J Fourier Anal Appl (2011) 17:991–1007
DOI 10.1007/s00041-010-9168-8

Null-controllability of a Hyperbolic Equation as Singular Limit of Parabolic Ones

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Received: 16 July 2010 / Revised: 9 December 2010 / Published online: 6 January 2011

Abstract This article considers a hyperbolic equation perturbed by a vanishing viscosity term depending on a small parameter $\varepsilon > 0$. We show that the resulting parabolic equation is null-controllable. Moreover, we provide uniform estimates, with respect to ε, for the parabolic controls and we prove their convergence to a control of the limit hyperbolic equation. The method we use is based on Fourier expansion of solutions and the analysis of a biorthogonal sequence to a family of complex exponential functions.

Keywords Null-controllability · Fourier expansion · Moment problem · Biorthogonal · Parabolic and hyperbolic equations

Mathematics Subject Classification (2000) 93B05 · 30E05 · 35E20

Communicated by L. Vega.

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1 Introduction

Given a time $T>0$, an initial data u_0 and a “profile” f, the null-controllability property of the simple 1-D parabolic problem

$$\begin{cases} u_t-\partial^2_{xx}u=f(x)v(t), & x\in(0,\pi),\ t>0,\\ u(t,0)=u(t,\pi)=0, & t>0,\\ u(0,x)=u_0(x), & x\in(0,\pi)\end{cases} \tag{1}$$

consists of finding a control v such that the corresponding solution u of (1) verifies $u(T)=0$. This problem has been well studied, by using nonharmonic Fourier analysis and moment problems, in the pioneering articles of Fattorini and Russell [3, 4]. Their method consists of constructing a control v by means of a biorthogonal family to the sequence of exponential functions $(e^{-\nu_n t})_{n\ge 1}$, where ν_n are the eigenvalues of the Dirichlet Laplace operator in $(0,\pi)$. We recall that a family of functions $(\phi_m)_{m\ge 1}\subset L^2(-\frac{T}{2},\frac{T}{2})$ with the property that

$$\int_{-\frac{T}{2}}^{\frac{T}{2}}\phi_m(t)e^{-\nu_n t}dt=\delta_{mn},\quad \forall m,n\ge 1, \tag{2}$$

is a *biorthogonal sequence to* $(e^{-\nu_n t})_{n\ge 1}$. Once a family $(\phi_m)_{m\ge 1}$ verifying (2) is given, a control $v(t)$ for (1) is obtained by considering linear combinations of functions ϕ_m. More precisely, if $u_0=\sum_{n\ge 1}a_n\sin(nx)$ and $f=\sum_{n\ge 1}\widehat{f_n}\sin(nx)$, then

$$v(t)=\sum_{n\ge 1}\frac{a_n}{\widehat{f_n}}\phi_n\left(t-\frac{T}{2}\right)e^{-\nu_n\frac{T}{2}},\quad t\in(0,T) \tag{3}$$

is a control for (1) in time T, if the series converges in $L^2(0,T)$. Like in [3], in order to ensure the density in $L^2(0,1)$ of the set of null-controllable initial data, we shall suppose that $\widehat{f_n}\ne 0$ for each $n\in\mathbb{N}^*$. Note that each biorthogonal element ϕ_n represents a control for one mode initial datum and, therefore, the study of the biorthogonal sequence's properties allows to obtain information about the control of any frequency or range of frequencies. Hence, one may deduce what frequencies are more difficult to control, estimate the magnitude of the control for any of them and characterize the spaces of controllable initial data. This is one of the advantages of the Fourier method's application in control.

An interesting fact is that, although the norm $\|\phi_n\|_{L^2(-\frac{T}{2},\frac{T}{2})}$ increases exponentially with n, the series (3) converges in $L^2(0,T)$ for a large class of Fourier coefficients $(a_n)_{n\ge 1}$. This is due to the presence of the negative exponentials $e^{-\nu_n\frac{T}{2}}$ which reflects the dissipative effect of the equation. Indeed, given any time $T>0$ and supposing that, for any $\eta>0$, $\liminf_{n\to\infty}|\widehat{f_n}|e^{\eta n}>0$, it is sufficient to prove the existence of two positive constants c and ω such that

$$\|\phi_n\|_{L^2(-\frac{T}{2},\frac{T}{2})}\le c\,e^{\omega\sqrt{|\nu_n|}}\quad \forall n\in\mathbb{N}^*, \tag{4}$$

in order to ensure the absolute convergence of the series in (3) in $L^2(0,T)$ for initial data u_0 in any negative Sobolev space in time T.

A way to obtain the controllability of the parabolic equation (1) from the controllability of a hyperbolic one has been introduced by Russell in [17]. This method was denominated *transmutation method*. It has been used in the last years to obtain controllability results for other types of equations and estimates for different controls norms (see, for instance, [13–15, 19, 20]). A similar problem was addressed in [10], where the null controllability property of the heat equation is obtained as limit of the exact controllability properties of singularly perturbed damped wave equations. Nevertheless, there are much fewer results concerning the passage from a parabolic equation to a hyperbolic one. As far as we know, the only papers addressing this type of questions are [2] and [5], where the controllability of the transport equation is consider by introducing a vanishing viscosity term. In [2] Carleman estimates are used to obtain a uniform bound for the sequence of controls. The same result was shown in [5], improving the control time, by means of nonharmonic Fourier analysis and biorthogonal technique. The main difficulty in this type of problems consists in the fact that the dissipation vanishes in the limit hyperbolic problem. Therefore, precise estimates on the behavior of the biorthogonal's norm $\|\phi_n\|$ are needed to study the convergence of the series (3) which gives the control.

In this article, we study the possibility to obtain controls of a hyperbolic equation as a limit of controls of parabolic ones. More precisely, we consider the equation

$$\begin{cases} u_t + i(-\partial^2_{xx})^{\frac{1}{2}}u - \varepsilon\partial^2_{xx}u = f(x)v_\varepsilon(t), & x \in (0,\pi),\ t > 0, \\ u(t,0) = u(t,\pi) = 0, & t > 0, \\ u(0,x) = u_0(x), & x \in (0,\pi), \end{cases} \tag{5}$$

and we address the following controllability problem: *given* $T > 0$*,* $f \in L^2(0,\pi)$ *and* $u_0 \in L^2(0,\pi)$*, is there a control* $v_\varepsilon \in L^2(0,T)$*, such that the corresponding solution* u *of (5) verifies* $u(T,x) = 0$*,* $x \in (0,\pi)$*? Moreover, what happens with the controls* v_ε *when* $\varepsilon \to 0^+$*?*

In order to give an answer to these questions we need some notation. Here and in what follows $(-\partial^2_{xx})^\alpha$ denotes the fractional power of order $\alpha > 0$ of the Dirichlet Laplacian in $(0,\pi)$. More precisely,

$$(-\partial^2_{xx})^\alpha : D((-\partial^2_{xx})^\alpha) \subset L^2(0,\pi) \to L^2(0,\pi),$$

$$D((-\partial^2_{xx})^\alpha) = \left\{ u \in L^2(0,\pi) : u = \sum_{n\geq 1} a_n \sin(nx) \text{ and } \sum_{n\geq 1} |a_n|^2 n^{4\alpha} < \infty \right\}, \tag{6}$$

$$u(x) = \sum_{n\geq 1} a_n \sin(nx) \longrightarrow (-\partial^2_{xx})^\alpha u(x) = \sum_{n\geq 1} a_n n^{2\alpha} \sin(nx).$$

For any $h \in L^2(0,\pi)$ we shall denote $\widehat{h}_n = \int_0^\pi h(x)\sin(nx)dx$. Let $f \in L^2(0,\pi)$ be such that

$$\widehat{f}_n \neq 0 \quad \forall n \geq 1 \tag{7}$$

and define the space

$$\mathcal{H}=\left\{h\in L^2(0,\pi):\sum_{n\in\mathbb{N}^*}\left|\frac{\widehat{h}_n}{\widehat{f}_n}\right|^2<\infty\right\}. \tag{8}$$

The main result of our paper is the following one.

Theorem 1.1 *Let $f\in L^2(0,\pi)$ be a function verifying (7). There exists a time $T>0$ with the property that, for any $u_0\in\mathcal{H}$ and $\varepsilon>0$, there exists a control $v_\varepsilon\in L^2(0,T)$ of (5) such that the sequence $(v_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $L^2(0,T)$ and any weak limit v of it, as $\varepsilon\to 0^+$, is a control in time T for the equation*

$$\begin{cases} u_t+i(-\partial^2_{xx})^{\frac{1}{2}}u=f(x)v(t), & x\in(0,\pi),\ t>0,\\ u(t,0)=u(t,\pi)=0, & t>0,\\ u(0,x)=u_0(x), & x\in(0,\pi). \end{cases} \tag{9}$$

Note that (5) is of parabolic type, whereas (9) is hyperbolic. Therefore, we are dealing with a singular problem in which we try to approximate controls of the limit equation (9) by controls of (5). From this point of view, this work is similar to the works [2, 5] mentioned above. Also, somehow related is [11], where the controllability of the semi-discrete wave equation is studied by adding a vanishing viscosity term (see, also, [23]).

We remark that the method we employ here is similar to the one used in [5]. More precisely, for each $\varepsilon>0$, we construct a control v_ε for the parabolic problem (5), and we obtain the uniform boundedness of the sequence $(v_\varepsilon)_{\varepsilon>0}$ in $L^2(0,T)$. Nextly, a classical (weak) limit argument allows us to obtain the desired result. As in [3, 4], the controls v_ε are given in terms of a biorthogonal sequence, by a formula similar to (3).

The eigenvalues of the operator corresponding to (5) are $\mu_n=in-\varepsilon n^2$, $n\geq 1$. Note that this family is slightly more complicated than the one in [5], where all the eigenvalues are purely real. The main difficulty in our study is related to the fact that $\lim_{\varepsilon\to 0}\Re(\mu_n)=0$ for each $n\geq 1$. Indeed, in order to prove the convergence result from Theorem 1.1, we need estimates of the biorthogonal's norm of the type

$$\|\phi_n\|_{L^2(-\frac{T}{2},\frac{T}{2})}\leq c\,e^{\omega|\Re(\mu_n)|}\quad \forall n\in\mathbb{N}^*, \tag{10}$$

where c and ω are positive constants independent of ε. Note that, for $n\leq\frac{1}{\sqrt{\varepsilon}}$, the real part of μ_n is uniformly bounded in n and ε and a similar property must be satisfies by $\|\phi_n\|_{L^2}$. Hence, estimates (10) are harder to prove than (4).

As in [3, 4], it is easy to show that there exist controls v_ε for (5), analytic in $(0,T)$. On the other hand, the controls v of (9) may belong to $L^2(0,T)$ only. Therefore, the approximation of v by v_ε may be very useful in certain numerical tasks.

Equation (5) is inspired from [9], where the fractional Schrödinger equation is introduced as a result of extending the Feynman path integral, from the Brownian-like to Lévy-like quantum mechanical paths. Instead of the second order space derivative as in the standard Schrödinger equation, [9] considers a space derivative of fractional order $r \in (1,2)$. Our article takes the limit case $r = 1$ and adds the vanishing viscosity term $-\varepsilon\partial_{xx}^2$ in order to stabilize the system.

The rest of the paper is organized as follows. Section 2 gives the equivalent characterization of the controllability property in terms of a moment problem. In Sect. 3, a biorthogonal sequence is constructed and evaluated. The final section is devoted to the proof of the main result and to some comments.

2 The Moment Problem

For the sake of completeness, we first present the main result concerning the well-posedness of (5).

Theorem 2.1 *Given any $T > 0$, $\varepsilon \geq 0$, $h \in L^1(0,T; L^2(0,\pi))$ and $u_0 \in L^2(0,\pi)$, there exists a unique weak solution $u \in C([0,T], L^2(0,\pi))$ of the problem*

$$\begin{cases} u_t + i(-\partial_{xx}^2)^{\frac{1}{2}}u - \varepsilon\partial_{xx}^2 u = h(t,x), & x \in (0,\pi),\ t \in (0,T), \\ u(t,0) = u(t,\pi) = 0, & t > 0, \\ u(0,x) = u_0(x), & x \in (0,\pi). \end{cases}$$

Proof Since the operator $(D(A), A)$, where $D(A) = D(-\partial_{xx}^2)$ if $\varepsilon > 0$ and $D(A) = (-\partial_{xx}^2)^{\frac{1}{2}}$ if $\varepsilon = 0$ and $A = i(-\partial_{xx}^2)^{\frac{1}{2}} - \varepsilon\partial_{xx}^2$, is maximal monotone in $L^2(0,\pi)$, we apply the classical semigroup theory. □

We can give now the characterization of the controllability property of (5) in terms of a moment problem. Based in Fourier expansion of the solution, the moment problems have been widely used in linear control theory. We refer to [1, 8, 22] for a quite complete discussion on the subject.

Theorem 2.2 *Let $T > 0$, $\varepsilon \geq 0$, $u_0 \in L^2(0,\pi)$ and $f \in L^2(0,\pi)$ such that*

$$u_0(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) \quad \textit{and} \quad f(x) = \sum_{n=1}^{\infty} \widehat{f_n} \sin(n\pi x).$$

There exists a control $v \in L^2(0,T)$ such that the solution u of (5) verifies $u(T,x) = 0$ for $x \in (0,\pi)$ if, and only if, $v \in L^2(0,T)$ is a solution of

$$\widehat{f_n} \int_{-\frac{T}{2}}^{\frac{T}{2}} v\left(s + \frac{T}{2}\right) e^{s\overline{\lambda}_n} ds = -e^{-\frac{T}{2}\overline{\lambda}_n} a_n \quad \forall n \geq 1, \tag{11}$$

where $\lambda_n = -in + \varepsilon n^2$.

Remark 2.1 If the function f introduced in Theorem 2.2 are such that $\widehat{f_n} = 0$ for some n, it is necessary to introduce an additional condition for the initial data u_0. In fact, in this case, (11) implies that the corresponding coefficient a_n must be equal to zero in order to ensure the controllability of u_0.

Proof We consider the equation

$$\begin{cases} -\varphi_t - i(-\partial^2_{xx})\varphi + \varepsilon(-\partial^2_{xx})\varphi = 0, & x \in (0,\pi),\ t \in (0,T), \\ \varphi(t,0) = \varphi(t,\pi) = 0, & t \in (0,T), \\ \varphi(T,x) = \varphi_0(x), & x \in (0,\pi), \end{cases} \tag{12}$$

and we multiply (5) by $\overline{\varphi}$ and integrate by parts over $(0,T)\times(0,\pi)$. It follows that $v \in L^2(0,T)$ is a control for (5) if, and only if, it verifies

$$\int_0^T v(t) \int_0^\pi f(x)\overline{\varphi}(t,x)dxdt = -\int_0^\pi u_0(x)\overline{\varphi}(0,x)dx \tag{13}$$

for any solution φ of (12). Since $(\sin(nx))_{n\geq 1}$ is a basis of $L^2(0,\pi)$ we have to check (13) only for the solutions of (12) of the form $\varphi(t,x) = e^{(t-T)\lambda_n}\sin(nx)$, $n \in \mathbb{N}^*$. Thus, v is a control for (5) if and only if it verifies (11). □

The notion of biorthogonal family is very useful in the study of moment problems. We recall that $(\theta_m)_{m\geq 1} \subset L^2(-\frac{T}{2},\frac{T}{2})$ is a *biorthogonal sequence to the family of exponential functions* $(e^{\lambda_n t})_{n\geq 1}$ *in* $L^2(-\frac{T}{2},\frac{T}{2})$ iff

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \theta_m(t) e^{\overline{\lambda}_n t} dt = \delta_{nm} \quad \forall n,m \geq 1. \tag{14}$$

It is easy to see from (11) that, if $(\theta_m)_{m\geq 1}$ is a biorthogonal sequence to the family of exponential functions $(e^{\lambda_n t})_{n\geq 1}$ in $L^2(-\frac{T}{2},\frac{T}{2})$, then a control v of (5) is given by

$$v(t) = \sum_{n=1}^{\infty} -\frac{a_n}{\widehat{f}_n} e^{-\frac{T}{2}\overline{\lambda}_n}\theta_n\left(t-\frac{T}{2}\right), \quad t \in (0,T), \tag{15}$$

provided that the right hand series converges in $L^2(0,T)$. Now, the main task is to show that there exists a biorthogonal sequence $(\theta_m)_{m\geq 1}$ and to evaluate its norm in order to prove the convergence of this series.

3 A Biorthogonal Sequence

The aim of this section is to construct and evaluate an explicit biorthogonal sequence to the family $(e^{\lambda_n t})_{n\geq 1}$ in $L^2(-\frac{T}{2},\frac{T}{2})$. In order to do that, we introduce a family $\Psi_m(z)$ of entire functions of exponential type (see, for instance, [21]) such that $\Psi_m(i\lambda_n) = \delta_{mn}$, where δ_{mn} is the Kronecker symbol. Nextly, Paley-Wiener theorem gives us a biorthogonal family θ_m as the inverse Fourier transform of Ψ_m. Each Ψ_m is obtained from a Weierstrass product P_m multiplied by a function M_ε with a suitable behavior on real axis. Such a method was used for the first time by Paley and Wiener [16]. The main difficulty in our analysis is to obtain good estimates for the behavior of P_m on the real axis, to construct an appropriate multiplier M_ε and to evaluate carefully $M_\varepsilon(i\lambda_m)$ in order to ensure (10).

For any $m \in \mathbb{N}^*$, we define the function

$$P_m(z) = \prod_{\substack{n\in\mathbb{Z}^*\\|n|\neq m}} \left(1+\frac{zi}{\lambda_n}\right)\left(\frac{\lambda_n}{\lambda_n-\lambda_m}\right), \tag{16}$$

where $\lambda_{-n} = \overline{\lambda}_n$ for $n > 0$. In what follows we study some of the properties of P_m.

Lemma 3.1 *P_m is an entire function of exponential type independent of ε such that*

$$P_m(i\lambda_n) = \delta_{mn}, \quad n \in \mathbb{N}^*. \tag{17}$$

Proof We evaluate the following expressions

$$E_m(z) = \prod_{\substack{n\in\mathbb{Z}^*\\|n|\neq m}} \left|1+\frac{zi}{\lambda_n}\right| \quad \text{and} \quad Q_m = \prod_{\substack{n\in\mathbb{Z}^*\\|n|\neq m}} \left|\frac{\lambda_n}{\lambda_n-\lambda_m}\right|.$$

For any $z \in \mathbb{C}$, we have that

$$\begin{aligned} E_m(z) &= \exp\left(\sum_{n=1}^{N_z} \ln\left|1-\frac{z^2}{|\lambda_n|^2}+2iz\Re\left(\frac{1}{\lambda_n}\right)\right|\right.\\ &\quad \left.+\sum_{n=N_z+1}^{\infty} \ln\left|1-\frac{z^2}{|\lambda_n|^2}+2iz\Re\left(\frac{1}{\lambda_n}\right)\right|\right)\\ &= \exp(A(z)+B(z)), \end{aligned} \tag{18}$$

where N_z is defined by

$$N_z = \max\left\{N \geq 1 : \left|2z\Re\left(\frac{1}{\lambda_n}\right)\right| \leq \frac{|z|^2}{|\lambda_n|^2},\ \forall n \leq N\right\}.$$

Then

$$\begin{aligned} A(z) &\leq \sum_{n=1}^{N_z} \ln\left(1+\frac{2|z|^2}{|\lambda_n|^2}\right) \leq \sum_{n=1}^{N_z} \ln\left(1+\frac{2|z|^2}{n^2}\right) \leq \int_0^\infty \ln\left(1+\frac{2|z|^2}{s^2}\right) ds\\ &= \frac{\sqrt{2}}{2}\pi|z| \end{aligned} \tag{19}$$

and

$$B(z) \leq \sum_{n=N_z+1}^{\infty} \ln\left(1+4|z|\left|\Re\left(\frac{1}{\lambda_n}\right)\right|\right) \leq \sum_{n=N_z+1}^{\infty} 4|z|\left|\Re\left(\frac{1}{\lambda_n}\right)\right| \leq 8|z|. \tag{20}$$

The last inequality from (20) follows from the fact that

$$\sum_{n\geq 1}\Re\left(\frac{1}{\lambda_n}\right)=\sum_{n\geq 1}\frac{\varepsilon n^2}{\varepsilon^2 n^4+n^2}\leq \sum_{n=1}^{[\frac{1}{\varepsilon}]}\frac{\varepsilon n^2}{n^2}+\sum_{n=[\frac{1}{\varepsilon}]+1}^{\infty}\frac{\varepsilon n^2}{\varepsilon^2 n^4}\leq 2.$$

Finally, for Q_m we obtain

$$\begin{aligned}
|Q_m|^2 &= \prod_{\substack{n\in\mathbb{N}^*\\ n\neq m}}\frac{(n^2+\varepsilon^2 n^4)^2}{[(n-m)^2+\varepsilon^2(n^2-m^2)^2][(n+m)^2+\varepsilon^2(n^2-m^2)^2]}\\
&= \prod_{\substack{n\in\mathbb{N}^*\\ n\neq m}}\frac{n^4(1+\varepsilon^2 n^2)^2}{(n+m)^2(n-m)^2[1+\varepsilon^2(n+m)^2][1+\varepsilon^2(n-m)^2]}\\
&= \prod_{\substack{n\in\mathbb{N}^*\\ n\neq m}}\frac{n^4}{(n-m)^2(n+m)^2}\prod_{\substack{n\in\mathbb{N}^*\\ n\neq m}}\frac{(1+\varepsilon^2 n^2)^2}{[1+\varepsilon^2(m+n)^2][1+\varepsilon^2(n-m)^2]}\\
&= 4\frac{1+4\varepsilon^2 m^2}{1+\varepsilon^2 m^2}.
\end{aligned}\tag{21}$$

This completes the proof. □

The following result is the key estimate which allows to construct the biorthogonal sequence we intend to obtain. Roughly speaking, we need to ensure that P_m is uniformly bounded inside a sufficiently long interval $I(\varepsilon)$, centered in 0, and that it does not increase very rapidly outside $I(\varepsilon)$.

Lemma 3.2 *The function P_m defined by (16) has the following behavior on the real axis*

$$|P_m(x)|\leq C\exp(\omega\varphi(x))\quad \forall x\in\mathbb{R},\tag{22}$$

where C and ω are two positive constants, independent of ε and m, and

$$\varphi(x)=\begin{cases}\varepsilon x^2 & \text{if } |x|\leq\frac{1}{\varepsilon},\\ \sqrt{\frac{|x|}{\varepsilon}} & \text{if } |x|>\frac{1}{\varepsilon}.\end{cases}\tag{23}$$

Proof Since $|P_m(x)|=E_m(x)Q_m$ and, from (21), $Q_m\leq 16$, it is enough to evaluate $E_m(x)$. In the sequel, c denotes a generic constant which may changes from one row to another but it is always independent of m and ε.

To begin with, we evaluate E_m on the real axis in the case $|x|<\frac{1}{\varepsilon}$. For any $x\neq|\lambda_n|$, $|x|<\frac{1}{\varepsilon}$, we set

$$(E_m)^2(x)=\prod_{\substack{n\in\mathbb{N}^*\\ n\neq m}}\left|1+\frac{xi}{\lambda_n}\right|^2\left|1+\frac{xi}{\overline{\lambda}_n}\right|^2$$

$$= \frac{|\lambda_m|^4}{(|\lambda_m|^2 - x^2)^2} \prod_{\substack{n\in\mathbb{N}^*\\ n\neq m}} \frac{(|\lambda_n|^2 - x^2)^2 + 4x^2(\Re(\lambda_n))^2}{(|\lambda_n|^2 - x^2)^2}$$

$$\times \prod_{n\in\mathbb{N}^*} \frac{(|\lambda_n|^2 - x^2)^2}{|\lambda_n|^4}. \tag{24}$$

Let x_ε be the unique positive solution of the equation $x^2 = x_\varepsilon^2 + \varepsilon^2 x_\varepsilon^4$. Firstly, note that

$$\frac{|\lambda_m|^4}{(|\lambda_m|^2 - x^2)^2} \le \frac{m^2}{(m - x_\varepsilon)^2}. \tag{25}$$

Let $A_m(x)$ be the first product from (24). Then,

$$\begin{aligned} A_m(x) &= \prod_{\substack{n\in\mathbb{N}^*\\ n\neq m}} \frac{(|\lambda_n|^2 - x^2)^2 + 4x^2(\Re(\lambda_n))^2}{(|\lambda_n|^2 - x^2)^2} \\ &\le \prod_{\substack{n=1\\ n\neq m}}^{[x_\varepsilon]} \left(1 + \frac{4\varepsilon^2 n^4 x^2}{(x^2 - n^2 - \varepsilon^2 n^4)^2}\right) \prod_{\substack{n=1+[x_\varepsilon]\\ n\neq m}}^{\infty} \left(1 + \frac{4\varepsilon^2 n^4 x^2}{(x^2 - n^2 - \varepsilon^2 n^4)^2}\right) \\ &= A_m^1(x) A_m^2(x). \end{aligned} \tag{26}$$

Here and in the sequel we denote by $[r]$ and $\{r\}$ the integer and fractional part of the real number r, respectively. Now we have

$$\begin{aligned} A_m^1(x) &= \prod_{\substack{n=1\\ n\neq m}}^{[x_\varepsilon]} \left(1 + \frac{4\varepsilon^2 n^4 x^2}{((x_\varepsilon^2 - n^2) + \varepsilon^2(x_\varepsilon^2 - n^4))^2}\right) \le \prod_{\substack{n=1\\ n\neq m}}^{[x_\varepsilon]} \left(1 + \frac{n^2}{(x_\varepsilon - n)^2} \frac{4\varepsilon^2 x_\varepsilon^2}{1 + \varepsilon^2 x_\varepsilon^2}\right) \\ &\le d_{1\varepsilon} e^{2\varepsilon x^2} \exp\left(\int_0^{x_\varepsilon} \ln\left(1 + \frac{s^2}{(x_\varepsilon - s)^2} \frac{4\varepsilon^2 x_\varepsilon^2}{1 + \varepsilon^2 x_\varepsilon^2}\right) ds\right) \le d_{1\varepsilon} \exp(c\varepsilon x^2), \end{aligned} \tag{27}$$

where $d_{1\varepsilon} = 1$ if $[x_\varepsilon] = m$ or $d_{1\varepsilon} = \frac{1}{\{x_\varepsilon\}^2}$ if $[x_\varepsilon] \neq m$.

Let us analyze $A_m^2(x)$.

$$\begin{aligned} A_m^2(x) &= \prod_{\substack{n=[x_\varepsilon]+1\\ n\neq m}}^{\infty} \left(1 + \frac{4\varepsilon^2 n^4 x^2}{(n - x_\varepsilon)^2 (n + x_\varepsilon)^2 (1 + \varepsilon^2(n^2 + x_\varepsilon^2))^2}\right) \\ &\le \prod_{\substack{n=[x_\varepsilon]+1\\ n\neq m}}^{[\frac{2}{\varepsilon}]} \left(1 + \frac{4\varepsilon^2 n^2 x^2}{(n - x_\varepsilon)^2 (1 + \varepsilon^2 n^2)^2}\right) \prod_{n=[\frac{2}{\varepsilon}]+1}^{\infty} \left(1 + \frac{4\varepsilon^2 n^2 x^2}{(n - x_\varepsilon)^2 (1 + \varepsilon^2 n^2)^2}\right) \\ &= A_m^{21}(x) A_m^{22}(x). \end{aligned} \tag{28}$$

Then,

$$\begin{aligned}A_m^{22}(x) &\le \prod_{n=[\frac{2}{\varepsilon}]+1}^{\infty}\left(1+\frac{4\varepsilon^2n^2x^2}{(n-x_\varepsilon)^2\varepsilon^4n^4}\right) \le \prod_{n=[\frac{2}{\varepsilon}]+1}^{\infty}\left(1+\frac{x^2}{(n-x_\varepsilon)^2}\right)\\ &\le \exp\left(\int_{[\frac{2}{\varepsilon}]}^{\infty}\ln\left(1+\frac{x^2}{(s-x_\varepsilon)^2}\right)ds\right) \le \exp\left(\int_{[\frac{2}{\varepsilon}]}^{\infty}\frac{x^2}{(s-x_\varepsilon)^2}ds\right)\\ &\le \exp\left(\frac{x^2\varepsilon}{1-\varepsilon}\right) \qquad (29)\end{aligned}$$

and

$$\begin{aligned}A_m^{21}(x) &\le \prod_{\substack{n=[x_\varepsilon]+1\\ n\neq m}}^{[\frac{2}{\varepsilon}]}\left(1+\frac{4\varepsilon^2n^2x^2}{(n-x_\varepsilon)^2}\right) \le d_{2\varepsilon}e^{2\varepsilon x^2}\exp\left(\int_{[x_\varepsilon]+1}^{[\frac{2}{\varepsilon}]}\ln\left(1+\frac{4\varepsilon^2s^2x^2}{(s-x_\varepsilon)^2}\right)ds\right)\\ &\le d_{2\varepsilon}e^{2\varepsilon x^2}\exp\left(\int_{x_\varepsilon}^{\frac{2}{\varepsilon}}\ln\left(1+\frac{4\varepsilon^2s^2x^2}{(s-x_\varepsilon)^2}\right)ds\right) \le d_{2\varepsilon}\exp(c\,\varepsilon\,x^2), \qquad (30)\end{aligned}$$

where $d_{2\varepsilon}=1$ if $[x_\varepsilon]=m-1$ or $d_{2\varepsilon}=\frac{1}{(1-\{x_\varepsilon\})^2}$ if $[x_\varepsilon]\neq m-1$.

From (26) up to (30) we deduce the existence of two positive constants C_1 and ω_1, independent of ε and m, such that

$$A_m(x) \le C_1\, d_{1\varepsilon}d_{2\varepsilon}\exp(\omega_1\varepsilon x^2) \quad \forall |x| \le \frac{1}{\varepsilon}. \qquad (31)$$

Now, let $B_m(x)$ be the second product from (24). We obtain that

$$\begin{aligned}B_m(x) &= \prod_{n\in\mathbb{N}^*}\frac{(|\lambda_n|^2-x^2)^2}{|\lambda_n|^4} = \prod_{n\in\mathbb{N}^*}\left(1-\frac{x_\varepsilon^2}{n^2}\right)^2\prod_{n\in\mathbb{N}^*}\left(1+\frac{\varepsilon^2x_\varepsilon^2}{1+\varepsilon^2n^2}\right)^2\\ &= \frac{\sin^2(\pi x_\varepsilon)}{\pi^2x_\varepsilon^2}\prod_{n\in\mathbb{N}^*}\left(1+\frac{\varepsilon^2x_\varepsilon^2}{1+\varepsilon^2n^2}\right)^2. \qquad (32)\end{aligned}$$

In order to estimate the last product from (32) we remark that

$$\begin{aligned}\prod_{n\in\mathbb{N}^*}\left(1+\frac{\varepsilon^2x_\varepsilon^2}{1+\varepsilon^2n^2}\right)^2 &\le \exp\left(\int_0^{\infty}\ln\left(1+\frac{\varepsilon^2x_\varepsilon^2}{1+\varepsilon^2s^2}\right)ds\right)\\ &\le \exp\left(\int_0^{\infty}\frac{\varepsilon^2x_\varepsilon^2}{1+\varepsilon^2s^2}ds\right) = \exp\left(\frac{\pi}{2}\varepsilon x_\varepsilon^2\right)\\ &\le \exp\left(\frac{\pi}{2}\varepsilon x^2\right). \qquad (33)\end{aligned}$$

From (32) and (33) we deduce the existence of two positive constants C_2 and ω_2, independent of ε and m, such that

$$B_m(x) \le C_2 \frac{\sin^2(\pi x_\varepsilon)}{x_\varepsilon^2} \exp(\omega_2 \varepsilon x^2) \quad \forall |x| \le \frac{1}{\varepsilon}. \tag{34}$$

Note that there exists a positive constant C_3, independent of ε and m, such that

$$d_{1\varepsilon} d_{2\varepsilon} \frac{\sin^2(\pi x_\varepsilon)}{x_\varepsilon^2} \frac{m^2}{(m - x_\varepsilon)^2} \le C_3. \tag{35}$$

Finally, estimates (25), (31), (34) and (35) ensure that there exist two positive constants C and ω, independent of ε and m, such that

$$E_m(x) \le C \exp(\omega \varepsilon x^2) \quad \forall |x| \le \frac{1}{\varepsilon}. \tag{36}$$

To conclude the proof it remains to evaluate the product $E_m(x)$ for the case $|x| \ge \frac{1}{\varepsilon}$. We note that

$$\begin{aligned}
(E_m)^2(x) &= \prod_{\substack{n\in\mathbb{N}^* \\ n\neq m}} \frac{(|\lambda_n|^2 - x^2)^2 + 4x^2(\Re(\lambda_n))^2}{|\lambda_n|^4} \\
&\le \prod_{n\in\mathbb{N}^*} \frac{|\lambda_n|^4 + x^4 + 4x^2(\Re(\lambda_n))^2}{|\lambda_n|^4} \\
&= \exp\left(\sum_{n\in\mathbb{N}^*} \ln\left(1 + \frac{x^4 + 4x^2(\Re(\lambda_n))^2}{|\lambda_n|^4}\right)\right) \\
&= \exp\left(\left(\sum_{n=1}^{\left[\frac{1}{2}\sqrt{\frac{|x|}{\varepsilon}}\right]} + \sum_{n=\left[\frac{1}{2}\sqrt{\frac{|x|}{\varepsilon}}\right]+1}^{\infty}\right) \ln\left(1 + \frac{x^4 + 4x^2(\Re(\lambda_n))^2}{|\lambda_n|^4}\right)\right) \\
&= \exp(F_m^1(x) + F_m^2(x)).
\end{aligned} \tag{37}$$

We first evaluate $F_m^2(x)$:

$$\begin{aligned}
F_m^2(x) &\le 8x^2 \sum_{n=\left[\frac{1}{2}\sqrt{\frac{|x|}{\varepsilon}}\right]+1}^{\infty} \frac{|\Re(\lambda_n)|^2}{|\lambda_n|^4} = 8x^2\varepsilon^2 \sum_{n=\left[\frac{1}{2}\sqrt{\frac{|x|}{\varepsilon}}\right]+1}^{\infty} \frac{1}{(1+\varepsilon^2 n^2)^2} \\
&\le 8x^2\varepsilon^2 \int_{\frac{1}{2}\sqrt{\frac{|x|}{\varepsilon}}}^{\infty} \frac{1}{(1+\varepsilon^2 x^2)^2} ds \le 8x^2\varepsilon^2 \int_{\frac{1}{2}\sqrt{\frac{|x|}{\varepsilon}}}^{\infty} \frac{1}{\varepsilon^4 s^4} ds = 3\sqrt{\frac{|x|}{\varepsilon}}.
\end{aligned} \tag{38}$$

Now we evaluate $F_m^1(x)$:

$$
\begin{aligned}
F_m^1(x) &\le \sum_{n=1}^{\left[\frac{1}{2}\sqrt{\frac{|x|}{\varepsilon}}\right]} \ln\left(1+\frac{2x^4}{|\lambda_n|^4}\right) \le \int_0^{\frac{1}{2}\sqrt{\frac{|x|}{\varepsilon}}} \ln\left(1+\frac{2x^4}{s^4(1+\varepsilon^2 s^2)^2}\right) ds \\
&\le \int_0^{\frac{1}{2\varepsilon}} \ln\left(1+\frac{2x^4}{s^4}\right) ds + \int_{\frac{1}{2\varepsilon}}^{\frac{1}{2}\sqrt{\frac{|x|}{\varepsilon}}} \ln\left(1+\frac{2x^4}{\varepsilon^4 s^8}\right) ds \\
&\le \int_0^{\frac{1}{2\varepsilon}} \frac{8x^4}{s^4+2x^4} ds + \frac{1}{2}\sqrt{\frac{|x|}{\varepsilon}} \ln(1+2^9) + \int_{\frac{1}{2\varepsilon}}^{\frac{1}{2}\sqrt{\frac{|x|}{\varepsilon}}} \frac{16x^4}{\varepsilon^4 s^8 + 2x^4} ds \\
&\le c\sqrt{\frac{|x|}{\varepsilon}}.
\end{aligned}
\tag{39}
$$

From (37)–(39) we deduce that there exist two positive constants C and ω, independent of ε and m, such that

$$
E_m(x) \le C \exp\left(\omega\sqrt{\frac{|x|}{\varepsilon}}\right) \quad \forall |x| > \frac{1}{\varepsilon}. \tag{40}
$$

By taking into account (36) and (40) the Lemma is proved. □

In order to complete the construction of the biorthogonal sequence we have to find a multiplier to compensate the growth of P_m on the real axis. We use an idea of Ingham [6], generalized by Redheffer [18].

Lemma 3.3 *There exists a function $M_\varepsilon : \mathbb{C} \to \mathbb{C}$ with the following properties*

1. *M_ε is an entire function of exponential type independent of ε and m*;
2. $|M_\varepsilon(x)| \le \exp(-Q\,\varphi(x))$, *for all $x \in \mathbb{R}$*;
3. $|M_\varepsilon(i\lambda_m)| \ge \exp(-R\,|\Re(\lambda_m)|)$, *for all $m \in \mathbb{N}^*$*,

where Q and R are positive constants independent of ε and m.

Proof Let $(\rho_n)_{n\ge 1}$ be the nonincreasing sequence defined by

$$
\rho_n = \begin{cases} e\varepsilon, & n \le \frac{1}{\varepsilon}, \\ e\sqrt{\frac{1}{\varepsilon n^3}}, & n > \frac{1}{\varepsilon}. \end{cases} \tag{41}
$$

Note that $\rho_n = e\frac{\varphi(n)}{n^2}$ and there exists a positive number $l > 0$ independent of ε such that

$$
\sum_{n\ge 1} \rho_n = e\sum_{n=1}^{[\frac{1}{\varepsilon}]} \varepsilon + e\sum_{n=[\frac{1}{\varepsilon}]+1}^{\infty} \sqrt{\frac{1}{\varepsilon n^3}} \le l < \infty.
$$

The function M_ε defined by

$$M_\varepsilon(z) = \prod_{n\geq 1} \frac{\sin(\rho_n z)}{\rho_n z}$$

is an entire function of exponential type less than l.

We pass to evaluate $M_\varepsilon(x)$ by considering the following two cases:

- $|x| > \frac{1}{\varepsilon}$. We consider $\nu = [\sqrt{\frac{|x|}{\varepsilon}}] \geq [\frac{1}{\varepsilon}]$ and we have that

$$|M_\varepsilon(x)| \leq \prod_{n=1}^{\nu} \frac{|\sin(\rho_n x)|}{|\rho_n x|} \leq \prod_{n=1}^{\nu} \frac{1}{\rho_n |x|} \leq \left(\frac{1}{\rho_\nu |x|}\right)^{\nu} \leq e^{-\nu} \leq e \exp\left(-\sqrt{\frac{|x|}{\varepsilon}}\right).$$

- $|x| \leq \frac{1}{\varepsilon}$. We consider $\nu = [\frac{1}{\varepsilon}]$ and note that

$$|M_\varepsilon(x)| \leq \prod_{n=1}^{\nu} \frac{|\sin(\rho_n x)|}{|\rho_n x|} = \left(\frac{|\sin(e\varepsilon x)|}{|e\varepsilon x|}\right)^{\nu}.$$

Since $e\varepsilon|x| \leq e$ and $\sin(t) \leq t - \frac{\sin(e)}{6e} t^3$ for all $t \in [0, e]$, it follows that

$$\begin{aligned}
|M_\varepsilon(x)| &\leq \left(1 - \frac{\sin(e)}{6e}(e\varepsilon|x|)^2\right)^{\nu} = \exp\left(\nu \ln\left(1 - \frac{\sin(e)}{6e}(e\varepsilon|x|)^2\right)\right) \\
&\leq \exp\left(-\nu \frac{\sin(e)}{6e}(e\varepsilon|x|)^2\right) \leq e \exp\left(-\frac{e \sin(e)}{6}\varepsilon|x|^2\right) \\
&\leq e \exp\left(-\frac{1}{6}\varepsilon|x|^2\right).
\end{aligned}$$

It follows that

$$|M_\varepsilon(x)| \leq C \exp\left(-\frac{1}{6}\varphi(x)\right) \quad \forall x \in \mathbb{R}, \tag{42}$$

and property 2. is proved with $Q = \frac{1}{6}$.

We pass now to evaluate $M_\varepsilon(i\lambda_m)$, $m \in \mathbb{N}^*$. Firstly, we remark that, if $|b| \geq |a|$,

$$\left|\frac{\sin(a + i\,b)}{a + i\,b}\right|^2 = \frac{e^{2b} + e^{-2b} - 2\cos(2a)}{4(a^2 + b^2)} = \frac{(e^b + e^{-b})^2 + 4\sin^2(a)}{4(a^2 + b^2)} \geq 1.$$

Thus, if $m \geq \frac{1}{\varepsilon}$, we get $|M_\varepsilon(i\lambda_m)| \geq 1$ and property 3. is verified by any $R > 0$.

Let us now analyze the case $m < \frac{1}{\varepsilon}$. We have that

$$|M_\varepsilon(i\lambda_m)| = \prod_{n=1}^{\infty} \left|\frac{\sin(i\,\rho_n\lambda_m)}{i\,\rho_n\lambda_m}\right| = \prod_{\rho_n|\lambda_m|\leq 1} \left|\frac{\sin(i\,\rho_n\lambda_m)}{i\,\rho_n\lambda_m}\right| \prod_{\rho_n|\lambda_m|>1} \left|\frac{\sin(i\,\rho_n\lambda_m)}{i\,\rho_n\lambda_m}\right|.$$

We consider first the case $\rho_n|\lambda_m| \le 1$ and note that

$$|\sin(i\,\rho_n\lambda_m)| \ge \sin(\rho_n|\lambda_m|) \ge \rho_n|\lambda_m| - \frac{(\rho_n|\lambda_m|)^3}{6}$$

and, consequently,

$$\begin{aligned}\prod_{\rho_n|\lambda_m|\le 1}\left|\frac{\sin(i\,\rho_n\lambda_m)}{i\,\rho_n\lambda_m}\right| &= \exp\left(\sum_{\rho_n|\lambda_m|\le 1}\ln\left|\frac{\sin(i\,\rho_n\lambda_m)}{i\,\rho_n\lambda_m}\right|\right)\\ &\ge \exp\left(\sum_{\rho_n|\lambda_m|\le 1}\ln\left(1-\frac{(\rho_n|\lambda_m|)^2}{6}\right)\right)\\ &\ge \exp\left(-\frac{|\lambda_m|^2}{3}\sum_{n\ge 1}\rho_n^2\right).\end{aligned}$$

Since

$$\sum_{n\ge 1}\rho_n^2 = \sum_{n\le[\frac{1}{\varepsilon}]} e^2\varepsilon^2 + \sum_{n>[\frac{1}{\varepsilon}]} e^2\frac{1}{\varepsilon n^3} \le e^2\varepsilon + \frac{e^2}{\varepsilon}\int_{[\frac{1}{\varepsilon}]}^{\infty}\frac{ds}{s^3} \le 4e^2\varepsilon$$

and $|\Re(\lambda_m)| = \varepsilon m^2 > \frac{\varepsilon}{2}m^2(1+\varepsilon^2 m^2) = \frac{\varepsilon}{2}|\lambda_m|^2$ we obtain

$$\prod_{\rho_n|\lambda_m|\le 1}\left|\frac{\sin(i\,\rho_n\lambda_m)}{i\,\rho_n\lambda_m}\right| \ge \exp\left(-\frac{4e^2}{3}\varepsilon|\lambda_m|^2\right) \ge \exp\left(-\frac{8e^2}{3}|\Re(\lambda_m)|\right). \tag{43}$$

Now, we pass to the case $\rho_n|\lambda_m| > 1$ which implies $|\lambda_m| > \frac{1}{\rho_n} \ge \frac{1}{e\varepsilon}$ and

$$\frac{|\Re(\lambda_m)|}{|\lambda_m|} = \frac{\varepsilon m}{\sqrt{1+\varepsilon^2 m^2}} \ge \frac{\varepsilon m}{\sqrt{2}} \ge \frac{1}{2e}.$$

Thus,

$$\prod_{\rho_n|\lambda_m|>1}\left|\frac{\sin(i\,\rho_n\lambda_m)}{i\,\rho_n\lambda_m}\right| \ge \prod_{\rho_n|\lambda_m|>1}\frac{2\rho_n\Re(\lambda_m)}{2\rho_n|\lambda_m|} = \prod_{\rho_n|\lambda_m|>1}\frac{\varepsilon m}{\sqrt{1+\varepsilon^2 m^2}} \ge \exp(-2\gamma)$$

where

$$\gamma = \#\left\{n\ge 1 \,:\, \rho_n > \frac{1}{|\lambda_m|}\right\}.$$

Since $|\lambda_m| > \frac{1}{e\varepsilon}$, $\gamma = [\sqrt[3]{\frac{e^2}{\varepsilon}|\lambda_m|^2}] \le \Re(\lambda_m)$ and, therefore,

$$\prod_{\rho_n|\lambda_m|>1}\left|\frac{\sin(i\,\rho_n\lambda_m)}{i\,\rho_n\lambda_m}\right| \ge \exp\left(-2\Re(\lambda_m)\right). \tag{44}$$

From (43) and (44) it follows that the function M_ε verifies property 3. with $R = \frac{8e^2}{3} + 2$ and the proof ends. □

Now, we have all the ingredients needed to show the existence of a biorthogonal sequence to the family $\{e^{\lambda_m t}\}_{m\in\mathbb{N}^*}$.

Theorem 3.1 *There exists $\widetilde{T} > 0$ independent of ε such that there exists a biorthogonal sequence $\{\theta_m\}_{m\in\mathbb{N}^*}$ to the family $\{e^{\lambda_m t}\}_{m\in\mathbb{N}^*}$ in $L^2(-\frac{\widetilde{T}}{2}, \frac{\widetilde{T}}{2})$ with the following property*

$$\|\theta_m\|_{L^2(-\frac{\widetilde{T}}{2}, \frac{\widetilde{T}}{2})} \le C \exp(\alpha |\Re(\lambda_m)|) \quad \forall m \in \mathbb{N}^* \tag{45}$$

where C and α are two positive constants independent of ε and m.

Proof For any $m \in \mathbb{N}^*$, we define the function

$$\Psi_m(z) = P_m(z) \left(\frac{M_\varepsilon(z)}{M_\varepsilon(i\lambda_m)} \right)^{\frac{\omega}{Q}} \frac{\sin(\delta(z - i\lambda_m))}{\delta(z - i\lambda_m)} \tag{46}$$

where $\delta > 0$ is an arbitrary constant and let

$$\theta_m(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \Psi_m(x) e^{ixt} dx. \tag{47}$$

From Lemmas 3.1 and 3.3 we deduce that there exists $\widetilde{T} > 0$ independent of ε such that Φ_m is an entire function of exponential type $\frac{\widetilde{T}}{2}$. Moreover, from the estimate of the function P_m on the real axis given by Lemma 3.2 and the properties of the function M_ε from Lemma 3.3, we obtain that

$$\begin{aligned} \int_{-\infty}^{\infty} |\Psi_m(x)|^2 dx &\le C^2 e^{\frac{\omega R}{Q}|\Re(\lambda_m)|} \int_{-\infty}^{\infty} \left| \frac{\sin(\delta(x - \lambda_m i))}{\delta(x - \lambda_m i)} \right|^2 dx \\ &= \frac{C^2}{\delta} e^{\frac{\omega R}{Q}|\Re(\lambda_m)|} \int_{-\infty}^{\infty} \left| \frac{\sin(t - i\delta\Re(\lambda_m))}{t - i\delta\Re(\lambda_m)} \right|^2 dt \le C e^{\alpha|\Re(\lambda_m)|}. \end{aligned}$$

From Paley-Wiener's Theorem we deduce that $\theta_m \in L^2(-\frac{\widetilde{T}}{2}, \frac{\widetilde{T}}{2})$ and Plancherel's Theorem gives that (45) holds. □

The following result is inspired in [7] (see also [8, 12, 20]) and it gives the existence of a new biorthogonal sequence with better norm properties.

Theorem 3.2 *There exist α, C and $T > 2\alpha$, three positive constants independent of ε, and a biorthogonal sequence $\{\zeta_m\}_{m\in\mathbb{N}^*}$ to the family $\{e^{\lambda_m t}\}_{m\in\mathbb{N}^*}$ in $L^2(-\frac{T}{2}, \frac{T}{2})$ with the following property*

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \left| \sum_{m\in\mathbb{N}^*} c_m \zeta_m(t) \right|^2 dt \le C \sum_{m\in\mathbb{N}^*} |c_m|^2 e^{2\alpha|\Re(\lambda_m)|} \tag{48}$$

for any finite sequence $(c_m)_{m\in\mathbb{N}^*}$.

Proof Since it is similar to that of Theorem 3.4 from [12], we only give the main ideas. Let $(\theta_m)_{m\geq 1}\subset(-\frac{\widetilde{T}}{2},\frac{\widetilde{T}}{2})$ be the biorthogonal family from Theorem 3.1. For any $a>0$ define $k_a=\frac{\sqrt{2\pi}}{a^2}(\chi_a*\chi_a)$, where χ_a represents the characteristic function $\chi_{[-a/2,a/2]}$. Evidently $\operatorname{supp}(k_a)\subset[-a,a]$. We introduce $\rho_m(x)=e^{ix\Im(\lambda_m)}k_a(x)$ and we define

$$\zeta_m=\frac{1}{2\pi\widehat{\rho}_m(i\overline{\lambda}_m)}\theta_m*\rho_m,\quad \forall m\geq 1 \tag{49}$$

where $\widehat{\rho}_m$ is the Fourier transform of ρ_m. Evidently, $\zeta_m\in L^2(-\frac{\widetilde{T}}{2}-a,\frac{\widetilde{T}}{2}+a)$. Let $T=\widetilde{T}+2a$. From the convolution's properties, it follows easily that $(\zeta_m)_{m\in\mathbb{N}^*}$ is a biorthogonal sequence to the family $\{e^{\lambda_m t}\}_{m\in\mathbb{N}^*}$ in $L^2(-\frac{T}{2},\frac{T}{2})$ and (48) holds. □

4 Controllability Results

We can prove now the main result of our paper.

Proof of Theorem 1.1 Let $T>2\alpha$ and $(\zeta_m)_{m\in\mathbb{N}^*}$ as in Theorem 3.2. We construct a control $v_\varepsilon\in L^2(0,T)$ of (5) corresponding to the initial data $u_0\in\mathcal{H}$ given by,

$$u_0(x)=\sum_{n\in\mathbb{N}^*}a_n\sin(nx) \tag{50}$$

as follows:

$$v_\varepsilon(t)=\sum_{n=1}^{\infty}-\frac{a_n}{\widehat{f}_n}e^{-\frac{T}{2}\overline{\lambda}_n}\zeta_n\left(t-\frac{T}{2}\right),\quad t\in(0,T). \tag{51}$$

From the properties of the biorthogonal sequence $(\zeta_m)_{m\in\mathbb{N}^*}$, it is easy to see that v_ε verifies (11). To conclude that v_ε is a control for (5), we only have to prove that the series from (51) converges in $L^2(0,T)$. This follows immediately from estimate (48) and the fact that $u_0\in\mathcal{H}$. Indeed, we have that

$$\int_0^T\left|\sum_{n=1}^{\infty}\frac{a_n}{\widehat{f}_n}e^{-\frac{T}{2}\overline{\lambda}_n}\zeta_n\left(t-\frac{T}{2}\right)\right|^2dt\leq C\sum_{n\in\mathbb{N}^*}\frac{|a_n|^2}{|\widehat{f}_n|^2}e^{(-T+2\alpha)|\Re(\lambda_n)|}\leq C.$$

The constant from the last inequality does not depend of ε. Thus, the sequence of controls $(v_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $L^2(0,T)$. Let v be a weak limit of this sequence. In order to show that v is a control for (9) we only have to pass to the limit as ε goes to zero in the characterization relation (11). □

Remark 4.1 The optimal time controllability for the limit equation (9) is $T_0=2\pi$. Probably, Theorem 1.1 is true for any $T\geq T_0$. From the proof of Theorem 1.1 we only deduce the existence of a uniform controllability time T independent of ε. Estimates on T can be explicitly obtained, but it would be difficult to show that T may be any number greater that T_0. This fact has already been noted in [5, 12], where similar methods are used. On the other side, if $T < T_0$, equation (9) is not even spectrally controllable, due to the fact that the family of complex exponentials $\{e^{\lambda_m t}\}_{m\geq 1}$ is not minimal in $L^2(0, T)$ (see, for instance, [1]).

Remark 4.2 The controllable space of initial data, $\mathcal{H}$, depends on the regularity of the "profile" f. Indeed, functions f with lower regularity will have smaller Fourier coefficients and consequently the controllable space $\mathcal{H}$ will be larger.

Acknowledgements The first author was partially supported by Grant MTM2008-03541 funded by MICINN (Spain) and Grant 420/2008 of CNCSIS (Romania). The second author was also partially supported by FONDAP and BASAL-CMM projects and the third author by CNPq (Brasil). Part of this work was done when S. Micu and J. Ortega were visiting IM/UFRJ and A.F. Pazoto and S. Micu were visiting Universidad de Chile with the partial support of grants Fondecyt 1070148 and 7080162.

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