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CONTROL AND STABILIZATION OF A FAMILY OF BOUSSINESQ SYSTEMS

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Abstract. This paper studies the internal controllability and stabilizability of a family of Boussinesq systems recently proposed by J. L. Bona, M. Chen and J.-C. Saut to describe the two-way propagation of small amplitude gravity waves on the surface of water in a canal. The space of the controllable data for the associated linear system is determined for all values of the four parameters. As an application of this newly established exact controllability, some simple feedback controls are constructed such that the resulting closed-loop systems are exponentially stable. When the parameters are all different from zero, the local exact controllability and stabilizability of the nonlinear system are also established.

1. **Introduction.**

Considered in this paper is a family of Boussinesq systems

$$\begin{cases} \eta_t + w_x + (\eta w)_x + a w_{xxx} - b\eta_{txx} = 0, \\ \\ w_t + \eta_x + w w_x + c\eta_{xxx} - d w_{txx} = 0, \end{cases} \tag{1.1}$$

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which were recently proposed by Bona, Chen and Saut [5, 6] to describe the two-way propagation of small amplitude gravity waves on the surface of water in a canal. The systems may also arise, for example, when modeling the propagation of long-crested waves on large lakes or on the oceans. Contrary to the classical Korteweg-de Vries equation which assumes that the waves travel only in one direction, system (1.1) is free of the presumption of unidirectionality and may have a wider range of applicability.

In (1.1), η is the elevation from the equilibrium position and $w = w_\theta$ is the horizontal velocity in the flow at height θh, where h is the undisturbed depth of the liquid. The parameters a, b, c and d in (1.1) are not completely independent. They are, in fact, required to fulfill the relations

$$a + b = \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right), \quad c + d = \frac{1}{2}\left(1 - \theta^2\right) \geq 0, \tag{1.2}$$

where $\theta \in [0, 1]$ specifies which horizontal velocity the variable w represents (cf. [5]). Consequently,

$$a + b + c + d = \frac{1}{3}.$$

A detailed analysis of this family of Boussinesq systems when posed on the whole real axis $\mathbb{R}$ has been conducted by Bona, Chen and Saut [5, 6], in which they have established, in particular, various well-posedness results for the initial value problem (IVP) for system (1.1). In this paper, we will focus on the system (1.1) posed on a finite interval $(0, L)$ with periodic boundary conditions imposed on η and w. As in [5], we will restrict our attention to system (1.1) with its parameters a, b, c, d satisfying one of the assumptions below.

C1. $b, d \geq 0$, $a \leq 0$, $c \leq 0$;
C2. $b, d \geq 0$, $a = c > 0$.

As it has been proved in [5], the IVP of the linear systems associated to (1.1) when posed on $\mathbb{R}$ is well-posed if either C1 or C2 is satisfied.

Since the length of the interval is irrelevant, we will assume $L = 2\pi$ through all our study. In this new setting, the well-posedness of (1.1) and their associated homogeneous and nonhomogeneous linear systems will be studied. However, the main task of this paper is to study control and stabilization problems for system (1.1) by means of some localized control actions. More precisely, we will consider the following nonhomogeneous systems

$$\begin{cases} \eta_t + w_x + (\eta w)_x + a w_{xxx} - b\eta_{txx} = f & \text{for } x \in (0, 2\pi),\ \ t \in (0, T), \\ w_t + \eta_x + w w_x + c\eta_{xxx} - d w_{txx} = g & \text{for } x \in (0, 2\pi),\ \ t \in (0, T) \end{cases} \tag{1.3}$$

with the periodic boundary conditions

$$\begin{cases} \frac{\partial^r \eta}{\partial x^r}(t, 0) = \frac{\partial^r \eta}{\partial x^r}(t, 2\pi) & \text{for } t \in (0, T),\ 0 \leq r \leq r_0, \\ \frac{\partial^q w}{\partial x^q}(t, 0) = \frac{\partial^q w}{\partial x^q}(t, 2\pi) & \text{for } t \in (0, T),\ 0 \leq q \leq q_0. \end{cases} \tag{1.4}$$

The number of boundary conditions depends on the values of the parameters. For instance, if $a = b = 0$ then $r_0 = 0$, if $a = 0$ and $b \neq 0$ then $r_0 = 1$ and if $a \neq 0$ then $r_0 = 2$. The values of q_0 depend on the parameters c and d in a similar way. The forcing functions f and g, which will be considered as control inputs, are assumed to be supported in ω, a nonempty open subinterval of $(0, 2\pi)$. We will be mainly interested in the following two problems for system (1.3)-(1.4).

Problem 1 (Exact controllability): *Given $T > 0$, the initial state (η^0, w^0) and the terminal state (η^1, w^1) in an appropriate space, can one find controls f and g in a suitable space such that (1.3) admits a solution $(\eta(x,t), w(x,t))$ satisfying the boundary conditions (1.4) and*

$$(\eta(x,0), w(x,0)) = (\eta^0(x), w^0(x)), \qquad (\eta(x,T), w(x,T)) = (\eta^1(x), w^1(x))?$$

Problem 2 (Stabilizability): *Can one find some (linear) feedback controls*

$$f = K_1(\eta, w), \qquad g = K_2(\eta, w)$$

such that the resulting closed-loop system

$$\begin{cases} \eta_t + w_x + (\eta w)_x + a w_{xxx} - b\eta_{txx} = K_1(\eta, w), & x \in (0, 2\pi), \ \ t \in (0, T), \\ \\ w_t + \eta_x + w w_x + c\eta_{xxx} - d w_{txx} = K_2(\eta, w), & x \in (0, 2\pi), \ \ t \in (0, T) \end{cases}$$

is stabilized, i.e., its solution (η, w) tends to zero in an appropriate space as $t \to \infty$?

Control and stabilization of the dispersive wave equations have been studied intensively in the last decade. In particular, for the Korteweg-de Vries (KdV) equation, the study of control and stabilization problems began with the works of Russell [36] and Zhang [43]. In [37, 38], Russell and Zhang considered the KdV equation posed on the finite interval $(0, 2\pi)$ with periodic boundary conditions and with localized control action. They showed in [37] that the associated linearized system is exactly controllable and exponentially stabilizable. Aided by then the newly discovered Bourgain smoothing property [8] for the KdV equation, those results were extended to the (nonlinear) KdV equation in [38] assuming its solutions have small amplitude.

In [31], Rosier studied the boundary controllability of the KdV equation posed on the finite interval $(0, L)$ with the Dirichlet type boundary conditions:

$$\begin{cases} u_t + u_x + u u_x + u_{xxx} = 0, \qquad x \in (0, L), \quad t \geq 0, \\ \\ u(0,t) = h_1(t), \quad u(L,t) = h_2(t), \quad u_x(L,t) = h_3(t), \qquad t \geq 0, \end{cases} \tag{1.5}$$

where the boundary value functions $h_j(t)$, $j = 1, 2, 3$ are considered as control inputs. Using only a single control input $h_2(t)$ (letting $h_1(t) = h_3(t) \equiv 0$) and assuming that the length L of the domain $(0, L)$ does not belong to the set

$$\mathcal{K} := \left\{ 2\pi\sqrt{\frac{k^2 + kl + l^2}{3}}; \quad k, \ l \in \mathbb{N}^* \right\}, \tag{1.6}$$

Rosier showed that the associated linear system is exactly controllable in the space $L^2(0, L)$ using the Hilbert Uniqueness Method (HUM), and that the nonlinear system is *locally* exactly controllable in the space $L^2(0, L)$. Interestingly, while Rosier showed in [31] that the linear system associated to (1.5) is *not* exactly controllable if $L \in \mathcal{K}$, it has been proved recently by Coron and Crépeau [10] (cf. also Crépeau [11]) that the nonlinear system (1.5) is locally exactly controllable in the space $L^2(0, L)$ for some values of L in the set $\mathcal{K}$.

Using all the three boundary control inputs and a very different approach, Zhang [46] showed that (1.5) is locally controllable in the space $H^s(0, L)$ for any $s \geq 0$

without any restriction on L. It should be pointed out that the exact controllability result presented in [46] is different from that in [31]; system (1.5) is shown to be exactly controllable in a neighborhood of any given smooth solution of the KdV equation in [46] while (1.5) is demonstrated in [31] to be exactly controllable in a neighborhood of the zero solution of the KdV equation.

To stabilize system (1.5), the simplest feedback control law to use is

$$h_1(t) = h_2(t) = h_3(t) \equiv 0, \qquad \forall\, t \geq 0. \tag{1.7}$$

With the help of an observability inequality established by Rosier in [31], Perla Menzala, Vasconcellos, and Zuazua [28] showed, assuming $L \notin \mathcal{K}$, that the linear system associated to (1.5) satisfying (1.7) is exponentially stable and that small amplitude solutions of the nonlinear system (1.5) satisfying (1.7) decay exponentially as $t \to \infty$. In order to stabilize large amplitude solutions and to remove the length restriction on L, Perla Menzala, Vasconcellos, and Zuazua [28] introduced an extra localized feedback control to (1.5) resulting in the following closed-loop system:

$$\begin{cases} u_t + u_x + uu_x + a(x)u + u_{xxx} = 0, \qquad x \in (0, L),\ t \geq 0, \\ \\ \qquad u(0,t) = 0, \quad u(L,t) = 0, \quad u_x(L,t) = 0,\ t \geq 0, \end{cases} \tag{1.8}$$

where $a = a(x)$ is a nonnegative smooth function supported in Ω, a subdomain of $[0, L]$. Under the assumption

$$0 \in \Omega, \quad L \in \Omega, \tag{1.9}$$

they showed that all solutions of (1.8) decay exponentially to 0 as $t \to \infty$. In addition, they conjectured that the assumption (1.9) could be removed. This is indeed the case; the conjecture has been confirmed recently by Pazoto [29] (see also Rosier and Zhang [34]). The stability result of Perla Menzala, Vasconcellos, and Zuazua has been extended by Rosier and Zhang [34] to the generalized KdV equation

$$\begin{cases} u_t + u_x + u^p u_x + a(x)u + u_{xxx} = 0, \qquad x \in (0, L),\ t \geq 0, \\ \\ u(0,t) = 0, \quad u(L,t) = 0, \quad u_x(L,t) = 0,\ t \geq 0 \end{cases} \tag{1.10}$$

with $1 \leq p < 4$.

Many other dispersive equations have also been studied for their control and stabilization problems (the reader is referred to [4, 12, 19, 22, 23, 25, 26, 32, 33, 38, 41, 44, 45, 47] and the references therein). Among them, the Benjamin-Bona-Mahony (BBM) equation

$$w_t + w_x + ww_x - w_{xxt} = 0$$

deserves a special attention. As an alternative of the KdV equation, the BBM equation is also considered as a model in nonlinear studies whenever one wishes to include and balance a weak nonlinearity and weak dispersive effects [3, 7]. However, while its mathematical theory such as the well-posedness, stability analysis and long time behavior is easier to establish than that of the KdV equation, its control theory seems harder to study. In contrast to some significant progress made for the KdV equation, the (nonlinear) BBM is still not known to possess any controllability. An exception is the boundary control of the linearized BBM equation

$$\begin{cases} w_t + w_x - w_{txx} = 0, \qquad x \in (0,1), \quad t \geq 0, \\ \\ u(0,t) = 0, \qquad u(1,t) = h(t) \qquad t \geq 0, \end{cases} \tag{1.11}$$

which has been shown by Micu [25] to have poor controllability; the system is not even *spectrally controllable* though it possesses approximate boundary controllability (cf. [43, 25]). It suggests that the BBM equation may not be exactly boundary controllable.

Depending on the values of its parameters, system (1.3) couples two equations that may be of KdV or BBM types. It is therefore interesting to see to which extent the controllability properties of each equation type are maintained and/or improved. Our strategy of study of (1.3) is to consider first the associated linear systems and prove the corresponding controllability and stabilizability properties. This linear problem is interesting by itself. As we have pointed out earlier, depending on the parameters a, b, c and d, system (1.3) may couple two KdV-type equations, two BBM-type equations, or one KdV and one BBM-type equations. It is not unusual that in the first case (KdV case) some good controllability properties are proved whereas in the second case (BBM case) there are no such controllability properties. What happens in the last case is *a priori* less clear. We will however prove that the system is controllable in that case. Different approaches will be used to establish the exact controllability depending on whether we employ a single control input or two control inputs. If only a single control action is used, the exact controllability will be established via the Hilbert Uniqueness Method (HUM) (cf. [20]). If both control actions are used, a stronger exact controllability result will be obtained by using the classical moment method (cf. [35]). As an application of the established exact controllability, some simple feedback controls are constructed such that the resulting closed-loop systems are shown to be exponential stable.

The contraction mapping principle will be used to extend the results obtained for the associated linear systems to the nonlinear systems. However, only the case when all the parameters of the system are different from 0 is considered here. Remark that (1.1) may be written in the following equivalent form

$$\begin{cases} \eta_t - \dfrac{a}{b}w_x + \left(1+\dfrac{a}{b}\right)(I-b\partial_x^2)_p^{-1}w_x = -(I-b\partial_x^2)_p^{-1}((\eta w)_x), \\[2ex] w_t - \dfrac{c}{d}\eta_x + \left(1+\dfrac{c}{d}\right)(I-d\partial_x^2)_p^{-1}\eta_x = -(I-d\partial_x^2)_p^{-1}(ww_x) \end{cases} \tag{1.12}$$

for $x\in(0,2\pi)$, $t\in(0,T)$. In (1.12), a very important smoothing effect on the nonlinear term may be noted if b and d are different from zero, which allows us to apply the contraction mapping principle.

If $\alpha>0$, the operator $(I-\alpha\partial_x^2)_p^{-1}$ from (1.12) is defined in the following way:

$$(I-\alpha\partial_x^2)_p^{-1}\varphi = v \Leftrightarrow \begin{cases} v-\alpha v_{xx}=\varphi \quad \text{in } (0,2\pi), \\[2ex] v(0)=v(2\pi), \quad v_x(0)=v_x(2\pi). \end{cases}$$

Since for any $\varphi\in L^2(0,2\pi)$, the elliptic equation from the right has a unique solution

$$v\in H_p^2(0,2\pi)=\{v\in H^2(0,2\pi)\big|\ v(0)=v(2\pi),\ v_x(0)=v_x(2\pi)\},$$

$(I-\alpha\partial_x^2)_p^{-1}$ is a well-defined, compact operator in $L^2(0,2\pi)$.

The rest of the paper is organized as follows. Section 2 is devoted to the study of the associated linear systems and is split into three subsections. The well-posedness problem will be discussed in subsection 2.1. Control and stabilization problems for the linear systems with a single control action will be addressed in subsection 2.2

while the same problems for the linear systems with two control actions will be investigated in subsection 2.3. The results obtained in section 2 will be extended to the nonlinear systems with nonzero parameters in section 3. The paper is finished with section 4 which provides some concluding remarks and some open problems for further study.

2. **Linear systems.** In this section we study the initial-boundary-value problem (IBVP) for the linear system associated to (1.3), namely

$$\begin{cases} \eta_t + w_x + aw_{xxx} - b\eta_{txx} = f & \text{for } x \in (0, 2\pi),\ \ t \in (0, T), \\ \\ w_t + \eta_x + c\eta_{xxx} - dw_{txx} = g & \text{for } x \in (0, 2\pi),\ \ t \in (0, T) \end{cases} \tag{2.1}$$

with the periodic boundary conditions

$$\begin{cases} \frac{\partial^r \eta}{\partial x^r}(t, 0) = \frac{\partial^r \eta}{\partial x^r}(t, 2\pi) & \text{for } t \in (0, T),\ 0 \le r \le r_0, \\ \\ \frac{\partial^q w}{\partial x^q}(t, 0) = \frac{\partial^q w}{\partial x^q}(t, 2\pi) & \text{for } t \in (0, T),\ 0 \le q \le q_0 \end{cases} \tag{2.2}$$

and the initial condition

$$\eta(0, x) = \eta^0(x), \qquad w(0, x) = w^0(x) \qquad \text{for } \ x \in (0, 2\pi). \tag{2.3}$$

Its well-posedness in a suitable classical Banach space will be investigated in subsection 2.1. Then, in subsections 2.2 and 2.3, considering f and g as control inputs acting only on a subdomain ω of $(0, 2\pi)$, we will study its control and stabilization problems. In particular, exact controllability and stabilizability will be established in subsection 2.2 for system (2.1) with a single control while subsection 2.3 will be devoted to study system (2.1) with two control inputs for its control and stabilization problems.

2.1. **Well-posedness.** We first introduce a few notations. Given any $v \in L^2(0, 2\pi)$ and $k \in \mathbb{Z}$, we denote by $\widehat{v}_k$ the $k-$Fourier coefficient of v,

$$\widehat{v}_k = \frac{1}{2\pi} \int_0^{2\pi} v(x) e^{-ikx} dx$$

and for any $m \in \mathbb{N}$ we define the space

$$H_p^m(0, 2\pi) = \left\{ v \in L^2(0, 2\pi) \ \middle| \ v = \sum_{k \in \mathbb{Z}} \widehat{v}_k e^{ikx},\ \sum_{k \in \mathbb{Z}} |\widehat{v}_k|^2 (1 + k^2)^m < \infty \right\}$$

which is a Hilbert space with respect to the inner product

$$(v, h)_m = \sum_{k \in \mathbb{Z}} \widehat{v}_k \overline{\widehat{h}_k} (1 + k^2)^m. \tag{2.4}$$

The norm corresponding to (2.4) is denoted by $||\ ||_m$. It may be seen that

$$H_p^m(0, 2\pi) = \left\{ v \in H^m(0, 2\pi) \ \middle| \ \frac{\partial^r v}{\partial x^r}(0) = \frac{\partial^r v}{\partial x^r}(2\pi),\ 0 \le r \le m - 1 \right\},$$

where $H^m(0, 2\pi)$ stands for the classical Sobolev space of exponent m in $(0, 2\pi)$. We also consider the closed subspace

$$H_{0,p}^m(0, 2\pi) = \left\{ v \in H_p^m(0, 2\pi) \ \middle| \ v = \sum_{k \in \mathbb{Z}} \widehat{v}_k e^{ikx},\ \widehat{v}_0 = 0 \right\}.$$

We may extend the definition of $H_p^m(0,2\pi)$ to the case $m=s$, a real number, by setting

$$H_p^s(0,2\pi)=\left\{v=\sum_{k\in\mathbb{Z}}\widehat{v}_k e^{ikx}\in H^s(0,2\pi)\;\middle|\;\sum_{k\in\mathbb{Z}}|\widehat{v}_k|^2(1+k^2)^s<\infty\right\},$$

where the convergence of the series is in the Sobolev space $H^s(0,2\pi)$. For any real number s, $H_p^s(0,2\pi)$ may also be seen as a Hilbert space with respect to the inner product defined by (2.4) with m replaced by s. In particular, for any $v\in H_p^s(0,2\pi)$,

$$\|v\|_s=\left(\sum_{k\in\mathbb{Z}}|\widehat{v}_k|^2(1+k^2)^s\right)^{\frac{1}{2}}.$$

The following map is a duality product between $H_p^s(0,2\pi)$ and $H_p^{-s}(0,2\pi)$ for any $s\geq 0$

$$\langle f,g\rangle_s=\sum_{k\in\mathbb{Z}}\widehat{f}_k\overline{\widehat{g}_k},\quad \forall f\in H_p^s(0,2\pi),\ g\in H_p^{-s}(0,2\pi). \tag{2.5}$$

It follows that $H_p^{-s}(0,2\pi)$ is the topological dual of $H_p^s(0,2\pi)$.

Assume that the initial data in (2.3) and the forcing terms in (2.1) are given by

$$(\eta^0,w^0)=\sum_{k\in\mathbb{Z}}\left(\widehat{\eta}_k^0,\widehat{w}_k^0\right)e^{ikx},\qquad (f,g)(t)=\sum_{k\in\mathbb{Z}}\left(\widehat{f}_k(t),\widehat{g}_k(t)\right)e^{ikx}.$$

At least formally, the solution of (2.1) may be written as

$$(\eta,w)(t,x)=\sum_{k\in\mathbb{Z}}(\widehat{\eta}_k(t),\widehat{w}_k(t))e^{ikx}$$

where $(\widehat{\eta}_k(t),\widehat{w}_k(t))$ fulfill

$$\begin{cases}(\widehat{\eta}_k)_t+ik\widehat{w}_k-iak^3\widehat{w}_k+bk^2(\widehat{\eta}_k)_t=\widehat{f}_k, & t\in(0,T),\\ (\widehat{w}_k)_t+ik\widehat{\eta}_k-ick^3\widehat{\eta}_k+dk^2(\widehat{w}_k)_t=\widehat{g}_k, & t\in(0,T),\\ \widehat{\eta}_k(0)=\widehat{\eta}_k^0,\qquad \widehat{w}_k(0)=\widehat{w}_k^0. \end{cases} \tag{2.6}$$

System (2.6) may be solved explicitly. If we set

$$\omega_1=\omega_1(k)=\frac{1-ak^2}{1+bk^2},\quad \omega_2=\omega_2(k)=\frac{1-ck^2}{1+dk^2},\quad A(k)=\begin{pmatrix}0&\omega_1\\ \omega_2&0\end{pmatrix},$$

it is easy to see that (2.6) is equivalent to

$$\begin{pmatrix}\widehat{\eta}_k\\ \widehat{w}_k\end{pmatrix}_t+ikA(k)\begin{pmatrix}\widehat{\eta}_k\\ \widehat{w}_k\end{pmatrix}=\begin{pmatrix}\frac{1}{1+bk^2}\widehat{f}_k\\ \frac{1}{1+dk^2}\widehat{g}_k\end{pmatrix},\quad \begin{pmatrix}\widehat{\eta}_k\\ \widehat{w}_k\end{pmatrix}(0)=\begin{pmatrix}\widehat{\eta}_k^0\\ \widehat{w}_k^0\end{pmatrix}.$$

Hence, the solution of (2.6) is given by

$$\begin{pmatrix}\widehat{\eta}_k\\ \widehat{w}_k\end{pmatrix}(t)=e^{-iktA(k)}\begin{pmatrix}\widehat{\eta}_k^0\\ \widehat{w}_k^0\end{pmatrix}+\int_0^t e^{-ik(t-s)A(k)}\begin{pmatrix}\frac{1}{1+bk^2}\widehat{f}_k(s)\\ \frac{1}{1+dk^2}\widehat{g}_k(s)\end{pmatrix}ds. \tag{2.7}$$

Note that the eigenvalues of the matrix $A(k)$ are $\pm\sigma(k)$, with $\sigma(k) = \sqrt{\omega_1(k)\omega_2(k)}$, and are always real. By a standard diagonalizing procedure we get that

$$e^{-iktA(k)} = \begin{pmatrix} \cos[k\sigma(k)t] & -i\sqrt{\frac{\omega_1}{\omega_2}}\sin[k\sigma(k)t] \\ & \\ -i\sqrt{\frac{\omega_2}{\omega_1}}\sin[k\sigma(k)t] & \cos[k\sigma(k)t] \end{pmatrix}.$$

Consequently, the solution of (2.6) is given by

$$\left\{\begin{array}{l} \widehat{\eta}_k(t) = \cos[k\sigma(k)t]\widehat{\eta}_k^0 - i\sqrt{\frac{\omega_1}{\omega_2}}\sin[k\sigma(k)t]\widehat{w}_k^0 - \\ \\ \displaystyle\int_0^t \left\{ \frac{\cos[k\sigma(k)(t-s)]}{1+bk^2}\widehat{f}_k(s) - i\sqrt{\frac{\omega_1}{\omega_2}}\frac{\sin[k\sigma(k)(t-s)]}{1+dk^2}\widehat{g}_k(s)\right\} ds, \\ \\ \widehat{w}_k(t) = -i\sqrt{\frac{\omega_2}{\omega_1}}\sin[k\sigma(k)t]\widehat{\eta}_k^0 + \cos[k\sigma(k)(t)]\widehat{w}_k^0 + \\ \\ \displaystyle\int_0^t \left\{ -i\sqrt{\frac{\omega_2}{\omega_1}}\frac{\sin[k\sigma(k)(t-s)]}{1+bk^2}\widehat{f}_k(s) + \frac{\cos[k\sigma(k)(t-s)]}{1+dk^2}\widehat{g}_k(s)\right\} ds. \end{array}\right. \tag{2.8}$$

Remark 2.1. The eigenvalues of system (2.1) are given by

$$\lambda_k = ik\sigma(k), \ k \in \mathbb{Z}. \tag{2.9}$$

Note that not all the eigenvalues in (2.9) are different. If we count only the distinct eigenvalues, we obtain the sequence $(\lambda_k)_{k\in\mathbb{I}}$, where $\mathbb{I} \subseteq \mathbb{Z}$ has the property that $\lambda_{k_1} \neq \lambda_{k_2}$ for any $k_1, k_2 \in \mathbb{I}$. For each $k_1 \in \mathbb{Z}$ set

$$I(k_1) = \{k \in \mathbb{Z} \,:\, k\sigma(k) = k_1\sigma(k_1)\}$$

and $|I(k_1)| = m(k_1)$. We have the following properties of $m(k_1)$:

- $m(k_1) \le 6$. This is a consequence of the fact that $m(k_1)$ is less than the number of entire roots of the equation $x\sigma(x) = \alpha$, where α is an arbitrary real number. The roots of this equation are also roots of a polynomial of degree less or equal to 6.
- If the sequence of eigenvalues tends to infinity, there exists $k_1^* \in \mathbb{N}$ such that $m(k_1) = 1$ for all $|k_1| > k_1^*$. This is a consequence of the fact that the function $x\sigma(x)$ is strictly increasing for $|x|$ large enough.

The number of the eigenfunctions corresponding to an eigenvalue $\lambda_{k_1} \neq 0$ is $2m(k_1)$ for any $k_1 \in \mathbb{I}$. These eigenfunctions read then

$$\left(e^{ikx}, -\frac{\sigma(k)}{\omega_1}e^{ikx}\right), \quad \left(e^{-ikx}, \frac{\sigma(k)}{\omega_1}e^{-ikx}\right), \qquad k \in I(k_1).$$

On the other hand, the zero eigenvalue has multiplicity two, unless there exists some $k_0 \in \mathbb{Z} \setminus \{0\}$ such that $a = c = \frac{1}{k_0^2}$ in which case it is of multiplicity six with associated eigenfunctions

$$(1,0), \quad (0,1), \quad \left(e^{ik_0x}, 0\right), \quad \left(0, e^{ik_0x}\right), \quad \left(e^{-ik_0x}, 0\right), \quad \left(0, e^{-ik_0x}\right).$$

$\square$

For each $s \in \mathbb{R}$, we define the space

$$V^s = \left\{(\eta, w) \in H_p^s(0,2\pi) \times H_p^{s-2}(0,2\pi) \;\Big|\; ||(\eta,w)||_{V^s}^2 := ||\eta||_s^2 + ||\mathcal{H}w||_s^2 < \infty \right\}$$

where $||\ ||_s$ is the $H_p^s(0,2\pi)$ norm and the operator $\mathcal{H}$ is defined in the following way

$$\mathcal{H}\left(\sum_{k\in\mathbb{Z}} \widehat{v}_k e^{ikx}\right) = \sum_{k\in\mathbb{Z}} \sqrt{\frac{\omega_1}{\omega_2}} \widehat{v}_k e^{ikx}.$$

V^s is a Hilbert space with respect to the inner product

$$((f_1, f_2), (g_1, g_2))_{V^s} = (f_1, g_1)_s + (\mathcal{H}f_2, \mathcal{H}g_2)_s.$$

The space V^s depends on the values of the parameters a, b, c and d and more precisely on the value of $\sqrt{\frac{\omega_1}{\omega_2}}$. Let us introduce the number $l \in \mathbb{Z}$ with the property that

$$\sqrt{\frac{\omega_1}{\omega_2}} \sim C|k|^l \text{ as } |k| \to \infty \tag{2.10}$$

where C is a positive constant not depending on k. We have that

$$V^s = H_p^s(0,2\pi) \times H_p^{s+l}(0,2\pi).$$

Note that, depending on the parameters a, b, c and d, we may have an asymmetry of regularity of the two components of the space V^s. Note also that, in any case, $V^s \subseteq H^s(0,2\pi) \times H^{s-2}(0,2\pi)$. The following result uses formula (2.8) and introduces a C_0 group associated with system (2.1).

Theorem 2.2. *The family of linear operators $(S(t))_{t\in\mathbb{R}}$ defined by*

$$S(t)(\eta^0, w^0)(\eta(t), w(t)) = \sum_{k\in\mathbb{Z}} (\widehat{\eta}_k(t), \widehat{w}_k(t))e^{ikx}, \tag{2.11}$$

where the Fourier coefficients of $(\eta(t), w(t))$ are obtained from the ones of (η^0, w^0) by

$$\begin{cases} \widehat{\eta}_k(t) = \cos[k\sigma(k)t]\widehat{\eta}_k^0 - i\sqrt{\frac{\omega_1}{\omega_2}}\sin[k\sigma(k)t]\widehat{w}_k^0, \\ \\ \widehat{w}_k(t) = -i\sqrt{\frac{\omega_2}{\omega_1}}\sin[k\sigma(k)t]\widehat{\eta}_k^0 + \cos[k\sigma(k)t]\widehat{w}_k^0, \end{cases} \tag{2.12}$$

is a group of isometries in V^s, for any $s \in \mathbb{R}$.

Proof. First, let us prove that $S(t)$ is a well-defined linear and continuous operator for any $t \in \mathbb{R}$. If $(\eta^0, w^0) = \sum_{k\in\mathbb{Z}}(\widehat{\eta}_k^0, \widehat{w}_k^0)e^{ikx} \in V^s$, then we claim that the series $\sum_{k\in\mathbb{Z}} (\widehat{\eta}_k(t), \widehat{w}_k(t))\, e^{ikx}$ converges in $C([0,\infty), V^s)$. This is equivalent to say that the sequence

$$\mathcal{P} = \left(\sum_{|k|\le N} (\widehat{\eta}_k(t), \widehat{w}_k(t))\, e^{ikx}\right)_{N\ge 1}$$

is a Cauchy sequence in $C([0,\infty), V^s)$.

It is clear that $\mathcal{P} \subset C([0,\infty), V^s)$ and that

$$\sup_{t\in[0,\infty)} \left|\left| \sum_{N<|k|\le N+p} (\widehat{\eta}_k(t), \widehat{w}_k(t))\, e^{ikx} \right|\right|^2_{V^s}$$

$$= \sup_{t\in[0,\infty)} \sum_{N<|k|\le N+p} \left(|\widehat{\eta}_k(t)|^2 + \frac{\omega_1}{\omega_2} |\widehat{w}_k(t)|^2 \right) (1+k^2)^s$$

$$= \sum_{N<|k|\le N+p} \left(|\widehat{\eta}^0_k|^2 + \frac{\omega_1}{\omega_2} |\widehat{w}^0_k|^2 \right) (1+k^2)^s.$$

Thus $\mathcal{P}$ is a Cauchy sequence in $C([0,\infty), V^s)$ since $(\eta^0, w^0) \in V^s$. Hence the operator $S(t)$ is well-defined in V^s and $S(\cdot)(\eta^0, w^0) \in C([0,\infty), V^s)$. Moreover, since

$$\left|\left| \sum_{|k|\le N} (\widehat{\eta}_k(t), \widehat{w}_k(t))\, e^{ikx} \right|\right|_{V^s} = \sqrt{\sum_{|k|\le N} \left(|\widehat{\eta}^0_k|^2 + \frac{\omega_1}{\omega_2} |\widehat{w}^0_k|^2 \right) (1+k^2)^s},$$

$(S(t))_{t\in\mathbb{R}}$ is a family of linear and continuous operators which are also isometries. It is easy to see that $S(0) = I$, $S(t)\circ S(s) = S(t+s)$ for any $t, s \in \mathbb{R}$ and, in addition,

$$\left|\left| S(t)(\eta^0, w^0) - (\eta^0, w^0) \right|\right|^2_{V^s}$$

$$= \sum_{k\in\mathbb{Z}} \left((\cos[k\sigma(k)t] - 1)^2 + \sin^2[k\sigma(k)t] \right) \left[|\widehat{\eta}^0_k|^2 + \frac{\omega_1}{\omega_2} |\widehat{w}^0_k|^2 \right] (1+k^2)^s$$

$$= 4 \sum_{k\in\mathbb{Z}} \sin^2\left[\frac{k\sigma(k)t}{2}\right] \left[|\widehat{\eta}^0_k|^2 + \frac{\omega_1}{\omega_2} |\widehat{w}^0_k|^2 \right] (1+k^2)^s.$$

Consequently $\lim_{t\to 0} S(t)(\eta^0, w^0) = (\eta^0, w^0)$ in V^s and the proof is complete. □

Another important parameter in our analysis will be the number $e \in \mathbb{Z}$ defined in the following way

$$\sqrt{\omega_1\omega_2} \sim C|k|^e \text{ as } |k| \to \infty \tag{2.13}$$

where C is a positive constant not depending on k.

Theorem 2.3. *The infinitesimal generator of the group $(S(t))_{t\in\mathbb{R}}$ is the unbounded operator $(D(\mathcal{A}), \mathcal{A})$ in V^s where $D(\mathcal{A}) = V^{s+(1+\max\{-1,\ e\})}$ and*

$$\mathcal{A}(\eta, w) = \begin{pmatrix} \dfrac{a}{b} w_x - \left(1 + \dfrac{a}{b}\right) (I - b\partial_x^2)_p^{-1} w_x \\ \dfrac{c}{d} \eta_x - \left(1 + \dfrac{c}{d}\right) (I - d\partial_x^2)_p^{-1} \eta_x \end{pmatrix}, \quad \forall (\eta, w) \in D(\mathcal{A}). \tag{2.14}$$

Proof. We show that

$$\lim_{t\to 0} \frac{S(t)(\eta, w) - (\eta, w)}{t} = \mathcal{A}(\eta, w), \tag{2.15}$$

if and only if $(\eta, w) \in V^{s+(1+\max\{-1,\ e\})}$.

This is equivalent to show that the derivative in zero of the series $\sum_{k\in\mathbb{Z}} (\widehat{\eta}_k(t), \widehat{w}_k(t)) e^{ikx}$, where $(\widehat{\eta}_k(t), \widehat{w}_k(t))$ is given by (2.12), is convergent to $\mathcal{A}(\eta, w)$ in V^s if and only if $(\eta, w) \in V^{s+(1+\max\{-1,\ e\})}$.

If we denote by

$$\mathcal{S}_N(t) = \sum_{|k|\leq N} (\widehat{\eta}_k(t), \widehat{w}_k(t))\, e^{ikx}$$

a partial sum of the series, a straightforward computation which takes into account (2.12) shows that

$$\left[\mathcal{S}_N\right]_t(0) = \mathcal{A}(\mathcal{S}_N)(0).$$

Both terms are convergent in V^s when N tends to infinity if $(\eta, w) \in V^{s+(1+\max\{-1,\, e\})}$ and

$$\left[\sum_{k\in\mathbb{Z}}(\widehat{\eta}_k(t), \widehat{w}_k(t))e^{ikx}\right]_t (0) = \mathcal{A}(\eta, w).$$

On the other hand, the derivative of the series $\sum_{k\in\mathbb{Z}}(\widehat{\eta}_k(t), \widehat{w}_k(t))e^{ikx}$ is convergent only if $(\eta, w) \in V^{s+(1+\max\{-1,\, e\})}$. Indeed, this may be easily seen if we compute

$$\|\, [S_{N+p}]_t(0) - [S_N]_t(0)\|^2_{V^s} = \sum_{N<|k|\leq N+p} k^2\sigma^2(k)\left(|\widehat{\eta}_k|^2 + \left|\frac{\omega_1(k)}{\omega_2(k)}\right| |\widehat{w}_k|^2\right)(1+k^2)^s$$

which is convergent if and only if $(\eta, w) \in V^{s+(1+\max\{-1,\, e\})}$. $\square$

System (2.1)-(2.3) may be written equivalently in the following form

$$\begin{pmatrix} \eta \\ w \end{pmatrix}_t (t) = \mathcal{A}\begin{pmatrix} \eta \\ w \end{pmatrix} + \begin{pmatrix} f^* \\ g^* \end{pmatrix}, \qquad \begin{pmatrix} \eta \\ w \end{pmatrix}(0) = \begin{pmatrix} \eta^0 \\ w^0 \end{pmatrix} \tag{2.16}$$

where $f^* = \sum_{k\in\mathbb{Z}} \frac{1}{1+bk^2}\widehat{f}_k e^{ikx}$ and $g^* = \sum_{k\in\mathbb{Z}} \frac{1}{1+dk^2}\widehat{g}_k e^{ikx}$.

The following existence and uniqueness result for the solutions of (2.1)-(2.3) is a direct consequence of the general theory for evolution equations associated with a group of isometries (see e.g. [9]).

Theorem 2.4. *Let $T > 0$ and $s \in \mathbb{R}$ be given. If $(\eta^0, w^0) \in V^s$ and $(f^*, g^*) \in L^1(0, T; V^s)$, then (2.16) admits a unique solution*

$$(\eta, w) \in C^1([0, T], V^{s-(1+\max\{-1,\, e\})}) \cap C([0, T], V^s).$$

Moreover, there exists a positive constant $C > 0$ depending only on s such that

$$\|(\eta, w)\|_{C([0,T];V^s)} \leq C\left[\|(f^*, g^*)\|_{L^1(0,T;V^s)} + \|(\eta^0, w^0)\|_{V^s}\right]. \tag{2.17}$$

Remark 2.5. System (2.1) has the following important regularity property: if $b \neq 0$ and $d \neq 0$ then $(f, g) \in L^1(0, T; V^s)$ implies $(f^*, g^*) \in L^1(0, T; V^{s+2})$. Hence, if $(\eta^0, w^0) \in V^{s+2}$ and $(f, g) \in L^1(0, T; V^s)$, then $(\eta, w) \in C([0, T]; V^{s+2})$. $\square$

Remark 2.6. If we define

$$\begin{cases} V^s_{0,0} = \{(\eta, w) \in V^s \,:\, \widehat{\eta}_0 = \widehat{w}_0 = 0\}, \\ V^s_{0,*} = \{(\eta, w) \in V^s \,:\, \widehat{\eta}_0 = 0\}, \\ V^s_{*,0} = \{(\eta, w) \in V^s \,:\, \widehat{w}_0 = 0\}, \end{cases}$$

we obtain that $V^s_{0,0}$, $V^s_{0,*}$ and $V^s_{*,0}$ are all closed subspaces of V^s. The group $(S(t))_{t\in\mathbb{R}}$ is well-defined in those spaces. Hence, for instance, if we consider that in Theorem 2.4 $\widehat{f}_0 = \widehat{g}_0 = 0$, then the corresponding solution of (2.1) is well defined in $V^s_{0,0}$. Remark that the elements of $V^s_{0,0}$ have the property that the mean of η and w are zero. This is a quantity which is conserved in system (2.1) if f and g also have zero mean. $\square$

To end this subsection we consider the following backward IBVP of the homogeneous adjoint system of (2.1):

$$\begin{cases} \xi_t + u_x + cu_{xxx} - b\xi_{txx} = 0 & \text{for } x \in (0, 2\pi),\ \ t \in (0, T), \\ u_t + \xi_x + a\xi_{xxx} - du_{txx} = 0 & \text{for } x \in (0, 2\pi),\ \ t \in (0, T), \\ \frac{\partial^r \xi}{\partial x^r}(t, 0) = \frac{\partial^r \xi}{\partial x^r}(t, 2\pi) & \text{for } t \in (0, T),\ 0 \le r \le r_0, \\ \frac{\partial^q u}{\partial x^q}(t, 0) = \frac{\partial^q u}{\partial x^q}(t, 2\pi) & \text{for } t \in (0, T),\ 0 \le q \le q_0, \\ \xi(T, x) = \xi^T(x), \qquad u(T, x) = u^T(x) & \text{for } x \in (0, 2\pi). \end{cases} \tag{2.18}$$

Letting $t' = T - t$, $x' = 2\pi - x$, one can easily see that (2.18) is (2.1)-(2.3) with f and g being zero and a and c exchanged.

Let

$$\widetilde{\omega}_1 = \widetilde{\omega}_1(k) = \frac{1 - ck^2}{1 + bk^2}, \quad \widetilde{\omega}_2 = \widetilde{\omega}_2(k) = \frac{1 - ak^2}{1 + dk^2}, \tag{2.19}$$

As before, for each $s \in \mathbb{R}$, we define the space

$$\widetilde{V}^s = \left\{ (\xi, u) \in H_p^s(0, 2\pi) \times H_p^{s-2}(0, 2\pi) \;\middle|\; ||(\xi, u)||^2_{\widetilde{V}^s} := ||\xi||_s^2 + ||\widetilde{\mathcal{H}}u||_s^2 < \infty \right\}$$

where $||\ ||_s$ is the $H_p^s(0, 2\pi)$ norm and the operator $\widetilde{\mathcal{H}}$ is defined in the following way

$$\widetilde{\mathcal{H}}\left(\sum_{k\in\mathbb{Z}} \widehat{v}_k e^{ikx} \right) = \sum_{k\in\mathbb{Z}} \sqrt{\frac{\widetilde{\omega_1}}{\widetilde{\omega_2}}} \widehat{v}_k e^{ikx}.$$

Introduce the number $\tilde{l} \in \mathbb{Z}$ with the property that

$$\sqrt{\frac{\widetilde{\omega_1}}{\widetilde{\omega_2}}} \sim C|k|^{\tilde{l}} \text{ as } |k| \to \infty$$

where C is a positive constant not depending on k. We have that

$$\widetilde{V}^s = H_p^s(0, 2\pi) \times H_p^{s+\tilde{l}}(0, 2\pi).$$

Theorem 2.7. *Let $T > 0$ and $s \in \mathbb{R}$ be given. If $(\xi^T, u^T) \in \widetilde{V}^s$, then (2.18) admits a unique solution (ξ, u) in $C^1([0, T], \widetilde{V}^{s-(1+\max\{-1,\, e\})}) \cap C([0, T], \widetilde{V}^s)$. Moreover, there exists a positive constant $C > 0$ depending only on s such that*

$$||(\xi, u)||_{C([0,T];\widetilde{V}^s)} \le C||(\xi^0, u^0)||_{\widetilde{V}^s}. \tag{2.20}$$

If

$$(\xi^T, u^T) = \sum_{k\in\mathbb{Z}} \left(\widehat{\xi}_k^T, \widehat{u}_k^T \right) e^{ikx},$$

then the solution of (2.18) may be written as

$$(\xi, u)(t, x) = \sum_{k\in\mathbb{Z}} (\widehat{\xi}_k(t), \widehat{u}_k(t)) e^{ikx}$$

where

$$\begin{cases} \widehat{\xi}_k = \frac{1}{2}\left(\widehat{\xi}_k^T + \sqrt{\frac{\tilde{\omega}_1}{\tilde{\omega}_2}}\widehat{u}_k^T\right) e^{ik\sigma(k)(T-t)} \\ \qquad\qquad +\frac{1}{2}\left(\widehat{\xi}_k^T - \sqrt{\frac{\tilde{\omega}_1}{\tilde{\omega}_2}}\widehat{u}_k^T\right) e^{-ik\sigma(k)(T-t)} \\ \widehat{u}_k = \frac{1}{2}\left(\sqrt{\frac{\tilde{\omega}_2}{\tilde{\omega}_1}}\widehat{\xi}_k^T + \widehat{u}_k^T\right) e^{ik\sigma(k)(T-t)} \\ \qquad\qquad +\frac{1}{2}\left(-\sqrt{\frac{\tilde{\omega}_2}{\tilde{\omega}_1}}\widehat{\xi}_k^T + \widehat{u}_k^T\right) e^{-ik\sigma(k)(T-t)}. \end{cases} \tag{2.21}$$

This formula will be very useful for the controllability problem, considered in the next section.

2.2. **Linear systems with a single control input.** In this subsection we study the control and stabilization of the following system with a single control input:

$$\begin{cases} \eta_t + w_x + aw_{xxx} - b\eta_{txx} = Qh & \text{for } x \in (0, 2\pi),\ \ t \in (0, T), \\ w_t + \eta_x + c\eta_{xxx} - dw_{txx} = 0 & \text{for } x \in (0, 2\pi),\ \ t \in (0, T), \\ \frac{\partial^r \eta}{\partial x^r}(t, 0) = \frac{\partial^r \eta}{\partial x^r}(t, 2\pi) & \text{for } t \in (0, T),\ 0 \le r \le r_0, \\ \frac{\partial^q w}{\partial x^q}(t, 0) = \frac{\partial^q w}{\partial x^q}(t, 2\pi) & \text{for } t \in (0, T),\ 0 \le q \le q_0, \\ \eta(0, x) = \eta^0(x), \qquad w(0, x) = w^0(x) & \text{for } x \in (0, 2\pi), \end{cases} \tag{2.22}$$

where the operator Q is defined by

$$[Qh](x, t) = q(x)h(x, t),$$

and $q \in L^2(0, 2\pi)$ is a given non-negative function supported in ω and such that $q(x) > C$ on a (nonempty) open set $\omega' \subset \omega$, $C > 0$ being some constant.

First, we look for a time $T > 0$ and a space $\mathcal{V}$ (to be made precise latter on) with the property that for $(\eta^0, w^0) \in \mathcal{V}$ there exists $h \in L^2((0, T) \times (0, 2\pi))$ such that the corresponding solution (η, w) of (2.22) satisfies

$$\eta(T, \cdot) = w(T, \cdot) = 0. \tag{2.23}$$

Property (2.23) represents a null-controllability. Since system (2.22) is conservative and time reversible, this property is equivalent to the exact controllability. Hence, any initial state in $\mathcal{V}$ may be led to any final state from $\mathcal{V}$ in time T. The controllability properties of (2.22) will depend on the values of the coefficients a, b, c and d. As we shall see latter on, a whole bunch of situations may occur, from non-spectral controllability to exact controllability in any time. This controllability problem will be solved in this subsection by using the Hilbert Uniqueness Method (HUM) introduced by J.-L. Lions (see [20]). The following duality product will play an important role.

$$\langle(\varphi, \zeta), (\psi, z)\rangle_D = \sum_{k\in\mathbb{Z}} \left(\widehat{\varphi_k}\overline{\widehat{\psi}_k}(1 + bk^2) + \widehat{\zeta}_k\overline{\widehat{z}_k}(1 + dk^2)\right) \tag{2.24}$$

for any $(\varphi, \zeta) \in \mathcal{V}$, $(\psi, z) \in \widetilde{V}^0_{*,0}$. The space $\mathcal{V}$ is defined in the following way

$$\mathcal{V} = \begin{cases} V^0_{*,0} & \text{if } b = 0, \\ \\ V^2_{*,0} & \text{if } b \neq 0. \end{cases} \tag{2.25}$$

Notice that $(\varphi, \zeta) \in \mathcal{V}$ if and only if the sum in (2.24) is well defined and its absolute value is less than $C||(\psi, z)||_{\widetilde{V}^0}$ (C being some constant) for any $(\psi, z) \in \widetilde{V}^0_{*,0}$. Moreover, the duality product (2.24) is found to be

$$\langle (\varphi, \zeta), (\psi, z) \rangle_D = \begin{cases} \int_0^{2\pi} \varphi(x)\overline{\psi}(x)dx + \langle \zeta, (I - d\partial_x^2)z \rangle_l & \text{if } b = 0, \\ \\ \int_0^{2\pi} (I - b\partial_x^2)\varphi(x)\overline{\psi}(x)dx + \langle \zeta, z \rangle_{2+l} & \text{if } b \neq 0,\ d = 0, \\ \\ \int_0^{2\pi} (I - b\partial_x^2)\varphi(x)\overline{\psi}(x)dx + \langle (I - d\partial_x^2)\zeta, z \rangle_l & \text{if } b \neq 0,\ d \neq 0, \end{cases}$$

where $\langle \cdot, \cdot \rangle_l$ is the duality product defined by (2.5).

The following proposition presents an equivalent condition for the controllability of (2.22).

Proposition 2.8. *Given $(\eta^0, w^0) \in \mathcal{V}$, there exists $h \in L^2((0, 2\pi) \times (0, T))$ such that the corresponding solution (η, w) of (2.22) satisfies (2.23) if and only if there exists $h \in L^2((0, 2\pi) \times (0, T))$ such that*

$$\langle (\eta^0, w^0), (\xi(0), u(0)) \rangle_D + \int_0^T \int_0^{2\pi} [Qh](x, t)\overline{\xi(x, t)}dxdt = 0 \tag{2.26}$$

for any $(\xi^T, u^T) \in \widetilde{V}^0_{*,0}$, where (ξ, u) is the solution of (2.18).

Proof. Note that it is sufficient to prove (2.26) for regular data. Multiplying the first and the second equation in (2.22) by $\overline{\xi}$ and $\overline{u}$ respectively, integrating in time and space over the domain $(0, T) \times (0, 2\pi)$ and adding the relations we obtain that

$$\begin{aligned} &\int_0^T \int_0^{2\pi} [Qh](x, t)\overline{\xi(x, t)}dxdt = \int_0^T \int_0^{2\pi} [\eta_t + w_x + aw_{xxx} - b\eta_{txx}] \overline{\xi}dxdt \\ &+ \int_0^T \int_0^{2\pi} [w_t + \eta_x + c\eta_{xxx} - dw_{txx}] \overline{u}dxdt \\ &= \int_0^{2\pi} \left[(\eta - b\eta_{xx})\overline{\xi} + (w - dw_{xx})\overline{u}\right] dx \Big|_0^T \\ &= \langle (\eta(T), w(T)), (\xi^T, u^T) \rangle_D - \langle (\eta^0, w^0), (\xi(0), u(0)) \rangle_D \end{aligned}$$

from which (2.26) follows immediately. □

A standard argument shows that the variational equality (2.26) has a solution if and only if there exists a constant $C > 0$ such that the following observation inequality holds true for any $(\xi^T, u^T) \in \widetilde{V}^0_{*,0}$

$$||(\xi(0), u(0))||^2_{\widetilde{V}^0} \leq C \int_0^T \int_0^{2\pi} [Q\xi](x, t)\overline{\xi(x, t)}dxdt,$$

where (ξ,u) is the solution of (2.18) with final data (ξ^T,u^T). Thus, because of the definition of the operator Q, it suffices to show

$$||(\xi(0),u(0))||^2_{\widetilde{V}^0} \le C\int_0^T\int_{\omega'}|\xi(x,t)|^2dxdt. \tag{2.27}$$

We show that (2.27) holds true for certain values of the parameters a,b,c and d by using the Fourier expansion of the solutions of (2.18).

Theorem 2.9 (Observability)**.** *Suppose that neither of the following two situations occur*

$$\begin{aligned} &\text{(S1)} \quad b\neq 0,\ d\neq 0 \ \textit{and}\ ac=0,\\ &\text{(S2)} \quad a=c=0 \ \textit{and}\ b^2+d^2\neq 0. \end{aligned}$$

Then there exist a time $T>0$ and a constant $C>0$ such that, for any $(\xi^T,u^T)\in \widetilde{V}^0_{*,0}$ the corresponding solution (ξ,u) of (2.18) satisfies the inequality (2.27).

Proof. Let $(\xi^T,u^T)\in\widetilde{V}^0_{*,0}$ be of the form

$$(\xi^T,u^T)=\sum_{k\in\mathbb{Z}}(\widehat{\xi}^T_k,\widehat{u}^T_k)e^{ikx}.$$

The corresponding solution of (2.18) is given by

$$(\xi,u)=\sum_{k\in\mathbb{Z}}(\widehat{\xi}_k,\widehat{u}_k)e^{ikx}$$

with

$$\begin{cases} \widehat{\xi}_k = \frac{1}{2}\left(\widehat{\xi}^T_k+\sqrt{\frac{\widetilde{\omega}_1}{\widetilde{\omega}_2}}\widehat{u}^T_k\right)e^{ik\sigma(k)(T-t)} \\ \qquad\qquad +\frac{1}{2}\left(\widehat{\xi}^T_k-\sqrt{\frac{\widetilde{\omega}_1}{\widetilde{\omega}_2}}\widehat{u}^T_k\right)e^{-ik\sigma(k)(T-t)}, \\ \widehat{u}_k = \frac{1}{2}\left(\sqrt{\frac{\widetilde{\omega}_2}{\widetilde{\omega}_1}}\widehat{\xi}^T_k+\widehat{u}^T_k\right)e^{ik\sigma(k)(T-t)} \\ \qquad\qquad +\frac{1}{2}\left(-\sqrt{\frac{\widetilde{\omega}_2}{\widetilde{\omega}_1}}\widehat{\xi}^T_k+\widehat{u}^T_k\right)e^{-ik\sigma(k)(T-t)}. \end{cases} \tag{2.28}$$

From the fact that a group of isometries is associated to (2.18), it follows that

$$||(\xi,u)(0)||^2_{\widetilde{V}^0}=\sum_{k\in\mathbb{Z}}\left(\left|\widehat{\xi}^T_k\right|^2+\frac{\widetilde{\omega}_1}{\widetilde{\omega}_2}\left|\widehat{u}^T_k\right|^2\right)=||(\xi^T,u^T)||^2_{\widetilde{V}^0}. \tag{2.29}$$

On the other hand,

$$\int_0^T \int_{\omega'} |\xi(x,t)|^2 \, dxdt = \frac{1}{4}\int_0^T \int_{\omega'} \left| \sum_{k\in\mathbb{Z}} \left[\left(\widehat{\xi}_k^T + \sqrt{\frac{\widetilde{\omega}_1}{\widetilde{\omega}_2}} \widehat{u}_k^T \right) e^{ik\sigma(k)t} \right.\right.$$
$$\left.\left. + \left(\widehat{\xi}_k^T - \sqrt{\frac{\widetilde{\omega}_1}{\widetilde{\omega}_2}} \widehat{u}_k^T \right) e^{-ik\sigma(k)t} \right] e^{ikx} \right|^2 dxdt$$
$$= \frac{1}{4}\int_{\omega'} \int_0^T \left| \sum_{k_1\in\mathbb{I}} \sum_{k\in I(k_1)} \left[\left(\widehat{\xi}_k^T + \sqrt{\frac{\widetilde{\omega}_1}{\widetilde{\omega}_2}} \widehat{u}_k^T \right) e^{ikx} \right.\right.$$
$$\left.\left. + \left(\widehat{\xi}_{-k}^T - \sqrt{\frac{\widetilde{\omega}_1}{\widetilde{\omega}_2}} \widehat{u}_{-k}^T \right) e^{-ikx} \right] e^{ik_1\sigma(k_1)t} \right|^2 dxdt.$$

Remark that, if a, b, c and d do not fulfill (S1) or (S2), then there exists a number $\gamma \in (0,\infty]$ such that

$$\liminf_{|k|\to\infty} |\lambda_{k+1} - \lambda_k| = \gamma > 0.$$

Moreover, the elements of the sequence $\left(e^{\lambda_k t}\right)_{k\in\mathbb{I}}$ are all different. By using a generalization of Ingham's inequality (see [13] and [2]) we deduce that, for any $T > \frac{2\pi}{\gamma}$, there exists a constant $C > 0$ such that

$$\int_0^T \int_{\omega'} |\xi(t,x)|^2 \, dxdt \geq C \int_{\omega'} \sum_{k_1\in\mathbb{I}} \left| \sum_{k\in I(k_1)} \left[\left(\widehat{\xi}_k^T + \sqrt{\frac{\widetilde{\omega}_1}{\widetilde{\omega}_2}} \widehat{u}_k^T \right) e^{ikx} \right.\right.$$
$$\left.\left. + \left(\widehat{\xi}_{-k}^T - \sqrt{\frac{\widetilde{\omega}_1}{\widetilde{\omega}_2}} \widehat{u}_{-k}^T \right) e^{-ikx} \right] \right|^2 dx.$$

Note that, in the hypothesis of the theorem, $\lim_{|k|\to\infty} |\lambda_k| = \infty$. From Remark 2.1 we obtain that $|I(k_1)| = m(k_1) = 1$ for k_1 large enough. For the remaining finite number of valued $k_1 \in \mathbb{I}$ we use the fact that ω' is a nonempty open interval and $|I(k_1)| \leq 6$ to deduce that there exists a constant $C > 0$, independent of k_1, such that

$$\int_{\omega'} \left| \sum_{k\in I(k_1)} \left[\left(\widehat{\xi}_k^T + \sqrt{\frac{\widetilde{\omega}_1}{\widetilde{\omega}_2}} \widehat{u}_k^T \right) e^{ikx} + \left(\widehat{\xi}_{-k}^T - \sqrt{\frac{\widetilde{\omega}_1}{\widetilde{\omega}_2}} \widehat{u}_{-k}^T \right) e^{-ikx} \right] \right|^2 dx$$
$$\geq C \sum_{k\in I(k_1)} \left[\left| \widehat{\xi}_k^T + \sqrt{\frac{\widetilde{\omega}_1}{\widetilde{\omega}_2}} \widehat{u}_k^T \right|^2 + \left| \widehat{\xi}_{-k}^T - \sqrt{\frac{\widetilde{\omega}_1}{\widetilde{\omega}_2}} \widehat{u}_{-k}^T \right|^2 \right].$$

This may also be seen as a consequence of the same generalized Ingham's inequality used before. It follows that there exists a constant $C > 0$ such that

$$\int_0^T \int_{\omega'} |\xi(x,t)|^2 \, dxdt \geq C \sum_{k\in\mathbb{Z}} \left(\left| \widehat{\xi}_k^T \right|^2 + \frac{\widetilde{\omega}_1}{\widetilde{\omega}_2} \left| \widehat{u}_k^T \right|^2 \right). \tag{2.30}$$

Hence, from (2.29) and (2.30) it follows that (2.27) holds and the proof ends. $\square$

Remark 2.10. If either (S1) or (S2) holds, the family of eigenvalues $(\lambda_k)_{k\in\mathbb{Z}}$ is bounded. It is well known that in this case Ingham's inequality does not hold and the system is not controllable. In [25] the BBM equation is studied in some details. The corresponding spectrum is equally bounded in that case and the equation is not controllable. Note that the non-controllability of (2.22) corresponds exactly to the cases in which two BBM-type equations are coupled. □

Remark 2.11. If

$$\begin{aligned} &(\mathrm{S3}) \quad b\,d = 0 \text{ and } ac \neq 0, \quad \text{or} \\ &(\mathrm{S4}) \quad b = d = 0,\ a^2 + c^2 \neq 0 \text{ and } a = 0 \text{ or } c = 0 \end{aligned} \tag{2.31}$$

we have that

$$\liminf_{k\to\infty}(\lambda_{k+1} - \lambda_k) = \infty$$

and inequality (2.27) holds for any $T > 0$. Hence, the system is controllable in any positive time. In the remaining cases the controllability time should be large enough. □

The following controllability result follows from Proposition 2.8 and Theorem 2.9,

Theorem 2.12 (Exact controllability)**.** *Suppose that neither of the following two situations occur*

$$\begin{aligned} &(\mathrm{S1}) \quad b \neq 0,\ d \neq 0 \text{ and } ac = 0, \\ &(\mathrm{S2}) \quad a = c = 0 \text{ and } b^2 + d^2 \neq 0. \end{aligned}$$

Then there exists a time $T > 0$ such that for given

$$(\eta^0, w^0) \in \mathcal{V}, \qquad (\eta^T, w^T) \in \mathcal{V},$$

one can find a control input $h \in L^2(0, T; L^2(0, 2\pi))$ such that (2.22) admits a unique solution

$$(\eta, w) \in C([0, T]; \mathcal{V})$$

satisfying

$$(\eta(\cdot, 0), w^0(\cdot, 0)) = (\eta^0(\cdot), w^0(\cdot)), \qquad (\eta(\cdot, T), w(\cdot, T)) = (\eta^T(\cdot), w^T(\cdot)) \qquad \text{in } \mathcal{V}.$$

Moreover, there exists a constant $C > 0$ such that

$$\|h\|_{L^2(0,T;L^2(0,2\pi))} \leq C \left(\|(\eta^0, w^0)\|_{\mathcal{V}} + \|(\eta^T, w^T)\|_{\mathcal{V}}\right).$$

Remark 2.13. The details of the controllability results are provided in Table 1. □

No.	b	d	a	c	Controllable space	Control time
1	$\neq$	$\neq$	$\neq$	$\neq$	$V^2_{*,0} = H^2_p \times H^2_{0,p}$	$T > 2\pi\sqrt{\frac{bd}{ac}}$
2	$\neq$	$\neq$	0	0	No	No
3	$\neq$	$\neq$	$\neq$	0	No	No
4	$\neq$	$\neq$	0	$\neq$	No	No
5	0	$\neq$	$\neq$	$\neq$	$V^0_{*,0} = L^2 \times H^1_{0,p}$	Arbitrarily small
6	0	$\neq$	0	0	No	No
7	0	$\neq$	0	$\neq$	$V^0_{*,0} = L^2 \times L^2_0$	$T > 2\pi\sqrt{-\frac{d}{c}}$
8	0	$\neq$	$\neq$	0	$V^0_{*,0} = L^2 \times H^2_{0,p}$	$T > 2\pi\sqrt{-\frac{d}{a}}$
9	$\neq$	0	$\neq$	$\neq$	$V^2_{*,0} = H^2_p \times H^1_{0,p}$	Arbitrarily small
10	$\neq$	0	0	0	No	No
11	$\neq$	0	0	$\neq$	$V^2_{*,0} = H^2_p \times L^2_0$	$T > 2\pi\sqrt{-\frac{b}{c}}$
12	$\neq$	0	$\neq$	0	$V^2_{*,0} = H^2_p \times H^2_{0,p}$	$T > 2\pi\sqrt{-\frac{b}{a}}$
13	0	0	0	0	$V^0_{*,0} = L^2 \times L^2_0$	$T > 2\pi$
14	0	0	$\neq$	$\neq$	$V^0_{*,0} = L^2 \times L^2_0$	Arbitrarily small
15	0	0	0	$\neq$	$V^0_{*,0} = L^2 \times H^{-1}_{0,p}$	Arbitrarily small
16	0	0	$\neq$	0	$V^0_{*,0} = L^2 \times H^1_{0,p}$	Arbitrarily small

TABLE 1. Controllability results for the linear system.

Remark 2.14. One may consider the control system (2.22) but with a control input acting on the second equation:

$$\begin{cases} \eta_t + w_x + a w_{xxx} - b\eta_{txx} = 0 & \text{for } x \in (0, 2\pi),\ \ t \in (0,T), \\ w_t + \eta_x + c\eta_{xxx} - d w_{txx} = Qh & \text{for } x \in (0, 2\pi),\ \ t \in (0,T), \\ \frac{\partial^r \eta}{\partial x^r}(t,0) = \frac{\partial^r \eta}{\partial x^r}(t,2\pi) & \text{for } t \in (0,T),\ \ 0 \le r \le r_0, \\ \frac{\partial^q w}{\partial x^q}(t,0) = \frac{\partial^q w}{\partial x^q}(t,2\pi) & \text{for } t \in (0,T),\ \ 0 \le q \le q_0, \\ \eta(0,x) = \eta^0(x), \qquad w(0,x) = w^0(x) & \text{for } x \in (0,2\pi). \end{cases} \tag{2.32}$$

Note that if we let $a \to c,\ b \to d,\ w \to \eta$, then the system (2.32) becomes (2.22). Thus one may produce the table of controllability for the system (2.32) from Table 1 by interchanging a and c, b and d as well as the order of the product spaces. $\square$

Next we turn to the stabilization problem for system (2.22). We seek a linear feedback control law such that the resulting closed-loop system is exponentially stable. Note that (2.22) can be written in the form:

$$\begin{cases} (\widehat{\eta}_k)_t + ik\widehat{w}_k - iak^3\widehat{w}_k + bk^2(\widehat{\eta}_k)_t = \widehat{q}_k, & t \in (0,T) \\ (\widehat{w}_k)_t + ik\widehat{\eta}_k - ick^3\widehat{\eta}_k + dk^2(\widehat{w}_k)_t = 0, & t \in (0,T) \\ \widehat{\eta}_k(0) = \widehat{\eta}^0_k, \qquad \widehat{w}_k(0) = \widehat{w}^0_k \end{cases} \tag{2.33}$$

for $-\infty < k < \infty$, where the $\widehat{q}_k$'s denote the Fourier coefficients of Qh.

Multiplying both sides of the first equation in (2.33) by $\overline{\widehat{\eta}_k}$ and the second equation by $\overline{\widehat{w}_k}$ if $a = c \ge 0$ or by $\frac{1-ak^2}{1-ck^2}$ if $a < 0,\ c < 0$, and then adding the resulting first equation to the conjugate of the resulting second equation, we obtain

$$\frac{d}{dt}(1+bk^2)\left(|\widehat{\eta}_k|^2 + \frac{\omega_1(k)}{\omega_2(k)}|\widehat{w}_k|^2\right) = 2\mathrm{Re}\,(q_k\overline{\widehat{\eta}_k})$$

for $-\infty < k < \infty$. Thus, if we define

$$E[\eta,w](t) = \int_0^{2\pi}\left(\left|(I-b\partial_x^2)^{1/2}\eta(x,t)\right|^2 + \left|(I-b\partial_x^2)^{1/2}\mathcal{H}w(x,t)\right|^2\right)dx, \quad (2.34)$$

then

$$\frac{d}{dt}E[\eta,w](t) = 2\mathrm{Re}\,\left(\int_0^{2\pi}[Qh](x,t)\overline{\eta(x,t)}dx\right).$$

Thus, if one chooses the feedback control law

$$h(x,t) = -\eta(x,t),$$

then the resulting closed loop system

$$\begin{cases} \eta_t + w_x + aw_{xxx} - b\eta_{txx} = -Q\eta & \text{for } x\in(0,2\pi),\ \ t\in(0,T), \\ w_t + \eta_x + c\eta_{xxx} - dw_{txx} = 0 & \text{for } x\in(0,2\pi),\ \ t\in(0,T), \\ \frac{\partial^r\eta}{\partial x^r}(t,0) = \frac{\partial^r\eta}{\partial x^r}(t,2\pi) & \text{for } t\in(0,T),\ 0\le r\le r_0, \\ \frac{\partial^q w}{\partial x^q}(t,0) = \frac{\partial^q w}{\partial x^q}(t,2\pi) & \text{for } t\in(0,T),\ 0\le q\le q_0, \\ \eta(0,x) = \eta^0(x), \qquad w(0,x) = w^0(x) & \text{for } x\in(0,2\pi) \end{cases} \quad (2.35)$$

has the property that

$$\frac{d}{dt}E[\eta,w](t) = -2\int_0^{2\pi}[Q\eta](x,t)\overline{\eta(x,t)}dx \le -C\int_{\omega'}|\eta(x,t)|^2dx$$

for any $t\ge 0$, where $C>0$ is a constant independent of η and w. Our main concern is whether the solution (η,w) tends to zero as $t\to\infty$ and if it does, how fast it decays?

Let us define the spaced $\mathcal{U}_{b,d}$ by

$$\mathcal{U}_{b,d} = \begin{cases} H_p^1\times H_{0,p}^1 & \text{if } b>0 \text{ and } d>0; \\ H_p^1\times L_0^2 & \text{if } b>0 \text{ and } d=0; \\ L^2\times H_{0,p}^1 & \text{if } b=0 \text{ and } d>0; \\ L^2\times L_0^2 & \text{if } b=0 \text{ and } d=0 \end{cases}$$

and the norm $||\cdot||_{\mathcal{U}_{b,d}}$ by

$$||(\eta,w)||^2_{\mathcal{U}_{b,d}} = \int_0^{2\pi}\left(|\eta|^2 + b|\eta_x|^2 + |w|^2 + d|w_x|^2\right)dx.$$

Theorem 2.15. *Assume that*

(i) *there exist constants* $c_1>0$ *and* $c_2>0$ *such that*

$$c_1E^{\frac{1}{2}}(u,v)\le\|(u,v)\|_{\mathcal{U}_{b,d}}\le c_2E^{\frac{1}{2}}(u,v)$$

for any $(u,v)\in\mathcal{U}_{b,d}$*;*

(ii) *the following open loop control system (the adjoint system of (2.22))*

$$\begin{cases} \chi_t + \zeta_x + c\zeta_{xxx} - b\chi_{txx} = Qh & \text{for } x \in (0,2\pi),\ \ t \in (0,T), \\ \zeta_t + \chi_x + a\chi_{xxx} - d\zeta_{txx} = 0 & \text{for } x \in (0,2\pi),\ \ t \in (0,T), \\ \frac{\partial^r \chi}{\partial x^r}(t,0) = \frac{\partial^r \chi}{\partial x^r}(t,2\pi) & \text{for } t \in (0,T),\ 0 \le r \le r_0, \\ \frac{\partial^q \zeta}{\partial x^q}(t,0) = \frac{\partial^q \zeta}{\partial x^q}(t,2\pi) & \text{for } t \in (0,T),\ 0 \le q \le q_0 \end{cases} \tag{2.36}$$

is exactly controllable in the space $\mathcal{U}_{b,d}$ in the time interval $(0,T)$ for some $T>0$ with control input h chosen in the space $L^2((0,2\pi)\times(0,T))$.

Then for any $(\eta^0, w^0) \in \mathcal{U}_{b,d}$, the closed-loop system (2.35) admits a unique solution

$$(\eta, w) \in C(\mathbb{R}^+; \mathcal{U}_{b,d}).$$

Moreover, there exist two constants $\delta > 0$ and $C > 0$ such that

$$\|(\eta(\cdot,t), w(\cdot,t))\|_{\mathcal{U}_{b,d}} \le C\|(\eta^0, w^0)\|_{\mathcal{U}_{b,d}} e^{-\delta t}$$

for any $t \ge 0$.

Proof. Let (η, w) be the solution (2.35) with initial value (η^0, w^0). By assumption (ii), there exists a control h in the space $L^2((0,2\pi)\times(0,T))$ such that the system (2.36) admits a solution

$$(\chi, \zeta) \in C([0,T]; \mathcal{U}_{b,d})$$

satisfying

$$(\chi,\zeta) = (0,0) \quad \text{at} \quad t = 0, \qquad (\chi,\zeta) = (\eta, w) \quad \text{at} \quad t = T.$$

Moreover, there exist positive constants C_1 and C_2 such that

$$\|h\|_{L^2((0,2\pi)\times(0,T))} \le C_1 \|(\eta(\cdot,T), w(\cdot,T))\|_{\mathcal{U}_{b,d}}$$

and

$$\sup_{0\le t\le T} \|(\chi(\cdot,t), \zeta(\cdot,t))\|_{\mathcal{U}_{b,d}} \le C_2 \|(\eta(\cdot,T), w(\cdot,T))\|_{\mathcal{U}_{b,d}}.$$

Multiplying the first and the second equation in (2.36) by $\overline{\eta}$ and $\overline{w}$ respectively, integrating in time and space and adding the relations we obtain that

$$\begin{aligned} & \|(\eta(\cdot,T), w(\cdot,T))\|^2_{\mathcal{U}_{bd}} \\ = \quad & \int_0^T \int_0^{2\pi} \overline{\eta(x,t)} [Qh](x,t)\,dxdt \\ & - \int_0^T \int_0^{2\pi} \overline{[Q\eta](x,t)} \chi(x,t)\,dxdt \\ = \quad & \int_0^T \int_0^{2\pi} \overline{[Q\eta](x,t)} (h(x,t) - \chi(x,t))\,dxdt \\ \le \quad & \|[Q\eta]\|_{L^2((0,2\pi)\times(0,T))} \|h - \chi\|_{L^2((0,2\pi)\times(0,T))} \\ \le \quad & C_3 \left(\int_0^T \int_0^{2\pi} [Q\eta](x,t)\overline{\eta(x,t)}\,dxdt \right)^{\frac{1}{2}} \|(\eta(\cdot,T), w(\cdot,T))\|_{\mathcal{U}_{b,d}} \end{aligned}$$

where $C_3 > 0$ is a constant. Consequently,

$$\int_0^T \int_0^{2\pi} [Q\eta](x,t)\overline{\eta(x,t)}dxdt \geq C_3^{-2}\|(\eta(\cdot,T), w(\cdot,T))\|^2_{\mathcal{U}_{b,d}}$$

Since

$$E[\eta,w](T) - E[\eta,w](0) = -2\int_0^T \int_0^{2\pi} [Q\eta](x,t)\overline{\eta(x,t)}dxdt,$$

there exists a constant $C > 0$ by assumption (i) such that

$$E[\eta,w](T) \leq E[\eta,w](0) - CE[\eta,w](T).$$

Thus

$$E[\eta,w](T) \leq \frac{1}{1+C}E[\eta,w](0).$$

The theorem follows then consequently. The proof is complete. □

Remark 2.16. If $a = c \geq 0$ or $a < 0$ and $c < 0$, then Assumption (i) of Theorem 2.15 is satisfied. Assumption (ii) of Theorem 2.15 is also satisfied if, in addition, $b = 0$. □

Remark 2.17. A similar theorem holds if the feedback control is acting on the second equation instead of the first one. □

The assumptions of Theorem 2.15 are quite restrictive. According to Table 1, the theorem may be applied only in the cases of No. 5, 13 and 14. In particular, it cannot be applied for the case where both b and d are not zero. This is mainly caused by the simple feedback control law we have used. If some more complicated linear feedback control laws are used, we will have much stronger stabilizability results. To see this, we rewrite system (2.22) as an abstract control system in the Hilbert space $\mathcal{V}$:

$$\frac{d}{dt}\vec{\eta} = \mathcal{A}\vec{\eta} + \mathcal{B}\vec{h}, \qquad \vec{\eta}(0) = \vec{\eta}_0 \tag{2.37}$$

with

$$\vec{\eta} = \begin{pmatrix} \eta \\ w \end{pmatrix}, \qquad \vec{\eta}_0 = \begin{pmatrix} \eta^0 \\ w^0 \end{pmatrix}, \qquad \vec{h} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \qquad \mathcal{B}\vec{h} = \begin{pmatrix} (I - b\partial_x^2)^{-1}Qh_1 \\ \\ 0 \end{pmatrix}$$

and the operator $\mathcal{A}$ is as given in (2.14), which is an infinitesimal generator of a C^0 group $\mathcal{S}(t)$ in the space $\mathcal{V}$ and $\mathcal{A}^* = -\mathcal{A}$. Note that $\mathcal{B}$ is a bounded linear operator from the space $L^2(0,2\pi)$ to the space $\mathcal{V}$. The following theorem is derived from Theorem 2.12 and a classical principle *exact controllability implies exponential stabilizability* for conservative control systems [40, 24].

Theorem 2.18. *Assume that the assumptions of Theorem 2.12 are satisfied. Then*

(i) *there exist a $T > 0$ and a $\delta > 0$ such that*

$$\int_0^T \|\mathcal{B}^* S^*(t)\vec{\eta}_0\|^2_{L^2(0,2\pi)}\, dt \geq \delta\|\vec{\eta}_0\|^2_{\mathcal{V}} \tag{2.38}$$

for any $\vec{\eta}_0 \in \mathcal{V}$.

(ii) *For any given $\alpha > 0$, there exists an operator $\mathcal{K} \in \mathcal{L}(\mathcal{V}, L^2(0, 2\pi))$ such that if one chooses*

$$\vec{h} = \mathcal{K}\vec{\eta}$$

in (2.37), then the resulting closed-loop system

$$\frac{d}{dt}\vec{\eta} = \mathcal{A}\vec{\eta} + \mathcal{B}\mathcal{K}\vec{\eta}, \qquad \vec{\eta}(0) = \vec{\eta}_0 \tag{2.39}$$

has the property that its solution satisfies

$$\|\vec{\eta}(t)\|_{\mathcal{Y}} \leq M\|\vec{\eta}_0\|_{\mathcal{V}} e^{-\alpha t} \tag{2.40}$$

for any $t \geq 0$ where M is a constant independent of $\vec{\eta}_0$.

Remark 2.19. According to Slemrod [40], one can choose

$$\mathcal{K} = -\mathcal{B}^* D_{T,\alpha}^{-1}$$

with

$$D_{T,\alpha}\vec{\eta} = \int_0^T e^{-2\alpha t} S(-t)\mathcal{B}\mathcal{B}^* S^*(-t)\vec{\eta}\, dt.$$

In addition, if one simply chooses $\mathcal{K} = -\mathcal{B}^*$, then there exists a $\nu > 0$ such that estimate (2.40) holds with α replaced by ν. □

2.3. **Linear systems with two control inputs.** In this subsection we study control and stabilization of the following linear system with two control inputs.

$$\begin{cases} \eta_t + w_x + a w_{xxx} - b\eta_{txx} = f & \text{for } x \in (0, 2\pi),\ \ t \in (0, T), \\ w_t + \eta_x + c\eta_{xxx} - d w_{txx} = g, & \text{for } x \in (0, 2\pi),\ \ t \in (0, T), \\ \frac{\partial^r \eta}{\partial x^r}(t, 0) = \frac{\partial^r \eta}{\partial x^r}(t, 2\pi), & \text{for } t \in (0, T),\ \ 0 \leq r \leq r_0, \\ \frac{\partial^q w}{\partial x^q}(t, 0) = \frac{\partial^q w}{\partial x^q}(t, 2\pi), & \text{for } t \in (0, T),\ \ 0 \leq q \leq q_0, \\ \eta(0, x) = \eta^0(x), \qquad w(0, x) = w^0(x), \quad x \in (0, 2\pi). \end{cases} \tag{2.41}$$

We assume throughout this subsection that

$$a = c, \qquad b = d. \tag{2.42}$$

In consideration of practical applications, we require that

$$[f(\cdot, t)] = \int_0^{2\pi} f(x, t)dx = 0, \qquad [g(\cdot, t)] = \int_0^{2\pi} g(x, t)dx = 0.$$

With this restriction on f and g, any smooth solution $(\eta(x, t), w(x, t))$ has the property that

$$\frac{d}{dt}\int_0^{2\pi} \eta(x, t)dx = 0, \qquad \frac{d}{dt}\int_0^{2\pi} w(x, t)dx = 0$$

for any t. Thus the quantities $[\eta(\cdot, t)]$ and $[w(\cdot, t)]$, the mean values of η and w, are conserved.

A more interesting case is obtained if some *a priori* restrictions are imposed on the applied control $f(x,t)$ and $g(x,t)$. Let us suppose that $\rho(x)$ is a nonnegative periodic function of period 2π supported in ω such that

$$[\rho] = \int_0^{2\pi} \rho(x)dx = 1.$$

For any function $h = h(x,t)$, we define the control operator G by

$$[Gh](x,t) = \rho(x)\left(h(x,t) - \int_0^{2\pi} \rho(y)h(y,t)dy\right). \tag{2.43}$$

Our control inputs f and g in (2.41) will take the form:

$$f(x,t) = [Gh_1](x,t), \qquad g(x,t) = [Gh_2](x,t) \tag{2.44}$$

and (2.41) becomes

$$\begin{cases} \eta_t + w_x + aw_{xxx} - b\eta_{txx} = Gh_1 & \text{for } x \in (0,2\pi),\ \ t \in (0,T) \\ w_t + \eta_x + a\eta_{xxx} - bw_{txx} = Gh_2 & \text{for } x \in (0,2\pi),\ \ t \in (0,T) \\ \frac{\partial^r \eta}{\partial x^r}(t,0) = \frac{\partial^r \eta}{\partial x^r}(t,2\pi) & \text{for } t \in (0,T),\ 0 \le r \le r_0 \\ \frac{\partial^q w}{\partial x^q}(t,0) = \frac{\partial^q w}{\partial x^q}(t,2\pi) & \text{for } t \in (0,T),\ 0 \le q \le q_0 \\ \eta(0,x) = \eta^0(x), \qquad w(0,x) = w^0(x) & \text{for } x \in (0,2\pi) \end{cases} \tag{2.45}$$

with $h_1(x,t)$ and $h_2(x,t)$ as our new control inputs. Note that

$$\int_0^{2\pi} [Gh](x,t)dx = \int_0^{2\pi} \rho(x)h(x,t)dx - \int_0^{2\pi} \rho(x)dx \int_0^{2\pi} \rho(y)h(y,t)dy = 0$$

because of the restriction on ρ. The mean values of both Gh_1 and Gh_2 are zero. The next two technique lemmas are useful in discussing control properties of system (2.45).

Lemma 2.20. *Let $s \in \mathbb{R}$ be given. There exists a constant C depending only on s such that*

$$\|Gh\|_{H_p^s} \le C\|h\|_{H_p^s} \tag{2.46}$$

for any $h \in H_p^s$. In addition,

$$\int_0^{2\pi} [G\xi](x)\overline{\xi(x)}dx = \int_0^{2\pi} \rho(x)\left|\xi(x) - \int_0^{2\pi} \rho(y)\xi(y)dy\right|^2 dx \tag{2.47}$$

for any $\xi \in L^2(0,2\pi)$.

Proof. If $s \ge 0$,

$$[Gh](x) = \rho(x)\left(h(x) - \int_0^{2\pi} \rho(y)h(y)dy\right).$$

Because $\rho = \rho(x)$ is assumed to be smooth, it is easy to see that the estimate (2.46) holds. If $s < 0$, for any $h \in H_p^s$, we define Gh by

$$[Gh](x) = \rho(x)h(x) - \rho(x)\langle \rho, h\rangle_{-s},$$

where $\langle \cdot,\cdot\rangle_s$ is defined in (2.5). For any $h \in H_p^s$, $\rho h \in H_p^s$ and $\rho\langle \rho, h\rangle_{-s} \in H_p^s$, it is straightforward to verify that (2.46) holds.

To see that (2.47) is true, note that

$$\int_0^{2\pi} \rho(y)dy = 1$$

and

$$\int_0^{2\pi} \rho(x) \int_0^{2\pi} \rho(y)\overline{\xi(y)}dy \left(\xi(x) - \int_0^{2\pi} \rho(y)\xi(y)dy \right) dx$$
$$= \left(1 - \int_0^{2\pi} \rho(x)dx\right) \left| \int_0^{2\pi} \rho(y)\xi(y)dy \right|^2 = 0.$$

(2.47) follows then by a direct computation. □

Lemma 2.21. *Let $\phi_j(x) = e^{ijx}$ for $j = \pm 1, \pm 2, \cdots$ and*

$$m_{j,k} = \frac{1}{2\pi} \int_0^{2\pi} G[\phi_j](x)\overline{\phi_k}(x)dx, \qquad j,\ k = \pm 1, \pm 2, \cdots .$$

In addition, for any given finite sequence of nonzero integers k_j, $j = 1, 2, 3, \cdots n$, let

$$A_n = \begin{pmatrix} m_{k_1,k_1} & \cdots & m_{k_1,k_n} \\ m_{k_2,k_1} & \cdots & m_{k_2,k_n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ m_{k_n,k_1} & \cdots & m_{k_n,k_n} \end{pmatrix}.$$

Then

(i) *there exists a constant $\mu > 0$ such that*

$$m_{k,k} \geq \mu \qquad \text{for any } k = \pm 1, \pm 2, \cdots; \tag{2.48}$$

(ii) *A_n is an invertible $n \times n$ hermitian matrix.*

Proof. Note that

$$m_{k,k} = \frac{1}{2\pi} \int_0^{2\pi} G[\phi_k](x)\overline{\phi_k}(x)dx = \frac{1}{2\pi} \left(\int_0^{2\pi} \rho(x)dx - \left| \int_0^{2\pi} \rho(x)\phi_k(x)dx \right|^2 \right) > 0.$$

for any $k \neq 0$, and

$$\lim_{k\to\infty} m_{k,k} = \frac{1}{2\pi} \int_0^{2\pi} \rho(x)dx.$$

Estimate (2.48) follows consequently. To see that (ii) is true, let Σ_n be the space spanned by ϕ_{k_j}, $j = 1, 2, \cdots, n$. In addition, let p_j be the projection of $G(\phi_{k_j})$ onto the space Σ_n, i.e.,

$$p_j = \sum_{l=1}^n m_{k_j,k_l}\phi_{k_l}$$

for $j = 1, 2, \cdots, n$. It suffices to show that p_j, $j = 1, 2, \cdots, n$ is a linearly independent set in the space Σ_n. Assume that there exist scalars λ_j, $j = 1, 2, \cdots, n$ such that

$$\lambda_1 p_1(x) + \lambda_2 p_2(x) + \cdots + \lambda_n p_n(x) \equiv 0.$$

Then, by the definition of p_j,

$$\sum_{j,l=1}^{n} \lambda_j m_{k_j,k_l} \phi_{k_l} = \sum_{l=1}^{n} \left\langle G\left(\sum_{j=1}^{n} \lambda_j \phi_{k_j}\right), \phi_{k_l}\right\rangle \phi_{k_l} \equiv 0.$$

Here $< f, g >$ stands for the inner product of f and g in the space $L^2(0, 2\pi)$. Since $\phi_{k_l},\ l = 1, 2, \cdots, n$ is a basis of Σ_n,

$$\left\langle G\left(\sum_{j=1}^{n} \lambda_j \phi_{k_j}\right), \phi_{k_l}\right\rangle = 0$$

for $l = 1, 2, \cdots, n$. As a result,

$$\left\langle G\left(\sum_{j=1}^{n} \lambda_j \phi_{k_j}\right), \sum_{l=1}^{n} \lambda_l \phi_{k_l}\right\rangle = 0,$$

which implies that

$$\sum_{l=1}^{n} \lambda_l \phi_{k_l} = 0,$$

and therefore

$$\lambda_l = 0, \qquad l = 1, 2, \cdots, n.$$

The proof is complete. □

Now we return to the study of the controllability of system (2.45). Consider the change of variables

$$\eta = v + u \qquad \text{and} \qquad w = v - u.$$

In terms of these new variables, the equations in (2.45) become

$$\partial_t \begin{pmatrix} v \\ u \end{pmatrix} + \mathcal{B} \begin{pmatrix} v \\ u \end{pmatrix} = \begin{pmatrix} \mathcal{H}_1 G(f^*) \\ \mathcal{H}_1 G(g^*) \end{pmatrix} \tag{2.49}$$

with

$$f^* = (h_1 + h_2)/2, \qquad g^* = (h_1 - h_2)/2,$$

where $\mathcal{B}$ is the skew-adjoint operator with symbol

$$ik \begin{pmatrix} \sigma(k) & 0 \\ 0 & -\sigma(k) \end{pmatrix}, \qquad \sigma(k) = \frac{1 - ak^2}{1 + bk^2}$$

and the symbol of $\mathcal{H}_1$ is $\frac{1}{1+bk^2}$.

We take $f^*(x, t)$ in (2.49) to have the following form

$$f^*(x, t) = \sum_{j=-\infty}^{\infty} f_j q_j(t) \phi_j(x)$$

where $\phi_j(x) = e^{ijx}$, $j = 0, \pm 1, \pm 2. \cdots$, f_j and $q_j(t)$ are to be determined later. Then

$$\frac{d}{dt} \hat{v}_k(t) + ik\sigma(k)\hat{v}_k(t) = \frac{1}{1 + bk^2} \sum_{j=-\infty}^{\infty} f_j q_j(t) m_{j,k}$$

where

$$m_{j,k} = \frac{1}{2\pi} \int_0^{2\pi} G(\phi_j)(x) \overline{\phi_k}(x) dx.$$

Thus,

$$\hat{v}_k(T) - e^{-ik\sigma(k)T}\hat{v}_k(0) = \frac{1}{1+bk^2}\sum_{j=-\infty}^{\infty} f_j m_{j,k}\int_0^T e^{-ik\sigma(k)(T-\tau)}q_j(\tau)d\tau$$

or

$$\hat{v}_k(T)e^{ik\sigma(k)T} - \hat{v}_k(0) = \frac{1}{1+bk^2}\sum_{j=-\infty}^{\infty} f_j m_{j,k}\int_0^T e^{ik\sigma(k)\tau}q_j(\tau)d\tau.$$

Let $p_k(t) = e^{-\lambda_k t} = e^{-ik\sigma(k)t}$. Recall that $\mathbb{I}$ is a subset of $\mathbb{Z}$ such that $\lambda_{k_1} \neq \lambda_{k_2}$ for any $k_1,\ k_2 \in \mathbb{I}$ with $k_1 \neq k_2$. If $a \neq 0$, then $k\sigma(k) \to \pm\infty$ as $k \to \pm\infty$ and there exists an integer $k^* > 0$ such that $k \in \mathbb{I}$ if $|k| > k^*$. Thus there are only finite many integers in $\mathbb{I}$, saying, $k_j,\ j = 1, 2, \cdots, n$, such that one can find another integer $k \neq k_j$ with $\lambda_k = \lambda_{k_j}$. Let

$$\mathbb{I}_j = \{k \in \mathbb{Z}, k \neq k_j,\ \lambda_k = \lambda_{k_j}\}, \quad j = 1, 2, \cdots, n.$$

Then

$$\mathbb{Z} = \mathbb{I} \cup \mathbb{I}_1 \cup \mathbb{I}_2 \cup \cdots \cup \mathbb{I}_n.$$

Note that each $\mathbb{I}_j$ contains at most two integers. Without loss of generality, we may assume that

$$\mathbb{I}_j = \{k_{j,1},\ k_{j,2}\} \quad j = 1, 2, \cdots, n$$

and rewrite k_j as $k_{j,0}$. Moreover, $\mathcal{P} \equiv \{p_k |\ k \in \mathbb{I}\}$ forms a Riesz basis for its closed span, P_T, in $L^2(0, T)$ if

$$T > \frac{2\pi}{\gamma} \qquad \text{where} \quad \gamma := \liminf_{k\to\infty}(\lambda_{k+1} - \lambda_k).$$

We let $\mathcal{L} \equiv \{q_j |\ j \in \mathbb{I}\}$ be the unique dual Riesz basis for $\mathcal{P}$ in P_T, i.e., the functions in $\mathcal{L}$ are the unique elements of P_T such that

$$\int_0^T q_j(t)\overline{p_k(t)}dt = \delta_{kj}, \qquad j,\ k \in \mathbb{I}.$$

In addition, we choose

$$q_k = q_{k_j} \quad \text{ if } k \in \mathbb{I}_j.$$

For such choice of $q_j(t)$, $\quad -\infty < j < \infty$, we have then, for any $k \in \mathbb{N}$,

$$\hat{v}_k(T)e^{ik\sigma(k)T} - \hat{v}_k(0) = \frac{1}{1+bk^2}f_k m_{k,k}, \text{ if } k \neq k_{j,l}, \quad l = 0, 1, 2, j = 1, 2, \cdots, n; \tag{2.50}$$

$$\begin{cases} \hat{v}_{k_{j,0}}(T)e^{ik_{j,0}\sigma(k_{j,0})T} - \hat{v}_{k_{j,0}}(0) = \frac{1}{1+bk_{j,0}^2}\sum_{l=0}^2 f_{k_{j,l}}m_{k_{j,l},k_{j,0}}, \\ \\ \hat{v}_{k_{j,1}}(T)e^{ik_{j,1}\sigma(k_{j,1})T} - \hat{v}_{k_{j,1}}(0) = \frac{1}{1+bk_{j,1}^2}\sum_{l=0}^2 f_{k_{j,l}}m_{k_{j,l},k_{j,1}}, \\ \\ \hat{v}_{k_{j,2}}(T)e^{ik_{j,2}\sigma(k_{j,2})T} - \hat{v}_{k_{j,2}}(0) = \frac{1}{1+bk_{j,2}^2}\sum_{l=0}^2 f_{k_{j,l}}m_{k_{j,l},k_{j,2}} \end{cases} \tag{2.51}$$

if $k = k_{j,l}$ for $j = 1, 2, \cdots, n$ and $l = 0, 1, 2$. According to Lemma 2.21, for given initial state v^0 and terminal state v^1 with zero mean, system (2.50)-(2.51) admits a unique solution $\vec{f}(\cdots, f_{-2}, f_{-1}, f_1, f_2, \cdots)$. In particular,

$$f_k = \frac{1+bk^2}{m_{k,k}}\left(\widehat{v}_k^1 e^{ik\sigma(k)T} - \widehat{v}_k^0\right), \quad \text{if } |k| \geq k^*.$$

Similarly, for given initial state u^0 and terminal state u^1 with zero mean, choose

$$g^*(x,t) = \sum_{j=-\infty}^{\infty} g_j q_{-j}(t)\phi_j(x)$$

with

$$g_k = \frac{1+dk^2}{m_{k,k}}\left(\widehat{u}_k^1 e^{ik\sigma(k)T} - \widehat{u}_k^0\right), \quad \text{if } |k| \geq k^*.$$

Then the system (2.49) admits a unique solution $(v(x,t), u(x,t))$ satisfying

$$(v(x,0), u(x,0)) = (v^0(x), u^0(x)), \qquad (v(x,T), u(x,T)) = (v^1(x), u^1(x))$$

for $x \in (0, 2\pi)$. This analysis leads to the following controllability result for the system (2.49).

Proposition 2.22. *Assume that the parameter $a \neq 0$ and $T > \frac{2\pi}{\gamma}$. Let $s,\ s' \in \mathbb{R}$ be given and let $n_1 = 2$ if $b \neq 0$ and $n_1 = 0$ if $b = 0$. Then, for any given initial state $(v^0, u^0) \in H^s_{0,p} \times H^{s'}_{0,p}$ and the terminal state $(v^1, u^1) \in H^s_{0,p} \times H^{s'}_{0,p}$, there exist*

$$f^* \in L^2(0,T; H^{s-n_1}_{0,p}) \qquad \textit{and} \qquad g^* \in L^2(0,T; H^{s'-n_1}_{0,p})$$

such that the system (2.49) admits a unique solution $(v,u) \in C([0,T]; H^s_{0,p} \times H^{s'}_{0,p})$ satisfying

$$(v(x,0), u(x,0)) = (v^0(x), u^0(x)) \qquad \textit{and} \qquad (v(x,T), u(x,T)) = (v^1(x), u^1(x)).$$

Moreover, there exists a constant $C > 0$ depending only on T, s and s' such that

$$\begin{aligned} &\|f^*\|_{L^2(0,T;H^{s-n_1}_{0,p})} + \|g^*\|_{L^2(0,T;H^{s'-n_1}_{0,p})} \\ \leq & C\left(\|(v^0,u^0)\|_{H^s_{0,p}\times H^{s'}_{0,p}} + \|(v^1,u^1)\|_{H^s_{0,p}\times H^{s'}_{0,p}}\right). \end{aligned}$$

Consequently we have the following exact controllability result for the original system (2.45).

Theorem 2.23. *Assume that the parameter $a \neq 0$ and $T > \frac{2\pi}{\gamma}$. Let $s \in \mathbb{R}$ be given. Then, for any (η^0, w^0) and (η^1, w^1) belonging to the space $H^s_{0,p} \times H^s_{0,p}$, there exist $h_1,\ h_2 \in L^2(0,T; H^{s-n_1}_{0,p})$ such that the system (2.45) admits a unique solution $(\eta, w) \in C([0,T]; H^s_{0,p} \times H^s_{0,p})$ satisfying*

$$\eta(\cdot, T) = \eta^1(\cdot), \qquad w(\cdot,T) = w^1(\cdot) \qquad \textit{in } H^s_{0,p}.$$

Remark 2.24. The following remarks are in order.

(i) The choice of the control inputs in Proposition 2.22 and Theorem 2.23 are based on the moment method instead of the HUM. For this approach, it is necessary to apply controls on both equations.

(ii) In contrast to using only one control input, the advantage of using two control inputs is that the system (2.45) is exactly controllable in $H^s_{0,p} \times H^s_{0,p}$ for any $s \in \mathbb{R}$. □

Next we turn to the stabilization issue. Again, we rewrite system (2.45) as an abstract control system in the space $V^s_{0,0}$:

$$\frac{d}{dt}\vec{\eta} = \mathcal{A}_1\vec{\eta} + \mathcal{B}_1\vec{h}, \qquad \vec{\eta}(0) = \vec{\eta}_0 \tag{2.52}$$

where $\mathcal{A}_1 = \mathcal{A}$ with $a = c$ and $b = d$, and

$$\mathcal{B}_1 \vec{h} = \begin{pmatrix} (I - b\partial_x^2)^{-1} G h_1 \\ \\ (I - b\partial_x^2)^{-1} G h_2 \end{pmatrix}.$$

The following stabilization result using two feedback controls will then be useful.

Theorem 2.25. *Assume that the parameter $a \neq 0$. Let $s \in \mathbb{R}$ be given. For any given $\alpha > 0$, there exists an operator $\mathcal{K}_1 \in \mathcal{L}(V^s_{0,0}, V^{s-n_1}_{0,0})$ such that if one chooses*

$$\vec{h} = \mathcal{K}_1 \vec{\eta}$$

in (2.52), then the resulting closed-loop system

$$\frac{d}{dt}\vec{\eta} = \mathcal{A}_1 \vec{\eta} + \mathcal{B}_1 \mathcal{K}_1 \vec{\eta}, \qquad \vec{\eta}(0) = \vec{\eta}_0 \tag{2.53}$$

has the property that its solution satisfies

$$\|\vec{\eta}(t)\|_{V^s} \leq M \|\vec{\eta}_0\|_{V^s} e^{-\alpha t} \tag{2.54}$$

for any $t \geq 0$ where M is a constant independent of $\vec{\eta}_0$.

Proof. The proof is the same as that of Theorem 2.18. □

3. **Nonlinear systems.**

In this section we consider the (nonlinear) Boussinesq systems posed on the finite interval $(0, 2\pi)$

$$\begin{cases} \eta_t + w_x + (\eta w)_x + a w_{xxx} - b\eta_{txx} = f & \text{for } x \in (0, 2\pi), \ \ t \in (0, T), \\ \\ w_t + \eta_x + w w_x + c\eta_{xxx} - d w_{txx} = g & \text{for } x \in (0, 2\pi), \ \ t \in (0, T) \end{cases} \tag{3.1}$$

with the periodic boundary conditions

$$\begin{cases} \frac{\partial^r \eta}{\partial x^r}(t, 0) = \frac{\partial^r \eta}{\partial x^r}(t, 2\pi) & \text{for } t \in (0, T), \ 0 \leq r \leq r_0, \\ \\ \frac{\partial^q w}{\partial x^q}(t, 0) = \frac{\partial^q w}{\partial x^q}(t, 2\pi) & \text{for } t \in (0, T), \ 0 \leq q \leq q_0, \end{cases} \tag{3.2}$$

and the initial condition

$$\eta(0, x) = \eta^0(x), \qquad w(0, x) = w^0(x), \qquad x \in (0, 2\pi). \tag{3.3}$$

Its well-posedness in a suitable Sobolev space will be studied in subsection 3.1. Its controllability and stabilizability will be investigated in subsection 3.2 and subsection 3.3, respectively.

3.1. **Well-posedness.**

The following lemma, whose proof can be found in [6], is needed in this study.

Lemma 3.1. *Let $s \geq -1$ be given. There exists a constant $C > 0$ depending only on s such that*

$$\|fg\|_{H^s_p} \leq C \|f\|_{H^{s+1}_p} \|g\|_{H^{s+1}_p}$$

for any $f, g \in H^{s+1}_p$.

We first consider the case when both parameters b and d in (3.1) are positive. Such systems are called *weakly dispersive systems* in [6]. Recall that l is an integer such that

$$h(k) = (\omega_1(k)/\omega_2(k))^{1/2} \sim C|k|^l, \qquad \text{as } |k| \to \infty$$

and

$$V^s = H_p^s \times H_p^{s+l}.$$

Theorem 3.2. *Assume $b > 0$ and $d > 0$. Let $T > 0$ and $s \geq 0$ be given such that $s \geq 0$ if $l \geq 0$ and $s \geq 1$ if $l = -1$. Then there exists a constant $r > 0$ such that for any $(\eta^0, w^0) \in V^s$ and any $(f, g) \in L^1(0, T; V^{s-2})$ satisfying*

$$\|(\eta^0, w^0)\|_{V^s} \leq r \quad \text{ and } \quad \|(f, g)\|_{L^1(0,T;V^{s-2})} \leq r, \tag{3.4}$$

the system (3.1) admits a unique solution $(\eta, w) \in C([0, T]; V^s)$ satisfying the boundary condition (3.2) and the initial condition (3.3). Moreover, the corresponding solution map is locally Lipschitz continuous and

$$\|(\eta, w)(t)\|_{V^s} \leq C\|(\eta^0, w^0)\|_{V^s} \tag{3.5}$$

for any $0 \leq t \leq T$ where C is a constant depending only on s, but independent of T and r.

Remark 3.3. It is also true that for any $r > 0$ there exists a $T > 0$ depending only on s and r such that for $(\eta^0, w^0) \in V^s$ satisfying (3.4), the system (3.1)-(3.3) admits a unique solution $(\eta, w) \in C([0, T]; V^s)$ satisfying the estimate (3.5). □

Proof. Rewrite (3.1)-(3.3) in its integral form:

$$(\eta, w)(t) = S(t)(\eta^0, w^0) + \int_0^t S(t-\tau)(f, g)(\tau)d\tau - \int_0^t S(t-\tau)((\eta w)_x, ww_x)(\tau)d\tau.$$

This suggests us, for given $(\eta^0, w^0) \in V^s$ and $(f, g) \in L^1(0, T; V^{s-2})$, to define a map

$$\Gamma : C([0, T]; V^s) \to C([0, T]; V^s)$$

by

$$\Gamma(u, v)(t) = S(t)(\eta^0, w^0) + \int_0^t S(t-\tau)(f, g)(\tau)d\tau - \int_0^t S(t-\tau)((uv)_x, vv_x)(\tau)d\tau \tag{3.6}$$

for any $(u, v) \in C([0, T]; V^s)$. According to Theorem 2.4, there exist constants $C_1 > 0$ and $C_2 > 0$ independent of T such that

$$\begin{aligned}
\|\Gamma(u, v)(t)\|_{V^s} &\leq C_1 \left(\|(\eta^0, w^0)\|_{V^s} + \|(f, g)\|_{L^1(0,T;V^{s-2})}\right) \\
&\quad + C_1 \int_0^T \|((uv)_x, vv_x)(\tau)\|_{V^{s-2}} d\tau \\
&\leq C_1 \left(\|(\eta^0, w^0)\|_{V^s} + \|(f, g)\|_{L^1(0,T;V^{s-2})}\right) + C_1 \int_0^T \|(uv, v^2/2)(\tau)\|_{V^{s-1}} d\tau \\
&\leq C_1 \left(\|(\eta^0, w^0)\|_{V^s} + \|(f, g)\|_{L^1(0,T;V^{s-2})}\right) + C_1 C_2 T \sup_{0 \leq t \leq T} \|(u, v)(t)\|^2_{V^s}
\end{aligned}$$

for any $0 \le t \le T$. Let $r > 0$ and $R > 0$ be chosen according to

$$R = 2C_1 r, \qquad 4C_1^2 C_2 r T = \frac{1}{2}, \tag{3.7}$$

then, if $\|(\eta^0, w^0)\|_{V^s} + \|(f,g)\|_{L^1(0,T;V^{s-2})} \le r$ and $\|(u,v)\|_{C([0,T];V^s)} \le R$, we have that

$$\|\Gamma(u,v)(t)\|_{V^s} \le C_1 r + 4C_1^2 r^2 C_1 C_2 T = \frac{3}{2} C_1 r < R.$$

Thus Γ maps the ball $B(0,R)$ in the space $C([0,T];V^s)$ into itself. Moreover, for any $(u,v) \in B(0,R)$ and any $(\mu,\nu) \in B(0,R)$,

$$\begin{aligned}
&\|\Gamma(u,v)(t) - \Gamma(\mu,\nu)(t)\|_{V^s} \le C_1 \int_0^T \|((uv)_x - (\mu\nu)_x, vv_x - \nu\nu_x)(\tau)\|_{V^{s-2}} d\tau \\
&\le C_1 \int_0^T \|(uv - \mu\nu, \frac{1}{2}v^2 - \frac{1}{2}\nu^2)(\tau)\|_{V^{s-1}} d\tau \\
&\le C_1 C_2 T \left(\|(u,v)\|_{C([0,T];V^s)} + \|(\mu,\nu)\|_{C([0,T];V^s)}\right) \|(u-\mu, v-\nu)\|_{C([0,T];V^s)} \\
&\le 4C_1^2 C_2 r T \|(u-\mu, v-\nu)\|_{C([0,T];V^s)} \\
&\le \quad \frac{1}{2} \|(u-\mu, v-\nu)\|_{C([0,T];V^s)}.
\end{aligned}$$

Thus, Γ is a contraction mapping on the ball $B(0,R)$ in the space $C([0,T];V^s)$. Its fixed point (η, w) is thus the desired solution of (3.1)-(3.3). The proof is complete. $\square$

Next we consider the system (3.1) with

$$b = d = 0, \qquad a \ne 0, \quad c \ne 0. \tag{3.8}$$

In view of the constraints (C1) and (C2), the only admissible case is when $a = c > 0$. The system (3.1) then takes the form

$$\begin{cases} \eta_t + w_x + (\eta w)_x + a w_{xxx} = f, \\ \\ w_t + \eta_x + w w_x + a \eta_{xxx} = g, \end{cases} \tag{3.9}$$

for $(x,t) \in (0,2\pi) \times (0,T)$. Introducing v and u by $\eta = v + u$ and $w = v - u$, we obtained the equivalent system

$$\begin{cases} v_t + v_x + a v_{xxx} + \frac{1}{2}\left[(v-u)(v+u)\right]_x + \frac{1}{2}(v-u)(v-u)_x = \frac{1}{2}(f+g), \\ \\ u_t - u_x - a u_{xxx} + \frac{1}{2}[(v-u)(v+u)]_x - \frac{1}{2}(v-u)(v-u)_x = \frac{1}{2}(f-g), \end{cases} \tag{3.10}$$

for $(x,t) \in (0,2\pi) \times (0,T)$. This is a system of two linear KdV-equations coupled through nonlinear terms. One can apply the theory developed by Kato [15] for the scalar KdV to obtain the following local well-posedness result.

Theorem 3.4. *Let $s > \frac{3}{2}$ and $T > 0$ be given and assume that*

$$b = d = 0, \qquad a = c > 0.$$

Then there exists a constant $r > 0$ such that for any $(\eta^0, w^0) \in V^s$ and any $(f, g) \in L^1(0, T; V^s)$ satisfying

$$\|(\eta^0, w^0)\|_{V^s} \leq r \quad \text{ and } \quad \|(f, g)\|_{L^1(0,T;V^s)} \leq r,$$

the system (3.1) admits a unique solution $(\eta, w) \in C([0, T]; V^s)$ satisfying the boundary condition (3.2) and the initial condition (3.3). Moreover, the corresponding solution map is continuous.

Next we turn to the other admissible Boussinesq systems (3.1) with the restrictions:

$$(C3) \qquad c = 0,\ b = 0, \quad a < 0, \quad d > 0;$$

or

$$(C4) \qquad a = c > 0, \quad b = 0, \quad d > 0;$$

or

$$(C5) \qquad a < 0, \quad c < 0, \quad b = 0, \quad d > 0.$$

By the same arguments used in the proofs of Theorem 3.1 and Theorem 3.5 in [6] we have the following local well-posedness result for the system (3.1)-(3.3).

Theorem 3.5. *Assume the parameters a, b, c and d in (3.1) satisfy one of the assumptions (C3), (C4) and (C5). Let $s \geq 1$ and $T > 0$ be given. Then there exists a constant $r > 0$ such that for any $(\eta^0, w^0) \in V^s$ and any $(f, g) \in L^2(0, T; H^s_p \times H^s_p)$ satisfying*

$$\|(\eta^0, w^0)\|_{V^s} \leq r \quad \text{ and } \quad \|(f, g)\|_{L^2(0,T;H^s_p \times H^s_p)} \leq r,$$

the system (3.1) admits a unique solution $(\eta, w) \in C([0, T]; V^s))$ satisfying the boundary condition (3.2) and the initial condition (3.3). Moreover, the corresponding solution map is continuous.

Now consideration is turned to systems (3.1) with

$$(C6) \qquad c = d = 0, \quad a < 0, \quad b > 0$$

or

$$(C7) \qquad a = b = 0, \quad d > 0, \quad c < 0,$$

or

$$(C8) \qquad a = c \geq 0, \quad d = 0, \quad b > 0.$$

Note that if either of $(C6)$ and $(C7)$ is satisfied, we have the number $l = 0$ whilst $l = -1$ if (C8) is satisfied.

Theorem 3.6. *Assume the parameters a, b, c and d in (3.1) satisfy (C6). Let $s \geq 2$ and $T > 0$ be given. Then there exists a constant $r > 0$ such that for any $(\eta^0, w^0) \in V^s$ and any $(f, g) \in L^2(0, T; H^{s-2}_p \times H^s_p)$ satisfying*

$$\|(\eta^0, w^0)\|_{V^s} \leq r \quad \text{ and } \quad \|(f, g)\|_{L^2(0,T;H^{s-2}_p \times H^s_p)} \leq r,$$

the system (3.1)-(3.3) admits a unique solution $(\eta, w) \in C([0, T]; V^s)$. Moreover, the corresponding solution map is continuous.

Remark 3.7. If assumption (C7) is satisfied instead of (C6), the theorem holds with $(f, g) \in L^2(0, T; H^s_p \times H^{s-2}_p)$ instead of $(f, g) \in L^2(0, T; H^{s-2}_p \times H^s_p)$. $\square$

Theorem 3.8. *Assume the parameters a, b, c and d in (3.1) satisfy (C8) . Let $s \geq 2$ and $T > 0$ be given. Then there exists a constant $r > 0$ such that for any $(\eta^0, w^0) \in V^s$ and any $(f, g) \in L^2(0, T; H_p^{s-1} \times H_p^{s-1})$ satisfying*

$$\|(\eta^0, w^0)\|_{V^s} \leq r \quad \text{and} \quad \|(f, g)\|_{L^2(0,T;H_p^{s-1}\times H_p^{s-1})} \leq r,$$

the system (3.1) admits a unique solution $(\eta, w) \in C([0, T]; V^s)$ satisfying the boundary condition (3.2) and the initial condition (3.3). Moreover, the corresponding solution map is continuous.

Both Theorem 3.6 and Theorem 3.8 can be proved by the same arguments as those in the proofs of Theorem 3.9 and Theorem 3.11 in [6] although the theorems in [6] are established for systems posed on the whole real line $\mathbb{R}$.

3.2. **Exact controllability.**

In this subsection, we study first the exact controllability of the nonlinear system with a single control input:

$$\begin{cases} \eta_t + w_x + (\eta w)_x + a w_{xxx} - b\eta_{txx} = Qh & \text{for } x \in (0, 2\pi), \ \ t \in (0, T), \\ w_t + \eta_x + w w_x + c\eta_{xxx} - d w_{txx} = 0 & \text{for } x \in (0, 2\pi), \ \ t \in (0, T), \\ \frac{\partial^r \eta}{\partial x^r}(t, 0) = \frac{\partial^r \eta}{\partial x^r}(t, 2\pi) & \text{for } t \in (0, T), \ 0 \leq r \leq 2, \\ \frac{\partial^q w}{\partial x^q}(t, 0) = \frac{\partial^q w}{\partial x^q}(t, 2\pi) & \text{for } t \in (0, T), \ 0 \leq q \leq 2, \\ \eta(0, x) = \eta^0(x), \qquad w(0, x) = w^0(x) & \text{for } x \in (0, 2\pi). \end{cases} \tag{3.11}$$

We assume that the coefficients a, b, c and d are all different from 0, hence

$$r_0 = q_0 = 2, \qquad \tilde{l} = l = e = 0.$$

According to Table 1, the associated linear system is exactly controllable in the space $V^2_{*,0} = H^2_p \times H^2_{0,p}$ with $T > 2\pi\sqrt{\frac{db}{ac}}$. The following theorem is one of the main results of this paper, which shows that the nonlinear system (3.11) is locally exactly controllable.

Theorem 3.9. *Assume that $a, b, c, d \neq 0$, and let $T > 2\pi\sqrt{\frac{db}{ac}}$ be given. Then there exists a constant $r > 0$ such that for any $(\eta^0, w^0), (\eta^1, w^1) \in V^2_{*,0}$ satisfying*

$$\left\|(\eta^0, w^0)\right\|_{V^2} \leq r, \qquad \left\|(\eta^1, w^1)\right\|_{V^2} \leq r,$$

one may find a control function $h \in L^2(0, T; L^2(0, 2\pi))$ such that (3.11) admits a unique solution

$$(\eta, w) \in C([0, T]; V^2_{*,0}) \cap C^1(0, T; V^1_{*,0})$$

satisfying

$$\eta(T, x) = \eta^1(x), \qquad w(T, x) = w^1(x), \qquad x \in (0, 2\pi). \tag{3.12}$$

Proof. In order to prove the above result, we rewrite (3.11) in its integral form:

$$(\eta, w)(t) = S(t)(\eta^0, w^0) + \int_0^t S(t - \tau)(Qh, 0)(\tau)d\tau - \int_0^t S(t - \tau)((\eta w)_x, w w_x)(\tau)d\tau.$$

For given $(\eta^0, w^0),\ (\eta^1, w^1) \in V^2_{*,0}$ and $(u,v) \in C([0,T]; V^2_{*,0})$, let

$$(p^0, q^0) = S(T)(\eta^0, w^0), \qquad (p^1, q^1) = \int_0^T S(T-\tau)((uv)_x, vv_x)(\tau)d\tau.$$

By the exact controllability results established in Section 2, there exists a $h \in L^2((0,2\pi)\times(0,T))$ such that

$$(\phi, \psi)(t) = \int_0^t S(t-\tau)(Qh, 0)(\tau)d\tau \tag{3.13}$$

satisfies

$$(\phi,\psi)(0) = (0,0), \qquad\qquad (\phi,\psi)(T) = (\eta^1, w^1) - (p^0, q^0) + (p^1, q^1). \tag{3.14}$$

Moreover there exists a constant $C_1 > 0$ depending only on T such that

$$\|h\|_{L^2((0,2\pi)\times(0,T))} \le C_1 \left(\|(\eta^0, w^0)\|_{V^2} + \|(\eta^1, w^1)\|_{V^2} + \|(p^1, q^1)\|_{V^2} \right). \tag{3.15}$$

It thus defines, for given (η^0, w^0) and (η^1, w^1) in V^2_0, a nonlinear map L from the space $C([0,T]; V^2_{*,0})$ to the space $L^2((0,2\pi)\times(0,T))$:

$$h := L(u,v), \qquad \text{for any } (u,v) \in C([0,T]; V^2_{*,0})$$

such that (3.13)-(3.15) hold.

For given $(\eta^0, w^0),\ (\eta^1, w^1) \in V^2_{*,0}$, let us define a nonlinear map

$$\Gamma : C([0,T]; V^2_{*,0}) \to C([0,T]; V^2_{*,0})$$

by

$$\begin{aligned} &\Gamma(u,v)(t) \\ =&S(t)(\eta^0, w^0) + \int_0^t S(t-\tau)(QL(u,v),0)(\tau)d\tau - \int_0^t S(t-\tau)((uv)_x, vv_x)(\tau)d\tau. \end{aligned}$$

By the definition of the operator L,

$$\Gamma(u,v)(0) = (\eta^0, w^0), \qquad \Gamma(u,v)(T) = (\eta^1, w^1).$$

Consequently, if we can show that Γ is a contraction map in a ball of the space $C([0,T]; V^2_{*,0})$, then its fixed point will solve the system (3.11) and satisfy (3.12). The proof will be then completed.

To this end, let r and R be given positive numbers (to be specified later), and assume that

$$||(\eta^0, w^0)||_{V^2} \le r, \qquad ||(\eta^1, w^1)||_{V^2} \le r, \qquad ||(u,v)||_{X_T} \le R,$$

where $X_T := C([0,T]; V^2_{*,0})$. There exist constants C_2 and C_3 independent of r and R such that

$$\begin{aligned} ||\Gamma(u,v)||_{X_T} &\le ||(\eta^0, w^0)||_{V^2} + C_2 T \|(u,v)\|^2_{X_T} \\ &\quad + C_1 C_2 T^{\frac12} \left(||(\eta^1, w^1)||_{V^2} + ||(\eta^0, w^0)||_{V^2} + C_3 \|(u,v)\|^2_{X_T} \right) \\ &\le (1 + 2C_1C_2T^{\frac12})r + (C_1C_3 + T^{\frac12})C_2T^{\frac12}R^2 =: A'r + B'R^2. \end{aligned}$$

Moreover, if (u^1, v^1) and (u^2, v^2) belong to the space X_T and

$$\|(u^1, v^1)\|_{X_T} \le R, \qquad \|(u^2, v^2)\|_{X_T} \le R,$$

then

$$\begin{aligned} ||\Gamma(u^1, v^1) - \Gamma(u^2, v^2)||_{X_T} &\le 2C_1C_2C_3T^{\frac{1}{2}}R\|(u^1 - u^2, v^1 - v^2)\|_{X_T} \\ &\quad +2C_2TR\|(u^1 - u^2, v^1 - v^2)\|_{X_T} \\ &\le 2C_2T^{\frac{1}{2}}(T^{\frac{1}{2}} + C_1C_3)R||(u^1 - u^2, v^1 - v^2)||_{X_T} \\ &:= B''R||(u^1 - u^2, v^1 - v^2)||. \end{aligned}$$

Choose r and R such that

$$A'r + B'R^2 \le R, \qquad B''R \le \frac{1}{2}. \tag{3.16}$$

Then

$$\|\Gamma(u, v)\|_{X_T} \le R$$

and

$$\|\Gamma(u^1, v^1) - \Gamma(u^2, v^2)\|_{X_T} \le \frac{1}{2}\|(u^1 - u^2, v^1 - v^2)\|_{X_T}.$$

Thus the map Γ is a contraction in the ball $B(0, R)$ of the space X_T if r and R are chosen according to (3.16). The proof is complete. $\square$

Next we study the exact controllability of the nonlinear system with two control inputs:

$$\begin{cases} \eta_t + w_x + (\eta w)_x + aw_{xxx} - b\eta_{txx} = Gh_1 & \text{for } x \in (0, 2\pi),\ \ t \in (0, T) \\ w_t + \eta_x + ww_x + c\eta_{xxx} - dw_{txx} = Gh_2, & \text{for } x \in (0, 2\pi),\ \ t \in (0, T) \\ \frac{\partial^r \eta}{\partial x^r}(t, 0) = \frac{\partial^r \eta}{\partial x^r}(t, 2\pi), & \text{for } t \in (0, T),\ \ 0 \le r \le 2 \\ \frac{\partial^q w}{\partial x^q}(t, 0) = \frac{\partial^q w}{\partial x^q}(t, 2\pi), & \text{for } t \in (0, T),\ \ 0 \le q \le 2 \\ \eta(0, x) = \eta^0(x), \qquad w(0, x) = w^0(x), \qquad x \in (0, 2\pi). \end{cases} \tag{3.17}$$

According to Theorem 2.23, if $a = c \ne 0$, $b = d$, and $T > \frac{2\pi}{\gamma}$, then the associated linear system is exactly controllable in the space $H^s_{0,p} \times H^s_{0,p}$ for any $s \in \mathbb{R}$. The theorem presented below extends this result partially to the nonlinear system (3.17).

Theorem 3.10. *Let $s \ge 0$ be given. Assume that $a = c \ne 0$ and $b = d \ne 0$, and $T > \frac{2\pi}{\gamma}$. Then there exists a constant $r > 0$ such that for any $(\eta^0, w^0), (\eta^1, w^1) \in V^s_{0,0}$ satisfying*

$$\left\|(\eta^0, w^0)\right\|_{V^s} \le r, \qquad \left\|(\eta^1, w^1)\right\|_{V^s} \le r,$$

one may find control functions $h_1,\ h_2 \in L^2(0, T; V^{s-2})$ such that (3.17) admits a unique solution $(\eta, w) \in C([0, T]; V^s_{0,0})$ satisfying

$$\eta(T, x) = \eta^1(x), \qquad w(T, x) = w^1(x), \qquad x \in (0, 2\pi). \tag{3.18}$$

Proof. The proof is very similar to that of Theorem 3.9 and is therefore omitted. □

3.3. **Stabilization.** In this subsection, for given $s \geq 0$, we consider the nonlinear closed-loop system

$$\frac{d}{dt}\vec{\eta} - \mathcal{A}\vec{\eta} + N(\vec{\eta}) = \widetilde{\mathcal{B}}\mathbf{K}\vec{\eta}, \qquad \vec{\eta}(0) = \vec{\eta}_0 \tag{3.19}$$

in the space V^s where

$$N(\vec{\eta}) = \begin{pmatrix} (I - b\partial_x^2)^{-1}(\eta w)_x \\ \\ (I - d\partial_x^2)^{-1} w w_x \end{pmatrix},$$

$\widetilde{\mathcal{B}} = \mathcal{B}$ or $\mathcal{B}_1$, $\mathbf{K}$ is a linear feedback operator bounded from the space V^s to the domain of $\widetilde{\mathcal{B}}$. As in subsection 3.2, we assume that all the coefficients a, b, c and d are not zero. Thus $l = 0$ and $V^s \sim H_p^s \times H_p^s$. Using the same argument as that in the proof of Theorem 3.2 yields the following well-posedness result for system (3.19).

Theorem 3.11. *Let $s \geq 0$ and $T > 0$ be given. Assume that a, b, c, d are all not zero. Then there exists a $r > 0$ such that for any $\vec{\eta}_0 \in V^s$ satisfying $\|\vec{\eta}_0\|_{V^s} \leq r$, (3.19) admits a unique solution $\vec{\eta} \in C([0,T]; V^s)$. Moreover, the solution map is locally Lipschitz continuous*

Next we show that, under the assumptions of Theorem 2.18 or Theorem 2.25, if $\mathbf{K}$ is $\mathcal{K}$ or $\mathcal{K}_1$, then small amplitude solutions of (3.19) exist for all time $t > 0$ and decay exponentially as $t \to \infty$. More precisely, we have the following two theorems.

Theorem 3.12. *Assume that $a \neq 0$, $c \neq 0$, $b > 0$, $d > 0$, $\widetilde{\mathcal{B}} = \mathcal{B}$. In addition, for a given $\alpha > 0$, let $\mathbf{K} = \mathcal{K}$ as chosen in Theorem 2.18. Then there exist $r > 0$ and $M > 0$ such that for any $\vec{\eta}_0 \in \mathcal{V}$ satisfying $\|\vec{\eta}_0\|_{\mathcal{V}} \leq r$, (3.19) admits a unique solution $\vec{\eta} \in C(\mathbb{R}^+; \mathcal{V})$. Moreover,*

$$\|\vec{\eta}(t)\|_{\mathcal{V}} \leq M\|\vec{\eta}_0\|_{\mathcal{V}} e^{-\alpha t}$$

for any $t \geq 0$.

Theorem 3.13. *Assume that $a = c \neq 0$, $b = d > 0$ and $\widetilde{\mathcal{B}} = \mathcal{B}_1$. In addition, for given $s \geq 0$ and $\alpha > 0$, let $\mathbf{K} = \mathcal{K}_1$ as chosen in Theorem 2.25. There exist $r > 0$ and $M > 0$ such that for any $\vec{\eta}_0 \in V^s_{0,0}$ satisfying $\|\vec{\eta}_0\|_{V^s} \leq r$, (3.19) admits a unique solution $\vec{\eta} \in C(\mathbb{R}^+; V^s_{0,0})$. Moreover,*

$$\|\vec{\eta}(t)\|_{V^s} \leq M\|\vec{\eta}_0\|_{V^s} e^{-\alpha t}$$

for any $t \geq 0$.

We only provide a proof of Theorem 3.13. The proof of Theorem 3.12 is similar and is therefore omitted.

Proof of Theorem 3.13: Let $S_F(t)$ be the C_0 semigroup generated by $\mathcal{A} + \mathcal{B}_1\mathcal{K}_1$. Then, by Theorem 2.25, there exists a $C_1 > 0$ such that

$$\|S_F(t)\vec{\eta}_0\|_{V^s} \leq C_1\|\vec{\eta}_0\|_{V^s} e^{-\alpha t} \qquad \forall\ t \geq 0. \tag{3.20}$$

In addition, let C_2 be a positive constant such that

$$\|N(\vec{u})\|_{V^s} \leq C_2\|\vec{u}\|^2_{V^s}$$

and

$$\|N(\vec{u})-N(\vec{v})\|_{V^s} \le C_2\Big(\|\vec{u}\|_{V^s}+\|\vec{v}\|_{V^s}\Big)\|\vec{u}-\vec{v}\|_{V^s}$$

for any $\vec{u}$, $\vec{v}\in V^s$. Let $Y_{s,\alpha}$ be the space

$$Y_{s,\alpha}=\{\vec{u}\in C_b([0,\infty);V^s);\ e^{\alpha t}\vec{u}\in C_b([0,\infty);V^s)\}$$

with the norm

$$\|\vec{u}\|_{Y_{s,\alpha}} := \sup_{0\le t<\infty}\|e^{\alpha t}\vec{u}(t)\|_{V^s}.$$

Rewrite (3.19) in its integral form:

$$\vec{\eta}(t)=S_F(t)\vec{\eta}_0-\int_0^t S_F(t-\tau)N(\vec{\eta})(\tau)d\tau. \tag{3.21}$$

For given $\vec{\eta}_0\in V^s$ with $\|\vec{\eta}_0\|_{V^s}=r$ ($r>0$ to be determined), consider the map

$$\Gamma(\vec{\eta})(t)=S_F(t)\vec{\eta}_0-\int_0^t S_F(t-\tau)N(\vec{\eta})(\tau)d\tau.$$

By (3.20),

$$\begin{aligned}\|\Gamma(\vec{\eta})(t)\|_{V^s} &\le C_1e^{-\alpha t}\|\vec{\eta}_0\|_{V^s}+C_1\int_0^t e^{-\alpha(t-\tau)}\|N(\vec{\eta})(\tau)\|_{V^s}d\tau\\ &\le C_1e^{-\alpha t}\|\vec{\eta}_0\|_{V^s}+C_1C_2e^{-\alpha t}\sup_{0\le\tau\le t}\|e^{\alpha\tau}\vec{\eta}(\tau)\|^2_{V^s}\end{aligned}$$

for any $t\ge 0$.

For $b>0$, let Q_b be the ball in the space $Y_{s,\alpha}$ centered at zero of radius b. Then for any $\vec{\eta}\in Q_b$, we have that

$$\|\Gamma(\vec{\eta})\|_{Y_{s,\alpha}}\le C_1\|\vec{\eta}_0\|_{V^s}+C_1C_2\|\vec{\eta}\|^2_{Y_{s,\alpha}}\le C_1r+C_1C_2b^2.$$

A similar calculation shows that for any $\vec{u}$, $\vec{v}\in Q_b$,

$$\|\Gamma(\vec{u})-\Gamma(\vec{v})\|_{Y_{s,\alpha}}\le 2bC_1C_2\|\vec{u}-\vec{v}\|_{Y_{s,\alpha}}.$$

Consequently, if one chooses

$$b=2C_1r,\quad \text{with}\quad r\le r_0:=\frac{1}{8C_1^2C_2},$$

then

$$\|\Gamma(\vec{\eta})\|_{Y_{s,\alpha}}\le b$$

and

$$\|\Gamma(\vec{u})-\Gamma(\vec{v})\|_{Y_{s,\alpha}}\le\frac{1}{2}\|\vec{u}-\vec{v}\|_{Y_{s,\alpha}}$$

for any $\vec{\eta}$, $\vec{u}$ and $\vec{v}$ in the ball Q_b. The map Γ is a contraction on the ball Q_b and thus admits a fixed point $\vec{\eta}\in Q_b$, which is the unique solution of (3.19) and satisfies

$$\|\vec{\eta}(t)\|_{V^s}\le 2C_1e^{-\alpha t}\|\vec{\eta}_0\|_{V^s}$$

for any $t\ge 0$. The proof is then complete. □

4. **Concluding remarks.** In this paper we have studied a family of Boussinesq systems

$$\begin{cases} \eta_t + w_x + (\eta w)_x + a w_{xxx} - b\eta_{txx} = f, \\ \\ w_t + \eta_x + w w_x + c\eta_{xxx} - d w_{txx} = g \end{cases} \tag{4.1}$$

posed on the finite interval $(0, 2\pi)$ with the periodic boundary conditions

$$\begin{cases} \frac{\partial^r \eta}{\partial x^r}(t,0) = \frac{\partial^r \eta}{\partial x^r}(t,2\pi) & \text{for } t \in (0,T),\ 0 \le r \le r_0, \\ \\ \frac{\partial^q w}{\partial x^q}(t,0) = \frac{\partial^q w}{\partial x^q}(t,2\pi) & \text{for } t \in (0,T),\ 0 \le q \le q_0 \end{cases} \tag{4.2}$$

for its control and stabilization problems. Here, the forcing functions f and g, both supported in a subinterval ω of $(0, 2\pi)$, are considered as control inputs. Investigation is conducted first for the associated linear systems

$$\begin{cases} \eta_t + w_x + a w_{xxx} - b\eta_{txx} = f, \\ \\ w_t + \eta_x + c\eta_{xxx} - d w_{txx} = g. \end{cases} \tag{4.3}$$

Different approaches are used to analyze the exact controllability of the linear system (4.3) depending on employing a single control input (f or g) or two control inputs. In the case of only one control action, the Hilbert Uniqueness Method is used to acquire a complete description of exact controllability of the linear system (4.3). System (4.3) is shown to be either exactly controllable or non-controllable based on the values of the system parameters a, b, c and d. If two control inputs are allowed to use, we first decouple the system (4.3), and then use the classical *moment method* to establish its exact controllability. However, there is a serious restriction in this approach though two controls are used instead of one; it is required that $a = c$ and $b = d$ in (4.3) in order to have the system decoupled. On the other hand, exact controllability results obtained using two controls via the *moment method* are stronger. For instance, if $a = c \neq 0$, $b = d > 0$, system (4.3) is shown to be exactly controllable in the space $H^s_p \times H^s_p$ for any $s \in \mathbb{R}$. By contrast, (4.3) is only shown to be exactly controllable in the space $H^2_p \times H^2_p$ if one uses just a single control input.

With the exact controllability results in our hands, we then turn to analyze stabilizability of system (4.3) by the classical approach *stabilizability via exact controllability* (cf . [40, 35]). Let

$$E[\eta, w](t) = \int_0^{2\pi} \left(|\eta|^2 + b|\eta_x|^2 + |w|^2 + d|w_x|^2\right) dx.$$

$$\frac{d}{dt}E[\eta, w](t) = 2Re \int_\omega \left(f(x,t)\overline{\eta(x,t)} + g(x,t)\overline{w(x,t)}\right) dx$$

for any smooth solution (η, w) of (4.2). If we choose

$$f(x,t) = -q(x)\eta(x,t), \qquad g(x,t) \equiv 0 \tag{4.4}$$

where q is nonnegative smooth function supported on ω, a subinterval of $(0, 2\pi)$, we have shown that the resulting closed-loop system is *exponentially stable* if

(i) the norm $\|\cdot\|_{\mathcal{U}_{bd}}$ in the space $\mathcal{U}_{bd}$ is equivalent to the norm $\|\cdot\|_{V^1}$ in the space V^1 and

(ii) the adjoint system of (4.3)

$$\begin{cases} \xi_t + u_x + cu_{xxx} - b\xi_{txx} = f, \\ \\ u_t + \xi_x + a\xi_{xxx} - du_{txx} = 0 \end{cases} \tag{4.5}$$

is exactly controllable in the space V^1.

It should be pointed out that assumptions (i) and (ii) are quite restrictive which limits a lot on the choice of the system parameters a, b, c and d. However, if more complicated linear feedback control laws are used, then the resulting closed-loop systems are exponentially stable as long as the corresponding open loop system is exactly controllable.

The results obtained for linear system (4.3) are extended to nonlinear system (4.1) with small amplitude solutions via *contraction mapping principle*. The key to this approach is that the associated linear system must possess certain strong smoothing property in order to recover the lost regularity of nonlinear system (4.1) caused by the nonlinear terms ww_x and $(\eta w)_x$. Because of this strict requirement, we have only succeeded in extending the linear exact controllability results to the nonlinear system (4.1) with its parameters a, b, c and d satisfying

$$b > 0, \quad d > 0, \quad a \neq 0, \quad c \neq 0. \tag{4.6}$$

In particular, we have shown that the system (4.1) is *locally* exactly controllable in the space $H_p^2 \times H_p^2$ if only a single control action is used. If both control actions are employed and assume that $a = c$, in addition to (4.6), then we have shown that (4.1) is *locally* exactly controllable in the space $H_p^s \times H_p^s$ for any $s \geq 0$ and that some linear feedback control can be constructed so that the resulting (nonlinear) closed-loop system is *locally* exponentially stable.

While we have made some progress in studying control and stabilization problems of system (4.1), there are still many problems left open for further study. We list below two of them with some remarks to end this paper.

Open questions:

(1) Exact controllability and stabilizability of nonlinear system (4.1) with the parameters $a = c = \frac{1}{6}$, $b = d = 0$, i.e.,

$$\begin{cases} \eta_t + w_x + (\eta w)_x + \frac{1}{6} w_{xxx} = f, \\ \\ w_t + \eta_x + ww_x + \frac{1}{6}\eta_{xxx} = g \end{cases} \tag{4.7}$$

posed on $(0, 2\pi)$ with periodic boundary conditions (4.2).

System (4.7) is a purely KdV type Boussinesq system which is transformed into the equivalent system

$$\begin{cases} v_t + v_x + \frac{1}{6} v_{xxx} + \frac{1}{2}\left[(v-u)(v+u)\right]_x + \frac{1}{2}(v-u)(v-u)_x = f^*, \\ \\ u_t - u_x - \frac{1}{6} u_{xxx} + \frac{1}{2}\left[(v-u)(v+u)\right]_x - \frac{1}{2}(v-u)(v-u)_x = g^* \end{cases} \tag{4.8}$$

by setting

$$f^* = \frac{f+g}{2}, \quad g^* = \frac{f-g}{2}, \quad v = \frac{\eta + w}{2}, \quad u = \frac{\eta - w}{2}.$$

This is a system of two linear KdV equations coupled through nonlinear terms. The associated linear system of (4.8) (or (4.7)) has been proved in this paper to have the exact controllability and stabilizability properties. The question is how to extend the linear results to the nonlinear system (4.8)? As pointed out earlier, a certain smoothing property of the associated linear system is needed in order to invoke the *contraction mapping principle*. In case of the scalar KdV equation, the smoothing property is provided by Bourgain (cf. [8, 39]). However, there are some difficulties in applying the arguments in [8, 39] to the coupled system (4.8) since the Bourgain spaces associated with $v_t+v_x+\frac{1}{6}v_{xxx}$ and with $u_t-u_x-\frac{1}{6}u_{xxx}$ are different. This is also the reason why the initial value problem for system (4.8) is known to be locally well-posed in the space $H^s_p\times H^s_p$ for $s>\frac{3}{2}$ only and, in sharp contrast, the initial value problem for the scalar KdV equation is (analytically) well-posed in the space H^s_p for $s\geq -\frac{1}{2}$.[1] In fact, it is an interesting open question itself whether the initial value problem for system (4.8) (or (4.7) equivalently) is locally well-posed in $H^s_p\times H^s_p$ for some values of $s\leq\frac{3}{2}$

(2) Extend exact controllability and stabilizability results obtained for linear system (4.3) to nonlinear system (4.1) where assumption (4.6) is not satisfied.

Naturally, one would try to look for some smoothing properties of the associated linear systems in order to use the *contraction mapping principle*. But the interested reader should be warned that for some linear systems such a smoothing property may not exist. One may have to seek an alternative approach [2].

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[1] The initial value problem for the KdV equation posed on a periodic domain is now known to be well-posed in the space H^s_p for $s\geq -1$ [14].

[2] Recently, a boundary damping mechanism to stabilize system (4.7) (with $f=g=0$) posed on an arbitrary bounded domain $(0,L)$ has been proposed in [30]

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