot, L) + az_{xx}(\cdot, L))\|^2_{H^{-1}} + \|abw_{xx}(\cdot, L) + z_{xx}(\cdot, L)\|^2_{H^{-1}})$$
$$+ \frac{1}{2}(\|b(w_x(\cdot, L) + az_x(\cdot, L))\|^2_{L^2} + \|abw_x(\cdot, L) + z_x(\cdot, L)\|^2_{L^2})$$
$$- \int_0^L [bu^1(x)w^1(x) + cv^1(x)z^1(x)]dx, \tag{95}$$

where $(w^1, z^1) \in (L^2(0, L))^2$ and (w, z) is the solution of the backward system (73) with initial data (w^1, z^1).

Let $(\widehat{w}^1, \widehat{z}^1) \in (L^2(0, L))^2$ be a minimizer of J. By differentiating J, we obtain that (77) is satisfied with $h_1, g_1 \in H^1_0(0, T)$ being the unique solutions of

$$\langle b\widehat{w}_{xx}(t, L) + a\widehat{z}_{xx}(t, L), \psi\rangle = \int_0^T (h_1)_t(t)\psi_t(t)dt, \quad \forall\, \psi \in H^1_0(0, T)$$
$$\langle ab\widehat{w}_{xx}(t, L) + \widehat{z}_{xx}(t, L), \psi\rangle = \int_0^T (g_1)_t(t)\psi_t(t)dt, \quad \forall\, \psi \in H^1_0(0, T)$$

and $h_2(t) = b\widehat{w}_x(t, L) + a\widehat{z}_x(t, L)$, $g_2(t) = ab\widehat{w}_x(t, L) + \widehat{z}_x(t, L)$. Above, $\langle \cdot, \cdot\rangle$ denotes the duality product between $H^{-1}(0, T)$ and $H^1_0(0, T)$.

Remark that $h_1, g_1 \in H^1_0(0, T)$ and $h_2, g_2 \in L^2(0, T)$. Hence, in order to get the controllability result it is sufficient to prove that J has at least one minimum point. But J is a continuous and convex function in $(L^2(0, L))^2$. Which guarantees that the map J has a minimum if it is coercive. This is a direct consequence of the observability inequality given by Theorem 3.1 and the proof ends. □

4. Boundary Controllability Result: The Nonlinear System

Now we can study the controllability of the nonlinear system (1). Note that the solutions of (1) can be written as

$$(u, v) = S(t)(u^0, v^0) + (\varphi, \psi) + (\eta, \zeta), \tag{96}$$

where $S(t)$ is the semigroup associated to the linear part of the system and $(\varphi, \psi), (\eta, \zeta)$ verify

$$\begin{cases} \varphi_t + \varphi_{xxx} + a_3\psi_{xxx} = f_1, & \text{in } (0, T) \times (0, L) \\ b_1\psi_t + r\psi_x + b_2a_3\varphi_{xxx} + \psi_{xxx} = f_2, & \text{in } (0, T) \times (0, L) \\ \varphi(t, 0) = 0, \quad \varphi(t, L) = 0, \quad \varphi_x(t, L) = 0, & \text{on } (0, T), \\ \psi(t, 0) = 0, \quad \psi(t, L) = 0, \quad \psi_x(t, L) = 0, & \text{on } (0, T), \\ \varphi(0, x) = 0, \quad \psi(0, x) = 0, & \text{on } (0, L), \end{cases} \tag{97}$$

and

$$\begin{cases} \eta_t + \eta_{xxx} + a_3\zeta_{xxx} = 0, & \text{in } (0,T)\times(0,L) \\ b_1\zeta_t + r\zeta_x + b_2a_3\eta_{xxx} + \zeta_{xxx} = 0, & \text{in } (0,T)\times(0,L) \\ \eta(t,0) = 0, \quad \eta(t,L) = h_1, \quad \eta_x(t,L) = h_2, & \text{on } (0,T), \\ \zeta(t,0) = 0, \quad \zeta(t,L) = g_1, \quad \zeta_x(t,L) = g_2, & \text{on } (0,T), \\ \eta(0,x) = 0, \quad \zeta(0,x) = 0, & \text{on } (0,L), \end{cases} \tag{98}$$

respectively, with $f_1 = -(uu_x + a_1vv_x + a_2(uv)_x)$ and $f_2 = -(vv_x + b_2a_2uu_x + b_2a_1(uv)_x)$.

In order to simplify the notation, we consider the following spaces:

$$\begin{aligned} X &= L^2(0,T,(H^1(0,L))^2)\cap C([0,T];(L^2(0,L))^2), \\ Y &= L^1(0,T,(L^2(0,L))^2), \\ Z &= L^2(0,T,(H^1(0,L))^2). \end{aligned} \tag{99}$$

The following results will be needed to study the solutions of (97).

Proposition 4.1. *If $u, v \in L^2(0,T;H^1(0,L))$, then uu_x, vv_x, uv_x, $u_xv \in L^1(0,T;L^2(0,L))$, the map*

$$(u,v)\in L^2(0,T,(H^1(0,L))^2) \to (uu_x, vv_x, uv_x, u_xv) \in L^1(0,T,(L^2(0,L))^4)$$

is continuous and there exists a constant $c > 0$ such that

$$\begin{aligned} &\|(uu_x, vv_x, uv_x, u_xv)\|_{L^1(0,T;(L^2(0,L))^4)} \\ &\qquad \le c[\|u\|^2_{L^2(0,T;H^1(0,L))} + \|v\|^2_{L^2(0,T;H^1(0,L))}]. \end{aligned} \tag{100}$$

Moreover, for any $(f_1, f_2) \in L^1(0,T,(L^2(0,L))^2)$ there exists a unique solution

$$(\varphi,\psi)\in L^2(0,T,(H^1(0,L))^2)\cap C([0,T];(L^2(0,L))^2)$$

of (97) and the map

$$\begin{aligned} &(f_1,f_2)\in L^1(0,T,(L^2(0,L))^2) \\ &\qquad \to (\varphi,\psi)\in L^2(0,T,(H^1(0,L))^2)\cap C([0,T];(L^2(0,L))^2) \end{aligned}$$

is continuous.

Proof. The proof is analogous to the proof of [24, Proposition 4.1] and therefore we omit it. □

Let us now define the maps

$$\Theta_1 : L^1(0,T;(L^2(0,L))^2) \to C([0,T];(L^2(0,L))^2)\cap L^2(0,T;(H^1(0,L))^2),$$
$$\Theta_1(f_1,f_2) = (\varphi,\psi),$$

where (φ,ψ) is the unique solution of (97) with nonhomogeneous term (f_1, f_2) and

$$\Theta_2 : (H^1_0(0,T))^2 \times (L^2(0,T))^2 \to C([0,T];(L^2(0,L))^2)\cap L^2(0,T;(H^1(0,L))^2),$$
$$\Theta_2((h_1,g_1),(h_2,g_2)) = (\eta,\zeta),$$

where (η,ζ) is the unique solution (98) with nonhomogeneous terms (h, g).

From Proposition 4.1 and Theorem 2.4, it follows that Θ_1 and Θ_2 are continuous, linear and well-defined maps. Therefore, the existence and uniqueness of solutions of the nonlinear system (1) can be proved easily if the initial data and the boundary conditions are small enough.

Theorem 4.1. *There exists* $\delta > 0$ *such that for any*

$$\|(u^0, v^0)\|_{(L^2(0,L))^2} \le \delta \quad \text{and} \quad \|(h_1, g_1)\|_{(H_0^1(0,T))^2}, \|(h_2, g_2)\|_{(L^2(0,T))^2} \le \delta,$$

system (1) *has a unique solution*

$$(u, v) \in C([0,T]; (L^2(0,L))^2) \cap L^2(0,T; (H^1(0,L))^2) \cap H^1(0,T; (H^{-2}(0,L))^2).$$

Proof. Let us define the map

$$\begin{aligned} G : L^2(0,T; (H^1(0,L))^2) \to L^2(0,T; (H^1(0,L))^2) \cap C([0,T]; (L^2(0,L))^2) \\ G(u,v) = S(\cdot)(u^0, v^0) + \Theta_1(f_1, f_2) + \Theta_2((h_1, g_1), (h_2, g_2)), \end{aligned} \tag{101}$$

where

$$\begin{aligned} f_1 &= -(uu_x + a_1 vv_x + a_2(uv)_x), \\ f_2 &= -(vv_x + b_2 a_2 uu_x + b_2 a_1 (uv)_x). \end{aligned} \tag{102}$$

By using the continuity of the maps Θ_1 and Θ_2 and estimate (100), it is easy to see that G has a unique fixed point $(u, v) \in L^2(0,T; (H^1(0,L))^2) \cap C([0,T]; (L^2(0,L))^2)$. This fixed point is the unique solution of system (1). □

Now we can prove the main result of this work.

Proof of Theorem 1.1. To prove this result we apply a fixed point argument for a suitable map. First, let us define

$$\begin{gathered} \Gamma : (L^2(0,L))^2 \to (H_0^1(0,T))^2 \times (L^2(0,T))^2, \\ \Gamma(u^1, v^1) = (\vartheta_1, \tau_1, \vartheta_2, \tau_2) \end{gathered}$$

where $\vartheta_1, \tau_1 \in H_0^1(0,T)$ and $\vartheta_2, \tau_2 \in L^2(0,T)$ are the controls given by Theorem 3.2 which lead the solution of (74)–(75) from the initial data $(0,0)$ to the final state (u^1, v^1).

More precisely, if $(\widehat{w}^1, \widehat{z}^1) \in (L^2(0,L))^2$ is the minimizer of the functional J defined in Theorem 3.2 and $(\widehat{w}, \widehat{z})$ is the solution of the backward system (73) with initial data $(\widehat{w}^1, \widehat{z}^1)$, then $\vartheta_1, \tau_1 \in H_0^1(0,T)$ and $\vartheta_2, \tau_2 \in L^2(0,T)$ are given by

$$\begin{aligned} \langle b\widehat{w}_{xx}(t,L) + a\widehat{z}_{xx}(t,L), \psi\rangle &= \int_0^T (\vartheta_1)_t(t)\psi_t(t)dt, \quad \forall\, \psi \in H_0^1(0,T), \\ \langle ab\widehat{w}_{xx}(t,L) + \widehat{z}_{xx}(t,L), \psi\rangle &= \int_0^T (\tau_1)_t(t)\psi_t(t)dt, \quad \forall\, \psi \in H_0^1(0,T), \\ \vartheta_2(t) &= b\widehat{w}_x(t,L) + a\widehat{z}_x(t,L), \\ \tau_2(t) &= ab\widehat{w}_x(t,L) + \widehat{z}_x(t,L). \end{aligned}$$

Since $J(\widehat{w}^1, \widehat{z}^1) \leq J(0,0) = 0$, from observability inequality (78) we deduce that Γ is continuous.

We define now the operator

$$F : L^2(0,T;(H^1(0,L))^2) \to L^2(0,T;(H^1(0,L))^2) \cap C([0,T];(L^2(0,L))^2)$$
$$F(u,v) = S(\cdot)(u^0,v^0) + \Theta_2 \circ \Gamma((u^1,v^1) - S(T)(u^0,v^0) + \Theta_1(-f_1,-f_2)(T,\cdot)) + \Theta_1(f_1,f_2),$$

where (f_1,f_2) are given by (102).

Remark that, if (u,v) is a fixed point of F, then (u,v) is a solution of (1) and satisfies

$$(u(T,x), v(T,x)) = (u^1, v^1),$$

that is, system (1) is controllable by (u^1,v^1).

We prove that there exists $\delta > 0$, small enough, such that if

$$\|(u^0,v^0)\|_{(L^2(0,L))^2} \leq \delta \quad \text{and} \quad \|(u^1,v^1)\|_{(L^2(0,L))^2} \leq \delta,$$

the map F has a fixed point. To do this, it is sufficient to show that there exists $R > 0$, with the following properties:

- $F(\overline{B}(0,R)) \subset \overline{B}(0,R) \subset L^2(0,T;(H^1(0,L))^2)$.
- There exits a constant $c \in (0,1)$ such that

$$\|F(u,v) - F(\widehat{u},\widehat{v})\|_X \leq c\|(u,v) - (\widehat{u},\widehat{v})\|_X, \quad \forall\, (u,v),(\widehat{u},\widehat{v}) \in \overline{B}(0,R)$$

where $\overline{B}(0,R)$ is the closed ball of radius R in $L^2(0,T;(H^1(0,L))^2)$. Since Θ_1, Θ_2 and Γ are continuous, there exist positive constants K_1, K_2, K such that

$$\begin{aligned}
\|\Theta_1(f_1,f_2)\|_X &\leq K_1\|(f_1,f_2)\|_Y, \\
\|\Theta_2((h_1,g_1),(h_2,g_2))\|_X &\leq K_2\|((h_1,g_1),(h_2,g_2))\|_{(H_0^1(0,T))^2 \times (L^2(0,T))^2}, \\
\|\Gamma(u^1,v^1)\|_{(H_0^1(0,T))^2 \times (L^2(0,T))^2} &\leq K\|(u^1,v^1)\|_{(L^2(0,T))^2}.
\end{aligned} \tag{103}$$

Let $R > 0$ (R will be chosen latter on) and let $(u,v) \in \overline{B}(0,R) \subset L^2(0,T;(H^1(0,L))^2)$. We have that

$$\begin{aligned}
\|F(u,v)\|_X &\leq \|(u^0,v^0)\|_{(L^2(0,L))^2} + K_2K\|(u^1,v^1) - S(T)(u^0,v^0) \\
&\quad + \Theta_1(-f_1,-f_2)(T,\cdot)\|_{(L^2(0,L))^2} + K_1\|(f_1,f_2)\|_Y \\
&\leq \delta + 2K_2K\delta + K_1KK_2C'\|(u,v)\|_X^2 + C'K_1\|(u,v)\|_X^2 \\
&\leq \delta + 2K_2K\delta + (KK_2+1)C'K_1R^2.
\end{aligned} \tag{104}$$

Therefore, $F(\overline{B}(0,R)) \subset \overline{B}(0,R)$ for any $R > 0$ such that

$$(1 + 2K_2K)\delta + (KK_2+1)K_1C'R^2 \leq R. \tag{105}$$

On the other hand, since

$$F(u,v) - F(\widehat{u},\widehat{v}) = \Theta_1((f_1,f_2) - (\widehat{f}_1,\widehat{f}_2)) + \Theta_2 \circ \Gamma(\Theta_1((\widehat{f}_1,\widehat{f}_2) - (f_1,f_2))),$$

we obtain

$$\begin{aligned}\|F(u,v) - F(\widehat{u},\widehat{v})\|_X &\leq K_1 C' \|(u,v) - (\widehat{u},\widehat{v})\|_X^2 + K_1 K_2 K C' \|(u,v) - (\widehat{u},\widehat{v})\|_X^2 \\ &\leq 2K_1 C' R(1 + KK_2)\|(u,v) - (\widehat{u},\widehat{v})\|_X. \end{aligned} \tag{106}$$

Hence, F is a contraction if R verifies

$$2RK_1C'(1 + KK_2) < 1. \tag{107}$$

Now, if R satisfies (107), by choosing

$$\delta = \frac{R}{2(1 + 2K_2K)}, \tag{108}$$

we have that (105) also holds. Thus, for every $(u^0, v^0), (u^1, v^1)$ such that

$$\|(u^0, v^0)\|_{(L^2(0,L))^2} \leq \delta, \quad \|(u^1, v^1)\|_{(L^2(0,L))^2} \leq \delta,$$

the map F has a fixed point and the proof ends.

5. Final Comments and Remarks

- In [18, 19], the authors consider a coupled system of Kuramoto–Sivashinsky (KS) equations in a bounded interval depending on a suitable parameter $\nu > 0$. Introducing appropriate boundary conditions, they show that the energy of the solutions of the corresponding damped model decays exponentially as $t \to +\infty$, uniformly with respect to the parameter $\nu > 0$. As ν tends to zero, they obtain a coupled system of Korteweg–de Vries (KdV) equations for which the energy tends to zero exponentially as well. The decay holds except when the length of the space interval L lies in a set of critical lengths. It would be interesting to pursue the analysis of [18, 19] in the context of the exact boundary controllability.
- Recently, Cerpa and Crépeau proved in [8] that, for some critical values of the length L, the corresponding nonlinear KdV equation is exact controllable. This suggests that the controllability of system (1) may hold with a smaller number of controls but this problem is open.
- We mention that if we consider the boundary conditions

  $$u(t,0) = h_1, u(t,L) = u_x(t,L) = 0 \quad \text{and} \quad v(t,0) = h_2, v(t,L) = v_x(t,L) = 0,$$

  with controls h_1 and h_2, it is possible to prove that the adjoint system satisfies a unique continuation property for any length $L > 0$. This implies an approximate controllability result for the associated linear system. However, we could not prove the exact boundary controllability for this case.

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