BOUNDARY CONTROLLABILITY OF A LINEAR HYBRID SYSTEM ARISING IN THE CONTROL OF NOISE

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Abstract. We consider a simple model arising in the control of noise. We assume that the two-dimensional cavity $\Omega = (0,1)\times(0,1)$ is occupied by an elastic, inviscid, compressible fluid. The potential ϕ of the velocity field satisfies the linear wave equation. The boundary of Ω is divided in two parts Γ_0 and Γ_1. The first one, Γ_0 is flexible and occupied by a vibrating string that obeys to the one-dimensional wave equation. On Γ_0 the continuity of the normal velocities of the fluid and the string is imposed. The subset Γ_1 of the boundary is assumed to be rigid and therefore, the normal velocity of the fluid vanishes. This constitutes a conservative system of two coupled wave equations in dimensions two and one respectively.

The control (an elastic force or an exterior source of noise) is assumed to act on the flexible part of the boundary. We are interested on the controllability problem: Given a large enough control time, what are the initial conditions we can drive to the equilibrium by means of, say, L^2- controls ? By using Fourier series the problem is decomposed into an infinite number of one-dimensional control problems that we solve by classical methods that combine HUM, multiplier techniques and Ingham type inequalities. Putting these one-dimensional results together we give a precise characterization of the space of controllable data in terms of Fourier series.

Key words. boundary control, hyperbolic system, aeromechanic structure interaction.

AMS subject classifications. 35B37, 93C20, 73K70.

1. Introduction.

Let Ω be the two-dimensional square $\Omega = (0,1)\times(0,1) \subset I\!\!R^2$.

We assume that Ω is filled with an elastic, inviscid, compressible fluid whose velocity field $\vec{v}$ is given by the potential $\phi = \phi(x,y,t) : \vec{v} = \nabla\phi$. By linearization we assume that the potential ϕ satisfies the linear wave equation in $\Omega\times(0,\infty)$.

The boundary $\Gamma = \partial\Omega$ of Ω is divided in two parts: $\Gamma_0 = \{(x,0) : x \in (0,1)\}$ and $\Gamma_1 = \Gamma\backslash\Gamma_0$. The subset Γ_1 is assumed to be rigid and we impose zero normal velocity of the fluid on it. The subset Γ_0 is supposed to be flexible and occupied by a flexible string that vibrates under the pressure of the fluid on the plane where Ω lies. The displacement of Γ_0, described by the scalar function $W = W(x,t)$, obeys the one-dimensional wave equation. On the other hand, on Γ_0 we impose the continuity of the normal velocities of the fluid and the string. The string is assumed to satisfy Neumann boundary conditions on its extremes. All deformations are supposed to be small enough so that linear theory applies. Under natural initial conditions for ϕ and W the linear motion of this system is described by means of the following

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coupled wave equations:

$$
(1.1) \qquad \begin{cases} \phi_{tt} - \Delta\phi = 0 & \text{in} \quad \Omega \times (0,\infty) \\ \frac{\partial \phi}{\partial \nu} = 0 & \text{on} \quad \Gamma_1 \times (0,\infty) \\ \frac{\partial \phi}{\partial y} = -W_t & \text{on} \quad \Gamma_0 \times (0,\infty) \\ W_{tt} - W_{xx} + \phi_t = 0 & \text{on} \quad \Gamma_0 \times (0,\infty) \\ W_x(0,t) = W_x(1,t) = 0 & \text{for} \quad t > 0 \\ \phi(0) = \phi^0, \phi_t(0) = \phi^1 & \text{in} \quad \Omega \\ W(0) = W^0, W_t(0) = W^1 & \text{on} \quad \Gamma_0. \end{cases}
$$

By ν we denote the unit outward normal to Ω.

In (1.1) we have chosen to take the various parameters of the system to be equal to one.

The system (1.1) is well-posed in the energy space $\mathcal{X} = H^1(\Omega) \times L^2(\Omega) \times H^1(\Gamma_0) \times L^2(\Gamma_0)$ for the variables (ϕ, ϕ_t, W, W_t). The energy

$$
(1.2) \qquad E(t) = \frac{1}{2} \int_\Omega \left[| \nabla\phi |^2 + | \phi_t |^2 \right] dxdy + \frac{1}{2} \int_{\Gamma_0} \left[| W_x |^2 + | W_t |^2 \right] dx
$$

remains constant along trajectories.

We study the controllability of system (1.1) under the action of an exterior force or source of noise on the flexible part of the boundary Γ_0. The control is given by a scalar function $\beta = \beta(x,t)$, and the controlled system reads as follows:

$$
(1.3) \qquad \begin{cases} \phi_{tt} - \Delta\phi = 0 & \text{in} \quad \Omega \times (0,\infty) \\ \frac{\partial \phi}{\partial \nu} = 0 & \text{on} \quad \Gamma_1 \times (0,\infty) \\ \frac{\partial \phi}{\partial y} = -W_t & \text{on} \quad \Gamma_0 \times (0,\infty) \\ W_{tt} - W_{xx} + \phi_t = \beta & \text{on} \quad \Gamma_0 \times (0,\infty) \\ W_x(0,t) = W_x(1,t) = 0 & \text{for} \quad t > 0 \\ \phi(0) = \phi^0, \phi_t(0) = \phi^1 & \text{in} \quad \Omega \\ W(0) = W^0, W_t(0) = W^1 & \text{on} \quad \Gamma_0. \end{cases}
$$

It is easy to see that the equilibria of these systems are of the form

$$
(1.4) \qquad (\phi, \phi_t, W, W_t) = (c_1, 0, c_2, 0),
$$

c_1 and c_2 being constant functions.

In view of the finite speep of propagation of the wave equation satisfied by ϕ, the geometry of Ω and the support of the control β (the subset Γ_0 of the boundary of Ω) the minimal controllability time for system (1.3) is $T_0 = 2$.

We choose the control β to be in the space $H^{-2}(0,T;L^2(\Gamma_0))$. Of course this is an arbitrary choice and many others make sense. However this is the most natural one when solving the control problem by means of J. L. Lions' HUM (see [10]), as we will do.

The problem of controllability can be formulated as follows: Given $T > 2$, find the space of initial data $(\phi^0, \phi^1, W^0, W^1)$ that can be driven to an equilibrium of the form (1.4) in time T by means of a suitable control $\beta \in H^{-2}(0,T;L^2(\Gamma_0))$.

The control set Γ_0 does not satisfy the necessary geometric conditions for controllability given by Bardos, Lebeau and Rauch in [6]. Indeed, any segment of the form $\{(x,\ell) : x \in (0,1)\}$ with $0 < \ell < 1$, constitutes a ray of geometric optics that never intersects the control region Γ_0. Therefore, we can not expect the space of controllable initial data to be an energy space.

In this paper we give a complete characterization of the controllable space in terms of Fourier series. This space consists on initial data whose Fourier coefficients, roughly, decay exponentially as the frequency increases.

The Fourier analysis of the system is possible because of the boundary conditions we have chosen for W. Indeed, W is assumed to satisfy Neumann type boundary conditions which are compatible with those of ϕ to develop solutions in Fourier series.

Indeed, let us decompose the control β, the solutions ϕ, W and the initial data in the following way

$$\left\{\begin{array}{l} \beta = \sum_{n=0}^{\infty} \beta_n(t)\cos(n\pi x), \\ \Phi = \sum_{n=0}^{\infty} \psi_n(y,t)\cos(n\pi x),\ (\phi^0, \phi^1) = \sum_{n=0}^{\infty} (\psi_n^0(y), \psi_n^1(y))\cos(n\pi x), \\ W = \sum_{n=0}^{\infty} V_n(t)\cos(n\pi x),\ (W^0, W^1) = \sum_{n=0}^{\infty} (V_n^0, V_n^1)\cos(n\pi x). \end{array}\right. \tag{1.5}$$

With this decomposition, system (1.3) can be split into the following sequence of one-dimensional controlled systems for $n = 0, 1, \ldots$:

$$\left\{\begin{array}{lll} \psi_{n,tt} - \psi_{n,yy} + n^2\pi^2\psi_n = 0 & \text{for} & (y,t) \in (0,1)\times(0,\infty) \\ \psi_{n,y}(1,t) = 0 & \text{for} & t > 0 \\ \psi_{n,y}(0,t) = -V_t(t) & \text{for} & t > 0 \\ V_{n,tt}(t) + n^2\pi^2 V_n(t) + \psi_{n,t}(0,t) = \beta_n(t) & \text{for} & t > 0 \\ \psi_n(0) = \psi_n^0, \psi_{n,t}(0) = \psi_n^1 & \text{in} & (0,1) \\ V_n(0) = V_n^0, V_{n,t}(0) = V_n^1. & & \end{array}\right. \tag{1.6}$$

First we will study the controllability of system (1.6) by using classical methods that combine HUM, multiplier techniques and Ingham type inequalities (see [9] and [8]). Combining these one-dimensional results with the Fourier decomposition (1.5), the controllability result for system (1.3) will be proved. Although the techniques we use are well known the obtention of sharp estimates for the controls requires the use of them in a rather refined way.

The control β we obtain is of the form $\beta = \frac{\partial^2}{\partial t^2}\gamma$, with $\gamma \in L^2(\Gamma_0 \times (0,T))$ having compact support in time. Therefore $\int_0^T \beta = 0$. Taking this fact into account it is easy to see that the constants c_1, c_2 of the equilibrium we reach at time $t = T$ are determined a priori by the initial data. Indeed, integrating the first equation of (1.3) in Ω we obtain that $\int_\Omega \phi_t dxdy - \int_{\Gamma_0} W dx$ remains constant in time. Therefore, necessarily,

$$c_2 = \int_{\Gamma_0} W^0 dx - \int_\Omega \phi^1 dxdy. \tag{1.7}$$

On the other hand, integrating the equation satisfied by W on $\Gamma_0 \times (0,T)$ and taking into accont that $\int_0^T \beta = 0$ we deduce that

$$\int_{\Gamma_0} W_t(T)dx + \int_{\Gamma_0} \phi(x,0,T)dx = \int_{\Gamma_0} W^1 dx + \int_{\Gamma_0} \phi^0(x,0)dx$$

and therefore

$$c_1 = \int_{\Gamma_0} \left(W^1 + \phi^0(x, 0) \right) dx. \tag{1.8}$$

In terms of the Fourier coefficients (1.5) these constants can be written in the following way:

$$c_1 = V_0^1 + \psi_0^0(0), \quad c_2 = V_0^0 - \int_0^1 \psi_0^1(y) dy. \tag{1.9}$$

Therefore, the constants c_1 and c_2 of the equilibrium we may reach are uniquely determined by the Fourier coefficients of the initial data corresponding to the frequency $n = 0$ in the x-variable.

This fact is related to the different nature of systems (1.6) for $n = 0$ and $n \geq 1$. While for any $n \geq 1$ system (1.6) is exactly controllable to zero at any time $T > 2$, when $n = 0$ we can only control the system to the equilibrium given by (1.9) in terms of the initial data.

The system under consideration can be viewed as a hybrid system coupling a fluid with an elastic structure. From a mathematical point of view the system couples a two-dimensional wave equation with a one-dimensional one. This type of systems is rather common when studying the vibrations of structures connecting several flexible bodies of different dimensions. Examples of this type can be found, for instance, in [11], [7] and [16]. However in all these cases the coupling is of a different nature since the continuity of displacements is imposed, and not the continuity of normal velocities.

The model under consideration is inspired in and related to that of H. T. Banks et al. in [5]. However, there are some important difference between these two models. In [5] the flexible part of the boundary Γ_0 is occupied by a flexible damped beam instead of a flexible string. But the main difference is related to the nature of the controls. In [5] the control acts on the system through a finite number of piezoceramic patches located on Γ_0. This restricts very much the set of admissible controls, that are essentially second derivatives of Heaviside functions, and much weaker controllability results have to be expected. In [5] the controllability problem is not addressed. Instead, they consider a quadratic optimal control problem. More recently in [3] a Riccati equation for the optimal control is derived. The problem of the controllability of one-dimensional beams with piezoelectric actuators has been succesfully addressed by M. Tucsnak [17]. However, to our knowledge, there are no rigorous results on the controllability of fluid-structure systems under such controls. To our knowledge the present paper represents the first attempt to solve the controllability problem for the two dimensional system although, as we said above, we do not address the problem in which the control is made through piezoelectric patches.

The authors in [13] have addressed the problem of the feedback stabilization of system (1.3) with a damping term concentrated on Γ_0. The results in [13] show that, in such a situation, every trajectory converges towards an equilibrium as time goes to infinity but that the decay rate is not uniform. A more detailled discussion on the lack of uniform decay can be found in [12]. More recently, in [2], the system introduced in [5] has been considered in which the condition $\frac{\partial \Phi}{\partial \nu} = -W_t$ on the continuity of the velocity fields has been replaced by a dissipative condition of the form $\frac{\partial \Phi}{\partial \nu} = -W_t + \Phi_t$. In [2] it is proved that when Ω is a general smooth bounded domain and the subset Γ_0 of the boundary is sufficiently large (in the spirit of the geometric conditions arising in the boundary stabilization of the wave equation), then the energy decays uniformly to zero. In [13] the existence of periodic solutions of this dissipative system on the presence of a periodic source of noise acting on the system through the flexible part of the boundary is considered too. Due to the very weak effect that the damping located on Γ_0 has on the fluid inside Ω, in order to guarantee the existence of such periodic solutions of finite energy, the exterior source of noise has to be assumed to belong to a rather small class of functions with rapidly decreasing Fourier coefficients. In this sense, this result is very close to the controllability one we present in this paper. For a detailled discussion see [12].

The rest of the paper is organized as follows. In section 2 we present rigorously the main results of this paper and make a discussion on their optimality. In section 3 we address the one-dimensional control problem (1.6). First, distinguishing the cases $n = 0$ and $n \geq 1$, we derive the necessary observability inequalities. Then, applying HUM, the one-dimensional controllability result is deduced. In section 4, combining the results of the previous one, we derive the controllability result for system (1.3).

In an Appendix at the end of the paper we give a detailled proof of a Ingham type inequality that provides explicit estimates of the constants appearing in it.

Acknowledgements. This work is part of the Doctoral dissertation of Sorin Micu at Universidad Complutense de Madrid in February 1996. Most of this work was done while this author was visiting Universidad Complutense with the financial support of the European Tempus Program "Matarou". The authors acknowledge the responsables of this Program and, in particular, Doina Cioranescu for their continuous support.

2. The main results: statements and discussion. As we said in the introduction the controllability problem of system (1.3) is reduced to study the one-parameter family of one-dimensional systems (1.6). When $n \geq 1$ we have the following controllability result for (1.6):

THEOREM 2.1. *Let $\mathcal{Y}$ be the space $H^1(0,1) \times L^2(0,1) \times I\!R \times I\!R$. Assume that $T > 2$ and $n \geq 1$. Then, for any $(\psi^1, \psi^0, V^1, V^0) \in \mathcal{Y}$, there exists a control $\beta \in H^{-2}(0,T)$ with compact support such that the solution (ψ, V) of (1.6) satisfies*

$$\psi(T) = \psi_t(T) \equiv 0 \text{ in } (0,1), \quad V(T) = V_t(T) = 0. \tag{2.1}$$

REMARK 1. *In the statement of Theorem 2.1 and in the sequel we drop the index n from the unknowns (ψ, V) to simplify the notation.*

The solution (ψ, V) is defined by transposition. Therefore (2.1) has to be understood in a suitable weak sense. We will return to this question in the proof of the theorem.

The proof of Theorem 2.1 provides the continuous dependence of the control β on the initial data. More precisely

$$\|\beta\|^2_{H^{-2}(0,T)} \leq C_n \left\{ \|(\psi^1, \psi^0, V^1, V^0)\|^2_{\mathcal{Y}'} + \mid \psi^0(0) \mid^2 \right\} \tag{2.2}$$

for any initial data $(\psi^0, \psi^1, V^0, V^1)$ as in the statement of Theorem 2.1. By $\mathcal{Y}'$ we denote the dual of the space $\mathcal{Y}$. The constant C_n in (2.2) will be evaluated in the next section (see also Remark 4).

As we said in the introduction, when $n = 0$ one can not expect the same cotrollability result due to the conservation of the quantities (1.9) along the trajectories. In this case the controllability result reads as follows:

THEOREM 2.2. *Assume that $T > 2$ and $n = 0$. Then, for any $\left(\psi^1, \psi^0, V^1, V^0\right) \in \mathcal{Y}$ there exists a control $\beta \in H^{-2}(0,T)$ with compact support such that the solution (ψ, V) of (1.6) satisfies:*

$$\psi(T) = V^1 + \psi^0(0), \psi_t(T) = 0 \text{ in } (0,1),\ V(T) = V^0 - \int_0^1 \psi^1 dy, V_t(T) = 0. \tag{2.3}$$

REMARK 2. *This result asserts that, when $n = 0$, any solution of (1.6) can be driven to an equilibrium configuration which is a priori determined by the initial data.*

Let us now state the controllability results for the two-dimensional system (1.3).

We use the Fourier decomposition method described in the Introduction. Thus we develop the initial data $\left(\phi^0, \phi^1, W^0, W^1\right)$ to be controlled in Fourier series:

$$\left\{ \begin{array}{ll} \phi^0 = \sum_{n=0}^{\infty} \psi_n^0(y)\cos(n\pi x), & \phi^1 = \sum_{n=0}^{\infty} \psi_n^1(y)\cos(n\pi x) \\ W^0 = \sum_{n=0}^{\infty} V_n^0 \cos(n\pi x), & W^1 = \sum_{n=0}^{\infty} V_n^1 \cos(n\pi x). \end{array} \right. \tag{2.4}$$

We assume that for every $n = 0, 1, \ldots$ the initial data satisfy the assumptions of Theorem 2.1 and Theorem 2.2. We set

$$\rho_n^0 = \psi_n^0, \rho_n^1 = -\psi_n^1 + V_n^0 \delta_0, \mu^0 = -V_n^0, \mu_n^1 = V_n^1 + \psi_n^0(0). \tag{2.5}$$

We introduce the following space of initial data:

$$\begin{array}{rl} H = & \left\{ \left(\phi^0, \phi^1, W^0, W^1\right) \in \mathcal{X} : \right. \\ & \left. : \sum_{n=0}^{\infty} C_n \left\| \left(\rho_n^1, \rho_n^0, \mu_n^1, \mu_n^0\right) \right\|_{\mathcal{Y}'}^2 = \left\| \left(\phi^0, \phi^1, W^0, W^1\right) \right\|_H^2 < \infty \right\} \end{array} \tag{2.6}$$

where the constants C_n are those appearing in (2.2).

THEOREM 2.3. *Assume that $T > 2$. Then, for every initial data $\left(\phi^0, \phi^1, W^0, W^1\right)$ in H there exists a control $\beta \in H^{-2}(0,T; L^2(0,1))$ such that the solution (ϕ, W) of (1.3) satisfies*

$$\left\{ \begin{array}{l} \phi(T) \equiv \mu^1 = \int_0^1 W^1(x)dx + \int_0^1 \psi^0(x,0)dx,\ \phi_t(T) \equiv 0 \\ W(T) \equiv <\rho^1, 1> = \int_0^1 W^0(x)dx - \int_0^1 \int_0^1 \psi^1(x,y)dxdy,\ W_t(T) \equiv 0. \end{array} \right. \tag{2.7}$$

Moreover there exists a constat $C > 0$ such that

$$\|\beta\|_{H^{-2}(0,T;L^2(0,1))} \leq C \| \left(\phi^0, \phi^1, W^0, W^1\right) \|_H. \tag{2.8}$$

REMARK 3. *The control time $T > 2$ is optimal. Indeed, when $T < 2$ it is easy to see that the set of controllable data is not dense in the space of finite energy data.*

Actually, when $T < 2$ none of the one-dimensional problems (1.6) is approximately controllable, i.e. the space of controllable data is no even dense in $\mathcal{Y}'$.

REMARK 4. *The developments of this article allow to show that $C_n = \mathcal{O}\left(e^{n^\alpha}\right)$ as $n \to \infty$ for any $\alpha > 1$. Thus, roughly speaking, the Fourier coefficients in the $x-$variable have to decay exponentially to guarantee the controlabillity. Let us explain with some more details why this result is natural.*

From the definition (2.6) of H and from the fact that C_n grows exponentially it is clear that there is no Sobolev space that might be contained in H (observe that Sobolev spaces correspond roughly to polynomial weights C_n). But this is known apriori. Indeed, as we said in the introduction, our control problem does not verify the geometric control property given in [6] and, as a consequence of this, no Sobolev space of initial data may be exactly controllable with β in $H^{-2}(0,T;L^2(0,1))$.

After the first version of this paper was written, B. Allibert in [1] has obtained some complementary results. In [1] it is proved that for any ε there exists $T(\varepsilon) > 0$ such that system (1.3) is controllable in time $T(\varepsilon)$ for all initial data in the space $H(\varepsilon)$ which is defined as in (2.6) but with $C_n = exp(\varepsilon n)$ as $n \to \infty$. Thus, the result in [1] shows roughly that as $t \to \infty$ the system is controllable in a larger and larger class of analytic functions. The results in [1] are an extention of previous results by the same author on the controllability of the classical wave equation in the square Ω and with control in Γ_0. Observe that all these problems have in common the fact that the geometric control condition of [6] is not satisfied. The structure of the set of controllable data in those situations is mainly unknown.

Since the constants C_n in our estimates are of order e^{n^α} we can control all the initial data which belong to the Gevrey classes of exponent $\alpha > 1$ in the $x-$variable.

REMARK 5. *Finally, let us mention that if a second control $\alpha \in L^2(0,T)$ is allowed to act in the system through the condition of continuity of the velocity fields*

$$\frac{\partial \Phi}{\partial y} = -W_t + \alpha, \quad in\ \Gamma_0 \times (0,T) \tag{2.9}$$

the same result hold with $C_n = \mathcal{O}(n^4 e^{2n\pi})$. This is a consequence of Proposition 3.2 bellow. From the proof of Proposition 3.2 it follows that this constant is sharp. However introducing controls of the form (2.9) does not seem to be realistic. This is the reason for using only the control β which requires important additional developments.

3. Controllability of the one-dimensional systems. This section is devoted to prove the controllability results for the one-dimensional systems (1.6) that are necessary to derive the controllability of system (1.3). In a first paragraph, by using classical multiplier techniques, we derive some hidden regularity results. In the second paragraph, with the same techniques we get the first observability inequalities. In a third paragraph, by using Ingham's inequalities, we obtain a refined version of these observability inequalities. Finally, in the last paragraph we apply HUM and prove the controllability result for (1.6).

3.1. Hidden regularity. Let us consider the system

$$\begin{cases} \eta_{tt} - \eta_{yy} + n^2\pi^2\eta = f & \text{in} \quad (0,1) \times (0,T) \\ \eta_y(1) = 0 & \text{for} \quad t \in (0,T) \\ \eta_y(0) = u_t & \text{for} \quad t \in (0,T) \\ u_{tt} + n^2\pi^2 u - \eta_t(0) = g & \text{for} \quad t \in (0,T) \\ \eta(0) = \eta^0, \eta_t(0) = \eta^1 & \text{in} \quad (0,1) \\ u(0) = u^0, u_t(0) = u^1. \end{cases} \tag{3.1}$$

System (3.1) is the adjoint of (1.6) . The unknowns are $\eta = \eta(y,t)$ and $u = u(t)$. Of course, since the coefficients of the system depend on $n = 0, 1 \dots$, solutions (η, u) depend on n too. However, in order to simplify the notations we will not use the index n to distinguish the solutions of (3.1) for the different values of n.

The energy space for system (3.1) is the Hilbert space $\mathcal{Y} = H^1(0,1) \times L^2(0,1) \times \mathbb{R} \times \mathbb{R}$.

It is easy to see that for any $(\eta^0, \eta^1, u^0, u^1) \in \mathcal{Y}$ and $(f, g) \in L^1(0, T; L^2(0,1) \times \mathbb{R})$ system (3.1) has a unique solution in the class

$$\eta \in C\left([0,T]; H^1(0,1)\right) \cap C^1\left([0,T]; L^2(0,1)\right); u \in C^1([0,T]; \mathbb{R}). \tag{3.2}$$

In other words $(\eta, \eta_t, u, u_t) \in C([0,T]; \mathcal{Y})$.

The energy of the system

$$F(t) = \frac{1}{2} \int_0^1 \left[\mid \eta_t \mid^2 + \mid \eta_y \mid^2 + n^2\pi^2\eta^2 \right] dy + \frac{1}{2} \left[\mid u_t \mid^2 + n^2\pi^2 \mid u \mid^2 \right] \tag{3.3}$$

satisfies

$$\frac{dF(t)}{dt} = \int_0^1 f(y,t)\eta_t(y,t)dy + g(t)u_t(t). \tag{3.4}$$

Therefore, when $f \equiv 0$ and $g \equiv 0$, the energy F remains constant along trajectories.

We observe that when $n \geq 1$ the square root of F defines a norm in $\mathcal{Y}$ equivalent to the canonical norm $\| \cdot \|_{\mathcal{Y}}$ of $\mathcal{Y}$:

$$\|(u, v, w, z,)\|_{\mathcal{Y}} = \left[\int_0^1 \left(\mid u_y \mid^2 + \mid u \mid^2 + \mid v \mid^2 \right) dy + w^2 + z^2 \right]^{1/2}. \tag{3.5}$$

However, when $n = 0$ this is not the case. Actually, for $n = 0, (\eta, u) = (c_1, c_2)$ with c_1, c_2 real constants are stationary solutions of (3.1) with $f \equiv 0, g \equiv 0$ for which the energy F vanishes.

We have the following "hidden regularity" result:

Proposition 3.1. *For any $T > 0$ there exists a constant $C(T) > 0$ independent of $n = 0, 1, \dots$ such that*

$$\begin{aligned} &\left(\int_0^T \mid u_{tt} \mid dt \right)^2 + \int_0^T \left[\mid u_t \mid^2 + (1 + n^4\pi^4)u^2 + (1 + n^2\pi^2)\eta^2(0,t) \right] dt \\ &\leq C\left(n^4 + 1\right) \left[\| \left(\eta^0, \eta^1, u^0, u^1\right) \|^2_{\mathcal{Y}} + \|f\|^2_{L^1(0,T;L^2(0,1))} + \|g\|^2_{L^1(0,T)} \right] \end{aligned} \tag{3.6}$$

for any $(\eta^0, \eta^1, u^0, u^1) \in \mathcal{Y}, f \in L^1(0, T; L^2(0,1))$ and $g \in L^1(0,T)$.

If $g \in L^2(0,T)$, then $u \in H^2(0,T)$ and we also have

$$\begin{aligned} &\int_0^T \mid u_{tt} \mid^2 dt \leq \\ &\leq C(n^4 + 1) \left[\|(\eta^0, \eta^1, u^0, u^1)\|^2_{\mathcal{Y}} + \|f\|^2_{L^1(0,T;L^2(0,1))} + \|g\|^2_{L^2(0,T)} \right]. \end{aligned} \tag{3.7}$$

Remark 6. *This proposition shows that u is more smooth than what (3.2) guarantees. This is due to the structure of the second order (in time) equations that u* *satisfies. The fact that the constant in (3.6) and (3.7) do not depend on the index n is worth mentioning.*

Proof of Proposition 3.1: It is enough to consider smooth solutions since a classical density argument allows to extend inequalities (3.6) and (3.7) to any solution with finite right hand side. We use a classical multiplier technique (see, for instance, [10]). We multiply the first equation in (3.1) by $(1-y)\eta_y$ and integrate over $(0,1)\times(0,T)$. Integrating by parts we obtain

$$\frac{1}{2}\int_0^T \left[\mid \eta_t \mid^2 + \mid \eta_y \mid^2 - n^2\pi^2\eta^2\right](0,t)dt = -\int_0^1 \eta_t(1-y)\eta_y dy\Big|_0^T +$$
$$+\frac{1}{2}\int_0^T\int_0^1 \left[\eta_t^2 + \eta_y^2 - n^2\pi^2\eta^2\right]dydt + \int_0^T\int_0^1 f(1-y)\eta_y dydt = X.$$

In this identity we use the notation $L\mid_0^T = L(T)-L(0)$. The right hand side of this identity can be easily bounded as follows

$$\mid X \mid \leq \frac{1}{2}\int_0^1 \left[\eta_t^2+\eta_y^2\right](y,0)dy + \frac{1}{2}\int_0^1 \left[\eta_t^1+\eta_y^2\right](y,T)dy + \int_0^T F(t)dt +$$

$$+\frac{1}{2}\left[\|f\|^2_{L^1(0,T;L^2(0,1))} + \|\eta_y\|^2_{L^\infty(0,T;L^2(0,1))}\right] \leq F(0)+F(T)+\int_0^T F(t)dt +$$

$$+\|F(t)\|_{L^\infty(0,T)} + \frac{1}{2}\|f\|^2_{L^1(0,T;L^2(0,1))} \leq C\left[\|F\|_{L^\infty(0,T)} + \|f\|^2_{L^1(0,T;L^2(0,1))}\right],$$

with $C>0$ independent of n.

In the sequel, if some constant in the inequalities depends on n, we will make it explicit by means of an index n on that constant.

On the other hand, from identity (3.4) and using Gronwall's inequality it is easy to deduce that

$$\|F\|^2_{L^\infty(0,T)} \leq C\left[\|f\|^2_{L^1(0,T;L^2(0,1))} + \|g\|^2_{L^1(0,T)} + F(0)\right].$$

Since $H^1(0,1)$ is continuously embedded in $C([0,1];\mathbb{R})$ we also have

$$\int_0^T \eta^2(0,t)dt \leq C\int_0^T F(t)dt \leq C\left[\|f\|^2_{L^1(0,T;L^2(0,1))} + \|g\|^2_{L^1(0,T)} + F(0)\right].$$

Combining these inequalities we deduce that

$$\begin{aligned}&\int_0^T \left[\mid \eta_t \mid^2 + \mid \eta_y \mid^2 + n^2\pi^2\eta^2\right](0,t)dt \\ &\leq C(n^2+1)\left[\|(\eta^0,\eta^1,u^0,u^1)\|^2_{\mathcal{Y}} + \|f\|^2_{L^1(0,T;L^2(0,1))} + \|g\|^2_{L^1(0,T)}\right].\end{aligned} \tag{3.8}$$

On the other hand

$$\begin{aligned}n^4\pi^4\int_0^T u^2(t)dt &\leq 2n^2\pi^2\int_0^T F(t)dt \\ &\leq Cn^4\left[\|(\eta^0,\eta^1,u^0,u^1)\|^2_{\mathcal{Y}} + \|f\|^2_{L^1(0,T;L^2(0,1))} + \|g\|^2_{L^1(0,T)}\right].\end{aligned} \tag{3.9}$$

Inequalities (3.6) and (3.7) are a direct consequence of (3.8) and (3.9) and the coupling conditions between η and u given in system (3.1), i.e.

$$\eta_y(0,t) = u_t(t); u_{tt}(t) = g(t) + \eta_t(0,t) - n^2\pi^2 u(t) \text{ for } t \in (0,T). \tag{3.10}$$

□

3.2. Observability inequalities. In this paragraph we consider the adjoint system (3.1) in the particular case where $f \equiv 0$ and $g \equiv 0$. More precisely, assume that η and u solve:

$$\begin{cases} \eta_{tt} - \eta_{yy} + n^2\pi^2\eta = 0 & \text{in} \quad (0,1)\times(0,T) \\ \eta_y(1,t) = 0 & \text{for} \quad t \in (0,T) \\ \eta_y(0,t) = u_t(t) & \text{for} \quad t \in (0,T) \\ u_{tt}(t) + n^2\pi^2 u(t) - \eta_t(0,t) = 0 & \text{for} \quad t \in (0,T) \\ \eta(0) = \eta^0, \eta_t(0) = \eta^1 & \text{in} \quad (0,1) \\ u(0) = u^0, u_t(0) = u^1. \end{cases} \tag{3.11}$$

We have the following observability result:

Proposition 3.2. *For any $T > 2$ there exists a constant $C > 0$ which is independent of $n = 0, 1, \ldots$ such that*

$$\begin{aligned} 2F(0) + \|\eta^0\|^2_{L^2(0,1)} + \mid u^0 \mid^2 \leq C e^{2n\pi} \int_0^T & \left[\mid u_{tt} \mid^2 + \mid u_t \mid^2 + \right. \\ & \left. +(1+n^4\pi^4) \mid u \mid^2 + (1+n^2\pi^2) \mid \eta(0,t) \mid^2\right] dt \end{aligned} \tag{3.12}$$

for any solution of (3.11).

Remark 7. *Let $\rho : (0,T) \to [0,1]$ be a non-negative smooth function with compact support and $\rho \equiv 1$ in $(\varepsilon, T-\varepsilon)$ with $\varepsilon > 0$ small enough such that $T - 2\varepsilon > 2$. In view of the time invariance of system (3.11) we deduce that*

$$\begin{aligned} 2F(\varepsilon) + \|\eta(\varepsilon)\|^2_{L^2(0,1)} + \mid u(\varepsilon) \mid^2 \leq C e^{2n\pi} \int_0^T \rho(t) & \left[\mid u_{tt} \mid^2 + (1+n^4\pi^2) \mid u \mid^2 + \right. \\ & \left. + (1+n^2\pi^2) \mid \eta(0,t) \mid^2\right] dt. \end{aligned}$$

Using the conservation of energy we deduce that

$$\begin{aligned} \|(\eta^0, \eta^1, u^0, u^1)\|^2_{\mathcal{Y}} & \leq 2F(0) + \|\eta^0\|^2_{L^2(0,1)} + \mid u^0 \mid^2 \\ & \leq C e^{2n\pi} \int_0^T \rho(t) \left[\mid u_{tt} \mid^2 + \mid u_t \mid^2 + (1+n^4\pi^4) \mid u \mid^2 + \right. \\ & \qquad \left. +(1+n^2\pi^2) \mid \eta(0,t) \mid^2\right] dt. \end{aligned} \tag{3.13}$$

This estimate will allow us to construct controls with compact support in time.

Proof of Proposition 3.2: The proof of this result is obtained by means of a genuinely one-dimensional method which consists roughly on viewing the wave equation in (3.11) as being an evolution equation with respect to y, while t plays the role of the space variable. This argument was used in [18] when studying the controllability of the one-dimensional semi-linear wave equation.

For any $0 \leq y \leq 1$ we define

$$G(y) = \frac{1}{2} \int_y^{T-y} \left[\mid \eta_t \mid^2 + \mid \eta_y \mid^2 + n^2\pi^2 \mid \eta \mid^2\right] (y,t) dt.$$

We have

$$G(0) = \frac{1}{2}\int_0^T \left[\mid \eta_t \mid^2 + \mid \eta_y \mid^2 + n^2\pi^2 \mid \eta \mid^2\right](0,t)dt. \tag{3.14}$$

On the other hand

$$\begin{aligned} G'(y) = & \int_y^{T-y} \left[\mid \eta_{yy}\eta_y + \eta_{ty}\eta_t + n^2\pi^2\eta_y\eta\right](y,t)dt \\ & -\frac{1}{2}\sum_{t=y,T-y}\left[\mid \eta_y(y,t)\mid^2 + \mid \eta_t(y,t)\mid^2 + n^2\pi^2 \mid \eta(y,t)\mid^2\right] \end{aligned}$$

and

$$\int_y^{T-y} \eta_{ty}(y,t)\eta_t(y,t)dt = -\int_y^{T-y}\eta_y(y,t)\eta_{tt}(y,t)dt + \eta_y(y,t)\eta_t(y,t)\Big|_{t=y}^{t=T-y}.$$

Therefore

$$\begin{aligned} G'(y) = & \int_y^{T-y}\left[\eta_{yy} - \eta_{tt} + n^2\pi^2\eta\right]\eta_y(y,t)dt + \eta_y(y,t)\eta_t(y,t)\Big|_{t=y}^{t=T-y} \\ & -\frac{1}{2}\sum_{t=y,T-y}\left[\mid \eta_y(y,t)\mid^2 + \mid \eta_t(y,t)\mid^2 + n^2\pi^2\mid \eta(y,t)\mid^2\right]. \end{aligned} \tag{3.15}$$

Using the first equation in (3.11) we have that

$$\int_y^{T-y}\left[\eta_{yy} - \eta_{tt} + n^2\pi^2\eta\right]\eta_y(y,t)dt = 2n^2\pi^2\int_y^{T-y}\eta\eta_y(y,t)dt$$

and on the other hand

$$\eta_y(y,t)\eta_t(y,t)\Big|_{t=y}^{t=T-y} - \frac{1}{2}\sum_{t=y,T-y}\left[\mid \eta_y(y,t)\mid^2 + \mid \eta_t(y,t)\mid^2 + n^2\pi^2\mid \eta(y,t)\mid^2\right] \le 0.$$

Combining these identities with (3.15) we deduce that

$$\begin{aligned} G'(y) & \le 2n^2\pi^2\int_y^{T-y}\eta\eta_y(y,t)dt \\ & \le n\pi\int_y^{T-y}\left[\mid \eta_y \mid^2 + n^2\pi^2\mid \eta \mid^2\right](y,t)dt \le 2n\pi G(y). \end{aligned}$$

Thus $G(y) \le e^{2n\pi}G(0)$, for all $y \in (0,1)$ and therefore $\int_0^1 G(y) \le e^{2n\pi}G(0)$.

In particular

$$\begin{aligned} (T-2)F(T) = & \int_1^{T-1} F(t)dt = \\ \frac{1}{2}\int_1^{T-1} & \left\{\left[\int_0^1 \mid \eta_y \mid^2 + \mid \eta_t \mid^2 + n^2\pi^2\eta^2\right]dy + \mid u_t \mid^2 + n^2\pi^2u^2\right\}dt \le \\ & \le \int_0^1 G(y)dy + \frac{1}{2}\int_1^{T-1}\left[\mid u_t \mid^2 + n^2\pi^2u^2\right]dt \le \end{aligned}$$

$$(3.16) \leq \frac{e^{2n\pi}}{2}\int_0^T \left[\mid \eta_y \mid^2 + \mid \eta_t \mid^2 + n^2\pi^2\eta^2\right](0,t)dt + \frac{1}{2}\int_0^T \left[\mid u_t \mid^2 + n^2\pi^2 u^2\right]dt.$$

Using the relations (3.11) at $y = 0$ we deduce that (3.12) holds when $n \geq 1$.

When $n = 0$, it is sufficient to add in (3.16) the extra quantity $\int_0^T \left[\mid \eta \mid^2 (0,t) + \mid u \mid^2 (t)\right] dt$ to deduce that (3.12) holds in that case too. □

REMARK 8. *When $n = 0$, inequality (3.12) shows that*

$$\|\eta_y^0\|^2_{L^2(0,1)} + \|\eta^1\|^2_{L^2(0,1)} + \mid u^1 \mid^2 \leq C\int_0^T \left[\mid u_{tt} \mid^2 + \mid u_t \mid^2\right] dt. \tag{3.17}$$

This inequality does not provide any estimate on u^0. This is related to the fact that, when $n = 0$, system (1.6) can not be driven exactly to zero but rather to the equilibrium given by the constants c_1, c_2 in (1.9).

3.3. Improved observability inequalities. The goal of this section is to obtain observability inequalities of the form (3.12) but, in which, the only term appearing in the right hand side is $\int_0^T \mid u_{tt} \mid^2 dt$. As we will see this is related to the controllability of system (1.6) using the sole control β. We have the following:

THEOREM 3.3. *Assume that $T > 2$. Then,*

(i) For any $n \geq 1$ there exists a constant $C = C(T,n) > 0$ such that

$$\|(\eta^0, \eta^1, u^0, u^1)\|^2_{\mathcal{Y}} \leq C(T,n)\int_0^T \mid u_{tt} \mid^2 dt \tag{3.18}$$

for any solution of (3.11). Moreover $C(T,n) = \mathcal{O}\left(e^{n^\alpha}\right)$, for any $\alpha > 1$.

(ii) If $n = 0$ there exists a constant $C = C(T) > 0$ such that

$$\|\eta_y^0\|^2_{L^2(0,1)} + \mid u^1 \mid^2 \leq C(T)\int_0^T \mid u_{tt} \mid^2 dt, \tag{3.19}$$

for any solution of (3.11).

REMARK 9. *As observed in Remark 7, in estimates (3.18) and (3.19) one can replace the right hand side by the quantity $\int_0^T \rho(t) \mid u_{tt}(t) \mid^2 dt$ where ρ is a smooth non-negative function with compact support in $(0,T)$ and such that $\rho \equiv 1$ in $(\varepsilon, T-\varepsilon)$ with $\varepsilon > 0$ small enough such that $T - 2\varepsilon > 2$.*

To prove Theorem 3.3 we need the following refined version of a result by A. Haraux [8] on non-harmonic Fourier series.

THEOREM 3.4. *Let $f = f(t)$ be of the form $f(t) = \sum_{n\in\mathbb{Z}} a_n e^{i\lambda_n t}$ where λ_n is a sequence of real numbers. We assume that there exist $N \in \mathbb{N}, \gamma > 0$ and $\gamma_\infty > 0$ such that*

$$\lambda_{n+1} - \lambda_n \geq \gamma_\infty > 0 \ \textit{if} \ \mid n \mid > N \tag{3.20}$$

$$\lambda_{n+1} - \lambda_n \geq \gamma > 0 \ \textit{for any} \ n \in \mathbb{Z}. \tag{3.21}$$

Let $J = [0,T] \subset \mathbb{R}$ be a finite interval with $T > \frac{2\pi}{\gamma_\infty}$. Then, there exist two positive constants $C_1, C_2 > 0$ such that

$$C_1 \sum_{n\in\mathbb{Z}} \mid a_n \mid^2 \leq \int_J \mid f(t) \mid^2 dt \leq C_2 \sum_{n\in\mathbb{Z}} \mid a_n \mid^2, \tag{3.22}$$

for all $(a_n)_n \in l^2$.

More precisely $C_1 = C_1(2N+1)$ *and* $C_2 = C_2(2N+1)$ *where* $C_i(j), i = 1,2$ *are given by the following recurrent formulas:*

$$
\left\{
\begin{array}{l}
C_1(j+1) = \left[\left(\dfrac{2C_2(j)}{\mid J \mid} + 1 \right) \dfrac{4}{C_1(j)(\mid J \mid \gamma_\infty - 2\pi)^2\gamma^2} + \dfrac{2}{\mid J \mid} \right]^{-1} \\
C_2(j+1) = 2\left(\mid J \mid (j+1) + C_2(0)\right),\ j = 0,1,\dots
\end{array}
\right.
\tag{3.23}
$$

and $C_1(0), C_2(0)$ *are such that (3.22) holds in the particular case in which* $\gamma_\infty = \gamma > 0$.

Remark 10. *(a) When* $\gamma_\infty = \gamma$, *a sequence on the conditions of Theorem 3.4 satisfies* $\lambda_{n+1} - \lambda_n \geq \gamma > 0\,,\ \forall n \in \mathbb{Z}$. *In this particular case the classical result by A. E. Ingham [9] shows the existence of* $c_1, c_2 > 0$ *such that (3.22) holds when* $\mid J \mid > \frac{2\pi}{\gamma}$.

Theorem 3.4 allows to deduce that (3.22) holds when the length of the interval J *is smaller. Indeed, it suffices* $\mid J \mid > 2\pi/\gamma_\infty, \gamma_\infty$ *being the "asymptotic gap" of the sequence* $\{\lambda_n\}$, *which is in general larger than* γ. *This relaxed gap condition was shown to be sufficient for (3.22) in [4]. Later on A. Haraux in [8] gave a constructive proof which allows to give explicit estimates on the constants* C_1 *and* C_2. *Following the construction in [8] one can easily see that (3.23) suffice. In the Appendix at the end of this paper we give all the details of this construction.*

(b) Clearly, the constants C_1 *and* C_2 *degenerate as* $N \to \infty$. *More precisely,* $C_2(N) = \mathcal{O}(N)$ *while* $\gamma^{2N}(C_1(N))^{-1} = \mathcal{O}\left(e^{N^\alpha}\right)$ *for any* $\alpha > 1$. *Indeed we have:*

$$
(C_1(N))^{-1} \leq \frac{2C_2(N)}{|J|} \frac{4}{(\mid J \mid \gamma_\infty - 2\pi)^2\gamma^2} (C_1(N-1))^{-1} = \frac{16N(C_1(N-1))^{-1}}{(\mid J \mid \gamma_\infty - 2\pi)^2\gamma^2}.
$$

Hence

$$
\gamma^{2N}(C_1(N))^{-1} \leq \left(\frac{16}{(\mid J \mid \gamma_\infty - 2\pi)^2\gamma^2} \right)^N N!\,(C_1(0))^{-1} \leq e^{N^\alpha}\,(C_1(0))^{-1}.
$$

In order to apply Theorem 3.4 and deduce that Theorem 3.3 holds we need precise estimates on the spectrum of (3.11). We look for solutions of (3.11) in separated variables of the form $(\eta, u) = e^{\nu t}(\varphi(y), \omega)$ with $\varphi = \varphi(y)$ and $\omega \in \mathbb{R}$. Due to the conservative character of the system we know that all eigenvalues ν are purely imaginary. On the other hand, the spectrum is symmetric with respect to the real axis. Thus, for any $n = 0,1,\dots$ there exists a sequence of eigenvalues $\nu_{n,m}$ with $\overline{\nu}_{n,m} = -\nu_{n,m} = \nu_{-n,m}$.

We have the following estimates:

Theorem 3.5. *(see [12] and [14]) For any* $n = 0,1,\dots$ *and* $m \in \mathbb{Z}$ *such that* $\mid m \mid > n$ *we have*

$$
\left\{
\begin{array}{lll}
\left| \nu_{n,m} - \sqrt{m^2 + n^2\pi i} \right| & \leq \dfrac{24}{\sqrt{m^2 + n^2\pi}} & \text{if } m > n \\
\left| \nu_{n,m} + \sqrt{m^2 + n^2\pi i} \right| & \leq \dfrac{24}{\sqrt{m^2 + n^2\pi}} & \text{if } m < -n.
\end{array}
\right.
\tag{3.24}
$$

Remark 11. *This theorem shows that, for sufficiently high frequences, the eigenvalues of (3.11) are uniformly close to the eigenvalues* $\lambda = \pm\sqrt{m^2 + n^2\pi i}$ *of the wave equation with Neumann boundary conditions*

$$
\left\{
\begin{array}{lll}
\eta_{tt} - \eta_{yy} + n^2\pi^2\eta = 0 & \mathit{in} & (0,1) \times (0,\infty) \\
\eta_y(0,t) = \eta_y(1,t) = 0 & \mathit{for} & t > 0.
\end{array}
\right.
\tag{3.25}
$$

Clearly, system (3.25) corresponds to the decomposition of the wave equation with Neumann boundary conditions in the square Ω following the development (1.5) in Fourier series. In other words, Theorem 3.5 asserts that the spectrum of the adjoint system of (1.1), i.e.

$$\begin{cases} \phi_{tt} - \Delta\phi = 0 & in \quad \Omega \times (0,\infty) \\ \frac{\partial \phi}{\partial \nu} = 0 & on \quad \Gamma_1 \times (0,\infty) \\ \frac{\partial \phi}{\partial y} = W_t & on \quad \Gamma_0 \times (0,\infty) \\ W_{tt} - W_{xx} - \phi_t = 0 & on \quad \Gamma_0 \times (0,\infty) \\ W_x(0,t) = W_x(1,t) = 0 & for \quad t > 0 \end{cases}$$

at high frequencies is uniformly close to the eigenvalues of the wave equation with Neumann boundary conditions on the whole boundary of the cavity Ω:

$$\begin{cases} \phi_{tt} - \Delta\phi = 0 & in \quad \Omega \times (0,\infty) \\ \frac{\partial \phi}{\partial \nu} = 0 & on \quad \partial\Omega \times (0,\infty). \end{cases}$$

This means roughly that the effect of the flexible boundary in the interior of the cavity is neglectible for high frequencies. However it is worth mentioning that the high frequency asymptotics are of a different nature in the region $\mid m \mid \leq n$.

From Theorem 3.5 it is easy to get explicit bounds on the gaps γ and γ_∞ associated to the sequence $\{\nu_{n,m}\}_{m\in\mathbb{Z}}$ for each $n = 0, 1, \ldots$

Proposition 3.6. *Given any $n = 0, 1, \ldots$ and $0 < \delta < \pi$ we have*

$$\mid \nu_{n,m+1} - \nu_{n,m} \mid \geq \pi - \delta \tag{3.26}$$

for any m with $\mid m \mid \geq N(n,\delta)$ where

$$N(n,\delta) = \max\left[\sqrt{\frac{96}{\pi\delta} - n^2}, \frac{2n\pi}{\delta} - n - \frac{1}{2}\right]. \tag{3.27}$$

On the other hand

$$\begin{cases} \mid \nu_{n,m+1} - \nu_{n,m} \mid \geq \frac{\pi}{4}, \forall m \in \mathbb{Z} & if \quad n = 0, 1 \\ \mid \nu_{n,m+1} - \nu_{n,m} \mid \geq \frac{\pi}{1+2n}, \forall m \in \mathbb{Z} & if \quad n \geq 2. \end{cases} \tag{3.28}$$

Furthermore, (3.22) holds for functions f of the form

$$f(t) = \sum_{m\in\mathbb{Z}^\star} a_{n,m} e^{-\nu_{n,m} t} + a_n^\star e^{-\nu_n^\star t} + a_n^{\star\star} e^{-\nu_n^{\star\star} t} \tag{3.29}$$

with $C_2 = C_2(2N(n,\delta)+1) = \mathcal{O}(n)$ and $(C_1)^{-1} = (C_1(2N(n,\delta)+1))^{-1}$. Moreover, $C_2 = \mathcal{O}(n)$ and $(C_1)^{-1} = \mathcal{O}\left(e^{n^\alpha}\right)$ for any $\alpha > 1$.

Proof: In view of (3.24) we have

$$\begin{aligned} &\mid \nu_{n,m+1} - \nu_{n,m} \mid \geq \\ &\geq \pi \left| \sqrt{(m+1)^2 + n^2} - \sqrt{m^2+n^2} \right| - \frac{24}{\pi}\left[\frac{1}{\sqrt{(m+1)^2+n^2}} + \frac{1}{\sqrt{m^2+n^2}}\right] \\ &\geq \frac{(2|m|+1)\pi}{(2|m|+1)+2n} - \frac{48}{\pi\sqrt{m^2+n^2}} \geq \pi - \left[\frac{48}{\pi\sqrt{m^2+n^2}} + \frac{2n\pi}{2|m|+1+2n}\right]. \end{aligned}$$

It is easy to see that when $| m | \geq N(n, \delta),$ where $N(n,\delta)$ is given by (3.27), then

$$\frac{48}{\pi\sqrt{m^2+n^2}} + \frac{2n\pi}{2 \mid m \mid +1+n} \leq \delta.$$

This concludes the proof of (3.26).

To prove (3.28) we observe that, for any $n = 0, 1, \ldots$ the eigenvalues $\nu_{n,m}$ with $m > 0$ are of the form

$$\nu_{n,m} = \sqrt{z_{n,m}^2 + n^2\pi^2}, \tag{3.30}$$

where $z_{n,m}$ are the zeros (ordered so that $z_{n,m}$ increases as m does) of the equation

$$tgz = \frac{z^2 + n^2\pi^2}{z^3}. \tag{3.31}$$

There are also two eigenvalues that we denote by $\nu_n^\star$ and $\nu_n^{\star\star}$ that do not satisfy (3.30). Indeed, they are given by

$$\nu_n^\star = \sqrt{n^2\pi^2 - (z_n^\star)^2}; \tag{3.32}$$

where $z_n^\star$ is the unique real positive solution of

$$e^{2z} = \frac{z^3 - z^2 + n^2\pi^2}{z^3 + z^2 - n^2\pi^2} \tag{3.33}$$

when $n \geq 1$ and $\nu_0^\star = 0,$ and $\nu_n^{\star\star} = \overline{\nu_n^\star}$.

By analyzing the graphs of the functions in (3.31) and (3.33) it is easy to see that (3.28) holds. We refer to [14] for a detailled proof.

To finish the proof we have to apply Theorem 3.4 for $\gamma = \min\left\{\frac{\pi}{4}, \frac{\pi}{1+2n}\right\}$ and $\gamma_\infty = \pi - \delta$. We obtain that (3.22) holds for functions f of the form (3.29).

In order to evaluate the constants we use the recurrent formulas (3.23). We have:

$$C_2 = C_2(2N(n,\delta)+1) = 2(T(N(n,\delta)+1) + C_2(0)) = \mathcal{O}(n).$$

On the other hand

$$(C_1)^{-1} = (C_1(2N(n,\delta)+1))^{-1} \leq \frac{8C_2(N(n,\delta)+1)(C_1(2N(n,\delta)))^{-1}}{T(T\gamma_\infty - 2\pi)^2\gamma^2} \leq$$

$$\leq Mn^3(C_1(2N(n,\delta)))^{-1} \leq M^{2N(n,\delta)+1}((N(n,\delta)+1)!)^3(C_1(0))^{-1} \leq C(\alpha)e^{n^\alpha},$$

where M is a positive constant and $\alpha > 1$. $\square$

Now we have all the ingredients to prove Theorem 3.3.

Proof of Theorem 3.3: Let us consider first the case $n \geq 1$. In view of Proposition 3.2 it is sufficient to show the existence of a constant C (depending on n and T) such that

$$\int_0^T \left[\mid u_t \mid^2 + n^4\pi^4 \mid u \mid^2 + n^2\pi^2 \mid \eta(0,t) \mid^2\right] dt \leq C \int_0^T \mid u_{tt} \mid^2 dt \tag{3.34}$$

holds for any solution of (3.11).

Let $U(t) = (\eta(t), \eta_t(t), u(t), u_t(t))$ be the vector valued unknown associated to (3.11) viewed as first order (in time) system. Let us denote by $\xi_\nu = (\xi_\nu^1, \xi_\nu^2, \xi_\nu^3, \xi_\nu^4)$ the vector valued eigenfunction of system (3.11) associated to the eigenvalue ν.

The solutions η and u of (3.11) can be written as follows

$$\eta(t) = \sum_{m\in \mathbb{Z}^\star} a_{n,m} e^{-\nu_{n,m}t}\xi^1_{n,m} + a_n^\star e^{-\nu_n^\star t}\xi^1_{\nu_n^\star} + a_n^{\star\star} e^{-\nu_n^{\star\star} t}\xi^1_{\nu_n^{\star\star}},$$
$$u(t) = \sum_{m\in \mathbb{Z}^\star} a_{n,m} e^{-\nu_{n,m}t}\xi^3_{n,m} + a_n^\star e^{-\nu_n^\star t}\xi^3_{\nu_n^\star} + a_n^{\star\star} e^{-\nu_n^{\star\star} t}\xi^3_{\nu_n^\star},$$

where the coefficients $\{a_{n,m}, a_n^\star, a_n^{\star\star}\}$ are those associated to the development of the initial data on the orthogonal basis generated by the eigenfunctions.

To get the bounds in (3.34) we first observe that

$$\eta(0,t) = \sum_{m\in \mathbb{Z}^\star} a_{n,m} e^{-\nu_{n,m}t}\xi^1_{n,m}(0) + a_n^\star e^{-\nu_n^\star t}\xi^1_{\nu_n^\star}(0) + a_n^{\star\star} e^{-\nu_n^{\star\star} t}\xi^1_{\nu_n^\star}(0)$$

and

$$\eta_t(0,t) = -\sum_{m\in \mathbb{Z}^\star} a_{n,m}\nu_{n,m} e^{-\nu_{n,m}t}\xi^1_{n,m}(0) - a_n^\star \nu_n^\star e^{-\nu_n^\star t}\xi^1_{\nu_n^\star}(0) - a_n^{\star\star}\nu_n^{\star\star} e^{-\nu_n^{\star\star} t}\xi^1_{\nu_n^{\star\star}}(0).$$

In view of Proposition 3.6 we can apply Theorem 3.4 to these series in any time interval $J = (0,T)$ with $T > 2$. Therefore, taking into account that $\mid \nu_n^\star \mid = \min\{\mid \nu_{n,m} \mid, \mid \nu_n^\star \mid, \mid \nu_n^{\star\star} \mid\}$, we have

$$\int_0^T \mid \eta(0,t) \mid^2 dt \leq C_2 \left(\sum_{m\in \mathbb{Z}^\star} \mid a_{n,m}\xi^1_{n,m}(0) \mid^2 + \mid a_n^\star \xi^1_{\nu_n^\star}(0) \mid^2 + \mid a_n^{\star\star}\xi^1_{\nu_n^{\star\star}}(0) \mid^2 \right)$$

$$\leq \frac{C_2}{\mid \nu_n^\star \mid^2} \left(\sum_{m\in \mathbb{Z}^\star} \mid a_{n,m}\xi^1_{n,m}(0)\,\nu_{n,m} \mid^2 + \mid a_n^\star \xi^1_{\nu_n^\star}(0)\,\nu_n^\star \mid^2 + \mid a_n^{\star\star}\xi^1_{\nu_n^{\star\star}}(0)\,\nu_n^{\star\star} \mid^2 \right)$$

$$\leq \frac{C_2}{C_1 \mid \nu_n^\star \mid^2} \int_0^T \mid \eta_t(0,t) \mid^2 dt.$$

On the other hand, from the equation that u satisfies in (3.11) we have

$$\int_0^T (\eta_t(0,t))^2\, dt \leq 2\int_0^T \left[\mid u_{tt} \mid^2 + n^4\pi^4 \mid u \mid^2\right] dt.$$

Thus, in order to conclude (3.34) it is sufficient to show that

$$\int_0^T \left[\mid u_t \mid^2 + n^4\pi^4 u^2\right] dt \leq C\int_0^T \mid u_{tt} \mid^2 dt$$

holds. The argument we have used to bound $\int_0^T \mid \psi(0,t) \mid^2 dt$ allows us to show that

$$\int_0^T \mid u \mid^2 dt \leq \frac{C_2}{C_1 \mid \nu_n^\star \mid^4}\int_0^T \mid u_{tt} \mid^2 dt \text{ and } \int_0^T \mid u_t \mid^2 dt \leq \frac{C_2}{C_1 \mid \nu_n^\star \mid^2}\int_0^T \mid u_{tt} \mid^2 dt.$$

Combining these results we deduce that (3.34) holds with a constant C of the order of

$$C = \frac{C_2}{C_1}\left\{\frac{1}{\mid \nu_n^\star \mid^2} + \frac{n^4\pi^4}{\mid \nu_n^\star \mid^4}\left(1 + \frac{2C_1}{C_1 \mid \nu_n^\star \mid^2}\right) + \frac{2C_2}{C_1 \mid \nu_n^\star \mid^2}\right\} \tag{3.35}$$

where $C_1 = C_1(2N+1), C_2 = C_2(2N+1)$ are given by (3.23) with $N = N(n,\delta)$ as in (3.27) and $\delta > 0$ such that $T = \frac{2\pi}{\pi-\delta}$.

We pass now to estimate the constant C of (3.35). In [14] we prove that $\nu_n^\star \sim n\pi$. On the other hand, from Proposition 3.3 we have that $C_2(C_1)^{-1} = \mathcal{O}\left(e^{n^\alpha}\right)$. Finally we obtain that $C = \mathcal{O}\left(e^{n^\alpha}\right)$ for all $\alpha > 1$.

Let us consider now the case $n = 0$. In view of (3.17) we have

$$\|\eta_y^0\|^2_{L^2(0,1)} + \|\eta^1\|^2_{L^2(0,1)} + \mid u^1 \mid^2 \leq \frac{1}{T-2}\int_0^T \left[\mid u_{tt} \mid^2 + 2 \mid u_t \mid^2\right] dt. \tag{3.36}$$

Therefore, it is sufficient to show that

$$\int_0^T \mid u_t \mid^2 dt \leq C \int_0^T \mid u_{tt} \mid^2 . \tag{3.37}$$

Proceeding as above we see that (3.37) holds with $C = C_2/C_1 \mid \nu_{n,1} \mid^2$ where $C_1 = C_1(2N+1), C_2 = C_2(2N+1)$ and $N = N(0,\delta)$ with $\delta > 0$ such that $T = \frac{2\pi}{\pi-\delta}$. $\square$

3.4. Controllability in one space dimension for $n \geq 1$: Proof of Theorem 2.1. In this section, applying HUM, we prove Theorem 2.1 as a consequence of the observability inequality (3.18).

Given any $(\eta^0, \eta^1, u^0, u^1) \in \mathcal{Y}$ we solve the adjoint system (3.11).

We fix, some non-negative smooth function $\rho : (0,T) \to I\!\!R$ with compact support such that $\rho \equiv 1$ in $(\varepsilon, T-\varepsilon)$ with $T - 2\varepsilon > 2$.

We then solve the backward system

$$\begin{cases} \psi_{tt} - \psi_{yy} + n^2\pi^2\psi = 0 & \text{in} \quad (0,1)\times(0,T) \\ \psi_y(1,t) = 0 & \text{for} \quad t \in (0,T) \\ \psi_y(0,t) = -V_t(t) & \text{for} \quad t \in (0,T) \\ V_{tt} + n^2\pi^2 V + \psi_t(0,t) = -\frac{d^2}{dt^2}(\rho(t)u_{tt}(t)) & \text{for} \quad t \in (0,T) \\ \psi(T) = \psi_t(T) = 0 & \text{in} \quad (0,1) \\ V(T) = V_t(T) = 0. \end{cases} \tag{3.38}$$

The solution of (3.38) is defined by transposition (see [10]). If we multiply in (3.38) by any solution $(\widetilde{\eta}, \widetilde{u})$ of (3.1) and integrate (formally) by parts we obtain the following identity:

$$\begin{gathered}\int_0^T \rho(t)u_{tt}(t)\widetilde{u}_{tt}(t)dt + \int_0^T\int_0^1 \widetilde{f}\psi dy dt - \int_0^T \widetilde{g}V dt = \int_0^1 [-\psi_t(0)\widetilde{\eta}(0) + \\ \psi(0)\widetilde{\eta}_t(0)]\, dy + V(0)\widetilde{\eta}(0,0) + \psi(0,0)\widetilde{u}(0) - V(0)\widetilde{u}_t(0) + V_t(0)\widetilde{u}(0).\end{gathered} \tag{3.39}$$

Notice that in the obtention of (3.39) we have used the fact that ρ and its first derivative vanish for $t = 0$ and T.

We adopt (3.39) as definition of weak solution in the sense of transposition of (3.38). More precisely we say that (ψ, V) solve (3.38) if (3.39) holds for any $(\widetilde{\eta}^0, \widetilde{\eta}^1, \widetilde{u}^0, \widetilde{u}^1) \in \mathcal{Y}$ and $(\widetilde{f}, \widetilde{g}) \in L^1(0,T; L^2(0,1) \times \mathbb{R})$.

We observe that (3.39) can be rewritten in the following way

$$\int_0^T \rho(t) u_{tt}(t) \widetilde{u}_{tt} dt - \int_0^T \int_0^1 \widetilde{f} \psi dy dt + \int_0^T \widetilde{g} V dt = - < \psi_t(0) + V(0)\delta_0, \widetilde{\eta}(0) > + \\ + < \psi(0), \widetilde{\eta}_t(0) > + (V_t(0) + \psi(0,0))\, \widetilde{u}(0) - V(0)\widetilde{u}_t(0) \tag{3.40}$$

where $< \cdot, \cdot >$ denotes both the duality pairing between $(H^1(0,1))'$ and $H^1(0,1)$ and the scalar product in $L^2(0,1)$ and $\delta_0 \in (H^1(0,1))'$ denotes the Dirac delta at $y = 0$.

We have the following existence and uniqueness result of solutions in the sense of transposition:

PROPOSITION 3.7. *System (3.38) has a unique solution in the sense of transposition. More precisely, for any solution (η, u) of (3.11) with initial data in $\mathcal{Y}$, there exists a unique $(\psi, V) \in C([0,T]; L^2(0,1)) \times L^2(0,T)$, $\rho^0 \in L^2(0,1)$, $\rho^1 \in (H^1(0,1))'$, $\mu^0 \in \mathbb{R}, \mu^1 \in \mathbb{R}$ satisfying*

$$\int_0^T \rho(t) u_{tt}(t) \widetilde{u}_{tt} dt = \int_0^T \int_0^1 \widetilde{f} \psi dy dt - \int_0^T \widetilde{g} V dt \\ + < \rho^1, \widetilde{\eta}(0) > + < \rho^0, \widetilde{\eta}_t(0) > + \mu^1 \widetilde{u}(0) + \mu^0 \widetilde{u}_t(0) \tag{3.41}$$

for any solution $(\widetilde{\eta}, \widetilde{u})$ of (3.1) with

$$(\widetilde{\eta}^0, \widetilde{\eta}^1, \widetilde{u}^0, \widetilde{u}^1) \in \mathcal{Y}, \widetilde{f} \in L^1\left(0,T; L^2(0,1)\right), \widetilde{g} \in L^2(0,1). \tag{3.42}$$

REMARK 12. *In the identity (3.41) ρ^0, ρ^1, μ^0 and μ^1 play respectively the role of $\psi(0), -\psi_t(0) + V(0)\delta_0, -V(0)$ and $V_t(0) + \psi(0,0)$. It is easy to see that, in the frame of smooth functions, there is a one to one correspondence between $\left(\rho^0, \rho^1, \mu^0, \mu^1\right)$ and $(\psi(0), \psi_t(0), V(0), V_t(0))$.*

Proof of Proposition 3.7: In view of Proposition 3.1 we know that the map

$$\left(\widetilde{\eta}^0, \widetilde{\eta}^1, \widetilde{u}^0, \widetilde{u}^1, \widetilde{f}, \widetilde{g}\right) \longrightarrow \int_0^T \rho(t) u_{tt}(t) \widetilde{u}_{tt}(t) dt$$

is linear and continuous from $\mathcal{Y} \times L^1\left(0,T; L^2(0,1)\right) \times L^2(0,T)$ into $\mathbb{R}$. This implies the existence and uniqueness of $\left(\rho^1, \rho^0, \mu^1, \mu^0\right) \times (\psi, V) \in \mathcal{Y}' \times L^\infty\left(0,T; L^2(0,1)\right) \times L^2(0,T)$ such that (3.41) holds. Moreover, there exists a constant $C > 0$ such that

$$\|(\psi, V)\|_{L^\infty(0,T;L^2(0,1)) \times L^2(0,T)} + \| \left(\rho^1, \rho^0, \mu^1, \mu^0\right) \|_{\mathcal{Y}'} \leq C \|u_{tt}\|_{L^2(0,T)} \\ \leq C \| \left(\eta^0, \eta^1, u^0, u^1\right) \|_{\mathcal{Y}'}. \tag{3.43}$$

The fact that $\psi \in C\left([0,T]; L^2(0,1)\right)$ can be deduced from (3.43) by a classical density argument. □

REMARK 13. *When the data of (3.11) are smooth, the solution (η, u) is smooth too. It is easy to see that (3.38) has a finite energy solution. In this case one can check that the element $\left(\rho^0, \rho^1, \mu^0, \mu^1\right) \in \mathcal{Y}'$ obtained in Proposition 3.7 is such that*

$$\rho^0 = \psi(0), \rho^1 = -\psi_t(0) + V(0)\delta_0, \mu^0 = -V(0), \mu^1 = V_t(0) + \psi(0,0).$$

By a density argument one can then deduce that the solution (ψ, V) obtained in Proposition 3.7 is such that the traces

$$\psi\,|_{t=0}, -\psi_t + V\delta_0|_{t=0}\,, -V\,|_{t=0}, V_t + \psi(0,t)|_{t=0}$$

are well defined and coincide with $\left(\rho^0, \rho^1, \mu^0, \mu^1\right)$.

The same arguments allows us to show that the traces are also well defined at $t = T$. This suffices to assert that the weak solution of (3.38) we have constructed by transposition is at rest at $t = T$.

We can now complete the proof of Theorem 2.1.

End of the proof of Theorem 2.1.

In view of Proposition 3.7 and Remark 13 we can define a linear and continuous map Λ from $\mathcal{Y}$ into $\mathcal{Y}'$ such that

$$\Lambda\left(\eta^0, \eta^1, u^0, u^1\right) = \left(-\psi_t + V\delta_0\,|_{t=0}, \psi(0), V_t + \psi(0,t)\,|_{t=0}, -V\,|_{t=0}\right).$$

Taking in (3.41), $\widetilde{f} \equiv 0, \widetilde{g} \equiv 0$ and $(\widetilde{\eta}, \widetilde{u}) = (\eta, u)$, we deduce that

$$< \Lambda\left(\eta^0, \eta^1, u^0, u^1\right), \left(\eta^0, \eta^1, u^0, u^1\right) >= \int_0^T \rho(t)\left|u_{tt}(t)\right|^2 dt$$

and in view of Theorem 3.3 and Remark 9 we deduce that there exists $C > 0$ such that

$$< \Lambda\left(\eta^0, \eta^1, u^0, u^1\right), \left(\eta^0, \eta^1, u^0, u^1\right) >\geq C\left\|\left(\eta^0, \eta^1, u^0 u^1\right)\right\|_{\mathcal{Y}}^2.$$

Actually, $C = \left[C(T,n)\right]^{-1}$, where $C(T,n)$ is as in (3.18).

This implies that Λ is an isomorphism.

This shows that given any $\left(\rho^1, \rho^0, \mu^1, \mu^0\right) \in \mathcal{Y}'$ there exists $\left(\eta^0, \eta^1, u^0, u^1\right) = \Lambda^{-1}\left(\rho^1, \rho^0, \mu^1, \mu^0\right)$ such that the corresponding solution of (3.38) in the sense of transposition satisfies

$$\psi(0) = \rho^0, -\psi_t + V\delta_0\left.\right|_{t=0} = \rho^1, -V\left.\right|_{t=0} = \mu^0, V_t + \psi(0,t)\left.\right|_{t=0} = \mu^1. \tag{3.44}$$

If we want this to be equivalent to the initial data of (1.6) we have to take

$$\rho^0 = \psi^0, \rho^1 = -\psi^1 + V^0\delta_0, \mu^0 = -V^0, \mu^1 = V^1 + \psi^0(0). \tag{3.45}$$

This makes sense when the data $\left(\psi^0, \psi^1, V^0, V^1\right)$ is in $\mathcal{Y}$.

The control we have obtained is of the form $\beta = -\dfrac{d^2}{dt^2}\left(\rho u_{tt}\right)$, where u corresponds to the solution (η, u) of (3.11) with data $\left(\eta^0, \eta^1, u^0, u^1\right) = \Lambda^{-1}\left(\rho^1, \rho^0, \mu^1, \mu^0\right)$, where $\left(\rho^0, \rho^1, \mu^0, \mu^1\right)$ is given by (3.44).

From the identities above we see that

$$\begin{aligned}\|\beta\|_{H^{-2}(0,T)}^2 \leq \|\rho u_{tt}\|_{L^2(0,T)}^2 &\leq C\|\left(\rho^1, \rho^0, \mu^1, \mu^0\right)\|_{\mathcal{Y}'}^2 \\ &\leq C\left\{\|\left(\psi^1, \psi^0, V^1, V^0\right)\|_{\mathcal{Y}'}^2 + \mid \psi^0(0)\mid^2\right\}\end{aligned}$$

where $C = C(T,n)$ is the constant obtained in (3.18). □

Remark 14. *In fact, in some sense, we obtain a stronger result since we prove that we can control the problem (3.41) for any initial data $\left(\rho^0, \rho^1, \mu^0, \mu^1\right) \in \mathcal{Y}'$. In order to give an interpretation of the control problem in terms of the initial data $\left(\psi^1, \psi^0, V^1, V^0\right)$ we have to assure that $\psi^0(0)$ makes sense. For this reason we consider that $\left(\psi^1, \psi^0, V^1, V^0\right) \in \mathcal{Y}$.*

3.5. Controllability in one space dimension for $n=0$: Proof of Theorem 2.2. First of all we observe that proving Theorem 2.2 is equivalent to showing that for any initial data as in the statement of Theorem 2.2 and satisfying the further assumptions

$$V^1+\psi^0(0)=0, V^0-\int_0^1 \psi^1(y)dy=0 \tag{3.46}$$

then, there exists a control β such that

$$\psi(T)=\psi_t(T)\equiv 0 \text{ in } (0,1), \quad V(T)=V_t(T)=0. \tag{3.47}$$

Indeed, this is an immediate consequence of the remark made in the introduction that shows that when β is of zero average the following identities hold

$$V_t(T)+\psi(0,T)=V^1+\psi^0(0), \quad V(T)-\int_0^1 \psi_t(y,T)dy=V^0-\int_0^1 \psi^1(y)dy. \tag{3.48}$$

Thus, in the sequel we focus on initial data $\left(\psi^0,\psi^1,V^0,V^1\right)$ satisfying (3.46). For the adjoint system

$$\begin{cases} \eta_{tt}-\eta_{yy}=0 & \text{in} & (0,1)\times(0,T) \\ \eta_y(1)=0 & \text{for} & t\in(0,T) \\ \eta_y(0)=u_t & \text{for} & t\in(0,T) \\ u_{tt}-\eta_t(0)=0 & \text{for} & t\in(0,T) \\ \eta(0)=\eta^0, \eta_t(0)=\eta^1 & \text{in} & (0,1) \\ u(0)=u^0, u_t(0)=u^1 & & \end{cases} \tag{3.49}$$

we consider initial data in the following subspace $\mathcal{Y}_0$ of $\mathcal{Y}$:

$$\mathcal{Y}_0=\left\{\left(\eta^0,\eta^1,u^0,u^1\right)\in\mathcal{Y}: u^1-\eta^0(0)=0, \int_0^1 \eta^1 dy+u^0=0\right\}. \tag{3.50}$$

It is easy to see that the subspace $\mathcal{Y}_0$ is invariant under the flow generated by (3.49).

Given $\left(\eta^0,\eta^1,u^0,u^1\right)\in\mathcal{Y}_0$ we solve first (3.49) and then the backward system:

$$\begin{cases} \psi_{tt}-\psi_{yy}=0 & \text{in} & (0,1)\times(0,T) \\ \psi_y(1,t)=0 & \text{for} & t\in(0,T) \\ \psi_y(0,t)=-V_t(t) & \text{for} & t\in(0,T) \\ V_{tt}(t)+\psi_t(0,t)=-\frac{d^2}{dt^2}\left(\rho(t)u_{tt}(t)\right) & \text{for} & t\in(0,T) \\ \psi(T)=\psi_t(T)=0 & \text{in} & (0,1) \\ V(T)=V_t(T)=0 & & \end{cases} \tag{3.51}$$

where ρ is as in the proof of Theorem 2.1.

Proceeding as in the proof of Proposition 3.7 one can show that (3.51) has a unique solution defined by transposition such that the traces (3.47) are well defined. On the other hand, integrating the equations in (3.51) we deduce that

$$\int_0^1 \rho^1(y)dy=0; \mu^1=0. \tag{3.52}$$

Let us denote by Z the subspace of $\mathcal{Y}'$ satisfying (3.52). More precisely.

$$Z = \left\{ \left(\rho^1, \rho^0, \mu^1, \mu^0\right) \in \mathcal{Y}' : \text{ (3.52) holds } \right\}. \tag{3.53}$$

It is easy to check that Z is actually the dual of $\mathcal{Y}_0$. Indeed, the dual of $\mathcal{Y}_0$ is a cocient space of $\mathcal{Y}'$ and there is a one-to-one correspondence between Z and this cocient space in the sense that, in Z, we have chosen the unique element of each class of the cocient space satisfying (3.52).

As in the proof of Theorem 2.1 we can define a linear and continuous operator $\Lambda : \mathcal{Y}_0 \longrightarrow Z$ that associates the trace $\left(\rho^1, \rho^0, \mu^1, \mu^0\right) \in Z$ in (3.41) to each $\left(\eta^0, \eta^1, u^0, u^1\right) \in \mathcal{Y}_0$.

We also have

$$< \Lambda\left(\eta^0, \eta^1, u^0, u^1\right), \left(\eta^0, \eta^1, u^0, u^1\right) >= \int_0^T \rho(t) \left|u_{tt}(t)\right|^2 dt.$$

In view of Theorem 3.3 and Remark 9 we deduce the existence of a constant $C > 0$ such that

$$< \Lambda\left(\eta^0, \eta^1, u^0, u^1\right), \left(\eta^0, \eta^1, u^0, u^1\right) >\geq C \left\|\left(\eta^0, \eta^1, u^0, u^1\right)\right\|^2_{\mathcal{Y}'}, \ \forall \left(\eta^0, \eta^1, u^0, u^1\right) \in \mathcal{Y}_0$$

since the quantity $\left[\|\eta^0_y\|^2_{L^2(0,1)} + \|\eta^1\|^2_{L^2(0,1)} + \mid u^1 \mid^2\right]^{1/2}$ defines a norm in $\mathcal{Y}_0$ which is equivalent to the norm induced by $\mathcal{Y}$.

We deduce that $\Lambda : \mathcal{Y}_0 \longrightarrow Z$ is an isomorphism.

Then, given initial data as in the statement of Theorem 2.2 and such that (3.46) holds we define $\left(\rho^1, \rho^0, \mu^1, \mu^0\right) \in Z$ by (3.45). The control we are looking for is $\beta = -\dfrac{d^2}{dt^2}\left(\rho(t) u_{tt}(t)\right)$ where u is the second component of the solution (η, u) of (3.49) with initial data $\left(\eta^0, \eta^1, u^0, u^1\right) = \Lambda^{-1}\left(\rho^1, \rho^0, \mu^1, \mu^0\right)$.

This concludes the proof of Theorem 3.5. $\square$

4. Controllability of the two-dimensional system: Proof of Theorem 2.3. In view of Theorems 2.1 and 2.2 for any $n = 0, 1, \ldots$ there exists a control $\beta_n \in H^{-2}(0,T)$ such that the solution (ψ_n, V_n) of (1.6) satisfies

$$\psi_n(T) \equiv \psi_{n,t}(T) = 0 \text{ in } (0,1), \quad V_n(T) = V_{n,t}(T) = 0 \tag{4.1}$$

for $n \geq 1$ and

$$\psi_0(T) = \mu^1, \psi_{0,t}(T) = 0 \text{ in } (0,1), \quad V_0(T) =< \rho^1, 1 >, V_{0,t}(T) = 0 \tag{4.2}$$

when $n = 0$.

On the other hand

$$\|\beta_n\|^2_{H^{-2}(0,T)} \leq C_n \left\|\left(\rho^1_n, \rho^0_n, \mu^1_n, \mu^0_n\right)\right\|^2_{\mathcal{Y}'}. \tag{4.3}$$

We construct the following control for the two-dimensional system:

$$\beta(x,t) = \sum_{n=0}^{\infty} \beta_n \cos(n\pi x). \tag{4.4}$$

We have, in view of (4.3),

$$\|\beta\|^2_{H^{-2}(0,T;L^2(0,1))} = \sum_{n=0}^{\infty} \|\beta_n(t)\|^2_{H^{-2}(0,T)}$$
$$\leq \sum_{n=0}^{\infty} C_n \left\| \left(\rho_n^1, \rho_n^0, \mu_n^1, \mu_n^0\right) \right\|^2_{\mathcal{Y}'} = \left\| \left(\psi^0, \psi^1, W^0, W^1\right) \right\|^2_H < \infty.$$

Therefore $\beta \in H^{-2}\left(0, T; L^2(0,1)\right)$. On the other hand,

$$\psi(x,y,t) = \sum_{n=0}^{\infty} \psi_n(y,t)\cos(n\pi x), \quad W(x,t) = \sum_{n=0}^{\infty} V_n(t)\cos(n\pi x)$$

solves (1.3) with the control β given in (4.4) and satisfies (2.7) at time $t = T$.

This concludes the proof of this Theorem. $\square$

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5. Appendix: Proof of Theorem 3.4. First of all we recall a classical result due to A.E. Ingham.

Theorem A. *(see Ingham [9], Theorems 1 and 2) Let* $f = f(t)$ *be of the form* $f(t) = \sum_{n\in\mathbb{Z}} a_n e^{i\lambda_n t}$ *where* λ_n *is a sequence of real numbers.*

We assume that there exists $\gamma > 0$ *such that*

$$\lambda_{n+1} - \lambda_n \geq \gamma,\ \forall n \in \mathbb{Z}. \tag{5.1}$$

Let $J = [0, T]$ *with* $T > \frac{2\pi}{\gamma}$*. Then, there exist two positive constants* $C_1^0, C_2^0 > 0$ *such that*

$$C_1^0 \sum_{n\in\mathbb{Z}} \mid a_n \mid^2 \leq \int_J \mid f(t) \mid^2 dt \leq C_2^0 \sum_{n\in\mathbb{Z}} \mid a_n \mid^2 \tag{5.2}$$

for all $a_n \in \ell^2$

Remark 15. *The constants* C_1^0 *and* C_2^0 *depend only on* $T - \frac{2\pi}{\gamma}$*.*

To prove (3.22) we follow the ideas of Haraux [8], paying special attention to the evaluation of the constants appearing there.

The second inequality of (3.22) results, with $C_2 = 2C_2^0 + 2|J|(2N+1)$, immediately using Theorem A. Indeed we have:

$$\int_J |f(t)|^2\ dt = \int_J \left| \sum_{|n|>N} a_n e^{i\lambda_n t} + \sum_{|n|\leq N} a_n e^{i\lambda_n t} \right|^2 \leq$$

$$\leq 2 \int_J \left(\left| \sum_{|n|>N} a_n e^{i\lambda_n t} \right|^2 + \left| \sum_{|n|\leq N} a_n e^{i\lambda_n t} \right|^2 \right).$$

Applying now Theorem A to the function $g(t) = \sum_{|n|\geq N} a_n e^{i\lambda_n t}$ we obtain:

$$\int_J |f(t)|^2\ dt \leq 2C_2^0 \sum_{|n|>N} |a_n|^2 + 2|J| \left(\sum_{|n|\leq N} |a_n| \right)^2 \leq$$

$$\leq 2C_2^0 \sum_{|n|>N} |a_n|^2 + 2|J|(2N+1) \sum_{|n|\leq N} |a_n|^2 \leq \left(2C_2^0 + 2|J|(2N+1)\right) \sum_{n\in\mathbb{Z}} |a_n|^2.$$

We pass now to prove the first inequality of (3.22). We do this by induction in p, the number of indexes $n \in \mathbb{Z}$ for which $\lambda_{n+1} - \lambda_n < \gamma_\infty$.

If $p = 0$ the result follows from Theorem A with $C_1 = C_1(0) = C_1^0$. Suppose now that $p > 0$.

We write the function f in the form $f(t) = \sum_{n\neq 0} a_n e^{i\lambda_n t} + a_0 e^{i\lambda_0 t}$ where λ_0 is one of those values for which $\lambda_{n+1} - \lambda_n < \gamma_\infty$. Moreover, without loss of generality, we may suppose that $\lambda_0 = 0$ (since we can consider the function $f(t)e^{-i\lambda_0 t}$ instead of $f(t)$). We apply now the induction hypotesis for the function $g(t) = \sum_{n\neq 0} a_n e^{i\lambda_n t}$ and we obtain that:

$$C_1(p-1) \sum_{n\neq 0} |a_n|^2 \leq \int_J |g(t)|^2 \leq C_2(p-1) \sum_{n\neq 0} |a_n|^2. \tag{5.3}$$

We know that $C_2(p-1) = 2C_2^0 + 2|J|(p-1)$. Let $\varepsilon > 0$ be so that $T' = T - \varepsilon > \frac{2\pi}{\gamma_\infty}$. We have

$$\int_0^\varepsilon (f(t+\eta) - f(t))\, d\eta = \sum_{n\neq 0} a_n \left(\frac{e^{i\lambda_n \varepsilon} - 1}{i\lambda_n} - \varepsilon \right) e^{i\lambda_n t}, \quad \forall t \in [0, T'].$$

Applying the induction hypotesis to the function $h(t) = \displaystyle\int_0^\varepsilon (f(t+\eta) - f(t))\, d\eta$ we obtain that:

$$C_1(p-1) \sum_{n\neq 0} \left| \frac{e^{i\lambda_n \varepsilon} - 1}{i\lambda_n} - \varepsilon \right| |a_n|^2 \leq \int_0^{T'} \left| \int_0^\varepsilon (f(t+\eta) - f(t))\, d\eta \right|^2. \tag{5.4}$$

We evaluate now the coefficients $\dfrac{e^{i\lambda_n \varepsilon} - 1}{i\lambda_n} - \varepsilon$. We have:

$$\left| e^{i\lambda_n \varepsilon} - 1 - \varepsilon i \lambda_n \right|^2 = |\cos(\lambda_n \varepsilon) - 1|^2 + |\sin(\lambda_n \varepsilon) - \lambda_n \varepsilon|^2 =$$

$$= 4\sin^4 \left(\frac{\lambda_n \varepsilon}{2} \right) + (\sin(\lambda_n \varepsilon) - \lambda_n \varepsilon)^2 \geq \begin{cases} 4 \left(\frac{\lambda_n \varepsilon}{2} \operatorname{arctg}\left(\frac{\pi}{2}\right) \right)^4, & \text{if } |\lambda_n| \varepsilon \leq \frac{\pi}{2} \\ (|\lambda_n| \varepsilon - 1)^2, & \text{if } |\lambda_n| \varepsilon > \frac{\pi}{2}. \end{cases}$$

Finally, taking into account that $|\lambda_n| \geq \gamma$, we obtain that, for ε small enough,

$$\left| \frac{e^{i\lambda_n \varepsilon} - 1}{i\lambda_n} - \varepsilon \right|^2 \geq \gamma^2 \varepsilon^4.$$

We return now to (5.4) and we get that:

$$\gamma^2 \varepsilon^4 C_1(p-1) \sum_{n\neq 0} |a_n|^2 \leq \int_0^{T'} \left| \int_0^\varepsilon (f(t+\eta) - f(t))\, d\eta \right|^2 \tag{5.5}$$

On the other hand

$$\int_0^{T'} \left| \int_0^\varepsilon (f(t+\eta) - f(t))\, d\eta \right|^2 \leq \int_0^{T'} \varepsilon \int_0^\varepsilon |f(t+\eta) - f(t)|^2\, d\eta \leq$$

$$\leq 2\varepsilon \int_0^{T'} \int_0^\varepsilon \left(|f(t+\eta)|^2 + |f(t)|^2 \right) d\eta \leq 2\varepsilon^2 \int_0^T |f(t)|^2 +$$

$$+ 2\varepsilon \int_0^\varepsilon \int_0^{T'} |f(t+\eta)|^2\, dt\, d\eta \leq 4\varepsilon^2 \int_0^T |f(t)|^2.$$

From (5.5) it follows that

$$\sum_{n\neq 0} |a_n|^2 \leq \frac{4}{\varepsilon^2 \gamma^2 C_1(p-1)} \int_0^T |f(t)|^2. \tag{5.6}$$

Observe that:

$$|a_0|^2 = \left| f(t) - \sum_{n\neq 0} a_n e^{i\lambda_n t} \right|^2 = \frac{1}{T}\int_0^T \left| f(t) - \sum_{n\neq 0} a_n e^{i\lambda_n t} \right|^2 dt \leq$$

$$\leq \frac{2}{T}\left(\int_0^T |f(t)|^2 + \int_0^T \left| \sum_{n\neq 0} a_n e^{i\lambda_n t} \right|^2 \right) \leq \frac{2}{T}\left(\int_0^T |f(t)|^2 + C_2(p-1)\sum_{n\neq 0} |a_n|^2 \right) \leq$$

$$\leq \left(\frac{2}{T} + \frac{8C_2(p-1)}{T\varepsilon^2\gamma^2 C_1(p-1)} \right) \int_0^T |f(t)|^2 .$$

From (5.6) we get that

$$\sum_{n\in \mathbb{Z}} |a_n|^2 \leq \left[\frac{4}{\varepsilon^2\gamma^2 C_1(p-1)} \left(\frac{2C_2(p-1)}{T} + 1 \right) + \frac{2}{T} \right] \int_0^T |f(t)|^2 .$$

We obtain the desired result and a recurrent formula to compute the constant $C_1(p)$:

$$C_1(p) = \left[\frac{4}{\varepsilon^2\gamma^2 C_1(p-1)} \left(\frac{2C_2(p-1)}{T} + 1 \right) + \frac{2}{T} \right]^{-1} .$$

□