ASYMPTOTICS FOR THE SPECTRUM OF A FLUID/STRUCTURE HYBRID SYSTEM ARISING IN THE CONTROL OF NOISE

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Abstract. We consider a simple model arising in the control of noise consisting of two coupled hyperbolic equations of dimensions two and one respectively. The one dimensional equation is assumed to be dissipative. We describe the asymptotic behavior of the eigenvalues and eigenfunctions of the system at high frequencies. Some other interesting features of the model like the exponential decay of solutions or the compactness of the damping term, are also studied.

Key words. Eigenvalues, eigenfunctions, high frequency asymptotics, hyperbolic system, aeromechanic structure interaction.

AMS subject classifications. 35P20, 35L20, 73K70.

1. Introduction.

Recently several works both in the mathematical and technical literature have dealt with the problem of the active control of noise generated in acoustic cavities by means of the vibrations of their flexible walls. Such studies were motivated, for instance by the development of a new class of turboprop engines which are very fuel efficient but also very loud. In this context the low frequency high magnitude acoustic fields produced by these engines cause vibrations in the fuselage which in turn generate unwanted interior noise.

In this article we analyse the spectral properties of a linear two-dimensional hybrid system arising in the development of these new technologies for noise reduction in the interior of a cavity (plane, car, etc.) which was proposed in a series of works by Banks et al. (see [3]).

Let us describe the system we study. We consider the two-dimensional square $\Omega = (0,1)\times(0,1)\subset I\!R^2$. We assume that Ω is filled with an elastic, inviscid, compressible fluid whose velocity field $\vec{v}$ is given by the potential $\phi = \phi(x,y,t)$, ($\vec{v} = \nabla\phi$). By linearization we assume that the potential ϕ satisfies the linear wave equation in $\Omega\times(0,\infty)$.

The boundary $\Gamma = \partial\Omega$ of Ω is divided in two parts: $\Gamma_0 = \{(x,0) : x\in(0,1)\}$ and $\Gamma_1 = \Gamma\backslash\Gamma_0$. The subset Γ_1 is assumed to be rigid and we impose zero normal velocity of the fluid on it. The subset Γ_0 is supposed to be flexible and occupied by a flexible string that vibrates under the pressure of the fluid on the plane where Ω lies. The displacement of Γ_0, described by the scalar function $W = W(x,t)$, obeys the one-dimensional dissipative wave equation. On the other hand, on Γ_0 we impose the continuity of the normal velocities of the fluid and the string. The string is assumed to satisfy Neumann boundary conditions on its extremes.

All deformations are supposed to be small enough so that linear theory applies.

Under natural initial conditions for ϕ and W the linear motion of this system is

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described by means of the following coupled wave equations:

$$
\left\{
\begin{array}{lll}
\phi_{tt} - \Delta\phi = 0 & \text{in} & \Omega \times (0,\infty) \\
\frac{\partial \phi}{\partial \nu} = 0 & \text{on} & \Gamma_1 \times (0,\infty) \\
\frac{\partial \phi}{\partial y} = -W_t & \text{on} & \Gamma_0 \times (0,\infty) \\
W_{tt} - W_{xx} + W_t + \phi_t = 0 & \text{on} & \Gamma_0 \times (0,\infty) \\
W_x(0,t) = W_x(1,t) = 0 & \text{for} & t > 0 \\
\phi(0) = \phi^0, \phi_t(0) = \phi^1 & \text{in} & \Omega \\
W(0) = W^0, W_t(0) = W^1 & \text{on} & \Gamma_0 .
\end{array}
\right. \tag{1.1}
$$

By ν we denote the unit outward normal to Ω.

In (1.1) we have chosen to take the various parameters of the system to be equal to one. This restricts the generality of our analysis. The dependence of the most interesting features of the spectrum with respect to the various parameters of the system will be studied elsewhere.

We remark also that, in (1.1), two wave equations, of dimensions two and one respectively and representing vibrations of different nature, are coupled. Therefore we say that (1.1) is a two-dimensional hybrid system. For examples of hybrid systems of dimension one, such as those coupling strings or beams with rigid bodies, see [10], [7] and [18].

System (1.1) is a modified version of the one introduced by H.T. Banks et al. in [3]. In [3] the flexible part of the boundary Γ_0 is assumed to be occupied by a flexible beam, leading to a fourth order one-dimensional equation on Γ_0. We have chosen to consider a one-dimensional wave equation instead to simplify the exposition. However, most of the relevant spectral properties remain unchanged considering a beam equation with appropriate boundary conditions.

We also remark that we choose Neumann boundary conditions for the string. This choice allows us to separate the variables and to obtain an explicit equation for the eigenvalues. In the case of Dirichlet boundary conditions, which are considered in [3], this is not longer possible. Nevertheless, using the information we get here about the eigenfunctions of system (1.1), it can be proved that the uniform decay fails and also that there exist solutions uniformly distributed in Ω with arbitrarily small decay (see [14]).

System (1.1) is well-posed in the energy space

$$\mathcal{X} = H^1(\Omega) \times L^2(\Omega) \times H^1(\Gamma_0) \times L^2(\Gamma_0)$$

for the variables (ϕ, ϕ_t, W, W_t).

The energy

$$E(t) = \frac{1}{2}\int_\Omega \left[| \nabla\phi |^2 + | \phi_t |^2 \right] dxdy + \frac{1}{2}\int_{\Gamma_0} \left[| W_x |^2 + | W_t |^2 \right] dx \tag{1.2}$$

satisfies

$$\frac{dE}{dt}(t) = -\int_{\Gamma_0} | W_t |^2 \, dx. \tag{1.3}$$

Hence, the system (1.1) is dissipative, the damping term being localized in the subset Γ_0 of the boundary.

Some of the properties of this system like existence, uniqueness, asymptotic behavior and existence of periodic solutions were studied in previous works (see [12] and [13]).

Our aim here is to characterize the asymptotic behavior of the eigenvalues and eigenfunctions of the differential operator corresponding to system (1.1) and to describe some interesting features of the model that are direct consequences of this analysis. The study is made by using separation of variables. In this way the system is reduced to an infinite number of one-dimensional systems depending on an integer parameter k which represents the frequency of vibration in x-direction. This allows us to obtain explicit equations for eigenvalues and to use Rouché's Theorem for their localization.

Let us describe briefly the most relevant results obtained in this paper:

a) Whenever the frequency of vibration in the $x-$direction is fixed the corresponding one-dimensional system does not decay uniformly. Indeed, at high frequencies, the real part of the one-parametric family of eigenvalues converges to zero. This is a typical situation in one-dimensional hybrid systems (see [7], [10] and [18]).

b) The effect of the damping term on the global dynamics of the system is almost negligible at high frequencies. Indeed, most of the eigenfunctions of the system (1.1) have their energy uniformly distributed in Ω while the real part of the eigenvalues converges to zero at high frequencies.

c) Among the two-parameter family of eigenvalues of the two-dimensional system only a one-parameter family of them is effectively damped so that their real parts remain uniformly away from zero. The corresponding eigenfunctions have their energy exponentially concentrated on the string Γ_0.

d) As a consequence of the previous property, the difference between the semigroup generated by the damped and undamped systems is not compact. This is in contrast with the results in [18] showing that the lack of uniform decay in damped one-dimensional hybrid systems is typically due to the compactness of the damping term. Thus, the non compactness result is genuinely two-dimensional.

Let us remark that the case we have addressed is not generic. Even in the case of surfaces of revolution the cylindrical case is a degenerate one. This was exhibited in the Thesis of B. Allibert in the frame of the classical wave equation with Dirichlet boundary conditions (see [1]). Nevertheless, in [11] we show that, in the case of a disk shaped cavity surrounded by a circular dissipative string the same phenomenon is present although all rays of geometric optics meet the boundary where the losses occur. This indicates that the same behavior can be expected for different kinds of geometries or boundary conditions (see also [14]).

The rest of the paper is organized as follows.

In Section 2 we present in detail the main results of this paper and we discuss some of their consequences. In Section 3 we localize the eigenvalues of the undamped system corresponding to (1.1) and describe its eigenfunctions. In Section 4 we obtain asymptotic estimates for the eigenvalues and eigenmodes of the damped system (1.1). This section is divided in two parts. In Subsection 4.1 we distinguish three types of eigenvalues which, at high frequencies, approach the imaginary axis. The corresponding eigenfunctions have the property that the energy concentrated in the string vanishes asymptotically. To complete the study, in Subsection 4.2 we prove that there exists a sequence of eigenvalues, tending to infinity, with uniformly negative real parts. The corresponding eigenfunctions have the property that the energy localized on the string does not vanish asymptotically. Moreover, as the frequency increases the whole energy is concentrated on the string at an exponential rate. These eigenfunctions span an infinite dimensional subspace of the energy space in which the decay rate of solutions is exponential.

In the last section we prove that the difference between the semigroup generated by the differential operator associated to the undamped system and that associated to the damped one is not compact as a consequence of the existence of an infinite-dimensional subspace in which the damping term is effective, i.e. it produces an exponential decay. We end up with an Appendix that contains some technical lemmas.

2. The main results: statements and discussion. As we said in the introdution the aim of this paper is the study of the spectrum of (1.1). In this section we state the main results concerning the eigenvalues and eigenfunctions of the system and some of their consequences.

In order to analyze the spectrum of (1.1) we look for solutions in separated variables of the form $(\phi, W) = (\psi(y,t), V(t))\cos(k\pi x)$.

We deduce that $(\psi(y,t), V(t))$ verifies the following one-dimensional system:

$$\left\{\begin{array}{lll} \psi_{tt} - \psi_{yy} + k^2\pi^2\psi = 0 & \text{in} & (0,1)\times(0,\infty) \\ \psi_y(1) = 0 & \text{for} & t\in(0,\infty) \\ \psi_y(0) = -V_t & \text{for} & t\in(0,\infty) \\ V_{tt} + k^2\pi^2 V + V_t + \psi_t(0) = 0 & \text{for} & t\in(0,\infty). \end{array}\right. \tag{2.1}$$

Now if we look for solutions of (2.1) of the form $(\psi(y,t), V(t)) = e^{\lambda\, t}(\psi(y), V)$, with $V \in I\!\!R$, it follows that the eigenvalues λ of system (1.1) are the roots of the equation:

$$e^{2\sqrt{\lambda^2+k^2\pi^2}} = -\frac{\lambda^2 - \sqrt{\lambda^2+k^2\pi^2}(\lambda^2+\lambda+k^2\pi^2)}{\lambda^2 + \sqrt{\lambda^2+k^2\pi^2}(\lambda^2+\lambda+k^2\pi^2)}. \tag{2.2}$$

The corresponding eigenfunctions are $\varphi_\lambda = \psi_\lambda \ \cos\ (k\pi x)$ where ψ_λ are the eigenfunctions of (2.1):

$$\psi_\lambda = \left(\begin{array}{l} \dfrac{1}{\lambda}\cosh\,(\sqrt{\lambda^2+k^2\pi^2}(y-1)) \\ \cosh\,(\sqrt{\lambda^2+k^2\pi^2}(y-1)) \\ \dfrac{\sqrt{\lambda^2+k^2\pi^2}}{\lambda^2}\ \sinh\,(\sqrt{\lambda^2+k^2\pi^2}) \\ \dfrac{\sqrt{\lambda^2+k^2\pi^2}}{\lambda}\ \sinh\,(\sqrt{\lambda^2+k^2\pi^2}) \end{array}\right). \tag{2.3}$$

We are interested in the asymptotic behavior of the eigenvalues λ when $|\lambda| \longrightarrow \infty$. For each $k \in I\!\!N$ we get a sequence of eigenvalues $(\lambda_{k,m})_{m\in Z\!\!\!Z^*}$ for the system (2.1) of modulus greater than $k\pi$ (that will be analyzed in Subsection 4.1) and two eigenvalues λ_k^* and λ_k^{**} with modulus less than $k\pi$ (that will be studied in Subsection 4.2). All these are the eigenvalues of system (1.1). For each k, $(\lambda_{k,m})_{m\in I\!\!N^*}$ are ordered such that $|\lambda_{k,m}|$ increases as m does and $\lambda_{k,-m} = \bar{\lambda}_{k,m}$ if $m \in I\!\!N^*$. The general result on the existence of eigenvalues is given in the following theorem.

THEOREM 2.1. *Let $k \in I\!\!N$ be fixed. The spectrum of the differential operator corresponding to system (2.1) consists of a sequence of eigenvalues $(\lambda_{k,m})_{m\in I\!\!N^*}\cup\{\lambda_k^*\}$ with positive imaginary part and another sequence of eigenvalues $(\lambda_{k,-m})_{m\in I\!\!N^*}\cup\{\lambda_k^{**}\}$ with the property that $\lambda_{k,-m} = \bar{\lambda}_{k,m}$ if $m > 0$ and $\lambda_k^{**} = \bar{\lambda}_k^*$. All these eigenvalues are zeros of the equation (2.2). If $k = 0$ then $\lambda_k^* = \lambda_k^{**} = 0$.*

REMARK 1. *We remark that the notations λ_k^* and λ_k^{**} are used for the eigenvalues with the smallest modulus of the system. We make this distinction since the properties of this wave numbers are different from the others, as we shall see in Theorem 2.9. Actually, this eigenvalues correspond to the eigenfunctions whose energy is concentrated on the string Γ_0 and decaying uniformly as $t \to \infty$.*

The asymptotic properties of the wave numbers and modes depend on the relation between k and m. Therefore we divide our analysis in four cases. First, in Theorems 2.2, 2.4 and 2.6, we characterize the eigenvalues that approach the imaginary axis as the wave number increases. These are the eigenvalues $(\lambda_{k,m})_{m\in\mathbb{Z}^*}$. Then, in Theorem 2.9, we study the eigenvalues λ_k^* and λ_k^{**} which have an uniformly negative real part.

THEOREM 2.2. *(Eigenvalues $\lambda_{k,m}$ with $|\lambda_{k,m}| \geq \sqrt{2}k\,\pi$) Let $k \in \mathbb{N}$ be fixed. The eigenvalues $\lambda_{k,m}$ of (2.1) with $|\,\lambda_{k,m}\,| > \sqrt{2}\pi\, k$ approach the imaginary axis when $|\,m\,| \to \infty$ and satisfy the following:*

$$
\begin{aligned}
&|\,\lambda_{k,m} - \sqrt{k^2+m^2}\,\pi\, i\,| \leq \frac{24}{\sqrt{m^2+k^2}\,\pi} \quad \text{if } \mathcal{I}m\,\lambda_{k,m} > 0,\ (m > k > 0),\\
&|\,\lambda_{k,m} + \sqrt{k^2+m^2}\,\pi\, i\,| \leq \frac{24}{\sqrt{m^2+k^2}\,\pi} \quad \text{if } \mathcal{I}m\,\lambda_{k,m} < 0,\ (m < -k < 0).
\end{aligned}
\tag{2.4}
$$

REMARK 2. *Theorem 2.2 shows that, when we fix the frequency of vibration in x-direction (k fixed) and we consider large frequencies in the y-direction (m large), the system behaves like the wave equation in Ω with homogeneous Neumann boundary conditions in all $\partial\Omega$:*

$$
\left\{
\begin{array}{lll}
\Phi_{tt} - \triangle\Phi = 0 & \textit{in} & \Omega \times (0,\infty)\\
\frac{\partial\Phi}{\partial\nu} = 0 & \textit{on} & \partial\Omega \times (0,\infty).
\end{array}
\right.
\tag{2.5}
$$

Therefore, the influence of the vibrating string on Γ_0 vanishes asymptotically.

REMARK 3. *Note that the existence of a sequence of eigenvalues $(\lambda_{k,m})_m$ which approach the imaginary axis when $|\,m\,| \to \infty$ implies that the decay rate of the energy of solutions of (1.1) is not exponential. It is known that, for linear problems, this is equivalent to a non uniform decay rate of solutions (see [9]).*

In fact we obtain that, for each $k \in \mathbb{N}$, the system (2.1) does not have an exponential decay. This is not the case in the classical wave equation with boundary dissipation:

$$
\left\{
\begin{array}{lll}
\Phi_{tt} - \triangle\Phi = 0 & \textit{in} & \Omega \times (0,\infty)\\
\frac{\partial\Phi}{\partial\nu} = 0 & \textit{on} & \Gamma_1 \times (0,\infty)\\
\frac{\partial\Phi}{\partial\nu} + \Phi_t = 0 & \textit{on} & \Gamma_0 \times (0,\infty).
\end{array}
\right.
\tag{2.6}
$$

In the context of (2.6), for k fixed, the corresponding one-dimensional systems have exponential decay, but the decay rate vanishes as $k \to \infty$. This is due to the fact that the region Γ_0 in which the damping is concentrated does not satisfy the necessary geometric control condition since there are rays of geometric optics that never intersect Γ_0 (see [5] and [17]). In our case the loss of uniform decay is even worse and it is due to the hybrid structure of the system or, equivalently, to the type of boundary condition we have imposed on Γ_0 and not only to the support Γ_0 of the damping term.

Moreover, as we mention in Remark 6, we can find a sequence of solutions of (1.1) with the energy uniformly distributed in all Ω and with arbitrarily small exponential *decay rate. This is not possible in the examples given in [5] and [17] where the energy of the solutions with non uniform decay concentrates on rays of geometric optics.*

REMARK 4. *The fact that the eigenvalues approach the imaginary axis is a consequence not only of the localization of the dissipative region but also of the hybrid structure of the system. In [11] we show that, in the case of a disk shaped cavity surrounded by a circular dissipative string the same phenomenon is present although all rays of geometric optics meet the boundary where the losses occur.*

This indicates that the same behavior can be expected for different kinds of geometries or boundary conditions (see also [14]).

We can now analyze the eigenfunctions corresponding to the wave numbers $\lambda_{k,m}$ of Theorem 2.2. Remark 2 indicates that one can expect the first two components of the eigenfunctions of (1.1) to behave like the eigenfunctions of (2.5). Therefore we define the function:

$$\psi_{k,m} = \begin{pmatrix} \dfrac{(-1)^{m+1} i}{\sqrt{k^2+m^2}\pi} \cos m\pi y \cos k\pi x \\ (-1)^{m+1} \cos m\pi y \cos k\pi x \\ 0 \\ 0 \end{pmatrix}. \tag{2.7}$$

Observe that the eigenmodes of (2.5) are the first two components of $\psi_{k,m}$.

THEOREM 2.3. *The eigenfunctions φ_λ, corresponding to the eigenvalues $\lambda = \lambda_{k,m}$ satisfying (2.4) have the following property:*

$$\| \varphi_\lambda - \psi_{k,m} \|_{\mathcal{X}} \leq \frac{c}{m} \tag{2.8}$$

where c is a constant which does not depend on m and k.

REMARK 5. *Theorem 2.5 indicates that the last two components of the eigenfunction φ_λ (which correspond to the string located in Γ_0) vanish asymptotically when the frequency increases. This implies that, at high frequencies (in the sense of (2.4)), the string does not play an important role in the dynamics of the system.*

REMARK 6. *The solutions of (1.1) corresponding to the eigenfunctions given by Theorem 2.3 form a sequence of solutions with the energy uniformly distributed in all Ω and with arbitrarily small exponential decay rate. This proves that the lack of the uniform decay of our system is related not only to the support of the dissipative mechanism but also to the nature of the boundary conditions or of the coupling between the different components of the system.*

The second range of frequencies is studied in the following theorem.

THEOREM 2.4. *(Eigenvalues $\lambda_{k,m}$ with $k\pi \leq |\lambda_{k,m}| \leq \sqrt{2}k\pi$, First part). For $k \in \mathbb{N}$ sufficiently large and $m = \pm 1, \pm 2, ..., \pm[\sqrt[3]{k}]$, the eigenvalues $\lambda_{k,m}$ of (1.1) satisfy:*

$$\begin{aligned} &\left| \lambda_{k,m} - \sqrt{k^2 + \left(\frac{2m-1}{2}\right)^2} \pi i \right| \leq \frac{2\pi}{\sqrt[3]{k}} \text{ if } \mathcal{I}m\, \lambda_{k,m} > 0, \ (1 \leq m \leq [\sqrt[3]{k}]), \\ &\left| \lambda_{k,m} + \sqrt{k^2 + \left(\frac{2m+1}{2}\right)^2} \pi i \right| \leq \frac{2\pi}{\sqrt[3]{k}} \text{ if } \mathcal{I}m\, \lambda_{k,m} < 0, \ (-[\sqrt[3]{k}] \leq m < 0). \end{aligned} \tag{2.9}$$

Remark 7. *Consider the following conservative wave equation:*

$$\left\{\begin{array}{lll}\Phi_{tt}-\triangle\Phi=0 & \text{in} & \Omega\times(0,\infty)\\ \frac{\partial\Phi}{\partial\nu}=0 & \text{on} & \Gamma_1\times(0,\infty)\\ \Phi=0 & \text{on} & \Gamma_0\times(0,\infty).\end{array}\right. \tag{2.10}$$

Its eigenvalues are exactly $\sqrt{k^2+\left(\frac{2m+1}{2}\right)^2}\,\pi\, i$. *Theorem 2.4 shows that, when we fix the frequency of vibration in the y-direction (m is fixed) and we consider large frequencies in the x-direction (k large), the eigenvalues of (1.1) behave like those of (2.10). The influence of the vibrating string on Γ_0 vanishes asymptotically in this range of eigenvalues. However, when comparing the behavior of these eigenvalues with those of Theorem 2.2, we observe that the boundary conditions for Φ on Γ_0 change.*

Let us analyze the eigenfunctions corresponding to the eigenvalues studied in Theorem 2.4. We consider first the function:

$$\tilde{\psi}_{k,m}=\begin{pmatrix}\dfrac{(-1)^{m+1}\, i}{\sqrt{k^2+\left(\frac{2m+1}{2}\right)^2}\,\pi}\sin\dfrac{2m+1}{2}\pi y\,\cos k\pi x\\ (-1)^{m+1}\sin\dfrac{2m+1}{2}\pi y\,\cos k\pi x\\ 0\\ 0\end{pmatrix} \tag{2.11}$$

and we remark that the first two components of it correspond to eigenfunctions of problem (2.10).

Theorem 2.5. *The eigenfunctions φ_λ corresponding to the eigenvalues $\lambda=\lambda_{k,m}$ of Theorem 2.4 satisfy:*

$$\|\,\varphi_\lambda-\tilde{\psi}_{k,m}\,\|_{\mathcal{X}}\le\frac{c}{\sqrt[3]{k}} \tag{2.12}$$

where c is a constant which does not depend on k and m.

Remark 8. *Remark 5 applies in this case too.*

The following theorem completes the study of the eigenvalues with real part tending to zero as the wave number increases.

Theorem 2.6. *(Eigenvalues $\lambda_{k,m}$ with $k\,\pi\le|\lambda_{k,m}|\le\sqrt{2}k\,\pi$, Second part). For all $k\in\mathbb{N}$ sufficiently large the eigenvalues $\lambda_{k,m}$ of (1.1) with $[\sqrt[3]{k}]<|m|\le k$ satisfy the following estimates:*

$$\begin{aligned}&\left|\lambda_{k,m}-\sqrt{\pi^2k^2+k^2\zeta_{k,m}^2}\,i\right|\le\frac{1}{\sqrt[5]{k}}\ \text{if}\ \mathcal{I}m\,\lambda_{k,m}>0,\ (k\ge m>[\sqrt[3]{k}]),\\ &\left|\lambda_{k,m}+\sqrt{\pi^2k^2+k^2\zeta_{k,m}^2}\,i\right|\le\frac{1}{\sqrt[5]{k}}\ \text{if}\ \mathcal{I}m\,\lambda_{k,m}<0,\ (-k\le m<-[\sqrt[3]{k}]),\end{aligned} \tag{2.13}$$

where $\zeta_{k,m} \in I\!R_+$ is the positive root of the equation:

$$\tan\, k\,\zeta \,=\, \frac{\pi^2}{k\,\zeta^3}, \tag{2.14}$$

which belongs to $\left(\frac{m}{k}\pi, \frac{2m+1}{2k}\pi\right)$.

REMARK 9. *When k remains bounded and m goes to infinity the roots $\zeta_{k,m}$ of the equation (2.14) behave like $\frac{m\pi}{k}$. This corresponds to the asymptotic behavior of the eigenvalues $\lambda_{k,m}$ studied in Theorem 2.2. On the other hand, when m remains bounded and k goes to infinity, the zeros $\zeta_{k,m}$ of (2.14) behave like $\frac{(2m+1)\pi}{2k}$. This agrees with the behavior of the eigenvalues $\lambda_{k,m}$ studied in Theorem 2.4.*

The eigenvalues $\lambda_{k,m}$ of Theorem 2.6 make the transition from one zone to another and still approach the imaginary axis at high frequencies.

The eigenfunctions corresponding to these eigenvalues have the same property as those of Theorems 2.3 and 2.5, i.e. the last two components vanish asymptotically.

THEOREM 2.7. *The eigenfunctions φ_λ corresponding to the eigenvalues of Theorem 2.6 satisfy:*

$$\lim_{|\lambda|\to\infty} \frac{||\varphi_\lambda^3||_{H^1(\Gamma_0)}}{||\varphi_\lambda||_{\mathcal{X}}} = 0, \qquad \lim_{|\lambda|\to\infty} \frac{||\varphi_\lambda^4||_{L^2(\Gamma_0)}}{||\varphi_\lambda||_{\mathcal{X}}} = 0, \tag{2.15}$$

where φ_λ^3 and φ_λ^4 are the third and the fourth components of φ_λ.

Until now we have obtained eigenvalues of system (1.1) approaching the imaginary axis when their modulus tends to infinity. The following result exhibits a sequence of eigenvalues with uniformly bounded negative real parts.

THEOREM 2.8. *(Eigenvalues λ_k with $|\lambda_k| \leq k\,\pi$). The equation (2.2) has, for sufficiently large k, two eigenvalues λ_k^* and λ_k^{**} with $\mathcal{I}m\,\lambda_k^* > 0$ and*

$$\left|\,\lambda_k^* - \sqrt{k^2(\alpha_1)^2 - k^2\pi^2}\,\right| \leq \frac{1}{k} \quad \text{and} \quad \lambda_k^* = \bar{\lambda}_k^{**}, \tag{2.16}$$

where α_1 is the root of

$$z^2 - \pi^2 + kz^3 + z\sqrt{z^2 - \pi^2} = 0 \tag{2.17}$$

with the following asymptotic behavior:

$$\alpha_1 = \sqrt[3]{\frac{\pi^2}{k}} - \frac{1}{3}\sqrt[3]{\frac{\pi}{k^2}}\,i + o\left(\frac{1}{\sqrt[3]{k^2}}\right), \text{ as } k \to \infty. \tag{2.18}$$

Therefore, λ_k^* satisfies:

$$\mathcal{R}e\,\lambda_k^* \longrightarrow -\frac{1}{3} \text{ when } k \longrightarrow \infty. \tag{2.19}$$

REMARK 10. *In Theorem 2.8 we prove the existence of two eigenvalues λ_k^* and λ_k^{**} with modulus less than $k\pi$. These are, for k fixed, the eigenvalues with smallest modulus and are the only ones uniformly dissipated by the system at large frequencies.*

The corresponding eigenfunctions λ_k^* can be written as:

$$\varphi_{\lambda_k^*} = \left(\begin{array}{c} \dfrac{\cosh\,(\sqrt{(\lambda_k^*)^2 + k^2\pi^2}(y-1))\,\cos k\pi x}{\sqrt{(\lambda_k^*)^2 + k^2\pi^2}\,\sinh\,(\sqrt{(\lambda_k^*)^2 + k^2\pi^2})} \\ -\dfrac{\lambda_k^*\cosh\,(\sqrt{(\lambda_k^*)^2 + k^2\pi^2}(y-1))\,\cos k\pi x}{\sqrt{(\lambda_k^*)^2 + k^2\pi^2}\,\sinh\,(\sqrt{(\lambda_k^*)^2 + k^2\pi^2})} \\ -\dfrac{1}{\lambda_k^*}\cos k\pi x \\ \cos k\pi x \end{array} \right)$$

and they have a different behavior.

THEOREM 2.9. *i) The sequence of eigenfunctions $\{\varphi_{\lambda_k^*}\}_k$ converge weakly to zero in $\mathcal{X}$ when k tends to infinity.*

ii) The sequences $\{\varphi^j_{\lambda_k^*}\}_k$ do not converge strongly to zero for any $j = 1, 2, 3, 4$ in the corresponding norms.

REMARK 11. *The eigenfunctions $\varphi_{\lambda_k^*}$ generate a subspace of the energy space of infinite dimension in which, in view of (2.19), the decay rate of the energy of the system is uniform. The energy of the solutions correponding to the eigenfunctions of Theorem 2.9 is concentrated in the string. Indeed, the estimates of Theorem 2.9 allows us to prove that:*

$$\int_0^1\int_\varepsilon^1 \left(||\varphi^1_{\lambda_k^*}||^2_{H^1(\Omega)} + ||\varphi^2_{\lambda_k^*}||^2_{L^2(\Omega)}\right) dx\,dy \le Ce^{-2\sqrt[3]{k^2\pi^2}\varepsilon}.$$

This indicates that the energy of the acoustic wave decays exponentially fast from Γ_0 to the interior of the domain.

Figure 1 describes the behavior of the different families of eigenvalues for each k.

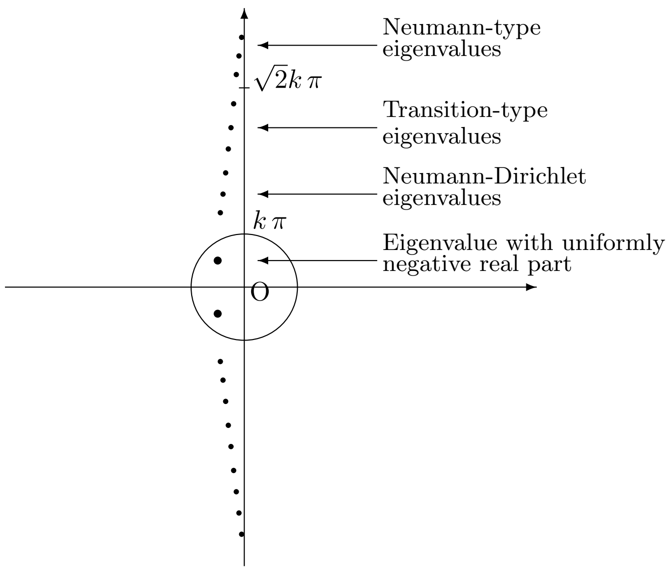

Fig.1: The sequence of eigenvalues for fixed k.

3. The conservative system. In this section we analyze the spectral properties of the conservative system corresponding to (1.1):

$$\left\{\begin{array}{lll}\phi_{tt}-\Delta\phi=0 & \text{in} & \Omega\times(0,\infty)\\ \frac{\partial\phi}{\partial\nu}=0 & \text{on} & \Gamma_1\times(0,\infty)\\ \frac{\partial\phi}{\partial y}=-W_t & \text{on} & \Gamma_0\times(0,\infty)\\ W_{tt}-W_{xx}+\phi_t=0 & \text{on} & \Gamma_0\times(0,\infty)\\ W_x(0,t)=W_x(1,t)=0 & \text{for} & t>0.\end{array}\right. \tag{3.1}$$

In (3.1) the dissipative term W_t of the equation of displacement of the string has been dropped. The energy of this system is defined by (1.2) too, but in this case we have that $\dfrac{dE}{dt}(t)=0$. This means that (3.1) is an undamped system.

The eigenvalues of (3.1) are characterized in the following theorem:

THEOREM 3.1. *System (3.1) has a two-parameter sequence of purely imaginary eigenvalues* $(\nu_{k,m})_{k\in\mathbb{N},\, m\in\mathbb{Z}^*}$ *given by:*

$$\nu_{k,m}=\sqrt{z_{k,m}^2+k^2\pi^2}\ i \quad \text{if } m>0 \quad \text{and} \quad \nu_{k,m}=-\nu_{k,-m} \quad \text{if } m<0\,, \tag{3.2}$$

where $(z_{k,m})_{m\in\mathbb{N}^*}$ *are the roots of the equation:*

$$\tan\, z=\frac{z^2+k^2\pi^2}{z^3}. \tag{3.3}$$

Moreover, there are another two eigenvalues of (3.1), ν_k^* *and* ν_k^{**}*, with the modulus less than* $k\,\pi$*, given by:*

$$\nu_k^*=\sqrt{k^2\pi^2-(z_k^*)^2}\ i\,,\quad \nu_k^{**}=\bar{\nu}_k^*\,, \tag{3.4}$$

where z_k^* *is the unique positive root of the equation:*

$$e^{2\,z}=\frac{z^3-z^2+k^2\pi^2}{z^3+z^2-k^2\pi^2}. \tag{3.5}$$

In the last case, $\nu_k^*=\nu_k^{**}=0$ *when* $k=0$

Proof: In order to study the spectrum of (3.1) we look for solutions of this system in separated variables: $(\phi,W)=e^{\nu t}(\psi,V)\cos(n\pi x)$ where $\psi=\psi(y)$ and $V\in\mathbb{R}$. It follows that the eigenvalues ν satisfy the following transcendental equation:

$$e^{2\sqrt{\nu^2+k^2\pi^2}}=-\frac{\nu^2-\sqrt{\nu^2+k^2\pi^2}(\nu^2+k^2\pi^2)}{\nu^2+\sqrt{\nu^2+k^2\pi^2}(\nu^2+k^2\pi^2)}. \tag{3.6}$$

Considering the change of variable $\nu=\sqrt{\zeta^2-k^2\pi^2}$ equation (3.6) becomes:

$$e^{2\zeta}=\frac{\zeta^3-\zeta^2+k^2\pi^2}{\zeta^3+\zeta^2-k^2\pi^2}. \tag{3.7}$$

Since the differential operator corresponding to (3.1) is conservative its eigenvalues will be all purely imaginary. Hence, we have to look only for those roots of (3.7) which are purely imaginary or real. It follows that the imaginary roots of (3.7) are the roots of the equation (3.3) and the real ones are roots of (3.5). □

We analyze now the eigenfunctions. By separation of variables, it is easy to see that the eigenfunctions have the following form:

$$\xi_\nu = \begin{pmatrix} \dfrac{-i}{\sqrt{z^2+k^2\pi^2}}\cos z(y-1)\,\cos k\pi x \\ -\cos z(y-1)\,\cos k\pi x \\ -\dfrac{z}{z^2+k^2\pi^2}\,\sin z\,\cos k\pi x \\ \dfrac{z\,i}{\sqrt{z^2+k^2\pi^2}}\,\sin z\,\cos k\pi x \end{pmatrix}. \tag{3.8}$$

THEOREM 3.2. *The eigenfunctions ξ_ν defined by (3.8) corresponding to the eigenvalues ν given by (3.3) have the following property:*

$$\lim_{|\nu|\to\infty}\frac{||\xi_\nu^3||_{H^1(0,1)}}{||\xi_\nu||_{\mathcal{X}}}=0,\qquad \lim_{|\nu|\to\infty}\frac{||\xi_\nu^4||_{L^2(0,1)}}{||\xi_\nu||_{\mathcal{X}}}=0$$

where ξ_ν^j is the j-th component of ξ_ν.

Proof: If ν is one of the eigenvalues of (3.1) with $|\nu| > k\pi$ it follows that $\zeta=\sqrt{\nu^2+k^2\pi^2}$ is a purely imaginary number. Therefore $\zeta = z\,i$ where $z\in I\!\!R$ is a solution of the equation (3.3).

Taking into account that z satisfies (3.3), a simple calculation gives us that:

$$||\xi_\nu^1||_{H^1}^2+||\xi_\nu^2||_{L^2}^2=\frac{1}{2}+\frac{1}{4(z^2+k^2\pi^2)}+\frac{(1+2k^2\pi^2)\,\sin 2z}{8z(z^2+k^2\pi^2)}=$$

$$=\frac{1}{2}+\frac{1}{4(z^2+k^2\pi^2)}+\frac{2z^3(z^2+k^2\pi^2)}{4(z^6+(z^2+k^2\pi^2)^2)},$$

$$||\xi_\nu^3||_{H^1}^2=\frac{z^2(1+k^2\pi^2)\,\sin^2 z}{2(z^2+k^2\pi^2)^2}=\frac{z^2(1+k^2\pi^2)}{2(z^6+(z^2+k^2\pi^2)^2)},$$

$$||\xi_\nu^4||_{L^2}^2=\frac{z^2\,\sin^2 z}{2(z^2+k^2\pi^2)}=\frac{z^2(z^2+k^2)}{2(z^6+(z^2+k^2\pi^2)^2)}.$$

We observe that if k remains bounded when $|\nu|\to\infty$ then, necessarily, $|z|\to\infty$. This remark allows us to conclude that

$$||\xi_\nu^1||_{H^1}^2+||\xi_\nu^2||_{L^2}^2\longrightarrow\frac{1}{2}\ \text{and}\ ||\xi_\nu^3||_{H^1}^2+||\xi_\nu^4||_{L^2}^2\longrightarrow 0,\ \text{when}\ \nu\longrightarrow\infty.$$

□

REMARK 12. *One can also see that ν_k^* does not have this property, i.e.:*

$$\liminf_{|\nu_k^*|\to\infty}\frac{||\xi_{\nu_k^*}^3||_{H^1(0,1)}}{||\xi_{\nu_k^*}||_{\mathcal{X}}}\neq 0\ \textit{and}\ \liminf_{|\nu|\to\infty}\frac{||\xi_{\nu_k^*}^4||_{L^2(0,1)}}{||\xi_{\nu_k^*}||_{\mathcal{X}}}\neq 0.$$

The proof of this fact is similar to that of Theorem 2.9 below.

4. The dissipative case. In this section we give the proofs of Theorems 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7, 2.8 and 2.9 which characterize the asymptotic behavior of the eigenvalues and the eigenfunctions of (1.1).

We begin with the proof of Theorem 2.1.

Proof of Theorem 2.1: Suppose first that $k \neq 0$. It is easy to see that the differential operator corresponding to (2.1) has compact resolvent (see [11]). Therefore, the spectrum of (2.1) consists of a sequence of complex eigenvalues $(\lambda_{k,m})_{m\in \mathbb{N}} \cup (\bar{\lambda}_{k,m})_{m\in \mathbb{N}}$ with the property that $\lim_{m\to\infty} |\lambda_{k,m}| = \infty$ and $\lambda_{k,m} \neq 0$ for all $m \in \mathbb{Z}$.

If $k = 0$, the operator has the same properties but the first two eigenvalues $\lambda_{0,0}$ and $\bar{\lambda}_{0,0}$ are equal to zero.

Moreover, since all the elements of the spectrum are eigenvalues of the operator it follows that they are roots of equation (2.2). $\square$

With the change of variable $\sqrt{\left(\frac{\lambda}{k}\right)^2 + \pi^2} = z$ equation (2.2) is reduced to

$$e^{2k\,z} = -\frac{z^2 - \pi^2 - kz^3 - z\sqrt{z^2-\pi^2}}{z^2 - \pi^2 + kz^3 + z\sqrt{z^2-\pi^2}}. \tag{4.1}$$

We present now four technical lemmas which give us the information we need about the poles of the function in the right hand side of (4.1). The proofs of these lemmas will be presented in an appendix at the end of this paper.

Lemma 4.1. *If α is a root of the equation:*

$$z^2 - \pi^2 + kz^3 + z\sqrt{z^2-\pi^2} = 0 \tag{4.2}$$

then, for k large enough, we have:

$$\frac{\pi}{2\sqrt[3]{k}} <| \alpha |< \frac{2\pi}{\sqrt[3]{k}}. \tag{4.3}$$

Lemma 4.2. *For k large enough, the equation (4.2) has three roots α_i, $i = 1, 2, 3$ with the property that:*

$$\left|\alpha_i - \sqrt[3]{\frac{\pi^2}{k}}\omega_i\right| \leq \frac{10}{\sqrt[3]{k^2}}, \tag{4.4}$$

where ω_i, $i = 1, 2, 3$ are the three cubic roots of unity.

Lemma 4.3. *The root α_1 of (4.2) satisfies:*

$$\alpha_1 = \sqrt[3]{\frac{\pi^2}{k}} - \frac{1}{3}\sqrt[3]{\frac{\pi}{k^2}}\, i + o\left(\frac{1}{\sqrt[3]{k^2}}\right), \quad \text{as } k \to \infty. \tag{4.5}$$

Lemma 4.4. *For k large enough, the equation*

$$z^2 - \pi^2 - kz^3 - z\sqrt{z^2-\pi^2} = 0 \tag{4.6}$$

has three roots β_i, $i = 1, 2, 3$ with the property that:

$$\left|\beta_i - \sqrt[3]{\frac{\pi^2}{k}}\tilde{\omega}_i\right| \leq \frac{10}{\sqrt[3]{k^2}}, \tag{4.7}$$

where $\tilde{\omega}_i = -\omega_i\,,\ i = 1,2,3.$

We can pass now to prove Theorems 2.2, 2.3, 2.4, 2.5, 2.6, 2.7, 2.8 and 2.9 . In the first subsection we analyze the case of the eigenvalues with real parts tending to zero, as the frequency increases (Theorems 2.2, 2.3, 2.4, 2.5, 2.6 and 2.7). In the second subsection we prove the existence of eigenvalues with uniformly negative real parts (Theorems 2.8 and 2.9).

4.1. Eigenvalues with real parts tending to zero.

Proof of Theorem 2.2: If we note $\sqrt{\lambda^2 + k^2\pi^2} = \mu$ we obtain that μ satisfies the following equation:

$$e^{2\mu} = -\frac{\mu^2 - k^2\pi^2 - \mu(\mu^2 + \sqrt{\mu^2 - k^2\pi^2})}{\mu^2 - k^2\pi^2 + \mu(\mu^2 + \sqrt{\mu^2 - k^2\pi^2})}. \tag{4.8}$$

We put the equation (4.8) in the form:

$$e^{2\mu} - 1 = -\frac{2(\mu^2 - k^2\pi^2)}{\mu^2 - k^2\pi^2 + \mu(\mu^2 + \sqrt{\mu^2 - k^2\pi^2})} \tag{4.9}$$

and we localize its roots applying Rouché 's Theorem.

In order to do this we consider the functions:

$$f(z) = e^{2z} - 1 \ \text{ and } \ g(z) = -\frac{2(z^2 - k^2\pi^2)}{z^2 - k^2\pi^2 + z(z^2 + \sqrt{z^2 - k^2\pi^2})}.$$

We remark that the equation $f(z) = 0$ has the roots $(\alpha_m)_{m\in\mathbb{Z}}$ with $\alpha_m = m\pi\, i$.

For each $m \in \mathbb{Z} \setminus \{0\}$ we define the square γ_m^1 of center α_m and side $2\varepsilon_m$ and the rectangle γ_m^2 defined by the lines $\mathcal{R}e\, z = \pm\delta_m$ and $\mathcal{I}m\, z = m\pi \pm \frac{3\pi}{4}$. Moreover, we consider the square γ^0 of center 0 and side $2M_k$ (see *Fig. 2*).

The constants ε_m, δ_m and M_k will be chosen in such a way that:

$$|\, f(z)\,| > |\, g(z)\,| \quad \text{for all } z \in \gamma_m^1 \cup \gamma_m^2 \cup \gamma^0. \tag{4.10}$$

First of all we have that, for all $z \in \mathbb{C}$:

$$\begin{aligned} |\, f(z)\,|^2 = |\, e^{2z} - 1\,|^2 = (e^{2\mathcal{R}e\, z} - \cos 2\,\mathcal{I}m\, z)^2 + (\sin 2\,\mathcal{I}m\, z)^2 \geq \\ \geq \max\{|\, e^{2\mathcal{R}e\, z} - 1\,|, |\sin 2\,\mathcal{I}m\, z\,|\}. \end{aligned} \tag{4.11}$$

In order to estimate g we consider the region G_1 of the complex plane defined by:

$$G_1 = \{z \in \mathbb{C} : |\, z\,| > \max\{k\,\pi, 4\}\} \tag{4.12}$$

where $g(z)$ is analytic in view of Lemma 4.2. We deduce that, for all $z \in G_1$:

$$\begin{aligned} |\, g(z)\,| = \left| \frac{2(z^2 - k^2\pi^2)}{z^2 - k^2\pi^2 + z(z^2 + \sqrt{z^2 - k^2\pi^2})} \right| \leq \frac{2}{|\, z\,| \left| \frac{z^2 + \sqrt{z^2 - k^2\pi^2}}{z^2 - k^2\pi^2} \right| - 1} \leq \\ \leq \frac{2}{|\, z\,| \frac{|z|^2 - |\sqrt{z^2 - k^2\pi^2}|}{|z|^2 + k^2\pi^2} - 1} \leq \frac{2}{|\, z\,| \frac{|z|^2 - \sqrt{2}|z|}{|z|^2 + k^2\pi^2} - 1} \leq \frac{2}{\frac{|z|}{4} - 1} \leq \frac{8}{|\, z\,| - 4}. \end{aligned} \tag{4.13}$$

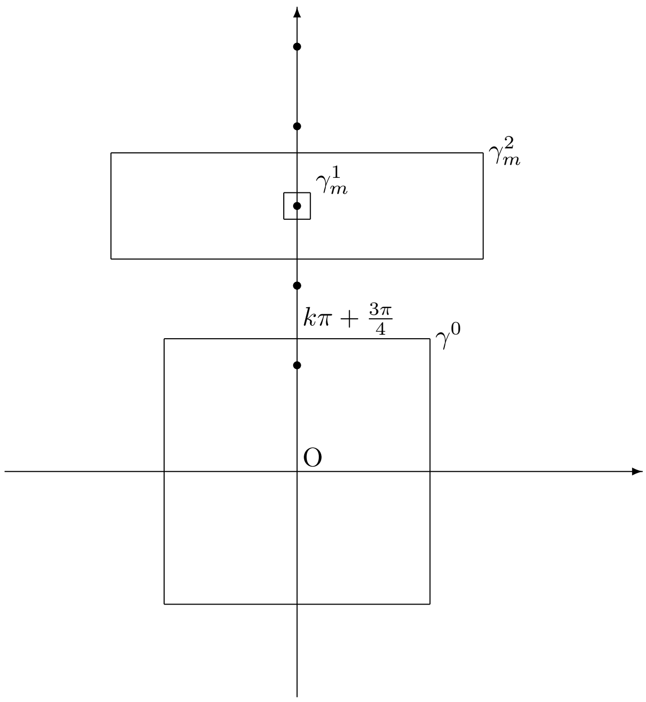

Fig. 2

We are now in conditions to determine the constants ε_m, δ_m and M_k such that (4.10) be satisfied.

If $z \in \gamma_m^1 \cap G_1$ we obtain that $\mid f(z) \mid > \varepsilon_m > \mid g(z) \mid$ if $\dfrac{16}{2m\pi - 9} < \varepsilon_m < \dfrac{1}{2}$.

Applying Rouché's Theorem it turns out that there exists a unique root of the equation (4.8) in each square γ_m^1 if $m \geq k+1$. We denote those roots by $\mu_{k,m}$.

If $z \in \gamma_m^2 \cap G_1$ we obtain that $\mid f(z) \mid > \frac{1}{2} > \mid g(z) \mid$ if $\delta_m > \frac{1}{2}$.

Since we did not impose any upper bound for δ_m we can apply again Rouché's Theorem and we obtain that, for each $m \geq k+1$ in the regions $|\mathcal{I}m\, z - m\pi| \leq \dfrac{3\pi}{4}$ the equation (4.8) has the same number of roots as $f(z) = 0$ does. This implies that the only roots of (4.8) in G_1 are those we found on i).

Finally, if we choose $M_k = k\,\pi + \frac{3\pi}{4}$ we obtain, like above, that, if $z \in \gamma^0 \cap G_1$, then $\mid f(z) \mid > 1/2 > \mid g(z) \mid$.

Applying Rouché's Theorem we deduce that the number of roots of (4.8) in γ^0 is equal to $2k+2$.

In order to obtain the roots of (2.2) we return to the variable λ.

First of all we remark that if λ solves (2.2) then $\bar{\lambda}$ is a solution too. Hence, it is sufficient to look for those λ with $\mathcal{I}m\,\lambda > 0$, the other eigenvalues being conjugates of these. On the other hand, when we pass from μ to λ we are interested in those values which have the property that $\mathcal{R}e\,\lambda < 0$, since the energy of the system decreases as t increases (see (1.2) and (1.3) above and [11] for a detailed discussion on this). Those remarks indicate that we can establish a bijective correspondence between the zeros of the equation (4.8) and those of the equation (2.2).

Since the previous analysis gives us the roots μ of (4.8) with the property that $\mid \mu \mid > \max\{k\pi, 4\}$, we obtain all the roots $\lambda = \sqrt{\mu^2 - k^2\pi^2}$ of (2.2) with the property that $\mid \lambda \mid > \sqrt{2}k\pi$. For those eigenvalues λ with $\mathcal{I}m\,\lambda > 0$ we have:

$$\mid \lambda - \sqrt{m^2+k^2}\,\pi\, i \mid = \mid \sqrt{\mu^2 - k^2\pi^2} - \sqrt{m^2+k^2}\,\pi\, i \mid =$$

$$= \frac{\mid \mu - m\pi\, i \mid \mid \mu + m\pi\, i \mid}{\sqrt{\mid \mathcal{I}m\sqrt{\mu^2-k^2\pi^2} + \sqrt{m^2+k^2}\pi \mid^2 + \mid \mathcal{R}e\sqrt{\mu^2-k^2\pi^2} \mid^2}} \leq$$

$$\leq \frac{\varepsilon_m \mid \mu + m\pi\, i \mid}{\mathcal{I}m\sqrt{\mu^2-k^2\pi^2} + \sqrt{m^2+k^2}\pi} \leq \frac{\varepsilon_m(\varepsilon_m + 2m\pi)}{\sqrt{k^2+m^2}\pi} \leq \frac{3m}{\sqrt{k^2+m^2}}\varepsilon_m.$$

It turns out that, for $m > k+1$, the eigenvalues $\lambda_{k,m}$ with $\mathcal{I}m\,\lambda_{k,m} > 0$ satisfy (2.4). The corresponding result for the case $\lambda_{k,m}$ with $\mathcal{I}m\lambda_{k,m} < 0$ can be obtained in the same way. $\square$

Remark 13. *Theorem 2.2 tells us that, for each $k \in I\!N$, and for each eigenvalue $(\lambda_{k,m})_{m\in\mathbb{Z}^*}$ with $|m| \geq k+1$ the index m is given by the nearest value $\pm\sqrt{k^2+m^2}\,\pi\, i$. The other eigenvalues, which belong to the circle centered in 0 and of radius $\sqrt{2}\,k\,\pi$, are ordered in the increasing way with respect to the modulus: λ_k^*, λ_k^{**}, $\lambda_{k,\pm1}$, $\lambda_{k,\pm2}$,..., $\lambda_{k,\pm k}$ (see Theorems 2.4, 2.6 and 2.8). Hence, for k fixed, λ_k^* and λ_k^{**} are the eigenvalues with the smallest modulus, while the modulus of $\lambda_{k,\pm k}$ approaches $\sqrt{2}k\pi$ when m increases.*

We prove now Theorem 2.3.

Proof of Theorem 2.3: From (2.4) we deduce that $\sqrt{\lambda^2+k^2\pi^2} = \mu = m\pi\, i + \alpha(m)$ with $|\alpha(m)| \leq \frac{1}{m}$.

The eigenfunction φ_λ can be decomposed as follows:

$$\varphi_\lambda = \begin{pmatrix} \frac{1}{\lambda}\cosh\,\sqrt{\lambda^2+k^2\pi^2}(y-1)\,\cos\,k\pi x \\ -\cosh\,\sqrt{\lambda^2+k^2\pi^2}(y-1)\,\cos\,k\pi x \\ -\frac{\sqrt{\lambda^2+k^2\pi^2}}{\lambda^2}\,\sinh\,(\sqrt{\lambda^2+k^2\pi^2})\,\cos\,k\pi x \\ \frac{\sqrt{\lambda^2+k^2\pi^2}}{\lambda}\,\sinh\,(\sqrt{\lambda^2+k^2\pi^2})\,\cos\,k\pi x \end{pmatrix} =$$

$$= \begin{pmatrix} (-1)^{m+1}\frac{i}{\lambda}\cosh\,\alpha(m)(y-1)\,\cos\,m\pi y\,\cos\,k\pi x \\ (-1)^m\, i\cosh\,\alpha(m)(y-1)\,\cos\,m\pi y\,\cos\,k\pi x \\ 0 \\ 0 \end{pmatrix} +$$

$$+\begin{pmatrix} (-1)^m \frac{1}{\lambda} \sinh\,\alpha(m)(y-1)\,\sin\,m\pi y\,\cos\,k\pi x \\ (-1)^{m+1}\,\sinh\,\alpha(m)(y-1)\,\sin\,m\pi y\,\cos\,k\pi x \\ (-1)^{m+1}\frac{\sqrt{\lambda^2+k^2\pi^2}}{\lambda^2}\,\sinh\,\alpha(m)\,\cos\,k\pi x \\ (-1)^m\frac{\sqrt{\lambda^2+k^2\pi^2}}{\lambda}\,\sinh\,\alpha(m)\,\cos\,k\pi x \end{pmatrix}.$$

We denote by φ_1 and φ_2 the two vector valued functions above.
We estimate first the norm of φ_2 in $\mathcal{X}$:

$$\|\varphi_2\|^2_{\mathcal{X}} = \int_0^1\int_0^1 \left\{\left(\left|\frac{1}{\lambda}\,\cos\,k\pi x\right|^2 + \left|\frac{k\pi}{\lambda}\,\sin\,k\pi x\right|^2\right)|\,\sinh\,\alpha(m)(y-1)\,\sin\,m\pi y|^2 + \right.$$

$$\left. + \left|\left(\frac{\alpha(m)}{\lambda}\cosh\alpha(m)(y-1)\sin m\pi y + \frac{m\pi}{\lambda}\sinh\alpha(m)(y-1)\cos m\pi y\right)\,\cos k\pi x\right|^2\right\} +$$

$$+\int_0^1\int_0^1 |\,\sinh\,\alpha(m)(y-1)\,\sin\,m\pi y\,\cos\,k\pi x|^2\,dx\,dy +$$

$$+\int_0^1\left\{\left|\frac{\sqrt{\lambda^2+k^2\pi^2}}{\lambda^2}\,\sinh\alpha(m)\cos k\pi x\right|^2 + \left|\frac{k\pi\sqrt{\lambda^2+k^2\pi^2}}{\lambda^2}\,\sinh\alpha(m)\sin k\pi x\right|^2\right\}dx +$$

$$+\int_0^1\left|\frac{\sqrt{\lambda^2+k^2\pi^2}}{\lambda}\,\sinh\,\alpha(m)\,\cos\,k\pi x\right|^2\,dx \le \int_0^1\left\{\left|\frac{1}{\lambda}\,\sinh\,\alpha(m)(y-1)\right|^2 + \right.$$

$$\left.\left|\frac{\alpha(m)}{\lambda}\,\cosh\,\alpha(m)(y-1)\right|^2 + \left(\frac{(k^2+m^2)\pi^2}{|\lambda|^2}+1\right)|\,\sinh\,\alpha(m)(y-1)|^2\right\}dy +$$

$$+\left(\left|\frac{\sqrt{\lambda^2+k^2\pi^2}}{\lambda^2}\right|^2 + \left|\frac{k\pi\sqrt{\lambda^2+k^2\pi^2}}{\lambda^2}\right|^2 + \left|\frac{\sqrt{\lambda^2+k^2\pi^2}}{\lambda}\right|^2\right)|\,\sinh\,\alpha(m)|^2 \le$$

$$\le 4\frac{|\,\alpha(m)\,|^2}{|\lambda|^2} + 4|\alpha(m)|^2 + 5\frac{|\,\alpha(m)\,|^2}{|\lambda|^2} + 4\frac{(k^2+m^2)\pi^2\ \ |\,\alpha(m)\,|^2}{|\lambda|^2} +$$

$$+4|\alpha(m)|^2\left|\frac{\lambda^2+k^2\pi^2}{\lambda^4}\right|^2\left|1+k^2\pi^2+\lambda^2\right|^2 \le 33|\alpha(m)|^2 \le \frac{c'}{m^2}.$$

where we take into account that $|\sinh\alpha(m)| \leq 2|\alpha(m)|$ and $|\cosh\alpha(m)| \leq 2$.

In this way we obtain that:

$$\|\varphi_2\|_{\mathcal{X}} \leq \frac{c'}{m}. \tag{4.14}$$

We estimate now

$$\|\varphi_1 - \psi_{k,m}\|_{\mathcal{X}} = \int_0^1\int_0^1 \left\{ \left| \frac{i}{\sqrt{m^2+k^2}\pi} + \frac{1}{\lambda}\cosh\alpha(m)(y-1) \right|^2 |\cos m\pi y \cos k\pi x|^2 + \right.$$

$$+ \left| \frac{i}{\sqrt{m^2+k^2}\pi} + \frac{1}{\lambda}\cosh\alpha(m)(y-1) \right|^2 |k\pi \cos m\pi y \sin k\pi x|^2 +$$

$$+ \left| \frac{i}{\sqrt{m^2+k^2}\pi} + \frac{1}{\lambda}\cosh\alpha(m)(y-1) \right|^2 |m\pi \sin m\pi y \cos k\pi x|^2 +$$

$$\left. + \left(\left| \frac{\alpha(m)}{\lambda}\sinh\alpha(m)(y-1) \right|^2 + |1-\cosh\alpha(m)(y-1)|^2 \right) |\cos m\pi y \cos k\pi x|^2 \right\} dxdy \leq$$

$$\leq \int_0^1 \left\{ \left| \frac{i}{\sqrt{m^2+k^2}\pi} + \frac{1}{\lambda}\cosh\alpha(m)(y-1) \right|^2 + \right.$$

$$+ \left| \frac{i(k^2+m^2)\pi^2}{\sqrt{m^2+k^2}\pi} + \frac{(k^2+m^2)\pi^2}{\lambda}\cosh\alpha(m)(y-1) \right|^2 +$$

$$\left. + \left| \frac{\alpha(m)}{\lambda}\sinh\alpha(m)(y-1) \right|^2 + |1-\cosh\alpha(m)(y-1)|^2 \right\} dy \leq$$

$$\leq \left| \frac{i}{\sqrt{m^2+k^2}\pi} + \frac{1}{\lambda} \right|^2 + \int_0^1 \left| \frac{1}{\lambda}(1-\cosh\alpha(m)(y-1)) \right|^2 dy +$$

$$+(k^2+m^2)\pi^2 \left| \frac{i}{\sqrt{m^2+k^2}\pi} + \frac{1}{\lambda} \right|^2 + (m^2+k^2)\pi^2 \int_0^1 \left| \frac{1}{\lambda}(1-\cosh\alpha(m)(y-1)) \right|^2 dy +$$

$$+ \int_0^1 \left| \frac{\alpha(m)}{\lambda}\sinh\alpha(m)(y-1) \right|^2 dy + \int_0^1 |1-\cosh\alpha(m)(y-1)|^2 dy \leq$$

$$\leq \frac{c''}{m^2} + 4|\alpha(m)|^2 + 2\frac{c''}{m^2} + 8|\alpha(m)|^2 + 4|\alpha(m)|^2 + 4|\alpha(m)|^2 \leq \frac{c'''}{m^2}$$

where we take into account that $|1-\cosh\alpha(m)| \leq 2\,|\,\alpha(m)\,|$.

We obtain that:

$$\| \varphi_1 - \psi_{k,m} \|_{\mathcal{X}} \leq \frac{c'''}{m}. \tag{4.15}$$

From estimates (4.14) and (4.15) we deduce that (2.8) holds. $\square$

Next we prove Theorem 2.4 which gives estimations for the eigenvalues $\lambda_{k,\pm1}$, $\lambda_{k,\pm2}$, ..., $\lambda_{k,\pm q}$ for $q = q(k) \leq [\sqrt[3]{k}]$. By $[\,\cdot\,]$ we denote the integer part function.

Proof of Theorem 2.4: If we consider the change of variable $\lambda = \sqrt{k^2 z^2 - k^2 \pi^2}$ the equation (2.2) is transformed in:

$$e^{2k\,z} = -\frac{z^2 - \pi^2 - kz^3 - z\sqrt{z^2 - \pi^2}}{z^2 - \pi^2 + kz^3 + z\sqrt{z^2 - \pi^2}}. \tag{4.16}$$

Let $k \in \mathbb{N}$ be sufficiently large so that Lemma 4.1 holds. We define the functions:

$$f(z) = e^{2kz} + 1, \quad g(z) = \frac{2(kz^3 + z\sqrt{z^2 - \pi^2})}{z^2 - \pi^2 + kz^3 + z\sqrt{z^2 - \pi^2}}.$$

For each integer m with $0 \leq |m| \leq [\sqrt[3]{k}]$ let $\gamma^1_{k,m}$ be the square of center $\dfrac{2m-1}{2k}\pi\, i$ and sides $\dfrac{3\pi}{2k\sqrt[3]{k}}$. For all $z \in \gamma^1_{k,m}$, we have

$$| f(z) | = | e^{2kz} + 1 | \geq \max \{ | e^{2k\,\mathcal{R}e\, z} - 1 |, |\ \sin\ 2k\,\mathcal{I}m\, z\, | \}$$

and since $|e^x - 1| > \dfrac{|x|}{2}$ and $|\sin x| > \dfrac{|x|}{2}$, for small x, we deduce that:

$$| f(z) | \geq \frac{3\pi}{4\sqrt[3]{k}}, \quad \forall z \in \gamma_{k,m}. \tag{4.17}$$

We now estimate g in the region $G^2 = \left\{ z \in \mathbb{C} : | z | \leq \dfrac{\pi}{\sqrt[3]{k^2}} \right\}$.
Lemma 4.1 implies that g is analytic in G^2.
For all $z \in G^2$ we have $|z|\,\sqrt[3]{k} \leq \pi$. Therefore we obtain that

$$\lim_{k\to\infty} kz^2 = \lim_{k\to\infty} z^2 = \lim_{k\to\infty} kz^3 = 0.$$

Hence, for all $z \in G^2$,

$$\lim_{k\to\infty} \left| \frac{kz^2 + \sqrt{z^2 - \pi^2}}{z^2 - \pi^2 + kz^3 + z\sqrt{z^2 - \pi^2}} \right| = \frac{1}{\pi},$$

which implies that, for k sufficiently large:

$$\left| \frac{kz^2 - \sqrt{z^2 - \pi^2}}{z^2 - \pi^2 + kz^3 - z\sqrt{z^2 - \pi^2}} \right| \leq 1, \quad \forall z \in G^2.$$

This result allows us to estimate the function g in G^2:

$$| g(z) | = 2 \mid z \mid \left| \frac{kz^2 + \sqrt{z^2 - \pi^2}}{z^2 - \pi^2 + kz^3 + z\sqrt{z^2 - \pi^2}} \right| \leq 2|z| \leq \frac{2\pi}{\sqrt[3]{k^2}}.$$

Finally, we obtain that $|f(z)| > |g(z)|$ for all $z \in \gamma^1_{k,m}$ if k is sufficiently large and $\gamma^1_{k,m} \subset G^2$. Remark that $\gamma^1_{k,m} \subset G^2$ if $|m| \leq [\sqrt[3]{k}]$.

Applying Rouché's Theorem we deduce that the equation (4.16) has a root $z_{k,m}$ in each square $\gamma^1_{k,m}$ if $|m| \leq [\sqrt[3]{k}]$. This root satisfies:

$$| \, z_{k,m+1} - \frac{1}{2k}(2m+1)\pi \, i \, | \leq \frac{3\sqrt{2}\pi}{4k\sqrt[3]{k}} \leq \frac{2\pi}{k\sqrt[3]{k}} \qquad \text{if } m \geq 0,$$

$$| \, z_{k,m} + \frac{1}{2k}(2m+1)\pi \, i \, | \leq \frac{3\sqrt{2}\pi}{4k\sqrt[3]{k}} \leq \frac{2\pi}{k\sqrt[3]{k}} \qquad \text{if } m < 0.$$

We deduce that the eigenvalues $\lambda_{k,m} = \sqrt{k^2 z^2_{k,m} - k^2\pi^2}$ with $0 < |m| \leq [\sqrt[3]{k}]$ satisfy (2.9). $\square$

Proof of Theorem 2.5: Estimates (2.9) imply that

$$\sqrt{\lambda^2 + k^2\pi^2} = \mu = \frac{2m+1}{2}\pi \, i + \alpha(k) \text{ with } |\alpha(k)| \leq \frac{2\pi}{\sqrt[3]{k}}.$$

We write the eigenfunction φ_λ in the following form:

$$\varphi_\lambda = \begin{pmatrix} \frac{1}{\lambda}\cosh\,(\sqrt{\lambda^2+k^2\pi^2}(y-1))\,\cos\,k\pi x \\ -\cosh\,(\sqrt{\lambda^2+k^2\pi^2}(y-1))\,\cos\,k\pi x \\ -\frac{\sqrt{\lambda^2+k^2\pi^2}}{\lambda^2}\,\sinh\,(\sqrt{\lambda^2+k^2\pi^2})\,\cos\,k\pi x \\ \frac{\sqrt{\lambda^2+k^2\pi^2}}{\lambda}\,\sinh\,(\sqrt{\lambda^2+k^2\pi^2})\,\cos\,k\pi x \end{pmatrix} =$$

$$= \begin{pmatrix} (-1)^m\frac{1}{\lambda}\cosh\,\alpha(k)(y-1)\,\sin\,\frac{2m+1}{2}\pi y\,\cos\,k\pi x \\ (-1)^{m+1}\cosh\,\alpha(k)(y-1)\,\sin\,\frac{2m+1}{2}\pi y\,\cos\,k\pi x \\ 0 \\ 0 \end{pmatrix} +$$

$$+ \begin{pmatrix} (-1)^{m+1}\frac{i}{\lambda}\sinh\,\alpha(k)(y-1)\,\cos\,\frac{2m+1}{2}\pi y\,\cos\,k\pi x \\ (-1)^m i\,\sinh\,\alpha(k)(y-1)\,\cos\,\frac{2m+1}{2}\pi y\,\cos\,k\pi x \\ (-1)^{m+1} i\,\frac{\sqrt{\lambda^2+k^2\pi^2}}{\lambda^2}\,\cosh\,\alpha(k)\,\cos\,k\pi x \\ (-1)^m i\,\frac{\sqrt{\lambda^2+k^2\pi^2}}{\lambda}\,\cosh\,\alpha(k)\,\cos\,k\pi x \end{pmatrix}.$$

Let φ_1 and φ_2 be the vector valued functions appearing in the decomposition of φ_λ above.

We evaluate first the norm of φ_2 in $\mathcal{X}$:

$$\|\varphi_2\|^2_{\mathcal{X}} = \int_0^1\int_0^1 \left|\frac{1}{\lambda}\cos k\pi x\right|^2 \sinh\alpha(k)(y-1)\cos\frac{2m+1}{2}\pi y +$$

$$+\int_0^1\int_0^1\left\{\left|\frac{k\pi}{\lambda}\sinh\alpha(k)(y-1)\cos\frac{2m+1}{2}\pi y\sin k\pi x\right|^2+\right.$$

$$+\left|\frac{\alpha(k)}{\lambda}\cosh\alpha(k)(y-1)\cos\frac{2m+1}{2}\pi y\cos k\pi x+\frac{(2m+1)\pi}{2\lambda}\sinh\alpha(k)(y-1)\times\right.$$

$$\left.\left.\times\sin\frac{2m+1}{2}\pi y\cos k\pi x\right|^2\right\}+\int_0^1\int_0^1\left|\sinh\alpha(k)(y-1)\cos\frac{2m+1}{2}\pi y\cos k\pi x\right|^2+$$

$$+\int_0^1\left\{\left|\frac{\sqrt{\lambda^2+k^2\pi^2}}{\lambda^2}\cosh\alpha(k)\cos k\pi x\right|^2+\left|\frac{k\pi\sqrt{\lambda^2+k^2\pi^2}}{\lambda^2}\cosh\alpha(k)\sin k\pi x\right|^2\right\}+$$

$$+\int_0^1\left|\frac{\sqrt{\lambda^2+k^2\pi^2}}{\lambda}\cosh\alpha(k)\cos k\pi x\right|^2\le\int_0^1\left\{\left|\frac{1}{\lambda}\sinh\alpha(k)(y-1)\right|^2+\right.$$

$$\left.+\left|\frac{\alpha(k)}{\lambda}\cosh\alpha(k)(y-1)\right|^2+\left(\left(k^2+\left(\frac{2m+1}{2}\right)^2\right)\frac{\pi^2}{|\lambda|^2}+1\right)|\sinh\alpha(k)(y-1)|^2\right\}+$$

$$+\left|\frac{\sqrt{\lambda^2+k^2\pi^2}}{\lambda^2}\cosh\alpha(k)\right|^2+\left|\frac{k\pi\sqrt{\lambda^2+k^2\pi^2}}{\lambda^2}\cosh\alpha(k)\right|^2+\left|\frac{\sqrt{\lambda^2+k^2\pi^2}}{\lambda}\cosh\alpha(k)\right|^2\le$$

$$\le 4\frac{|\alpha(k)|^2}{|\lambda|^2}+5\frac{|\alpha(k)|^2}{|\lambda|^2}+\left(k^2+\left(\frac{2m+1}{2}\right)^2\right)\frac{|\alpha(k)|^2\pi^2}{|\lambda|^2}+4|\alpha(k)|^2+$$

$$5(k^2\pi^2+1)\left|\frac{\sqrt{\lambda^2+k^2\pi^2}}{\lambda^2}\right|^2+5\left|\frac{\sqrt{\lambda^2+k^2\pi^2}}{\lambda}\right|^2\le 14|\alpha(k)|^2+60|\alpha(k)|^2\le\frac{c'}{\sqrt[3]{k}},$$

where we take into account that, for k large enough,

$$\left|\frac{\sqrt{\lambda^2+k^2\pi^2}}{\lambda}\right|\le 2|\alpha(k)|,\ |\sinh\alpha(k)|\le 2|\alpha(k)| \text{ and } |\cosh\alpha(k)|\le 5.$$

We obtain that:

$$\|\varphi_2\|_{\mathcal{X}} \leq \frac{c'}{\sqrt[3]{k}}. \tag{4.18}$$

We compute now:

$$\| \varphi_1 - \tilde{\psi}_{k,m} \|_{\mathcal{X}}^2 =$$

$$\int_0^1 \int_0^1 \left| \frac{i}{\sqrt{\left(\frac{2m+1}{2}\right)^2 + k^2\pi}} + \frac{1}{\lambda} \cosh \alpha(k)(y-1) \right|^2 \left| \sin \frac{2m+1}{2}\pi y \cos k\pi x \right|^2 +$$

$$\int_0^1 \int_0^1 \left\{ \left| \frac{i}{\sqrt{\left(\frac{2m+1}{2}\right)^2 + k^2\pi}} + \frac{1}{\lambda} \cosh \alpha(k)(y-1) \right|^2 |k\pi \ \sin \frac{2m+1}{2}\pi y \sin k\pi x|^2 + \right.$$

$$+ \left| \frac{\alpha(k)}{\lambda} \sinh \alpha(k)(y-1) \ \sin \frac{2m+1}{2}\pi y \cos k\pi x \right|^2 +$$

$$+ \left| \left(\frac{i}{\sqrt{\left(\frac{2m+1}{2}\right)^2 + k^2\pi}} + \frac{1}{\lambda} \cosh \alpha(k)(y-1) \right) \frac{2m+1}{2}\pi \cos \frac{2m+1}{2}\pi y \cos k\pi x \right|^2 +$$

$$\left. + \left| (1 - \cosh \alpha(k)(y-1)) \sin \frac{2m+1}{2}\pi y \cos k\pi x \right|^2 \right\} dxdy \leq$$

$$\leq \int_0^1 \left(\left(\frac{(2m+1)\pi}{2} \right)^2 + k^2\pi^2 + 1 \right) \left| \frac{i}{\sqrt{\left(\frac{2m+1}{2}\right)^2 + k^2\pi}} + \frac{1}{\lambda} \cosh \alpha(k)(y-1) \right|^2 +$$

$$+ \int_0^1 \left| \frac{\alpha(m)}{\lambda} \sinh \alpha(m)(y-1) \right|^2 + |1 - \cosh \alpha(k)(y-1)|^2 \leq$$

$$\leq \left(\left(\frac{(2m+1)\pi}{2} \right)^2 + k^2\pi^2 + 1 \right) \left| \frac{i}{\sqrt{\left(\frac{(2m+1)}{2}\right)^2 + k^2\pi}} + \frac{1}{\lambda} \right|^2 +$$

$$+ \left(\left(\frac{2m+1}{2} \right)^2 \pi^2 + k^2\pi^2 + 1 \right) \int_0^1 \left| \frac{1}{\lambda} (1 - \cosh \alpha(k)(y-1)) \right|^2 +$$

$$+\int_0^1 \left|\frac{\alpha(k)}{\lambda} \sinh \alpha(k)(y-1)\right|^2 + \int_0^1 |1- \cosh \alpha(m)(y-1)|^2 \leq$$

$$\leq \frac{2\pi}{\sqrt[3]{k}} + 4|\alpha(k)|^2 + 2\frac{2\pi}{\sqrt[3]{k}} + 8|\alpha(m)|^2 + 4|\alpha(k)|^2 + 4|\alpha(k)|^2 \leq \frac{c''}{\sqrt[3]{k}}$$

since, for k large enough ($k > (2\pi)^3$), $|1- \cosh \alpha(k)| \leq 2 \mid \alpha(k) \mid$.

We obtain that:

$$\| \varphi_1 - \tilde{\psi}_{k,m} \|_{\mathcal{X}} \leq \frac{c'''}{\sqrt[3]{k}}. \tag{4.19}$$

The estimates (4.18) and (4.19) imply that (2.12) holds. □

We pass now to the analysis of the roots of (2.2) $\lambda_{k,\pm(q+1)}$, $\lambda_{k,\pm(q+2)}$, ..., $\lambda_{k,\pm k}$, with $q = [\sqrt[3]{k}]$, which make the transition from the eigenvalues studied in Theorem 2.2 to those studied in Theorem 2.4. First we prove the following Lemma:

LEMMA 4.5. *For each $k \in \mathbb{N}^*$, the equation*

$$e^{2k\,z} = \frac{\pi^2 + kz^3}{-\pi^2 + kz^3} \tag{4.20}$$

has a sequence of roots $\pm\zeta_{k,m}\, i$, $m \in \mathbb{N}^*$, where $\zeta_{k,m} \in \mathbb{R}_+$ is the positive root of the equation (2.14) which belongs to $\left(\frac{m}{k}\pi, \frac{2m+1}{2k}\pi\right)$.

Proof: We look for roots of (4.20) of the form $z = \zeta\, i$. Hence, ζ is a root of the equation:

$$e^{2k\,\zeta\, i} = \frac{\pi^2 - k\,\zeta^3\, i}{-\pi^2 - k\,\zeta^3\, i}. \tag{4.21}$$

Consequently, z is a root of (4.20) if ζ satisfies:

$$-\pi^2 \cos k\zeta + k\zeta^3 \sin k\zeta = 0 \tag{4.22}$$

which is equivalent to (2.14).

It is easy to see that (4.22) has a zero in each interval $\left(\frac{m}{k}\pi, \frac{2m+1}{2k}\pi\right)$ that we denote by $\zeta_{k,m}$. □

We pass now to study the eigenvalues $\lambda_{k,\pm([\sqrt[3]{k}]+1)}$, $\lambda_{k,\pm([\sqrt[3]{k}]+2)}$, ..., $\lambda_{k,\pm k}$.

Proof of Theorem 2.6: We saw that the change of variables $\lambda = \sqrt{k^2z^2 - k^2\pi^2}$ transforms (2.2) in (4.16).

We define the region of the complex plane

$$G^3 = \left\{ z \in \mathbb{C}\colon \frac{\pi}{2\sqrt[3]{k^2}} \leq |z| \leq 2\pi, \quad |\mathcal{R}e\, z| \leq \frac{1}{k}, \quad \mathcal{I}m\, z > 0 \right\}$$

and we prove that (4.16) has a set of zeros $z_{k,m}$ in G^3 satisfying the estimate

$$|z_{k,m} - \zeta_{k,m}\, i| \leq \frac{1}{k\sqrt[5]{k}}, \quad m \in \{[\sqrt[3]{k}] + 1, ..., k\}, \tag{4.23}$$

where $\zeta_{k,m}$ are the zeros of (2.14).

We remark that, if $m \in \{[\sqrt[3]{k}]+1, ..., k\}$, then $\zeta_{k,m}$ belongs to G^3.

We write (4.16) in the following form:

$$e^{2k\,z} - \frac{\pi^2 + kz^3}{-\pi^2 + kz^3} = -\frac{2z(kz^4 + \pi^2\sqrt{z^2 - \pi^2})}{(-\pi^2 + kz^3)(z^2 - \pi^2 + kz^3 + z\sqrt{z^2 - \pi^2})},$$

and applying Rouché's Theorem we prove that the zeros of (4.16) approach to those of (4.20).

We consider first the function:

$$g(z) = -\frac{2z(kz^4 + \pi^2\sqrt{z^2 - \pi^2})}{(-\pi^2 + kz^3)(z^2 - \pi^2 + kz^3 + z\sqrt{z^2 - \pi^2})}$$

and we obtain an upper bound for g in G^3.

To do this we evaluate first the denominator of g:

$$|2z(kz^4 + \pi^2\sqrt{z^2 - \pi^2})| \leq 2k|z|^5 + 2\pi^2|z|^2 + 4\pi^3|z|.$$

We obtain that:

$$|2z(kz^4 + \pi^2\sqrt{z^2 - \pi^2})| \leq \begin{cases} 6k|z|^5, \text{ if } \dfrac{\pi}{2\sqrt[3]{k^2}} \leq |z| \leq \dfrac{\pi}{\sqrt[4]{k}}, \\ 6\pi^3|z|, \text{ if } \dfrac{\pi}{\sqrt[4]{k}} \leq |z| \leq 2\pi \end{cases}$$

We estimate now the numerator of g:

$$|(-\pi^2 + kz^3)(z^2 - \pi^2 + kz^3 + z\sqrt{z^2 - \pi^2})| \geq$$

$$\geq |-\pi^2 + kz^3|\,(|-\pi^2 + kz^3| - |z|(|z| + \sqrt{|z|^2 + \pi^2})) \geq$$

$$\geq |-\pi^2 + kz^3|^2 \,-\, 5\pi|z|\,|-\pi^2 + kz^3|.$$

If $\dfrac{\pi}{2\sqrt[3]{k^2}} \leq |z| \leq \dfrac{\pi}{\sqrt[4]{k}}$ and $|\mathcal{R}e\, z| \leq \frac{1}{k}$ we have that, for k sufficiently large:

$$|-\pi^2 + kz^3| \geq \mathcal{R}e\,(-\pi^2 + kz^3) \geq \frac{\pi}{2}.$$

If $\dfrac{\pi}{\sqrt[4]{k}} \leq |z| \leq 2\pi$ we have that:

$$|-\pi^2 + kz^3|\,(|-\pi^2 + kz^3| - |z|(|z| + \sqrt{|z|^2 + \pi^2})) \geq \sqrt{k}k|z|^4(\sqrt{k}|z|^2 - \frac{2\pi}{\sqrt{k}}).$$

From the last two inequalities, we deduce that, for k sufficiently large, the following estimate holds:

$$|(-\pi^2 + kz^3)(z^2 - \pi^2 + kz^3 + z\sqrt{z^2 - \pi^2})| \geq \begin{cases} c_1, \text{ if } \dfrac{\pi}{2\sqrt[3]{k^2}} \leq |z| \leq \dfrac{\pi}{\sqrt[4]{k}}, \\ c_2\sqrt{k}\,k\,|z|^4, \text{ if } \dfrac{\pi}{\sqrt[4]{k}} \leq |z| \leq 2\pi \end{cases}$$

where c_1 and c_2 are two positive constants which do not depend on k.

Going back to the function g we obtain that, for k sufficiently large,

$$|g(z)| \leq \frac{c}{\sqrt[4]{k}}, \qquad \text{for all } z \text{ in } G^3,$$

where c is a positive constant which does not depend on k.

We study now the function:

$$f(z) = e^{2k\,z} - \frac{\pi^2 + kz^3}{-\pi^2 + kz^3}.$$

For each $m \in I\!N^*$ we consider the circle $\gamma^2_{k,m}$ of center $\zeta_{k,m}\,i$ and radius $r_{k,m} = \dfrac{1}{k\sqrt[5]{k}}$ and the circle $\widehat{\gamma}^2_{k,m}$ with the same center but with radius $R_{k,m} = \dfrac{1}{k}$.

In G^3 the function f is analytic. Applying Taylor's formula at $\zeta_{k,m}\,i$ we obtain that:

$$\begin{aligned} f(z) &= f(\zeta_{k,m}\,i) + (z - \zeta_{k,m}\,i) f'(\zeta_{k,m}\,i) \\ &+ \frac{(z-\zeta_{k,m}\,i)^2}{2\pi\, i} \int_{\widehat{\gamma}^2_{k,m}} \frac{f(\zeta)\,d\zeta}{(\zeta-\zeta_{k,m}\,i)^2(\zeta - z)}. \end{aligned} \tag{4.24}$$

We look for an upper bound for the error term on the circumference $\gamma^2_{k,m}$. We have

$$\left| \frac{(z-\zeta_{k,m}\,i)^2}{2\pi\, i} \int_{\widehat{\gamma}^2_{k,m}} \frac{f(\zeta)\,d\zeta}{(\zeta-\zeta_{k,m}\,i)^2(\zeta - z)} \right| \leq \frac{r^2_{k,m}}{2\pi} \frac{2\pi R_{k,m} M}{R^2_{k,m}(R_{k,m} - r_{k,m})} = \frac{M}{\sqrt[5]{k}(\sqrt[5]{k}-1)}$$

where M is an upper bound for f on the circumference $\widehat{\gamma}^2_{k,m}$.

On the other hand

$$|f(z)| = \left| e^{2k\,z} - \frac{\pi^2 + kz^3}{-\pi^2 + kz^3} \right| \leq |e^{2k\,z}| + \left| \frac{\pi^2 + kz^3}{-\pi^2 + kz^3} \right| \leq e^{2k\,|\mathcal{R}e\,z|} + 1 + \frac{2\pi^2}{|\pi^2 - k\,z^3|}.$$

Since $|\mathcal{R}e\,z| < \dfrac{1}{k}$ in G^3, we obtain that $|\pi^2 - k\,z^3| > 1$ and $|f(z)| < M = e^2 + 1 + 2\pi^2$. Therefore the error term in Taylor's formula on $\gamma^2_{k,m}$ satisfies:

$$\left| \frac{(z-\zeta_{k,m}\,i)^2}{2\pi\, i} \int_{\widehat{\gamma}^2_{k,m}} \frac{f(\zeta)\,d\zeta}{(\zeta-\zeta_{k,m}\,i)^2(\zeta - z)} \right| \leq \frac{M}{\sqrt[5]{k}(\sqrt[5]{k}-1)}.$$

On the other hand,

$$|(z-\zeta_{k,m}\,i)f'(\zeta_{k,m}\,i)| = r_{k,m} \left| 2k \frac{\pi^2 - k\zeta^3_{k,m}\,i}{-\pi^2 - k\,\zeta^3_{k,m}\,i} - \frac{6\pi^2 k \zeta^2_{k,m}}{(-\pi^2 - k\,\zeta^3_{k,m}\,i)^2} \right| \geq 2kr_{k,m}.$$

Going back to Taylor's formula (4.24), we deduce that, if z belongs to the circumference $\gamma^2_{k,m}$, then:

$$|f(z)| \geq |(z-\zeta_{k,m}\,i)f'(\zeta_{k,m}\,i)| - \left| \frac{(z-\zeta_{k,m}\,i)^2}{2\pi\, i} \int_{\widehat{\gamma}^2_{k,m}} \frac{f(\zeta)\,d\zeta}{(\zeta-\zeta_{k,m}\,i)^2(\zeta - z)} \right| \geq$$

$$\geq 2kr_{k,m} - \frac{20}{\sqrt[5]{k}(\sqrt[5]{k}-1)} \geq \frac{C}{\sqrt[5]{k}}.$$

Finally, we obtain that $|f(z)| > |g(z)|$ for all z in $\gamma^2_{k,m}$.

Applying Rouché's Theorem we deduce that the equation (4.16) has a unique zero $z_{k,m}$ which satisfies (4.23) in each circle $\gamma^2_{k,m}$.

Taking into account that $\lambda = \sqrt{k^2\pi^2 + k^2 z^2}$ we deduce immediately the desired result. $\square$

Proof of Theorem 2.7: The eigenvalues $\lambda_{k,m}$ studied in Theorem 2.6 approach to $\sqrt{\pi^2\, k^2 + k^2\, \zeta^2_{k,m}}\, i$, where $\zeta_{k,m}$ are the roots of the equation:

$$\tan\, k\,\zeta \;= \frac{\pi^2}{k\,\zeta^3}. \tag{4.25}$$

By a similar method one can prove that $\lambda_{k,m}$ satisfy the estimates

$$\begin{aligned} &\left|\lambda_{k,m} - \sqrt{\pi^2\, k^2 + k^2\, \varrho^2_{k,m}}\, i\right| \leq \frac{1}{\sqrt[5]{k}} \text{ if } \mathcal{I}m\, \lambda_{k,m} > 0,\ (k \geq m > [\sqrt[3]{k}]),\\ &\left|\lambda_{k,m} + \sqrt{\pi^2\, k^2 + k^2\, \varrho^2_{k,m}}\, i\right| \leq \frac{1}{\sqrt[5]{k}} \text{ if } \mathcal{I}m\, \lambda_{k,m} < 0,\ (-k \leq m < -[\sqrt[3]{k}]), \end{aligned} \tag{4.26}$$

where $\varrho_{k,m}$ is the root of the equation:

$$\tan k\, \varrho \;= \frac{\pi^2 + \varrho^2}{k\varrho^3} \tag{4.27}$$

which belongs to the interval $\left(\frac{m}{k}\pi, \frac{2m+1}{2k}\pi\right)$.

Taking into account the estimates of Theorem 3.1 for the eigenvalues $\nu_{k,m}$ of the conservative problem, we deduce that, for the eigenvalues $\lambda_{k,m}$ studied in Theorem 2.6, we have:

$$|\lambda_{k,m} - \nu_{k,m}| \leq \frac{1}{\sqrt[5]{k}} \text{ for } [\sqrt[3]{k}] < |m| \leq k. \tag{4.28}$$

Since the eigenfunctions $\varphi_{\lambda_{k,m}}$ and $\xi_{\nu_{k,m}}$ have the same form, we deduce that:

$$||\varphi_{\lambda_{k,m}} - \xi_{\nu_{k,m}}||_{\mathcal{X}} \leq \frac{1}{\sqrt[5]{k}}.$$

The properties of the eigenfunctions $\varphi_{\lambda_{k,m}}$ are obtained from the corresponding properties of $\xi_{\nu_{k,m}}$ (see Theorem 3.2). $\square$

4.2. Eigenvalues with uniform negative real parts. The eigenvalues obtained in Theorems 2.2, 2.4 y 2.6 have in common the fact that their real parts tend to zero when the modulus increases. On the other hand, the last two components of the corresponding eigenfunctions vanish asymptotically.

Next we prove that there exists a sequence of eigenvalues $(\lambda^*_k)_k$ of modulus less than $k\pi$ with completely different properties.

Proof of Theorem 2.8: We consider again equation (4.16) and we look for the roots with real part going to infinity.

In the circle δ_1 of center $\sqrt[3]{\frac{\pi^2}{k}}$ and radius $\frac{10}{\sqrt[3]{k^2}}$ the function $h(z) = z^2 - \pi^2 - kz^3 - z\sqrt{z^2 - \pi^2}$ does not vanish (the three roots of this function are $\sqrt[3]{\frac{\pi^2}{k}}\tilde{\omega}_i$, where $\tilde{\omega}_i$ are the cubic roots of -1 as we saw in Lemma 4.3).

We write the equation (4.16) in the form:

$$e^{-2k\,z} = -\frac{z^2 - \pi^2 + kz^3 + z\sqrt{z^2 - \pi^2}}{z^2 - \pi^2 - kz^3 - z\sqrt{z^2 - \pi^2}}. \tag{4.29}$$

If z belongs to the circle δ_1 we have that $\mathcal{R}e\, z > \frac{\pi}{2\sqrt[3]{k}}$ and hence:

$$\left|e^{-2kz}\right| = e^{-2k\mathcal{R}e\, z} \leq e^{-2k\frac{\pi}{2\sqrt[3]{k}}} = e^{-\pi\sqrt[3]{k^2}}.$$

We consider now the circle $\mathcal{C}'$ centered in α_1 and of radius $\frac{1}{k^2}$ (see *Fig. 3*).

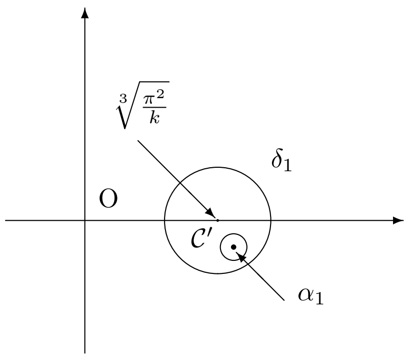

Fig. 3

Since the circle $\mathcal{C}'$ is contained in δ_1 we have that:

$$\left|e^{-2kz}\right| =\leq e^{-2k\frac{\pi}{2\sqrt[3]{k}}} = e^{-\pi\sqrt[3]{k^2}}, \quad \forall z \in \mathcal{C}'. \tag{4.30}$$

In $\mathcal{C}'$ the function $u(z) = z^2 - \pi^2 + kz^3 + z\sqrt{z^2 - \pi^2}$ is analytic and it has a unique zero α_1.

Since

$$|u'(\alpha_1)| \geq |3k\alpha_1^2| - \left(\left|\sqrt{\alpha_1^2 - \pi^2}\right| + |\alpha_1|\left|2 + \frac{\alpha_1}{\sqrt{\alpha_1^2 - \pi^2}}\right|\right) >$$

$$> 3k\frac{\pi^2}{4\sqrt[3]{k^2}} - \left(|\alpha_1| + \pi + |\alpha_1|\left|2 + \frac{\alpha_1}{\sqrt{\alpha_1^2 - \pi^2}}\right|\right) > \sqrt[4]{k},$$

for k sufficiently large, applying Taylor's Theorem we obtain

$$|u(z) - u'(\alpha_1)(z - \alpha_1)| \leq a|z - \alpha_1|^2$$

where a is a constant depending on k.

Nevertheless, we have that $|a| \leq \sup \{|u''(z)| \,:\, z \in \mathcal{C}'\} < k$.

We obtain that, if z belongs to the circumference of $\mathcal{C}'$,

$$|u(z)| \geq |u'(\alpha_1)||z - \alpha_1| - a|z - \alpha_1|^2 > \frac{1}{k^2}.$$

Hence:

$$\left| -\frac{z^2 - \pi^2 + kz^3 + z\sqrt{z^2 - \pi^2}}{z^2 - \pi^2 - kz^3 - z\sqrt{z^2 - \pi^2}} \right| \geq \frac{|u(z)|}{|\, kz^3 + \pi^2 \,| + |z||z - \sqrt{z^2 - \pi^2}|} \geq \frac{1}{k^3}.$$

Thus, for k sufficiently large and z on the circumference of $\mathcal{C}'$, we have:

$$\left| e^{-2k\,z} \right| < \left| \frac{z^2 - \pi^2 + kz^3 + z\sqrt{z^2 - \pi^2}}{z^2 - \pi^2 - kz^3 - z\sqrt{z^2 - \pi^2}} \right|.$$

Applying Rouché's Theorem we deduce that the equation (4.29) has a unique root z_k^* in $\mathcal{C}'$. Remark that, if z_k^* is a root of (4.29), then $z_k^{**} = \bar{z}_k^*$, $-z_k^*$ and $-z_k^{**}$ are roots of this equation too.

Since $z_k^* \in \mathcal{C}'$ it follows that $z_k^* = \alpha_1 + \mathcal{O}\left(\frac{1}{k^2}\right)$. Hence, Lemma 4.3 ensures that:

$$z_k^* = \sqrt[3]{\frac{\pi^2}{k}} - \frac{1}{3}\sqrt[3]{\frac{\pi}{k^2}}\, i + o\left(\frac{1}{\sqrt[3]{k^2}}\right). \tag{4.31}$$

We go back to the equation (2.2) and we obtain two roots λ_k^* and λ_k^{**} setting $\lambda_k^* = \sqrt{k^2(z_k^*)^2 - k^2\pi^2}$ and $\lambda_k^{**} = \sqrt{k^2(z_k^{**})^2 - k^2\pi^2}$.

We have:

$$\left| \lambda_k^* - \sqrt{k^2(\alpha_1)^2 - k^2\pi^2} \right| = \frac{\left|(\lambda_k^*)^2 - (k^2(\alpha_1)^2 - k^2\pi^2)\right|}{\left|\lambda_k^* + \sqrt{k^2(\alpha_1)^2 - k^2\pi^2}\right|} = \frac{\left|k^2(z_k^*)^2 - k^2(\alpha_1)^2)\right|}{\left|\lambda_k^* + \sqrt{k^2(\alpha_1)^2 - k^2\pi^2}\right|} =$$

$$= \frac{k^2\,|(z_k^* - \alpha_1)(z_k^* + \alpha_1)|}{\left|\lambda_k^* + \sqrt{k^2(\alpha_1)^2 - k^2\pi^2}\right|} \leq \frac{|z_k^* + \alpha_1|}{k\left|\sqrt{(z_k^*)^2 - \pi^2} + \sqrt{(\alpha_1)^2 - \pi^2}\right|} \leq \frac{1}{|k|}.$$

A similar result is obtained for λ_k^{**}.

We now prove (2.19). Remark first that if $\zeta = \sqrt{a + b\,i}$, $\quad a, b \in I\!R$ then $(\mathcal{R}e\,\zeta)^2 = \frac{1}{2}(a + \sqrt{a^2 + b^2})$. We deduce that:

$$\begin{aligned}(\mathcal{R}e\,\lambda_k^*)^2 = \frac{1}{2}\Big(&-k^2\pi^2 + k^2((\mathcal{R}e\,z_k^*)^2 - (\mathcal{I}m\,z_k^*)^2) + \\ &+\sqrt{\left(-k^2\pi^2 + k^2((\mathcal{R}e\,z_k^*)^2 - (\mathcal{I}m\,z_k^*)^2)\right)^2 + \left(2k^2\mathcal{R}e\;z_k^*\,\mathcal{I}m\,z_k^*\right)^2}\Big).\end{aligned} \tag{4.32}$$

Since z_k^* satisfies (4.31) we deduce from the relation (4.32) that:

$$(\mathcal{R}e\,\lambda_k^*)^2 = \frac{1}{2}\left(-k^2\pi^2 + k^2((\mathcal{R}e\,z_k^*)^2 - (\mathcal{I}m\,z_k^*)^2) +\right.$$

$$+\sqrt{(-k^2\pi^2+k^2((\mathcal{R}e\, z_k^*)^2-(\mathcal{I}m\, z_k^*)^2))^2+(2k^2\mathcal{R}e\, z_k^*\,\mathcal{I}m\, z_k^*)^2}\Bigg)=$$

$$=2k^4(\mathcal{R}e\, z_k^*\,\mathcal{I}m\, z_k^*)^2\Bigg[k^2\pi^2-k^2((\mathcal{R}e\, z_k^*)^2-(\mathcal{I}m\, z_k^*)^2)+$$

$$+\sqrt{(-k^2\pi^2+k^2((\mathcal{R}e\, z_k^*)^2-(\mathcal{I}m\, z_k^*)^2))^2+(2k^2\mathcal{R}e\, z_k^*\,\mathcal{I}m\, z_k^*)^2}\ \Bigg]^{-1}.$$

Finally, taking into account the asymptotic expression for z_k^*, (4.31), we obtain that (2.19) holds. $\square$

Proof of Theorem 2.9: i) The weak convergence of $\{\varphi_{\lambda_k^*}\}_k$ is a direct consequence of the equation they satisfy.

ii) We prove first that $\{\varphi^3_{\lambda_k^*}\}_k$ does not tend strongly to zero in $H^1(0,1)$. We have:

$$||\varphi^3_{\lambda_k^*}||_{H^1(0,1)}=\frac{1}{|\lambda_k^*|^2}\left(\int_0^1|\cos\ k\pi x|^2+\int_0^1|k\pi\sin\ k\pi x|^2\right)=\frac{1+k^2\pi^2}{2|\lambda_k^*|^2}.$$

Since $(\lambda_k^*)^2=-k^2\pi^2+k^2\alpha_1+\mathcal{O}(k)=-k^2\pi^2+\mathcal{O}(k)$ we obtain that $\varphi^3_{\lambda_k^*}$ does not tend to zero in $H^1(0,1)$. Evidently, $\varphi^4_{\lambda_k^*}$ does not tend to zero in $L^2(0,1)$.

We pass now to the study of $\varphi^1_{\lambda_k^*}$. We evaluate first the expression:

$$|a_k|^2=\left|\frac{1}{\sqrt{(\lambda_k^*)^2+k^2\pi^2}\ \sinh\ (\sqrt{(\lambda_k^*)^2+k^2\pi^2})}\right|^2=$$

$$=\frac{1}{|(\lambda_k^*)^2+k^2\pi^2|\ \left(|\sinh\ \mathcal{R}e\sqrt{(\lambda_k^*)^2+k^2\pi^2}|^2+|\sin\ \mathcal{I}m\sqrt{(\lambda_k^*)^2+k^2\pi^2}|^2\right)}.$$

Now,

$$||\cosh\ (\sqrt{(\lambda_k^*)^2+k^2\pi^2}(y-1))\ \cos\ k\pi x||^2_{H^1(\Omega)}=$$

$$=\frac{1}{2}\int_0^1\Bigg(|\cosh\ (\sqrt{(\lambda_k^*)^2+k^2\pi^2}(y-1))|^2+k^2\pi^2|\cosh\ (\sqrt{(\lambda_k^*)^2+k^2\pi^2}(y-1))|^2+$$

$$+((\lambda_k^*)^2+k^2\pi^2)|\sinh\ (\sqrt{(\lambda_k^*)^2+k^2\pi^2}(y-1))|^2\Bigg)=$$

$$=\frac{1}{4}\int_0^1\Bigg((-|(\lambda_k^*)^2+k^2\pi^2|+k^2\pi^2+1)\cos\ (\mathcal{I}m\sqrt{(\lambda_k^*)^2+k^2\pi^2}(y-1))+$$

$$+(|(\lambda_k^*)^2+k^2\pi^2|+k^2\pi^2+1)\sinh\ (2\mathcal{R}e\sqrt{(\lambda_k^*)^2+k^2\pi^2}(y-1))\Bigg)=$$

$$\frac{(-|(\lambda_k^*)^2+k^2\pi^2|+k^2\pi^2+1)\ \sin\ 2\mathcal{I}m\sqrt{(\lambda_k^*)^2+k^2\pi^2}}{8\mathcal{I}m\sqrt{(\lambda_k^*)^2+k^2\pi^2}}+$$

$$+\frac{(|(\lambda_k^*)^2+k^2\pi^2|+k^2\pi^2+1)\ \sinh\ 2\mathcal{R}e\sqrt{(\lambda_k^*)^2+k^2\pi^2}}{8\mathcal{R}e\sqrt{(\lambda_k^*)^2+k^2\pi^2}}.$$

Taking into account that $\sqrt{(\lambda_k^*)^2+k^2\pi^2}=kz_k^*=\sqrt[3]{k^2\pi^2}-\frac{1}{3}\sqrt[3]{k\pi}\,i+o(\sqrt[3]{k})$ we obtain that

$$||\varphi^1_{\lambda_k^*}||^2_{H^1(\Omega)}\longrightarrow\frac{\sqrt[3]{\pi^2}}{4}.$$

Similarly it turns out that $||\varphi^2_{\lambda_k^*}||_{L^2(\Omega)}$ does not tend to zero. □

5. A non compactness result. The following result is a direct application of the existence of a sequence of eigenvalues with modulus tending to infinity and uniformly negative real parts.

It is well known that, in the context of one-dimensional hybrid systems, the dissipative term is often a compact perturbation of the differential operator associated to the corresponding conservative system. This argument was used to prove that the decay rate of the energy of those systems is not uniform (see [18]). Nevertheless, in our case, this kind of arguments can not be used since the dissipative term $(0,0,0,W_t)$ is, at least apparently, a bounded but not compact perturbation of the conservative operator. It is natural to study whether this term produces a compact perturbation of the underlying conservative system.

A way to do this consists in analyzing whether the difference between the semigroup generated by the conservative operator and the semigroup generated by the dissipative one is compact or not. The existence of the sequence $(\lambda_k^*)_k$ of eigenvalues implies that the answer is negative.

THEOREM 5.1. *Let $\{S_D(t)\}_{t\geq 0}$ be the semigroup generated by the dissipative operator and let $\{S_C(t)\}_{t\geq 0}$ be the semigroup generated by the conservative system. Then, for all $t>0$, the difference $(S_D-S_C)(t)$ is not a compact operator in $\mathcal{X}$.*

Proof: Suppose that there exists $t_0>0$ such that $(S_D-S_C)(t_0)$ is compact. Theorem 2.9 implies that there exists a sequence of eigenfunctions $\{\varphi_{\lambda_k^*}\}_k$, corresponding to the eigenvalues λ_k^*, which converges weakly to zero in $\mathcal{X}$. So:

$$||(S_C(t_0)-S_D(t_0))\varphi_{\lambda_k^*}||_{\mathcal{X}}\longrightarrow 0\text{ when }k\longrightarrow\infty.$$

Since $\varphi_{\lambda_k^*}$ is an eigenfunction of the dissipative problem we have that:

$$S_D(t_0)\varphi_{\lambda_k^*}=e^{\lambda_k^*t_0}\varphi_{\lambda_k^*}.$$

Hence,

$$||S_C(t_0)\varphi_{\lambda_k^*}-e^{\lambda_k^*t_0}\varphi_{\lambda_k^*}||_{\mathcal{X}}\longrightarrow 0\text{ when }k\longrightarrow\infty. \tag{5.1}$$

Since the conservative operator generates a group of isometries we get that

$$||S_C(t_0)\varphi_{\lambda_k^*}||_{\mathcal{X}}=||\varphi_{\lambda_k^*}||_{\mathcal{X}},$$

and therefore

$$(5.2)\quad ||\varphi_{\lambda_k^*}||_{\mathcal{X}} = ||S_C(t_0)\varphi_{\lambda_k^*}||_{\mathcal{X}} \leq ||S_C(t_0)\varphi_{\lambda_k^*} - e^{\lambda_k^* t_0}\varphi_{\lambda_k^*}||_{\mathcal{X}} + \left|e^{\lambda_k^* t_0}\right| ||\varphi_{\lambda_k^*}||_{\mathcal{X}}.$$

In view of Theorem 2.8 we have that the sequence $(\lambda_k^*)_k$ has the property that $\mathcal{R}e\,\lambda_k^* \to -\frac{1}{3}$, when $k \to \infty$ and hence, there exists $k_1 \in I\!N$ such that $\mathcal{R}e\,\lambda_k^* < -\frac{1}{4}$ for all $k > k_1$.

We deduce that, for all $t > 0$, there exists a constant ε, depending on t but independent of k, such that:

$$\left|e^{\lambda_k^* t}\right| = e^{\mathcal{R}e\,\lambda_k^* t} < 1 - \varepsilon.$$

Let us take $t = t_0$ in the last equality. Going back to (5.2), we obtain:

$$\varepsilon ||\varphi_{\lambda_k^*}||_{\mathcal{X}} \leq ||S_C(t_0)\varphi_{\lambda_k^*} - e^{\lambda_k^* t_0}\varphi_{\lambda_k^*}||_{\mathcal{X}}. \tag{5.3}$$

Remark that (5.1) and (5.3) imply that $||\varphi_{\lambda_k^*}||_{\mathcal{X}}$ goes to zero when $k \to \infty$ and this is a contradiction with the result of Theorem 2.9.

Finally, we obtain that $(S_D - S_C)(t)$ is not a compact operator fot any $t > 0$. □

Remark 14. *In order to compare the noncompactness result of Theorem 5.1 for our 2-d case with analoguous 1-d models we consider the following problem (see [18]):*

$$\begin{cases} u_{tt} - u_{xx} = 0, & x \in (0,1), \quad t > 0, \\ u(t,0) = 0, & t > 0, \\ u_{tt}(t,1) + u_t(t,1) = -u_x(t,1), & t > 0. \end{cases} \tag{5.4}$$

This is a "string-mass" model since it couples the vibrations of a string with a rigid body at the end $x = 1$ (see [9] and [18]).

The natural energy space corresponding to (5.4) is

$$\mathcal{Z} = V \times L^2(0,1) \times I\!R,$$

where $V = \left\{v \in H^1(0,1) : v(0) = 0\right\}$.

Observe that, if we define the energy of a solution u of (5.4) by

$$E(t) = \frac{1}{2}\int_0^1 \left(|u_t|^2 + |u_x|^2\right)\,dx + \frac{1}{2}|u_t(t,1)|^2, \tag{5.5}$$

we obtain that

$$\frac{dE}{dt}(t) = -(u_t(t,1))^2 \leq 0. \tag{5.6}$$

Therefore we are dealing with a dissipative hybrid system, $u_t(t,1)$, in the last relation of (5.4), being the damping term.

Let us now consider the vector valued unknown $U = (u, u_t, u(\cdot,1))$ and write equation (5.4) in the following abstract form:

$$\begin{cases} U_t + A_D(U) = 0, \quad t > 0, \\ U(0) = U_0. \end{cases} \tag{5.7}$$

The operator A_D in (5.7) is an unbounded operator in $\mathcal{Z}$ defined by

$$\begin{gathered}\mathcal{D} := \mathcal{D}(A_D) = \{U \in \mathcal{Z} : A_D(U) \in \mathcal{Z}\} = \\ = \left\{U = (u, v, p) \in H^2(0,1) \cap V \times V \times I\!R : u(1) = p\right\}, \\ A_D(u, v, p) = (-v, -u_{xx}, v(1) + u_x(1)).\end{gathered}$$

It is easy to show that $(\mathcal{D}, A_D)$ is a maximal monotone operator in $\mathcal{Z}$.
Let us now consider the projection operator

$$B : \mathcal{Z} \to \mathcal{Z}, \quad B(u, v, p) = (0, 0, p).$$

Observe that B is a compact operator in $\mathcal{Z}$ and $A_C = A_D - B$ is the conservative operator corresponding to (5.4).

Let $\{T_D(t)\}_{t\geq 0}$ be the strongly continuous semigroup generated by the dissipative operator A_D and let $\{T_C(t)\}_{t\geq 0}$ be the strongly continuous semigroup generated by the conservative operator A_C.

For (5.4) all the eigenvalues of the operator A_D approach the imaginary axis when the frequency increases. This is one of the consequences of the fact that A_D is obtained from A_C by a compact perturbation B. In the case of our system (1.1) this is not the case; the perturbation term is only bounded in the energy space. This is one of the major differences between one and two-dimensional hybrid systems.

Moreover, since B is a compact operator, it can be shown that, for all $t \geq 0$, the difference $(T_D - T_C)(t)$ is a compact operator in $\mathcal{Z}$.

6. Comments. Our results indicate that the interaction between the fluid and structure in this type of models is very weak at high frequencies. As a consequence of this, if we try to change the dynamics of the system acting only on the string located on Γ_0, we have to impose very restrictive conditions on the data of the system. This explains the results obtained in the context of the controllability of these systems and concerning the existence of periodic solutions (see [12] and [13]).

The weak interaction of the string and the fluid is a consequence of both the hybrid structure of the system and of the localization of the string in a relatively small part of the boundary of Ω.

In [11] we analyze a slightly different model in which the domain Ω is a ball of $I\!R^2$ and the dissipation acts on the whole boundary. We prove that the energy does not decay uniformly. This clearly shows that the very weak interaction between fluid and structure at high frecuencies is due to the hybrid structure of the system.

From our study the property of completeness of the eigenfuncions of the differential operator associated to (1.1) is easy to prove. The question of whether these eigenfunctions form a Riesz basis is open (for the notions of completness and Riesz basis see [6]). For the one-dimensional systems, obtained by separation of variables fixing the number of oscillations in the x-variable, we can prove that the eigenfunctions do form a Riesz basis. However our estimates are not enough to give an answer to this question in the context of the two-dimensional problem.

We also remark that we have been able to obtain very precise informations about the eigenvalues because we had the explicit equation they satisfy. We got this equation by separation of variables, which was possible since we considered Neumann boundary conditions for the string. The analysis in the case of Dirichlet boundary conditions for the string is much more difficult. Partial results, like the non uniform decay of the energy of the system, were obtained in [11] (see also [14]).

The analysis of the rate of decay of low frequencies is a relevant problem for applications. Obviously, the techniques developed in this paper do not allow to answer to this question. This problem requires different approaches.

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7. Appendix. We present here the proofs of Lemmas 4.1, 4.2, 4.3 and 4.4.

Proof of Lemma 4.1: If α is a root of (4.2), then:

$$| k || \alpha |^3 = | \alpha^2 - \pi^2 + \alpha\sqrt{\alpha^2 - \pi^2} | \leq 2 | \alpha |^2 + \pi | \alpha | + \pi^2 \leq \max \{4 | \alpha |^2, 4\pi^2\}.$$

We obtain that:

$$| \alpha | \leq \max \{\frac{4}{k}, \sqrt[3]{\frac{4\pi^2}{k}}\} < \frac{2\pi}{\sqrt[3]{k}} \text{ for all } k \geq 1. \tag{7.1}$$

On the other hand we have:

$$| k || \alpha |^3 = | \alpha^2 - \pi^2 + \alpha\sqrt{\alpha^2-\pi^2} | = \frac{\pi^2 | \alpha^2 - \pi^2 |}{| \alpha^2 - \pi^2 - \alpha\sqrt{\alpha^2-\pi^2} |} \geq$$

$$\geq \frac{\pi^2 | \alpha^2 - \pi^2 |}{| \alpha |^2 + \pi^2 + | \alpha | \sqrt{| \alpha |^2 + \pi^2}} | \geq \frac{\pi^2(\pi^2 - | \alpha |^2)}{| \alpha |^2 + \pi^2 + | \alpha | (| \alpha | + \pi)}.$$

In view of (7.1) we obtain that, if $k > 8\pi^3$, then:

$$k | \alpha |^3 > \frac{\pi^2(\pi^2-1)}{2+\pi^2+\pi} > \frac{\pi^3}{8}.$$

□

Proof of Lemma 4.2: We study the relation between the roots of (4.2) and those of the equation:

$$kz^3 - \pi^2 = 0. \tag{7.2}$$

The last equation has three roots $a_i = \sqrt[3]{\frac{\pi^2}{k}}\omega_i$, $i = 1,2,3$, where ω_i are the three cubic roots of unity.

We consider the functions $u(z) = kz^3 - \pi^2$ and $v(z) = z\sqrt{z^2-\pi^2} + z^2$ defined in the circle δ_0 of center 0 and radius $\frac{2\pi}{\sqrt[3]{k}}$, where both are analytic.

In the circle δ_0 we have:

$$| v(z) | = | z\sqrt{z^2-\pi^2} + z^2 | \leq | z | (\sqrt{| z |^2 + \pi^2} + | z |) \leq \frac{10\pi^2}{\sqrt[3]{k}}$$

and hence

$$| v(z) | < \frac{10\pi^2}{\sqrt[3]{k}} \text{ if } | z | \leq \frac{2\pi}{\sqrt[3]{k}}. \tag{7.3}$$

On the other hand,

$$| u(z) | = | kz^3 - \pi^2 | = k \left| z - \sqrt[3]{\frac{\pi^2}{k}}\omega_1 \right| \left| z - \sqrt[3]{\frac{\pi^2}{k}}\omega_2 \right| \left| z - \sqrt[3]{\frac{\pi^2}{k}}\omega_3 \right|.$$

If z belongs to the circumference δ_0 we have that

$$\left| z - \sqrt[3]{\frac{\pi^2}{k}}\omega_i \right| \geq |z| - \left|\sqrt[3]{\frac{\pi^2}{k}}\omega_i\right| = \frac{2\pi}{\sqrt[3]{k}} - \sqrt[3]{\frac{\pi^2}{k}} > \sqrt[3]{\frac{\pi^3}{k}}, \quad i = 1,2,3.$$

Hence

$$| u(z) | > \pi^3 \text{ if } | z | = \frac{2\pi}{\sqrt[3]{k}}. \tag{7.4}$$

The inequalities (7.3) and (7.4) imply that $| u(z) | > | v(z) |$, for all z on the circumference δ_0.

Applying Rouché's Theorem we obtain that (4.2) has the same number of roots as (7.2) in the circle δ_0. It follows that (4.2) has three roots which satisfy (4.3). The inequality (7.3) is still valid in δ_i, $i = 1, 2, 3$. On the other hand, for all z on the circumference δ_i:

$$| u(z) | = | kz^3 - \pi^2 | = k \mid z - \sqrt[3]{\frac{\pi^2}{k}}\omega_1 \mid \mid z - \sqrt[3]{\frac{\pi^2}{k}}\omega_2 \mid \mid z - \sqrt[3]{\frac{\pi^2}{k}}\omega_3 \mid >$$

$$> k\frac{10}{\sqrt[3]{k^2}}\left(\frac{\pi}{\sqrt[3]{k}}\right)^2 = \frac{10\pi^2}{\sqrt[3]{k}}.$$

Applying Rouché's Theorem we deduce that the roots of (4.2) are located in the circles δ_i and the estimate (4.5) holds. □

Proof of Lemma 4.3: Step 1: We prove first that the equation:

$$-\pi^2 + kz^3 + \pi z\, i = 0 \tag{7.5}$$

has a unique solution p_k in the circle δ_1 of center $\sqrt[3]{\frac{\pi^2}{k}}$ and radius $\frac{10}{\sqrt[3]{k^2}}$ and hence:

$$p_k = \sqrt[3]{\frac{\pi^2}{k}} - \frac{1}{3}\sqrt[3]{\frac{\pi}{k^2}}\, i + o\left(\frac{1}{\sqrt[3]{k^2}}\right).$$

The existence of the root p_k in δ_1 follows applying the estimates obtained in Lemma 4.2 to the functions $u(z) = -\pi^2 + kz^3$ and $v(z) = \pi z\, i$.

We define now $r_k = p_k - \sqrt[3]{\frac{\pi^2}{k}}$ and we deduce that r_k satisfies:

$$kr_k^3 + 3kr_k^2\sqrt[3]{\frac{\pi^2}{k}} + 3kr_k\sqrt[3]{\frac{\pi^4}{k^2}} + \pi r_k\, i + \pi\sqrt[3]{\frac{\pi^2}{k}}\, i = 0.$$

Multiplying the last equation by $\sqrt[3]{k}$ we deduce that:

$$\sqrt[3]{k}\left(3kr_k\sqrt[3]{\frac{\pi^4}{k^2}} + \pi\sqrt[3]{\frac{\pi^2}{k}}\, i\right) = \sqrt[3]{k}\left(-kr_k^3 - 3kr_k^2\sqrt[3]{\frac{\pi^2}{k}} - \pi r_k\, i\right).$$

Since $|r_k| = \left|p_k - \sqrt[3]{\frac{\pi^2}{k}}\right| \leq \frac{10}{\sqrt[3]{k^2}}$ we have that $r_k = -\frac{1}{3}\sqrt[3]{\frac{\pi}{k^2}}\, i + o\left(\frac{1}{\sqrt[3]{k^2}}\right)$.

Hence, $p_k = \sqrt[3]{\frac{\pi^2}{k}} - \frac{1}{3}\sqrt[3]{\frac{\pi}{k^2}}\, i + o\left(\frac{1}{\sqrt[3]{k^2}}\right)$.

Step 2: We prove now that the root α_1 of (4.2) belongs to the circle $\mathcal{C}$ centered in p_k and of radius $s_k = \frac{1}{\sqrt[4]{k^3}}$ (see *Fig.* 4). This implies immediately that α_1 satisfies (4.5).

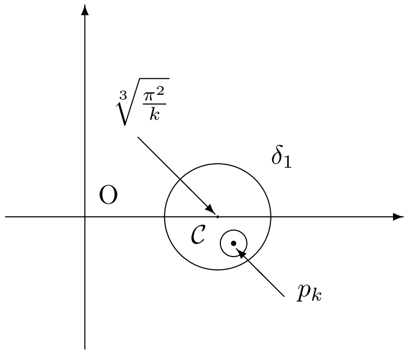

Fig. 4

We use again Rouché's Theorem considering the functions:

$$u(z) = -\pi^2 + kz^3 + \pi z\, i, \quad v(z) = -z^2 - z\sqrt{z^2 - \pi^2} + \pi z\, i.$$

For z in δ_1 we have:

$$|v(z)| = |z|^2 \left| -1 - \frac{z}{\sqrt{z^2 - \pi^2} + \pi\, i} \right| \leq 2|z|^2 \leq \frac{100}{\sqrt[3]{k^2}}.$$

On the other hand, applying Taylor's formula in the point p_k, we get

$$u(z) = u'(p_k)(z - p_k) - \frac{(z - p_k)^2}{2\pi\, i} \int_{\widehat{\gamma}} \frac{u(\zeta)\, d\zeta}{(\zeta - p_k)^2(\zeta - z)},$$

where $\widehat{\gamma}$ is the circle of center p_k and radius $S_k = \dfrac{1}{\sqrt[3]{k}}$.

We estimate first the quantity

$$\left| \frac{(z - p_k)^2}{2\pi\, i} \int_{\widehat{\gamma}} \frac{u(\zeta) d\,\zeta}{(\zeta - p_k)^2(\zeta - z)} \right| \leq \frac{s_k^2}{2\pi} \, \frac{M}{S_k^2(S_k - s_k)} \, 2\pi S_k \, \leq 2M \sqrt[12]{\frac{1}{k^{10}}},$$

where M is an upper bound for u in $\widehat{\gamma}$.

On the other hand

$$|z - p_k|\, |u'(p_k)| = s_k |3kp_k^2 + \pi\, i| \geq s_k\, (3k|p_k|^2 - \pi) \geq \frac{1}{\sqrt[4]{k^3}} (3k \sqrt[3]{\frac{\pi^4}{k^2}} - \pi) \geq \frac{1}{2} \sqrt[12]{\frac{1}{k^5}}.$$

We obtain that, for k sufficiently large and z on the circumference $\mathcal{C}$:

$$|u(z)| > |z - p_k|\, |u'(p_k)| - 2M \sqrt[12]{\frac{1}{k^{10}}} \geq \frac{1}{2} \sqrt[12]{\frac{1}{k^5}} - 2M \sqrt[12]{\frac{1}{k^{10}}} > \frac{1}{4 \sqrt[12]{k^5}} > \frac{100}{\sqrt[3]{k^2}}.$$

It results that, for k sufficiently large, $|u(z)| > |v(z)|$ on the circumference of $\mathcal{C}$. Applying Rouché's Theorem we deduce that α_1 satisfies (4.5). □

Proof of Lemma 4.4: We simply remark that, making $z = -s$, the equation (4.6) is transformed into (4.2). □