PERIODIC SOLUTIONS FOR A BIDIMENSIONAL HYBRID SYSTEM ARISING IN THE CONTROL OF NOISE

SORIN MICU*

Abstract. We consider a simple model arising in the control of noise. We assume that the two-dimensional cavity $\Omega = (0,1) \times (0,1)$ is occupied by an elastic, inviscid, compressible fluid. The potential ϕ of the velocity field satisfies the linear wave equation. The boundary of Ω is divided in two parts Γ_0 and Γ_1. The first one, Γ_0 is flexible and occupied by a dissipative vibrating string. The transversal displacement of the string, W, satisfies a non homogeneous one-dimensional wave equation. On Γ_0 the continuity of the normal velocities of the fluid and the string is imposed. The subset Γ_1 of the boundary is assumed to be rigid and therefore, the normal velocity of the fluid vanishes. This constitutes a non homogeneous dissipative system of two coupled wave equations in dimensions two and one respectively.

The non homogeneous term acting on the flexible part of the boundary (an elastic force or an exterior source of noise) is assumed to be periodic. We are interested on the existence of periodic solutions of this system. Due to the localization of the damping term in a relatively small part of the boundary and to the effect of the hybrid structure of the system, the existence of periodic solutions holds for a restricted class of non homogeneous terms. Some resonance-type phenomena are also exhibited.

Key words. periodic solutions, non homogeneous hyperbolic system, aeromechanic structure interaction, resonance.

AMS subject classifications. 35B10, 93C20, 73K70.

1. Introduction. Let Ω be the two-dimensional square $\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$.

We assume that Ω is filled with an elastic, inviscid, compressible fluid whose velocity field $\vec{v}$ is given by the potential $\phi = \phi(x,y,t) : \vec{v} = \nabla\phi$. By linearization we assume that the potential ϕ satisfies the linear wave equation in $\Omega \times (0,\infty)$.

The boundary $\Gamma = \partial\Omega$ of Ω is divided in two parts: $\Gamma_0 = \{(x,0) : x \in (0,1)\}$ and $\Gamma_1 = \Gamma \backslash \Gamma_0$. The subset Γ_1 is assumed to be rigid and we impose zero normal velocity of the fluid on it. The subset Γ_0 is supposed to be flexible and occupied by a flexible string that vibrates, on the plane where Ω lies, under the pressure of the interior fluid. The string is considered to be dissipative. We suppose also that an exterior force f is acting on the flexible part of the boundary. The displacement of Γ_0, described by the scalar function $W = W(x,t)$, obeys the one-dimensional wave equation with a non-homogeneous term f. On the other hand, on Γ_0 we impose the continuity of the normal velocities of the fluid and the string. The string is assumed to satisfy Neumann boundary conditions on its extremes. All deformations are supposed to be small enough so that linear theory applies.

Under natural initial conditions for ϕ and W the linear motion of this system is described by means of the following coupled wave equations:

$$
\left\{
\begin{array}{lll}
\phi_{tt} - \Delta\phi = 0 & \text{in} & \Omega \times (0,\infty) \\
\frac{\partial\phi}{\partial\nu} = 0 & \text{on} & \Gamma_1 \times (0,\infty) \\
\frac{\partial\phi}{\partial y} = -W_t & \text{on} & \Gamma_0 \times (0,\infty) \\
W_{tt} - W_{xx} + W_t + \phi_t = f & \text{on} & \Gamma_0 \times (0,\infty) \\
W_x(0,t) = W_x(1,t) = 0 & \text{for} & t > 0.
\end{array}
\right. \tag{1.1}
$$

*Departamento de Matemática Aplicada, Facultad de Ciencias Matemáticas, Universidad Complutense, 28040 Madrid, Spain and Facultatea de Matematica-Informatica, Universitatea din Craiova, 1100, Romania, (sorin@sunma4.mat.ucm.es). Supported by grant 5006/1996 (Romania) and CHRX-CT94-0471 of the European Union.

By ν we denote the unit outward normal to Ω.

In (1.1) we have chosen to take the various parameters of the system to be equal to one.

The system (1.1) is well-posed in the energy space $\mathcal{X} = H^1(\Omega)\times L^2(\Omega)\times H^1(\Gamma_0)\times L^2(\Gamma_0)$, for the variables (ϕ, ϕ_t, W, W_t), and for functions $f \in L^1(0,T; L^2(0,1))$.

The energy of the system is defined by:

$$E(t) = \frac{1}{2}\int_\Omega \left[\mid \nabla\phi \mid^2 + \mid \phi_t \mid^2\right] dxdy + \frac{1}{2}\int_{\Gamma_0} \left[\mid W_x \mid^2 + \mid W_t \mid^2\right] dx. \tag{1.2}$$

Observe that

$$\frac{dE}{dt}(t) = -\int_{\Gamma_0} (W_t)^2 + \int_{\Gamma_0} f\, W_t. \tag{1.3}$$

The term W_t in the string equation produces the dissipativity of the system. An important feature of this problem is the localization of the damping term on the part Γ_0 of the boundary. In [12] and [13] we investigated the asymptotic behavior of solutions of the homogeneous system corresponding to (1.1) i.e. with $f \equiv 0$. The main result is the convergence in $\mathcal{X}$ of each trajectory to an equilibrium point as the time goes to infinity. Nevertheless this convergence is not uniform depending on the initial data of the system. This phenomenon, due to the hybrid structure of the system (in other terms, to the nature of the coupling condition), is telling us that the dissipation is to weak at high frequency to ensure the uniform decay.

In this paper we study the existence of periodic solutions of system (1.1) under the assumption that the non homogeneous term f is periodic in time with period T:

$$f(t+T) = f(t), \text{ for all } t > 0. \tag{1.4}$$

We remark that this a natural hypothesis when f models an exterior source of noise which is a frequent situation in the problems of noise reduction.

Observe that, since the dissipation is weak at high frequencies, one can expect that some additional conditions on the Fourier modes of the function f have to be imposed in order to ensure the existence of finite energy periodic solutions. Our aim is to give a complete characterization, in terms of Fourier series, of the space of functions f for which equation (1.1) admits a periodic solution. This space consists on functions in $H^1(0,T; L^2(0,1))$ whose Fourier coefficients, roughly, decay exponentially as the frequency of vibration in the $x-$direction increases.

The Fourier analysis of the system is possible because of the boundary conditions we have chosen for W. Indeed, W is assumed to satisfy Neumann type boundary conditions which are compatible with those of ϕ to develop solutions in Fourier series.

Let us decompose the function f in the following way

$$f = \sum_{n=0}^{\infty} f_n(t)\cos(n\pi x). \tag{1.5}$$

With this decomposition, system (1.1) can be split into the following sequence of one-dimensional systems for $n = 0, 1, \ldots$:

$$\begin{cases} (\psi_n)_{tt} - (\psi_n)_{yy} + n^2\pi^2\psi_n = 0 & y \in (0,1),\ t > 0 \\ (\psi_n)_y(1,t) = 0 & t > 0 \\ (\psi_n)_y(0,t) = -(V_n)_t(t) & t > 0 \\ (V_n)_{tt}(t) + n^2\pi^2 V_n(t) + (V_n)_t + (\psi_n)_t(0,t) = f_n(t) & t > 0. \end{cases} \tag{1.6}$$

First we show that (1.6) has a finite energy periodic solution if $f_n \in H^1(0,T)$. Notice that we need one more derivative on f then what is needed to ensure the well-posedness of the initial boundary-value problem. Combining the one-dimensional results with the Fourier decomposition (1.5), an existence result of periodic solutions for (1.1) is obtained. Nevertheless, in order to ensure the convergence of the series which define the periodic solutions, we need to impose several conditions on the regularity of the non homogeneous term f. First of all, due to the hybrid structure of the system as happens for the 1-d problem (1.6), we ask one more time-derivative for f: $f \in H^1(0,T;L^2(0,1))$. On the other hand, since the damping term is concentrated on a relatively small part of the boundary, f has to belong to a class of analitic functions with respect to the variable x.

We remark that the existence of periodic solutions is equivalent to the boundedness of all trajectories and the absence of the resonance phenomenon. In Section 6 we prove a result of non existence of periodic solutions. This provides an example in which the resonance phenomenon occurs.

The model under consideration is inspired in and related to that of H. T. Banks et al. in [1]. However, there are some important differences between these two models. In [1] the flexible part of the boundary Γ_0 is occupied by a flexible damped beam instead of a flexible string. We have chosen to consider a one-dimensional wave equation instead to simplify the exposition. However, most of the relevant properties remain unchanged considering a beam equation with appropriate boundary conditions (see [14]).

We also remark that we choose Neumann boundary conditions for the string. This choice allows us to separate the variables and to obtain a sequence of one-dimensional systems (1.6) which are easier to study. In the case of Dirichlet boundary conditions, which are considered in [1], this method cannot be applied (see [16] for more detailes).

Other properties of (1.1) like the controllability and the behavior of the spectrum at high frequencies were studied in previous works like [15] and [17].

The rest of the paper is organized as follows. In Section 2 we present rigorously the main results of this paper and make a discussion on their optimality. In Sections 3 and 4 we analyze the one-dimensional problem (1.6) distinguishing the cases $n \geq 1$ and $n = 0$. In Section 5, combining the results of the previous ones, we derive the result of existence of periodic solutions for system (1.1). In Section 6 we discuss the possibility that the resonance phenomenon occurs. In the final Appendix we recall some classical results in semigroup theory that are used in the paper and we prove a convergence theorem for periodic solutions.

Acknowledgements. This work is part of the Doctoral dissertation of the author at Universidad Complutense de Madrid in February 1996. Most of this work was done while he was visiting Universidad Complutense with the financial support of the European Tempus Program "Matarou". The author acknowledges the responsables of this Program and, in particular, Doina Cioranescu for their continuous support. The author is also grateful to Eduard Feireisl and Enrique Zuazua for stimulating discussions on the subject of this paper.

2. The main results: statements and discussion.

2.1. Existence results. As we said in the introduction the problem of existence of periodic solutions of system (1.1) is reduced to study the one-parameter family of one-dimensional systems (1.6). For system (1.6) we have the following result:

THEOREM 2.1. *Let $n \geq 1$. If $f_n \in H^1(0,T)$ is a periodic function of period T then (1.6) has a unique T-periodic solution $(\psi_n, V_n) \in \mathcal{C}^1([0,\infty); L^2(0,1) \times \mathbb{R}) \cap$*

$\mathcal{C}([0,\infty); H^1(0,1) \times I\!R)$.

REMARK 1. *Observe that problem (1.6) is well possed for functions* $f_n \in L^1(0,T)$. *In Theorem 2.1 we have imposed an extra assumption on* f_n. *This is due to the hybrid boundary conditions we are dealing with, which are, roughly, of second order. This is also related to the fact that the spectrum of system (1.6) approches the imaginary axis at high frequencies. We shall make a detailed study of the case in which* $f \notin H^1(0,T)$ *in the last section of the paper.*

Let $\mathcal{X} = H^1(\Omega) \times L^2(\Omega) \times H^1(\Gamma_0) \times L^2(\Gamma_0)$. For each $(\phi^0, \phi^1, W^0, W^1) \in \mathcal{X}$ there is a unique weak solution $(\phi, \phi_t, W, W_t) \in \mathcal{C}([0,T]; \mathcal{X})$ of system (1.1) for which the energy function

$$E(t) = \frac{1}{2}\int_\Omega \left[\mid \nabla\phi \mid^2 + \mid \phi_t \mid^2\right] dxdy + \frac{1}{2}\int_{\Gamma_0} \left[\mid W_x \mid^2 + \mid W_t \mid^2\right] dx$$

belongs to $\mathcal{C}^1[0,T]$.

The following result of stability of solutions of the homogeneous system corresponding to (1.1) was proved in [13] (see also [12]). Let us recall it.

THEOREM 2.2. *Suppose that* $f = 0$. *For each initial data* $(\phi_0, \phi_1, W_0, W_1)$ *in* $\mathcal{X}$ *the corresponding weak solution of (1.1) has the property that*

$$2E(t) = ||(\phi - c_1,\, \phi_t,\, W - c_2,\, W_t)||_{\mathcal{X}} \longrightarrow 0 \ \textit{as}\ t \to \infty, \tag{2.1}$$

where $c_1 = \int_\Omega \phi_1 - \int_{\Gamma_0} W_1 + \int_{\Gamma_0} \phi_0$ *and* $c_2 = \int_{\Gamma_0} W_0 - \int_{\Gamma_0} \phi_1$.

Moreover, the convergence to the equilibrium of the trajectories is not uniform, i.e. there are no constants $\omega > 0$ *and* $M > 0$ *such that*

$$|E(t)| \leq ME(0)\ exp\,(-\omega\, t),\ \textit{for all}\ t \geq 0\ \textit{and for all solution of (1.1)}. \tag{2.2}$$

REMARK 2. *Let us make now some remarks about the proof of Theorem 2.1. The existence of periodic solutions is often obtained by a classical fixed point method for the Poincaré map. More precisely, let us consider the abstract equation*

$$U_t + \mathcal{A}U = F, \tag{2.3}$$

where F *is a* $T-$*periodic function.*

Suppose that $\mathcal{A}$ *is a maximal-monotone operator in a Hilbert space* X, *generating a semigroup of contractions* $\{S(t)\}_{t\geq 0}$ *with exponential decay rate, i.e. such that there exist two positive constants* M *and* ω *with the property that*

$$||S(t)||_X \leq M\ exp\,(-\omega\, t), \quad \forall t > 0. \tag{2.4}$$

We define the map

$$\mathcal{J} : X \longrightarrow X, \quad \mathcal{J}U_0 = U(T),$$

where $U(T)$ *is the solution of (2.3) with the initial data* U_0 *evaluated in* $t = T$. *Observe that* $U(T) = S(T)U_0$.

Due to the property (2.4) of the semigroup we obtain that $\mathcal{J}^n$ *is a contraction for* n *large enough. It follows that* $\mathcal{J}^n$ *has a unique fixed point which implies that* $\mathcal{J}$ *has a unique fixed point. But a fixed point of the map* $\mathcal{J}$ *gives a periodic solution of (2.3).*

In our case this method cannot be applied directly since, as we mentioned before, the semigroup associated to (1.1) (and the one corresponding to (1.6)) does not have an exponential decay (see Theorem 2.2). The idea of the proof of Theorem 2.1 consists on introducing an artificial dissipation which produces the exponential decay of the energy. This allows to prove the existence of a periodic solution for the perturbed system. Finally we prove the convergence to a periodic solution of the initial problem when the perturbative term goes to zero.

In order to give a result for the two dimensional case we need some estimates of the periodic solutions given by Theorem 2.1.

THEOREM 2.3. *Let $n \geq 1$. If $f_n \in H^1(0,T)$ is a $T-$periodic function then the periodic solution of (1.6) given by Theorem 2.1, (ψ_n, V_n), satisfies*

$$\int_0^T \left[((V_n)_t)^2 + n^2\pi^2 (V_n)^2)\right] \leq c \int_0^T (1+n^2\pi^2)(f_n)^2 \tag{2.5}$$

$$\begin{aligned} &\int_0^T \left[((\psi_n)_t)^2 + ((\psi_n)_y)^2 + n^2\pi^2(\psi_n)^2)(y)\right] \leq \\ &\leq c' \int_0^T \left[(1+n^2\pi^2)((f_n)_t)^2 + (1+n^4\pi^4+n^6\pi^6)(f_n)^2\right] exp\left(\frac{3n\pi}{2}y\right) \end{aligned} \tag{2.6}$$

where c and c' are two constants independent of n.

REMARK 3. *The exponential constant in n appearing in (2.6) indicates that we can have solutions of (1.1) for which the energy concentrated on the string decays exponentially fast as the frequency of vibration in the $x-$direction increases.*

We return now to equation (1.6) considering the case $n = 0$. In view of Theorem 2.2 if a particular solution of (1.6) is not bounded as $t \to \infty$ then periodic solutions do not exist. Observe that, if f_0 is a constant function then $(\psi, V)(t) = (f_0\, t, 0)$ is an unbounded solution of (1.6). Therefore, in order to ensure the existence of periodic solutions, an extra assumption on f_0 is necessary. We have the following result:

THEOREM 2.4. *Let $n = 0$. If $f_0 \in H^1(0,T)$ is a periodic function of period T and $\int_0^T f_0(s)\, ds = 0$ then (1.6) has a periodic solution. Moreover, if $(\bar{\psi}_0, \bar{V}_0)$ is another periodic solution then $(\psi_0, V_0) - (\bar{\psi}_0, \bar{V}_0) = (c_1, c_2)$ where c_1 and c_2 are two constant functions.*

REMARK 4. *The techniques used to prove Theorem 2.4 are different from those of Theorem 2.1. This is due to the fact that the semigroup corresponding to equation (1.6) is not a semigroup of contractions in the case $n = 0$. Therefore, some of the estimates we obtained in the case $n \geq 1$ are not longer valid. The proof of Theorem 2.4 is based on the construction of explicit solutions for $f_0 = e^{i\,\nu_m\, t}$, with $\nu_m \in \mathbb{R}$.*

Let us now state the result of existence of periodic solutions for the two-dimensional system (1.1).

We use the Fourier decomposition method described in the Introduction. Thus we develop the non-homogeneous term f in Fourier series:

$$f(t,x) = \sum_{n=0}^{\infty} f_n(t) \cos(n\pi x). \tag{2.7}$$

The main result concerning the existence of periodic solutions of (1.1) is the following.

THEOREM 2.5. *If $f \in H^1((0,T);L^2(0,1))$ is a periodic function of period T and satisfies:*

$$\sum_{n=1}^{\infty} n \int_0^T ((f_n)_t)^2\, dt \;\; exp\left(\frac{3\pi}{2} n\right) < \infty, \tag{2.8}$$

$$\sum_{n=1}^{\infty} n^5 \int_0^T (f_n)^2\, dt \;\; exp\left(\frac{3\pi}{2} n\right) < \infty, \tag{2.9}$$

$$\int_0^T f_0(s)ds = 0, \tag{2.10}$$

then (1.1) has a periodic solution of finite energy.

REMARK 5. *If (ψ_n, V_n), $n \geq 0$ are the periodic solutions given by Theorems 2.1 and 2.4 then $\sum_{n=0}^{\infty} (\psi_n, V_n) \cos(n\pi x)$ is a periodic solution for (1.1) if this series is convergent in the energy space. Taking into account the estimates of Theorem 2.3 we can easily deduce that this series is convergent if (2.8) and (2.9) are satisfied.*

REMARK 6. *Taking into account the results of Theorem 2.2 we can discuss now the uniqueness of the periodic solutions of (1.1). Suppose that (1.1) has two periodic solutions (ϕ, W) and $(\bar{\phi}, \bar{W})$. Then $(\phi - \bar{\phi}, W - \bar{W})$ is a periodic solution of the homogeneous system. On the other hand Theorem 2.2 implies that $(\phi - \bar{\phi}, W - \bar{W})(t)$ tends to a constant function (c_1, c_2) as t tends to infinity. This implies that $(\phi, W)(t) - (\bar{\phi}, \bar{W})(t) = (c_1, c_2)$ and the two periodic solutions differ by a constant function. Observe that this agrees with Theorem 2.4.*

REMARK 7. *We have obtained that equation (1.1) has a periodic solution if the Fourier coefficients in the $x-$variable of the function f decay exponentially. This is due to the fact that there are solutions of (1.1) for which the energy concentrated on the string decays exponentially fast as the frequency of vibration in the $x-$direction increases. This phenomenon is due to the fact that the support of the dissipation term is relatively small and therefore there are rays of the geometric optics which never intersect Γ_0 (for instance, every segment $\{(x, y_0) : 0 < x < 1\}$ is such a ray for $y_0 \in (0,1)$). In this sense this result is strongly related to our previous results on the controllability of the system (see [15]).*

2.2. A non existence result. The existence of periodic solutions is equivalent to the boundedness of all the trajectories. Indeed, one of the implications is a direct consequence of Theorem 2.2. The other one relies on the following special case of the Browder-Petrysyn fixed-point theorem (see [8], Lemma IV.2.2.2, p. 157 and [4]):

THEOREM 2.6. *Let $\mathcal{H}$ be a real Hilbert space and $\mathcal{T} : \mathcal{H} \longrightarrow \mathcal{H}$ be such that:*

$$\forall (u,v) \in \mathcal{H} \times \mathcal{H}, \quad |\mathcal{T}u - \mathcal{T}v| \leq |u - v|$$
$$\exists u_0 \in \mathcal{H}, \quad Sup\{|\mathcal{T}^n u_0|\} < +\infty.$$

Then there exists $\bar{u} \in \mathcal{H}$ such that $\bar{u} = \mathcal{T}(\bar{u})$.

Let us show that the existence of a bounded solution of (1.1) implies the existence of a periodic solution of that system. Indeed, considering the map $\mathcal{T} = S(T)U_0$ like in Remark 2 and applying Theorem 2.6 it follows that $\mathcal{T}$ has a fixed point. Hence, a periodic solution exists.

When periodic solutions do not exist, the dissipation is so inefficient that a bounded signal $f(t)$ induces unbounded (in time) solutions of the problem. This phenomenon is usually called resonance and is due to the interaction between the "spectrum" of the differential operator and the one of $f(t)$.

It is easy to see that the eigenvalues of the conservative system corresponding to (1.6) (obtained by dropping the term V_t in the last equation) are the roots of the following algebraic equation:

$$\tan\sqrt{\zeta^2 - n^2\pi^2} = \frac{\zeta^2}{(\zeta^2 - n^2\pi^2)\sqrt{\zeta^2 - n^2\pi^2}}. \tag{2.11}$$

Let us denote by E the set of the real roots of (2.11) (for a detailed analysis of these eigenvalues see [17]). We have the following result:

THEOREM 2.7. *Suppose the period T is such that a sequence $(m_k)_{k\geq 0} \subset \mathbb{Z}$ exists with $|m_k| \to \infty$ and*

$$dist\left(\frac{2\pi m_k}{T}, E\right) = \mathcal{O}\left(\frac{1}{m_k^2}\right) \quad as\ k \to \infty. \tag{2.12}$$

Let f be a $T-$periodic function such that $f \in L^2(0,T) \setminus H^1(0,T)$. Then there is no periodic solution of finite energy of problem (1.6).

REMARK 8. *In many conservative systems the resonance phenomenon is produced when the frequency of vibration of the non homogeneous term coincides with some eigenvalue of the system. This is not true for our system. Indeed, if we consider a function $f \in H^1(0,1)$ of period $T = \frac{2\pi}{\nu}$, where ν is an eigenvalue of the conservative system corresponding to (1.6), Theorem 2.1 shows that the resonance phenomenon can not occur. Condition (2.12) indicates that the resonance may only be produced when integer multiples of the frequency $\frac{2\pi}{T}$ approach the set of the wave numbers of the conservative system at a given rate.*

REMARK 9. *With the same notation as in Remark 2 let us consider that the non homogeneous term, F, has the following expantion:*

$$F(t,x) = \sum_{m\in\mathbb{Z}} F_m(x) e^{i\,\frac{2m\pi}{T}t} \tag{2.13}$$

and suppose that the series is convergent in $L^2(0,T;X)$.

A periodic solution of (2.3) can be written as

$$U(t,x) = \sum_{m\in\mathbb{Z}} U_m(x) e^{i\,\frac{2m\pi}{T}t} \tag{2.14}$$

where $U_m = R\left(\frac{2m\pi\, i}{T} : \mathcal{A}\right) F_m$, $R(\lambda : \mathcal{A})$ being the resolvent of $\mathcal{A}$ in λ.

By Theorem 7.7 and Remark 17 it follows that the existence of a finite-energy periodic solution is equivalent to

$$\sum_{m\in\mathbb{Z}} ||U_m||_X^2 < \infty. \tag{2.15}$$

It is easy to see that (2.15) is satisfied when the semigruop generated by $\mathcal{A}$, $\{S(t)\}_{t\geq 0}$, *satisfies (2.4). Indeed, it follows that*

$$\left|\left|R\left(\frac{2m\pi\, i}{T}:\mathcal{A}\right)F_m\right|\right| = \left|\left|\int_0^\infty S(t)F_m\,dt\right|\right| \leq \frac{M}{\omega}||F_m||.$$

Therefore, the converge of (2.13) in $L^2(0,T;X)$ *implies (2.15).*

The semigroup corresponding to our system does not have an exponential decay and in fact the eivenvalues of $\mathcal{A}$ *approach the imaginary axis as the frequency increases.*

Let us show that, in this case, the the converge of (2.13) in $L^2(0,T;X)$ *does not imply (2.15). To do this, we consider, for each* $m\in\mathbb{Z}$, *an element* $F_m\in X$ *such that*

$$\left|\left|R\left(\frac{2m\pi\, i}{T}:\mathcal{A}\right)F_m\right|\right|^2 \geq \left|\left|R\left(\frac{2m\pi\, i}{T}:\mathcal{A}\right)\right|\right|\,||F_m||^2-||F_m||^2.$$

It follows that

$$\sum_{m\in\mathbb{Z}}||U_m||_X^2 \geq \sum_{m\in\mathbb{Z}}\left(\left|\left|R\left(\frac{2m\pi\, i}{T}:\mathcal{A}\right)\right|\right|\,||F_m||^2-||F_m||^2\right)\geq$$

$$\geq \sum_{m\in\mathbb{Z}}\left(\frac{1}{dist^2\left(\frac{2m\pi\, i}{T};\Sigma(\mathcal{A})\right)}-1\right)||F_m||^2,$$

where $\Sigma(\mathcal{A})$ *is the spectrum of* $\mathcal{A}$.

It the distance $dist\left(\frac{2m\pi\, i}{T},\Sigma(\mathcal{A})\right)$ *tends to zero as m goes to infinity (as it happens in our case) the convergence of (2.13) in* $L^2(0,T;X)$ *does not imply (2.15).*

Remark 10. *The conditions of Theorem 2.7 are very restrictive and the existence of periods* T *satisfying (2.12) seems to be a difficult problem. For instance, in the simplest case,* $n=0$, *the elements of the set* E, $(\nu_m)_{m\in\mathbb{Z}}$, *have the following asymptotic expantion (see Olver [18], p. 12):*

$$\nu_m = m\pi + \frac{1}{m\pi} + \mathcal{O}\left(\frac{1}{m^3}\right).$$

It follows that (2.12) is satisfied iff there is a sequence, $(m_p)_{p\geq 0}\subset\mathbb{N}^*$, *such that*

$$\frac{2\pi}{T}-\frac{m_p}{m_k}\pi-\frac{1}{m_p m_k\pi}=\mathcal{O}\left(\frac{1}{m_k^3}\right). \tag{2.16}$$

The existence of numbers T *with the property (2.16) is not an easy problem in number theory and it seems to be open.*

The following theorem shows that the rational periods can not verify (2.12).

Theorem 2.8. *If* $T\in\mathbb{Q}$ *and* $f\in L^2(0,T)$ *is a* $T-$*periodic function, system (1.6) has a periodic solution.*

3. The one dimensional problem: the case $n \neq 0$. In this section we shall give the proofs of Theorems 2.1 and 2.3. In order to prove the existence of periodic solutions for the one-dimensional sistems (1.6) we introduce a perturbative term in the equation of ψ:

$$(3.1)\quad \begin{cases} \psi_{tt} - \psi_{yy} + \varepsilon\psi_t + n^2\pi^2\psi = 0 & \text{for} \quad (y,t) \in (0,1) \times (0,\infty) \\ \psi_y(1,t) = 0 & \text{for} \quad t > 0 \\ \psi_y(0,t) = -V_t(t) & \text{for} \quad t > 0 \\ V_{tt}(t) + n^2\pi^2 V(t) + V_t + \psi_t(0,t) = f(t) & \text{for} \quad t > 0. \end{cases}$$

Remark 11. *In system (3.1) and in the sequel we drop the index n from the unknowns (ψ, V) and the non-homogeneous term f to simplify the notation.*

First of all we define the Hilbert space $\mathcal{Y} = H^1(0,1) \times L^2(0,1) \times I\!\!R \times I\!\!R$ with the inner product defined by:

$$(3.2)\qquad (f,g) = \int_0^1 \left((f_1)_y (g_1)_y + n^2\pi^2 f_1\, g_1\right) + \int_0^1 (f_2 g_2) + f_3 g_3 + f_4 g_4$$

where $f = (f_1, f_2, f_3, f_4)$ and $g = (g_1, g_2, g_3, g_4)$ are two elements of $\mathcal{Y}$.

Next we consider the operator

$$(3.3)\qquad \begin{gathered} \mathcal{A}^1_\varepsilon : \mathcal{D}(\mathcal{A}^1_\varepsilon) \subset \mathcal{Y} \longrightarrow \mathcal{Y}, \\ \mathcal{A}^1_\varepsilon(\psi, \xi, w, v) = (-\xi, -\psi_{xx} + \varepsilon\xi + n^2\pi^2\psi, -v, n^2\pi^2 w + v + \xi(0)) \end{gathered}$$

with domain

$$\mathcal{D}(\mathcal{A}^1_\varepsilon) = \left\{(\psi, \xi, w, v) \in \mathcal{Y} : \mathcal{A}^1_\varepsilon(\psi, \xi, w, v) \in \mathcal{Y}, \partial\psi/\partial y(1) = 0,\ \partial\psi/\partial y(0) = -v\right\}.$$

Theorem 3.1. *The operator $(\mathcal{D}(\mathcal{A}^1_\varepsilon), \mathcal{A}^1_\varepsilon)$ is maximal-monotone and it generates a strongly continuous semigroup of contractions in $\mathcal{Y}$, $\{S_\varepsilon(t)\}_{t\geq 0}$. Moreover, the domain $\mathcal{D}(\mathcal{A}^1_\varepsilon)$ is dense and compact in $\mathcal{Y}$.*

Proof. It is easy to see that $\mathcal{A}^1_\varepsilon$ is monotone since

$$(\mathcal{A}^1_\varepsilon(\psi, \xi, w, v), (\psi, \xi, w, v)) = \varepsilon \int_0^1 \xi^2 + v^2 \geq 0.$$

The fact that $\mathcal{A}^1_\varepsilon$ is maximal is an immediate consequence of Lax-Milgram's Lemma. It results that $\mathcal{A}^1_\varepsilon$ is maximal-monotone and that $\mathcal{D}(\mathcal{A}^1_\varepsilon)$ is dense in $\mathcal{Y}$.

Classical results of regularity of the Laplace operator imply that the domain $\mathcal{D}(\mathcal{A}^1_\varepsilon) \subset H^2(0,1) \times H^1(0,1) \times I\!\!R \times I\!\!R$ and therefore it is compact in $\mathcal{Y}$. □

We write now the equation (3.1) in the following abstract form:

$$(3.4)\qquad \begin{cases} U_t(t) + \mathcal{A}^1_\varepsilon U(t) = F, \quad \forall t \geq 0 \\ U(0) = U_0 \in \mathcal{D}(\mathcal{A}^1_\varepsilon) \\ U(t) \in \mathcal{D}(\mathcal{A}^1_\varepsilon), \quad \forall t \geq 0 \end{cases}$$

with $U = (\psi, \psi_t, V, V_t)$ and $F = (0,0,0,f)$.

Remark 12. *If we define the energy of a solution $U = (\psi, \psi_t, V, V_t)$ of (3.4) by*

$$(3.5)\qquad E_n(t) = \frac{1}{2}\int_0^1 (\psi_t^2 + \psi_y^2 + n^2\pi^2\psi^2) + \frac{1}{2}(w_t^2 + n^2\pi^2 w^2)$$

then we obtain that

$$\frac{dE_n}{dt}(t) = -\varepsilon \int_0^1 (\psi_t)^2 \, - (V_t)^2(t) + f(t)V_t(t). \tag{3.6}$$

Observe that, when $\varepsilon > 0$, an extra dissipative term has been introduced in the system.

The following theorem is essential in order to apply a fixed point method as we shall do.

THEOREM 3.2. *Let $\varepsilon > 0$ and $f = 0$. There are two constants $M > 1$ and $\omega > 0$, independent of the initial data of the system (3.4) but depending on the ε and n, such that*

$$E_n(t) \leq M E_n(0) e^{-\omega t}, \quad \forall t \geq 0 \tag{3.7}$$

for each weak solution of (3.4).

Proof. First step: Suppose that $U = (\psi, \psi_t, V, V_t)$ is a classical solution of (3.4).

Let $\delta > 0$ and define

$$F_n(t) = E_n(t) + \delta \left(\int_0^1 \psi_t \psi \, + w_t w + \psi(0) w \right), \quad \forall t \geq 0. \tag{3.8}$$

It follows that

$$| F_n(t) - E_n(t) | \leq \delta \left(\int_0^1 | \psi_t \psi | \, dy + | w_t w | + | \psi(0) w | \right) \leq \delta C_1 E(t)$$

where C_1 is a constant which does not depend on ε.

For $\delta < \dfrac{1}{2C_1}$ it follows that

$$\frac{1}{2} E_n(t) < F_n(t) < \frac{3}{2} E_n(t), \quad \forall t \geq 0. \tag{3.9}$$

We compute now

$$(F_n)_t(t) = (E_n)_t(t) + \delta \int_0^1 (\psi_t^2 + \psi \psi_{tt}) \, dy \, + \delta(w_t^2 + w w_{tt}) + \delta(\psi(0) w_t + \psi_t(0) w) =$$

$$= -\varepsilon \int_0^1 (\psi_t)^2 \, dt \, - (w_t)^2 + \delta \left(\int_0^1 (\psi_t)^2 \, dt \, - \int_0^1 (\psi_y)^2 \, dt \, - \psi_y(0)\psi(0) - \right.$$

$$\left. -\varepsilon \int_0^1 \psi_t \psi \, dt \, - n \int_0^1 \psi^2 \, dt \, + (w_t)^2 - n w^2 - \psi_t(0) w + w_t \psi(0) + w \psi_t(0) \right) \leq$$

$$\leq - \left(\varepsilon - \delta - \frac{\varepsilon \delta}{2\mu_2} \right) \int_0^1 (\psi_t)^2 \, dt \, - \delta \left(1 - \frac{C_1 \mu_1}{2} - \frac{C_1 \mu_4}{2} \right) \int_0^1 (\psi_y)^2 \, dt \, -$$

$$-\delta \left(n - \frac{C_1 \mu_1}{2} - \frac{\varepsilon \mu_2}{2} - \frac{C_1 \mu_4}{2} \right) \int_0^1 \psi^2 \, dt \, -$$

$$-\left(1-\delta-\frac{\delta}{2\mu_1}-\frac{\delta}{2\mu_3}-\frac{\delta}{2\mu_4}\right)(w_t)^2-\delta\left(n-\frac{\mu_3}{2}\right)w^2.$$

If we choose $\mu_3=\mu_2=\delta$ and $\mu_1=\mu_4=4\delta$ we obtain that

$$(F_n)_t(t)=-\left(\frac{\varepsilon}{2}-\delta\right)\int_0^1(\psi_t)^2\,dt\ -\delta(1-4C_1\delta)\int_0^1(\psi_y)^2\,dt\ -$$

$$-\delta\left(n-\delta(\frac{\varepsilon}{2}+4C_1)\right)\int_0^1\psi^2\,dt\ -\left(\frac{1}{4}-\delta\right)(w_t)^2-\delta\left(n-\frac{\delta}{2}\right)w^2.$$

For $\delta<\ \min\left\{\frac{\varepsilon}{2},\frac{1}{4},\frac{1}{4C_1},2n,\frac{n}{4C_1+\frac{\varepsilon}{2}}\right\}$ it follows that

$$(F_n)_t(t)\le-\beta E_n(t),$$

with $\beta=\ \min\left\{\frac{\varepsilon}{2}-\delta,\,\delta(1-4C_1\delta),\,\delta\left(n-\delta(\frac{\varepsilon}{2}+4C_1)\right),\,\frac{1}{4}-\delta,\,\delta\left(n-\frac{\delta}{2}\right)\right\}.$

Taking (3.9) into account we obtain

$$E_n(t)\le ME_n(0)e^{-\omega t},\quad\forall t\ge0, \tag{3.10}$$

where $\omega=\frac{2}{3}\beta$.

Second step: Suppose now that $U=(\psi,\xi,w,v)$ is a weak solution of (3.4) corresponding to the initial data $U^0=(\psi^0,\psi^1,w^0,w^1)\in\mathcal{Y}$. Since $\mathcal{D}(\mathcal{A}^1_\varepsilon)$ is dense in $\mathcal{Y}$ we can apply a standard density argument to deduce that (3.7) is satisfied for this type of solutions too. The constants M and ω are the same as in the previous step. □

Remark 13. *We remark that the hypothesis $\varepsilon>0$ is essential in the proof of Theorem 3.2. We have that, for ε small enough, $\omega<\frac{\varepsilon}{3}$. Therefore, when ε tends to zero, ω tends to zero. In fact, as we have proved in [12] and [13], the semigroup $\{S_0(t)\}_{t\ge0}$, corresponding to $\varepsilon=0$, does not have an exponential decay.*

In the case $\varepsilon>0$ we can apply a fixed point method in order to deduce the existence of a periodic solution of (3.1). We have the following result

Theorem 3.3. *Let $\varepsilon>0$. If $f\in L^1(0,T)$ is a periodic function of period T, (3.1) has a periodic weak solution $U_\varepsilon\in\mathcal{C}([0,\infty),\mathcal{Y})$.*

Moreover, if $f\in W^{1,1}(0,T)$, $U_\varepsilon\in\mathcal{C}^1([0,\infty),\mathcal{Y})\cap\mathcal{C}([0,\infty),\mathcal{D}(\mathcal{A}^1_\varepsilon))$ and the periodic solution is a classical one.

Proof. We define the map

$$\mathcal{J}:\mathcal{Y}\longrightarrow\mathcal{Y},\quad\mathcal{J}(\psi^0,\psi^1,w^0,w^1)=(\psi(T),\psi_t(T),V(T),V_t(T)) \tag{3.11}$$

where (ψ,ψ_t,V,V_t) is the solution of (3.4) with $U^0=(\psi^0,\psi^1,V^0,V^1)$. Then

$$\|\mathcal{J}^nU^0-\mathcal{J}^nU^1\|_{\mathcal{Y}}=\|S_\varepsilon(kT)(U^0-U^1)\|_{\mathcal{Y}}\le$$

$$\le\|S_\varepsilon(kT)\|_{\mathcal{L}(\mathcal{Y},\mathcal{Y})}\|U^0-U^1\|_{\mathcal{Y}}=c\|U^0-U^1\|_{\mathcal{Y}}$$

with $\|S_\varepsilon(kT)\|_{\mathcal{L}(\mathcal{Y},\mathcal{Y})}=c<1$ for k large enough since, by (3.7),

$$\|S_\varepsilon(t)\|\le Me^{-\omega t},\ \forall t\ge0.$$

It follows that $\mathcal{J}^k : \mathcal{Y} \longrightarrow \mathcal{Y}$ is a contraction and it has a unique fixed point in $\mathcal{Y}$, that we denote by U^0_ε.

We prove now that $\mathcal{J}^k(U^0_\varepsilon)$ is the unique fixed point of $\mathcal{J}$. We have

$$\mathcal{J}^k(U^0_\varepsilon) = U^0_\varepsilon \Rightarrow \mathcal{J}(\mathcal{J}^k(U^0_\varepsilon)) = \mathcal{J}(U^0_\varepsilon) \Rightarrow \mathcal{J}^k(\mathcal{J}(U^0_\varepsilon)) = \mathcal{J}(U^0_\varepsilon).$$

It follows that $\mathcal{J}(U^0_\varepsilon)$ is a fixed point of $\mathcal{J}^k$ and from the uniqueness of this fixed point we deduce that $\mathcal{J}(U^0_\varepsilon) = U^0_\varepsilon$. Hence U^0_ε is a fixed point for $\mathcal{J}$.

From the uniqueness of the fixed point of $\mathcal{J}^k$ it follows that $\mathcal{J}$ has a unique fixed point.

The initial data U^0_ε gives us a periodic solution of (3.4), denoted by U_ε.

We prove now that $U^0_\varepsilon \in \mathcal{D}(\mathcal{A}^1_\varepsilon)$ which implies that the corresponding periodic solution U_ε is a classical solution if $f \in W^1(0,T)$. To obtain this it is sufficient to prove that $S_\varepsilon(t)$ has an exponential decay not only in $\mathcal{Y}$ norm but also in the norm of $\mathcal{D}(\mathcal{A}^1_\varepsilon)$.

For $U^0 \in \mathcal{D}(\mathcal{A}^1_\varepsilon)$ we have

$$\| S_\varepsilon(t)U^0 \|_{\mathcal{D}(\mathcal{A}^1_\varepsilon)} = \| S_\varepsilon(t)U^0 \|_{\mathcal{Y}} + \| \mathcal{A}^1_\varepsilon S_\varepsilon(t)U^0 \|_{\mathcal{Y}} =$$

$$= \| S_\varepsilon(t)U^0 \|_{\mathcal{Y}} + \| S_\varepsilon(t)\mathcal{A}^1_\varepsilon U^0 \|_{\mathcal{Y}} \leq$$

$$\leq \| S_\varepsilon(t) \|_{\mathcal{L}(\mathcal{Y},\mathcal{Y})} (\| U^0 \|_{\mathcal{Y}} + \| \mathcal{A}^1_\varepsilon U^0 \|_{\mathcal{Y}}) = \| S_\varepsilon(t) \|_{\mathcal{L}(\mathcal{Y},\mathcal{Y})} \| U^0 \|_{\mathcal{D}(\mathcal{A}^1_\varepsilon)} .$$

So, we can argue as before with $\mathcal{D}(\mathcal{A}^1_\varepsilon)$ instead of $\mathcal{X}$. It follows that

$$\| S_\varepsilon(t) \|_{\mathcal{L}(\mathcal{D}(\mathcal{A}^1_\varepsilon),\mathcal{D}(\mathcal{A}^1_\varepsilon))} \leq \| S_\varepsilon(t) \|_{\mathcal{L}(\mathcal{Y},\mathcal{Y})} .$$

Hence $\| S_\varepsilon(t) \|_{\mathcal{L}(\mathcal{D}(\mathcal{A}^1_\varepsilon),\mathcal{D}(\mathcal{A}^1_\varepsilon))}$ has an exponential decay.

It follows that the fixed point of $\mathcal{J}$ belongs to $\mathcal{D}(\mathcal{A}^1_\varepsilon)$. The regularity of the periodic solution is obtained now by Theorem 7.4. □

We prove now some estimates for the periodic solution given in Theorem 3.3.

THEOREM 3.4. *If $f \in H^2(0,1)$ then the periodic solution of (3.4) given by Theorem 3.2, $U_\varepsilon = (\psi_\varepsilon, (\psi_\varepsilon)_t, V_\varepsilon, (V_\varepsilon)_t)$, satisfies*

$$\int_0^T (((V_\varepsilon)_t)^2 + n^2\pi^2(V_\varepsilon)^2)\, dt \ \leq c \int_0^T (1+n^2\pi^2) f^2\, dt, \tag{3.12}$$

$$\begin{aligned} &\int_0^T (((\psi_\varepsilon)_t)^2 + ((\psi_\varepsilon)_y)^2 + n^2\pi^2(\psi_\varepsilon)^2)(y)\, dt \ \leq \\ &\leq c' \int_0^T [(1+n^2\pi^2)(f_t)^2 + (1+n^4\pi^4+n^6\pi^6)(f)^2]\, dt\, exp\,(\frac{3n\pi}{2}y), \quad \forall y \in (0,1) \end{aligned} \tag{3.13}$$

where c and c' are two constants which do not depend on ε and n.

Proof. Integrating in time the derivative of the energy of the system and the derivative of the energy of the system obtained by taking a time-derivative in (3.1), we get that:

$$\begin{aligned} \int_0^T ((V_\varepsilon)_t)^2\, dt \ &\leq \int_0^T f^2\, dt, \\ \int_0^T ((V_\varepsilon)_{tt})^2\, dt \ &\leq \int_0^T (f_t)^2\, dt. \end{aligned} \tag{3.14}$$

The last equation of (3.1) implies that $n^2\pi^2 \int_0^T V_\varepsilon = \int_0^T f$. Applying Poincaré's inequality we deduce that:

$$\int_0^T V_\varepsilon^2 \le C\left(\frac{1}{n^2\pi^2}+1\right)\int_0^T f^2. \tag{3.15}$$

We go back to the equation for V_ε in (3.1) and we obtain:

$$\int_0^T ((\psi_\varepsilon)_t)^2(t,0)\,dt \ \le C\left((n^2\pi^2+1)\int_0^T f^2\,dt\ + \int_0^T (f_t)^2\,dt\right). \tag{3.16}$$

Since $\int_0^T \psi_\varepsilon\,(t,y)dt\ = 0$, for all $y \in (0,1)$ it follows from Poincaré's inequality that

$$\int_0^T (\psi_\varepsilon)^2(t,0)\,dt\ \le C\int_0^T ((\psi_\varepsilon)_t)^2(t,0)\,dt. \tag{3.17}$$

We can apply now multiplier techniques (see [10], Lemma 1.3, p.139). Multiplying the first equation of (3.1) by $(1-y)(\psi_\varepsilon)_y$, integrating by parts and applying Gronwall's Lemma we obtain that

$$\begin{aligned}&\int_0^T (((\psi_\varepsilon)_t)^2+((\psi_\varepsilon)_y)^2+n^2\pi^2(\psi_\varepsilon)^2)\,dt\ (y) \le\\ &\int_0^T (((\psi_\varepsilon)_t)^2+((\psi_\varepsilon)_y)^2+n^2\pi^2(\psi_\varepsilon)^2)\,dt\ (0)\ \ exp((\frac{3\pi n}{2}+\frac{\varepsilon}{2})y).\end{aligned} \tag{3.18}$$

With the estimates (3.14), (3.16) and (3.17) for $\psi_\varepsilon(t,0)$, $(\psi_\varepsilon)_y(t,0)$ and $(\psi_\varepsilon)_t(t,0)$ in $L^2(0,T)$, we deduce that, for all $y \in (0,1)$,

$$\begin{aligned}&\int_0^T (((\psi_\varepsilon)_t)^2+((\psi_\varepsilon)_y)^2+n^2\pi^2(\psi_\varepsilon)^2)(t,y)\,dt\ \le\\ &\le c'\int_0^T [(1+n^2\pi^2)(f_t)^2+(1+n^2\pi^2+n^4\pi^4)(f)^2]\,dt\,exp(\frac{3\pi n}{2}y).\end{aligned} \tag{3.19}$$

□

The last estimates show that the set of periodic solutions corresponding to each ε, $\{(\psi_\varepsilon,(\psi_\varepsilon)_t,V_\varepsilon,(V_\varepsilon)_t)\}_{\varepsilon>0}$, is bounded in $L^2(0,T;Y)$. Since our problem is linear we can pass to the limit when $\varepsilon \to 0$ and we obtain for each $n \in \mathbb{N}$ a weak periodic solution of equation (1.6). The result is stated in Theorem 2.1.

Proof of Theorem 2.1: Suppose that $f \in H^2(0,T)$ (the case $f \in H^1(0,T)$ can be obtained easily by density).

Let $U_\varepsilon = (\psi_\varepsilon,(\psi_\varepsilon)_t,V_\varepsilon,(V_\varepsilon)_t)$ be the periodic solution of (3.1) for each $\varepsilon > 0$.

By Theorem 3.4, $(\psi_\varepsilon)_{\varepsilon>0}$ is bounded in $L^2(0,T;H^1(0,1))\cap H^1(0,T;L^2(0,1))$ and $(V_\varepsilon)_{\varepsilon>0}$ is bounded in $H^1(0,T)$ (and even in $H^2(0,T)$).

It follows that U_ε are uniformly bounded in $L^2(0,T;\mathcal{Y})$. Moreover, U_ε satisfies

$$(U_\varepsilon)_t + \mathcal{A}_0^1 = F_\varepsilon \tag{3.20}$$

where $F_\varepsilon = (0, -\varepsilon\psi_t, 0, f)$. Observe that, since $(\psi_\varepsilon)_t$ is bounded in $L^2(0,T;L^2(0,1))$, F_ε has the property that

$$F_\varepsilon \longrightarrow F \text{ in } L^2(0,T;\mathcal{Y}) \text{ when } \varepsilon \to 0.$$

We can now apply Theorem 7.7 and deduce the existence of a periodic solution of (1.6) □

With the estimates of Theorem 3.4 the proof of Theorem 2.3 follows immediately.

Proof of Theorem 2.3: Since the periodic solutions $(\psi_\varepsilon, V_\varepsilon)$ satisfy (3.12) and (3.13) it follows that the limit periodic function (ψ, V) satisfies (2.5) and (2.6). □

4. The one-dimensional case: $n = 0$**.** In this section we prove the existence of periodic solutions for system (1.6) in the case $n = 0$.

We begin with the following remark: if (ψ, V) is a periodic solution of (1.6) with $n = 0$ then, integrating the last equation between 0 and T we deduce that $\int_0^T f_0(s)ds = 0$. Therefore, this is a necessary condition for the existence of periodic solutions.

On the other hand we observe that in the case $n = 0$ the expression (3.2) does not define an inner product in $\mathcal{Y}$. Moreover, the operator corresponding to (1.6) is no longer maximal monotone. We can find a subspace of $\mathcal{Y}$ in which this operator is maximal monotone but, since we are interested in a non homogeneous problem this is useless (a non homogeneous term $(0,0,0,f_0)$ belongs to that subspace for each t iff $f_0 = 0$). This remark indicates that it is not possible to apply in this case the same kind of arguments as in the previous section.

Remark 14. *In the rest of the section we drop the index* $_0$ *from the unknowns* (ψ, V) *and the non homogeneous term* f *to simplify the notation.*

We prove now Theorem 2.4

Proof of Theorem 2.4: Since f is a continuous function of period T we can decompose it in the following way

$$f(t) = \sum_{m\in\mathbb{Z}} a_m e^{i\nu_m t} \tag{4.1}$$

where $\nu_m = \frac{2m\pi}{T}$ and $a_m \in \mathbb{C}$.

The exponentials $\left\{e^{i\,\nu_m\,t}\right\}_{m\in\mathbb{Z}}$ form a complete orthogonal sequence in $L^2(0,T)$.

It follows that

$$\begin{aligned} &f \in L^2(0,T) \text{ iff } \sum_{m\in\mathbb{Z}} |a_m|^2 < \infty \\ &f \in H^1(0,T) \text{ iff } \sum_{m\in\mathbb{Z}} |a_m|^2|\nu_m|^2 < \infty. \end{aligned} \tag{4.2}$$

We also remark that $\int_0^T f = 0$ iff $a_0 = 0$.

The idea of the proof of the Theorem consists on finding, for each $n \in \mathbb{Z}\setminus\{0\}$, an explicit periodic solution of the system:

$$\begin{cases} \phi_{tt} - \phi_{yy} = 0 & \text{for} \quad (y,t) \in (0,1)\times(0,\infty) \\ \phi_y(1,t) = 0 & \text{for} \quad t > 0 \\ \phi_y(0,t) = -u_t(t) & \text{for} \quad t > 0 \\ u_{tt}(t) + u_t + \phi_t(0,t) = a_m e^{i\,\nu_m\,t} & \text{for} \quad t > 0 \end{cases} \tag{4.3}$$

and then adding all them in order to get a periodic solution for (1.6).

It is easy to see that (4.3) has a periodic solution of the form

$$\phi_m = c_m a_m \cosh\left(i\,\nu_m(1-y)\right) e^{i\,\nu_m\,t}, \quad u_m = c_m a_m \sinh\left(i\,\nu_m\right) e^{i\,\nu_m\,t} \tag{4.4}$$

where the constants c_m are given by

$$c_m = \frac{1}{(-\nu_m^2+\nu_m i)\sinh(\nu_m\,i) - \nu_m\cosh(\nu_m\,i)}. \tag{4.5}$$

We show that, under the hypotesis $f \in H^1(0,T)$, the series $\sum_{m\in\mathbb{Z}} \phi_m$ and $\sum_{m\in\mathbb{Z}} u_m$ are convergent in $L^2(0,T,H^1(0,1)) \cap H^1(0,T,L^2(0,1))$ and in $H^1(0,T)$ respectively.

First, we remark that $\sum_{m\in\mathbb{Z}} \phi_m$ converges in $L^2(0,T,H^1(0,1)) \cap H^1(0,T,L^2(0,1))$ iff

$$\sum_{m\in\mathbb{Z}} |a_m|^2|c_m|^2|\nu_m|^2 < \infty$$

while $\sum_{m\in\mathbb{Z}} u_m$ converges in $H^1(0,T)$ iff

$$\sum_{m\in\mathbb{Z}} |a_m|^2|c_m|^2|\nu_m|^2|\sinh(\nu_m\,i)|^2 < \infty.$$

Observe that the first condition implies the second one.

We have that:

$$|c_m|^2 = \frac{1}{(-\nu_m^2+\nu_m\,i)\sin(\nu_m) - \nu_m\,i\cos(\nu_m)} =$$

$$= \frac{1}{\nu_m^2\sin^2(\nu_m) + \nu_m^2(-\cos(\nu_m)+\nu_m\sin(\nu_m))^2}.$$

We claim that

$$\liminf_{|m|\to\infty} \left(\nu_m^2\sin^2(\nu_m) + \nu_m^2(-\cos(\nu_m)+\nu_m\sin(\nu_m))^2\right) > 0. \tag{4.6}$$

Assuming that (4.6) holds it follows that $|c_m|$ is bounded and therefore

$$\sum_{m\in\mathbb{Z}} |a_m|^2|c_m|^2|\nu_m|^2 < \infty \text{ iff } \sum_{m\in\mathbb{Z}} |a_m|^2|\nu_m|^2 < \infty,$$

which it is true since $f \in H^1(0,T)$.

Therefore, to complete the proof it is sufficient to show that (4.6) holds.

In order to prove (4.6) suppose that there is a subsequence, that we shall denote by the same index, $(\nu_m)_m$, such that

$$|\nu_m(-\cos(\nu_m)+\nu_m\sin(\nu_m))| \longrightarrow 0 \text{ and } |\nu_m\sin(\nu_m)| \longrightarrow 0$$

as $|m|$ tends to infinity.

It follows that $|\cos(\nu_m)| \to 0$ and $|\sin(\nu_m)| \to 0$ which it is not possible. $\square$

5. The two-dimensional case: Proof of Theorem 2.5. First of all we put system (1.1) in an abstract form. To do this we define the space:

$$\mathcal{D}(\mathcal{A}) = \{(\phi^0, \phi^1, W^0, W^1) \in H^2(\Omega) \times H^1(\Omega) \times H^2(\Gamma_0) \times H^1(\Gamma_0) :$$

$$\frac{\partial \phi^0}{\partial \nu} = 0 \text{ on } \Gamma_1, \ \frac{\partial \phi^0}{\partial y} = -W^1 \text{ on } \Gamma_0, \ W_x^0(0) = W_x^0(1) = 0\}$$

and the operador $\mathcal{A}$ defined in $\mathcal{D}(\mathcal{A})$ by:

$$\mathcal{A}(\phi, \psi, W, V) = (-\psi \ , -\Delta\phi \ , -V \ , -W_{xx} + V + \psi). \tag{5.1}$$

With these notations system (1.1) can be written as

$$\left\{ \begin{array}{l} U_t(t) + \mathcal{A}U(t) = F, \quad \forall t \geq 0 \\ U(0) = U_0 \in \mathcal{D}(\mathcal{A}) \\ U(t) \in \mathcal{D}(\mathcal{A}), \quad \forall t \geq 0 \end{array} \right. \tag{5.2}$$

for the variable $U = (\phi, \phi_t, W, W_t)$ and with $F = (0, 0, 0, f)$.

We can now pass to prove Theorem 2.5.

Proof of Theorem 2.5: For each $n \geq 0$, Theorems 2.1 and 2.4 give us a periodic solution of system (1.6), $(\psi_n(t, y), V_n(t))$. We define:

$$\phi(t, x, y) = \sum_{n=0}^{\infty} \psi_n(t, y) \cos(n\pi x), \ W(t, x) = \sum_{n=0}^{\infty} V_n(t) \cos(n\pi x)$$

and we show that the series converge in $L^2(0, T; H^1(\Omega)) \cap H^1(0, T; L^2(\Omega))$ and in $H^1(0, T; H^1(\Gamma_0))$ respectively. In this case (ϕ, W) is a periodic solution of (1.1) of finite energy.

We have that $\sum_{n=0}^{\infty} \psi_n(t, y) \cos(n\pi x)$ converges in $L^2(0, T; H^1(\Omega)) \cap H^1(0, T; L^2(\Omega))$ if and only if

$$\left|\left| \sum_{n=0}^{\infty} \psi_n(t, y) \cos(n\pi x) \right|\right|^2_{L^2(0,T;H^1(\Omega))} =$$
$$\sum_{n=0}^{\infty} \| \psi_n(t, y) \|^2_{L^2(0,T;H^1(0,1))} \| \cos(n\pi x) \|^2_{H^1(0,1)} < \infty,$$

$$\left|\left| \sum_{n=0}^{\infty} \psi_n(t, y) \cos(n\pi x) \right|\right|^2_{H^1(0,T;L^2(\Omega))} =$$
$$\sum_{n=0}^{\infty} \| \psi_n(t, y) \|^2_{H^1(0,T;L^2(0,1))} \| \cos(n\pi x) \|^2_{L^2(0,1)} < \infty.$$

Taking into account (3.19) we have that the last condition is satisfied if

$$\sum_{n=0}^{\infty} \int_0^T \left((1 + n^2\pi^2)(f_t)^2 + (1 + n^2\pi^2 + n^6\pi^6)(f)^2 \right) dt \ \frac{2}{3n\pi} \left(\exp \frac{3n\pi}{2} \right) < \infty.$$

It follows that the series $\phi = \sum_{n=0}^{\infty} \psi_n(t,y)\cos(n\pi x)$ is converges in the space $L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$ if (2.8) and (2.9) are satisfied.

On the other hand, $\sum_{n=0}^{\infty} V_n(t)\cos(n\pi x)$ converges in $H^1(0,T;H^1(\Gamma_0))$ if and only if

$$\sum_{n=0}^{\infty} \| w_n(t)\cos(n\pi x) \|^2_{H^1(0,T;H^1(\Gamma_0))} = \sum_{n=1}^{\infty} \| w_n(t) \|^2_{H^1(0,T)} \| \cos(n\pi x) \|^2_{H^1(0,1)} < \infty.$$

It follows that, if (2.8) and (2.9) are satisfied, then $W(t,x) = \sum_{n=1}^{\infty} w_n(t)\cos(n\pi x)$ converges in $H^1(0,T;H^1(\Gamma_0))$. $\square$

We remark that (2.8) and (2.9) are satisfied, for instance, if $f(t)$ is a function with a finite number of non zero Fourier coefficients.

Nevertheless, one can prove that (2.8) and (2.9) are also satisfied when f belongs to some Gevrey class. Let us recall first the definition of this type of functions (see [9] p. 146 and [11]).

DEFINITION 5.1. *A function $p : \mathbb{R} \longrightarrow \mathbb{R}$ in $\mathcal{C}^\infty(\mathbb{R})$ belongs to Gevrey's class of exponent δ (we write $p \in \gamma^\delta$) if for every compact $K \subset \mathbb{R}$ and every $\theta > 0$, there is a positive constant $C_\theta > 0$, such that*

$$|\partial_x^j p(x)| \le C_\theta \theta^j \left(j^j\right)^\delta, \quad \textit{for } j = 1,2,3,...$$

when $x \in K$.

DEFINITION 5.2. *A function $q : [0,\infty) \times \mathbb{R} \longrightarrow \mathbb{R}$ belongs to Gevrey's class of exponent δ in the second variable (we write $q(.,x) \in \gamma^\delta$) if $\partial_x^j q(t,x)$ is continuous in $[0,\infty) \times \mathbb{R}$ for every $j \ge 0$ and, for every compact $K \subset [0,\infty) \times \mathbb{R}$ and every $\theta > 0$ there is a positive constant $C_\theta > 0$ such that*

$$|\partial_x^j q(t,x)| \le C_\theta \theta^j \left(j^j\right)^\delta, \quad \textit{for } j = 1,2,3,...$$

when $(t,x) \in K$.

We have the following result:

PROPOSITION 5.3. *If $f \in H^1(0,T;L^2(0,1))$, supp $f(t,x) \subset (0,1)$ for all t and $f(\cdot,x)$ and $f_t(\cdot,x) \in \gamma^1$ then (2.8) and (2.9) are satisfied.*

Proof. The function f can be written as

$$f(t,x) = \sum_{n=0}^{\infty} f_n(t)\cos(n\pi x).$$

Integrating by parts k times and taking into account that supp $f(t,x) \subset (0,1)$ we obtain that:

$$| f_n(t) | = 2 \left| \int_0^1 f(t,x)\cos n\pi x \, dx \right| = \frac{2}{(n\pi)^k} \left| \int_0^1 \partial_x^k f(t,x) \cos\left(n\pi x + \frac{n\pi}{2}\right) dx \right| \le$$

$$\le \frac{2}{(k\pi)^k} \| \partial_x^k f \|_\infty \le \frac{2}{(n\pi)^k} C_\theta \theta^k k^k, \quad \forall \theta > 0,$$

where $||\cdot||_\infty$ denotes the norm in $\mathcal{C}[0,1]$.

For a fixed n we can choose $k=n$ and we obtain

$$| f_n(t) | \leq \frac{2}{(n\pi)^n} C_\theta \theta^n n^n = 2C_\theta \frac{\theta^n}{\pi^n}, \quad \forall \theta > 0$$

Since this is true for every $\theta > 0$ we can consider $\theta = exp(-\frac{3\pi}{4})$ and obtain that (2.9) is satisfied.

Since

$$f_t(t,x) = \sum_{n=0}^{\infty} (f_n)_t(t) \cos(n\pi x),$$

we can apply a similar argument for (2.8) $\square$

6. A resonance-type result. In this section we analyze the conditions in which the resonance phenomenon can occur. We consider only the one-dimensional system (1.6) since it is easier to study and all the properties are transmited to the two-dimensional case by Fourier decomposition. We have seen in the previous section that the condition $f \in H^1(0,T)$ is sufficient to ensure the existence of a periodic solution. Therefore, in this case, the resonance cannot occur. Theorem 2.7 gives the conditions in which this phenomenon can occur.

Proof of Theorem 2.7: Like in the proof of Theorem 2.4 the function f can be decomposed in the following way

$$f(t) = \sum_{m \in \mathbb{Z}} a_m e^{i\,\nu_m t} \tag{6.1}$$

where $\nu_m = 2m\pi/T$ and $a_m \in \mathbb{C}$.

The exponentials $\left\{e^{i\,\nu_m\,t}\right\}_{m \in \mathbb{Z}}$ form a complete orthogonal sequence in $L^2(0,T)$. It follows that

$$f \in L^2(0,T) \setminus H^1(0,T) \text{ iff } \sum_{m \in \mathbb{Z}} |a_m|^2 < \infty \text{ and } \sum_{m \in \mathbb{Z}} |a_m|^2 |\nu_m|^2 = \infty. \tag{6.2}$$

Suppose that (1.6) has a periodic solution (ψ, V) of finite energy.

It follows that (ψ, V) can be decomposed as

$$(\psi, V) = \sum_{m \in \mathbb{Z}} (\phi_m, u_m) e^{i\,\nu_m\,t}$$

and the convergence of the series is uniform in $\mathcal{C}^1([0,T]; L^2(0,1) \times \mathbb{R})$.

Moreover, (ϕ_m, u_m) has to satisfy the following equation

$$\begin{cases} (-\nu_m^2 + n^2\pi^2)\phi_m - (\phi_m)_{yy} = 0 \text{ for } y \in (0,1) \\ (\phi_m)_y(1) = 0 \\ (\phi_m)_y(0) = -\nu_m\, i\, u_m \\ (-\nu_m^2 + \nu_m\, i + n^2\pi^2) u_m + \nu_m\, i\, \phi_m(0) = a_m. \end{cases} \tag{6.3}$$

It is easy to see that the solutions of (6.3) are

$$\begin{aligned} \phi_m &= c_m a_m \cosh\left(\sqrt{\nu_m^2 - n^2\pi^2}\, i\,(1-y)\right), \\ u_m &= c_m a_m \frac{\sqrt{\nu_m^2 - n^2\pi^2}}{\nu_m} \sinh\left(\sqrt{\nu_m^2 - n^2\pi^2}\, i\right) \end{aligned} \tag{6.4}$$

where the constants c_m are given by

$$c_m = \nu_m\, i\, \Big[(-\nu_m^2 + \nu_m\, i + n^2\pi^2)\sqrt{\nu_m^2 - n^2\pi^2}\, i \sinh\big(\sqrt{\nu_m^2 - n^2\pi^2}\, i\big) - \\ -\nu_m^2 \cosh\big(\sqrt{\nu_m^2 - n^2\pi^2}\, i\big)\Big]^{-1}. \tag{6.5}$$

We show that, under the hypotesis $f \notin H^1(0,T)$, the series $\sum_{m\in\mathbb{Z}} \phi_m e^{i\,\nu_m\, t}$ cannot be convergent in $H^1(0,T,L^2(0,1))$.

First, we remark that $\sum_{m\in\mathbb{Z}} \phi_m\, e^{i\,\nu_m\, t}$ converges in $H^1(0,T,L^2(0,1))$ if and only if

$$\sum_{m\in\mathbb{Z}} |a_m|^2 |c_m|^2 |\nu_m|^2 < \infty. \tag{6.6}$$

Indeed we have

$$||\phi_m\, e^{i\,\nu_m\, t}||^2_{H^1(0,T,L^2(0,1))} = \sum_{m\in\mathbb{Z}} |\nu_m|^2 ||\phi_m||^2_{L^2(0,1)} = \\ \frac{1}{2}\sum_{m\in\mathbb{Z}} |\nu_m|^2 |c_m|^2 |a_m|^2 \left|1 - \frac{1}{2\nu_m}\sin(2\nu_m)\right|.$$

Nevertheless, under the hypothesis of the theorem, $c_{m_k} = \mathcal{O}(1)$ as $k \to \infty$. In order to prove this let us put $|c_{m_k}|^2 = |\nu_{m_k}|^2 (\alpha(\nu_{m_k})^2 + \beta(\nu_{m_k})^2)^{-1}$ where

$$\alpha(\nu_{m_k}) = (\nu_{m_k}^2 - n^2\pi^2)\sqrt{\nu_{m_k}^2 - n^2\pi^2}\times \\ \sin\big(\sqrt{\nu_{m_k}^2 - n^2\pi^2}\big) - \nu_{m_k}^2 \cos\big(\sqrt{\nu_{m_k}^2 - n^2\pi^2}\big) \\ \beta(\nu_{m_k}) = \nu_{m_k}\sqrt{\nu_{m_k}^2 - n^2\pi^2}\sin\big(\sqrt{\nu_{m_k}^2 - n^2\pi^2}\big).$$

It follows that

$$|\beta(\nu_{m_k})| \le \frac{\nu_{m_k}^2}{|\nu_{m_k}^2 - n^2\pi^2|}\left(\frac{|\alpha(\nu_{m_k})|}{|\nu_{m_k}|} + |\nu_{m_k}|\right).$$

On the other hand, if ζ is a root of (2.11), we have that $\alpha(\zeta) = 0$. Applying Taylor's Theorem if follows that

$$\alpha(\nu_{m_k}) = \alpha(\zeta) - \alpha'(\xi)(\zeta - \nu_{m_k})$$

where ξ is a real number between ζ and ν_{m_k}. Hence

$$|\alpha(\nu_{m_k})| = |\alpha'(\xi)|\,|\zeta - \nu_{m_k}| \le c|\nu_{m_k}|^3 |\zeta - \nu_{m_k}|,$$

where c is a positive constant depending only on n.

Therefore we have that

$$\frac{|\alpha(\nu_{m_k})|}{|\nu_{m_k}|} \le c|\nu_{m_k}|^2 \text{ dist } (\nu_{m_k}, E).$$

Taking into account (2.12) we deduce that $\frac{|\alpha(\nu_{m_k})|}{|\nu_{m_k}|}$ and $\frac{|\beta(\nu_{m_k})|}{|\nu_{m_k}|}$ remain bounded as $k \to \infty$.

It follows that $c_{m_k} = \mathcal{O}(1)$ as $k \to \infty$ and (6.6) cannot be satisfied if $f \notin H^1(0,T)$.

□

Proof of Theorem 2.8: The arguments used in the proofs of Theorems 2.4 and 2.7 indicate that

$$|c_m|^2 = \mathcal{O}\left(\frac{1}{|\nu_m|^2}\right) \text{ as } m \to \infty \tag{6.7}$$

ensures the existence of periodic solution even in the case $f \in L^2(0,T)$.

Observe first that

$$|c_m|^2|\nu_m|^2 = \frac{|\nu_m|^4}{|\alpha(\nu_m)|^2 + |\beta(\nu_m)|^2} \leq \frac{c|\nu_m|^4}{|\alpha(\nu_m)|^2}.$$

Hence (6.7) is satisfied if

$$\liminf_{|m|\to\infty} \frac{|\alpha(\nu_m)|}{|\nu_m|^2} > 0. \tag{6.8}$$

Let us suppose that there is a sequence $(m_k)_{k\geq 0}$ such that $|m_k| \to \infty$ and $\lim\limits_{m\to\infty} \frac{|\alpha(\nu_{m_k})|}{|nu_{m_k}|^2} = 0$. This implies that

$$\lim_{k\to\infty} \left(\sqrt{\nu_{m_k}^2 - n^2\pi^2}\sin\left(\sqrt{\nu_{m_k}^2 - n^2\pi^2}\right) - \cos\left(\sqrt{\nu_{m_k}^2 - n^2\pi^2}\right)\right) = 0. \tag{6.9}$$

Let $T = \frac{q}{p} \in \mathbb{Q}$ with $p, q \in \mathbb{N}$.

For each $k \geq 0$ there is $s_k \in \mathbb{N}$ and $r_k \in \mathbb{N}$, $0 \leq r_k < p$ such that

$$\nu_{m_k} = \frac{2m_k q\pi}{T} = s_k\pi + \frac{r_k\pi}{p}.$$

It follows that

$$\sin\left(\sqrt{\nu_{m_k}^2 - n^2\pi^2}\right) = \sin\left(s_k\pi + \frac{r_k\pi}{p} - \frac{n^2\pi^2}{\nu_{m_k} + \sqrt{\nu_{m_k}^2 - n^2\pi^2}}\right) =$$

$$= \pm\sin\left(\frac{r_k\pi}{p} - \frac{n^2\pi^2}{\nu_{m_k} + \sqrt{\nu_{m_k}^2 - n^2\pi^2}}\right).$$

Therefore, condition (6.9) implies that $r_k = 0$ for all $k \geq 0$. Hence $\nu_{m_k} = s_k\pi$, with $s_k \in \mathbb{Z}$. But in this case

$$\lim_{k\to\infty} \sqrt{\nu_{m_k}^2 - n^2\pi^2}\sin\left(\sqrt{\nu_{m_k}^2 - n^2\pi^2}\right) = \frac{n^2\pi^2}{2},$$

$$\lim_{k\to\infty} \cos\left(\sqrt{\nu_{m_k}^2 - n^2\pi^2}\right) = \pm 1.$$

The last results contradict (6.9). So (6.7) must be true and the proof is completed.

□

7. Appendix. In this section we recall some classical notions and results that have been used in this paper. For a comprehensive introduction to this topic see [5]. At the end we give a result on the convergence of periodic solutions.

Let X be a Banach space and let $\mathcal{A}$ be a linear unbounded operator defined in a subspace $\mathcal{D}(\mathcal{A}) \subset \mathcal{X}$.

Definition 7.1. *The operator $\mathcal{A}$ is called monotone if $||x + \lambda \mathcal{A}x|| \geq ||x||$ for all $x \in \mathcal{D}(\mathcal{A})$ and for all $\lambda > 0$.*

Definition 7.2. *The operator $\mathcal{A}$ is called maximal-monotone if it is monotone and there exists a real number $\lambda > 0$ such that $R(\mathcal{I} + \lambda\mathcal{A}) = \mathcal{X}$.*

Remark 15. *If $\mathcal{X}$ is a Hilbert space with the inner product defined by $(\,\cdot\,,\,\cdot\,)$ then it can be shown that $\mathcal{A}$ is monotone iff $(\mathcal{A}x, x) \geq 0$ for all $x \in \mathcal{D}(\mathcal{A})$.*

Theorem 7.3. *If $\mathcal{A}$ is maximal-monotone operator then it generates a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ of contractions in $\mathcal{X}$.*

We consider now the following Cauchy problem:

$$\begin{cases} U_t + \mathcal{A}U = F \\ U(0) = U_0 \\ U(t) \in \mathcal{D}(\mathcal{A}). \end{cases} \tag{7.1}$$

We have the following classical result of existence of solutions of (7.1) (see [5], Cap. 4, pp. 51-53):

Theorem 7.4. *Let $\mathcal{A}$ be a maximal-monotone operator generating a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ in $\mathcal{X}$. Then:*

1. Weak solutions: If $F \in L^1(0, T; \mathcal{X})$ then (7.1) has a unique weak solution $U \in \mathcal{C}([0, T]; \mathcal{X})$ which satisfies

$$U(t) = S(t)U_0 + \int_0^t S(t-s)F(s)ds, \; \forall t \in [0, T]. \tag{7.2}$$

2.Strong solutions: If $U_0 \in \mathcal{D}(\mathcal{A})$ and $F \in W^{1,1}(0, T; \mathcal{X})$ or $F \in L^1(0, T; \mathcal{D}(\mathcal{A}))$ then U given by (7.2) is a classical solution of (7.1)

$$U \in \mathcal{C}^1([0, T], \mathcal{X}) \cap \mathcal{C}([0, T], \mathcal{D}(\mathcal{A})). \tag{7.3}$$

We recall now the following result of extrapolation (see [5] Proposition 2.3.1, pp. 27-28)

Theorem 7.5. *Let $\mathcal{A}$ be a maximal monotone operator in $\mathcal{X}$ with compact resolvent. Then there are a Banach space $\mathcal{X}_{-1}$ and a maximal monotone operator $\mathcal{B}$ such that:*

(i) $\mathcal{X}$ is a dense subspace of $\mathcal{X}_{-1}$

(ii) $||x||_{\mathcal{X}_{-1}} = ||(\mathcal{I} + \mathcal{A})^{-1}x||_{\mathcal{X}}, \quad \forall x \in \mathcal{X}$

(iii) $\mathcal{D}(\mathcal{B}) = \mathcal{X}$ with equivalent norms

(iv) $\mathcal{A}x = \mathcal{B}x, \quad \forall x \in \mathcal{D}(\mathcal{A})$

(v) $x \in \mathcal{X}$ and $\mathcal{B}x \in \mathcal{X}$ implies $x \in \mathcal{D}(\mathcal{A})$.

Remark 16. *If $\mathcal{A}$ is an operator with compact resolvent then $\mathcal{X}$ is a compact subspace of $\mathcal{X}_{-1}$. Indeed, let $(x_m)_{m\geq 0}$ be a sequence in $\mathcal{X}$ such that $x_m \rightharpoonup x$. It* *follows that*

$$||x_m - x||_{X_{-1}} = ||(\mathcal{I}+\mathcal{A})^{-1}(x_m - x)||_{\mathcal{X}} \longrightarrow 0$$

where the first equality holds by definition and the second one is due to the compactness of the resolvent of $\mathcal{A}$.

With the aid of this result we can give another interpretation to the weak solutions of (7.1) (see [5] Corollary 3.3.2, p. 41 and Corollary 4.1.7, p. 53).

THEOREM 7.6. *Under the hypotesis of Theorem 7.5 let $\{S(t)\}_{t\geq 0}$ and $\{T(t)\}_{t\geq 0}$ be the semigroups generated by $\mathcal{A}$ and $\mathcal{B}$ respectively. Then*

(i) $T(t)x = S(t)x, \quad \forall x \in \mathcal{X}$

(ii) If $F \in L^1(0,T;\mathcal{X})$ and $U_0 \in \mathcal{X}$ then the weak solution of (7.1) given by (7.2) is the strong solution of the problem

$$\begin{cases} U_t + \mathcal{B}U = F \\ U(0) = U_0 \\ U(t) \in \mathcal{C}^1([0,T],\mathcal{X}_{-1}) \cap \mathcal{C}([0,T],\mathcal{X}). \end{cases} \tag{7.4}$$

The following theorem is used to prove the existence of periodic solutions in Theorems 2.1, 2.4 and 2.5.

THEOREM 7.7. *Let $(\mathcal{A},\mathcal{D}(\mathcal{A}))$ be a maximal monotone operator in a Hilbert space $\mathcal{X}$ with compact resolvent. Suppose that $F_m \in L^2(0,T;\mathcal{X})$, $m \in \mathbb{Z}$, are periodic functions of period T and suppose that the problem*

$$U_t + \mathcal{A}U = F_m \tag{7.5}$$

has, for each $m \in \mathbb{Z}$, a unique periodic solution $U_m \in \mathcal{C}([0,\infty);\mathcal{X})$ of period T.

Moreover, suppose that F_m converges to F in $L^2(0,T;\mathcal{X})$.

If the sequence $(U_m)_{m\geq 0}$ is bounded in $L^2(0,T;\mathcal{X})$ then (7.1) has a periodic solution $U \in \mathcal{C}([0,\infty);\mathcal{X})$.

Proof. We shall give the proof in three steps:

Step 1. $(U_m)_{m\geq 0}$ is bounded in $\mathcal{C}([0,T];\mathcal{X})$.

Step 2. $(U_m)_{m\geq 0}$ is bounded in $\mathcal{C}^1([0,T];\mathcal{X}_{-1})$.

Step 3. There is a subsequence of $(U_m)_{m\geq 0}$ which converges in $\mathcal{C}([0,T];\mathcal{X}_{-1})$ and the limit, U, belongs to $\mathcal{C}([0,T];\mathcal{X})$ and it is a weak periodic solution of (7.1).

Let us proceed with the proof of these steps.

Step 1. Since U_m is a weak solution of (7.5) it satisfies the relation (7.2). It follows that for all $0 \leq \tau \leq T \leq t \leq 2T$

$$||U_m(t)||^2_{\mathcal{X}} = \left|\left| S(t)U_{0,m} + \int_0^t S(t-s)F_m(s)ds \right|\right|^2_{\mathcal{X}} \leq$$

$$\leq 2||S(\tau)U_{0,m}||^2_{\mathcal{X}} + 2T^2||F_m||^2_{L^2(0,T;\mathcal{X})}.$$

Integrating in τ between 0 and T we obtain that

$$T||U_m(t)||^2_{\mathcal{X}} \leq 2\int_0^T \left|\left| U_m(\tau) - \int_0^\tau S(\tau-s)F_m(s)ds \right|\right|^2 d\tau + 2T^3||F_m||^2_{L^2(0,T;\mathcal{X})} \leq$$

$$4||U_m||^2_{L^2(0,T;\mathcal{X})} + (4T^2 + 2T^3)||F_m||^2_{L^2(0,T;\mathcal{X})}, \quad \forall t,\ T \le t \le 2T.$$

Since $(U_m)_{m\geq 0}$ is bounded in $L^2(0,T;\mathcal{X})$ and F_m converges to F in $L^2(0,T;\mathcal{X})$ and taking into account that U_m is a periodic function of period T it follows that U_m is bounded in $\mathcal{C}([0,T];\mathcal{X})$.

Step 2. Let $\mathcal{B}$ and $\mathcal{X}_{-1}$ be the extentions of the operator $\mathcal{A}$ and space $\mathcal{X}$ of Theorem 7.5. It follows that U_m is a strong solution of (7.4) and belongs to the class $\mathcal{C}^1([0,T];\mathcal{X}_{-1})$. Moreover, since $\mathcal{B}$ is a continuous operator from $\mathcal{X}$ to $\mathcal{X}_{-1}$ we deduce that

$$||(U_m)_t(t)||_{\mathcal{X}_{-1}} = ||\mathcal{B}(U_m)(t)||_{\mathcal{X}_{-1}} \le c||U_m(t)||_{\mathcal{X}}.$$

Therefore $(U_m)_{m\geq 0}$ is bounded in $\mathcal{C}^1([0,T];\mathcal{X}_{-1})$.

Step 3. Since $(U_m)_{m\geq 0}$ is uniformly bounded in $\mathcal{C}^1([0,T];\mathcal{X}_{-1})$ and $\mathcal{X}_{-1}$ is compact in $\mathcal{X}$ we can apply the Theorem of Ascoli-Arzela and obtain that there is a subsequence of $(U_m)_{m\geq 0}$, that we shall denote in the same way, which is convergent to a function U in $\mathcal{C}([0,T];\mathcal{X}_{-1})$. U can be extended to a periodic function in $\mathcal{C}([0,\infty);\mathcal{X}_{-1})$ and it is a weak solution of (7.4):

$$U(t) = T(t)U_0 + \int_0^T T(t-s)F(s)ds, \tag{7.6}$$

where $\{T(t)\}_{t\geq 0}$ is the semigroup generated by $\mathcal{B}$ and U_0 is the limit of the sequence $U_{0,m}$ in $\mathcal{X}_{-1}$.

From Step 1 we know that $U_{0,m}$ is bounded in $\mathcal{X}$. It follows that there are a subsequence of $U_{0,m}$, that we shall also denote in the same way, and an element $\widehat{U}_0$, such that $U_{0,m} \rightharpoonup \widehat{U}_0$ in $\mathcal{X}$. Since $\mathcal{X} \subset \mathcal{X}_{-1}$ with compactness we deduce that $\widehat{U}_0 = U_0$ and therefore, $U_0 \in \mathcal{X}$. Now, taking into account that $U_0 \in \mathcal{X}$, $F \in L^2(0,T;\mathcal{X})$ and $U(t)$ satisfies (7.6) we deduce by Theorem 7.6 i) that

$$U(t) = S(t)U_0 + \int_0^T S(t-s)F(s)ds.$$

It follows that $U \in \mathcal{C}([0,T];\mathcal{X})$ is the weak periodic solution of problem (7.1) and the proof is completed.

□

Remark 17. *Suppose now that F has the following expantion:*

$$F(t,x) = \sum_{m\in\mathbb{Z}} F_m(x)e^{i\,\frac{2m\pi}{T}t} \tag{7.7}$$

and suppose that the series is convergent in $L^2(0,T;X)$.

Let U be a periodic solution of

$$U_t - \mathcal{A}U = F. \tag{7.8}$$

Since $\left\{e^{i\,\frac{2m\pi\,t}{T}}\right\}_{m\in\mathbb{Z}}$ is an orthogonal basis for $L^2(0,T)$ it follows that U can be written as

$$U(t,x) = \sum_{m\in\mathbb{Z}} U_m(x)e^{i\,\frac{2m\pi}{T}t} \tag{7.9}$$

where U_m *verifies,*

$$\frac{2m\pi}{T}\, i\, U_m + \mathcal{A}U_m = F_m. \tag{7.10}$$

Theorem 7.7 shows that the convergence of (7.7) in $L^2(0,T;X)$ *ensures the existence of a finite-energy periodic solution of (7.8).*

The above considerations imply that the convergence of (7.7) in $L^2(0,T;X)$ *is also a necessary condition for the existence of a finite-energy periodic solution of (7.8).*

REFERENCES

[1] H. T. Banks, W. Fang, R. J. Silcox and R. C. Smith, *Approximation Methods for Control of Acustic/Structure Models with Piezoceramic Actuators*, Journal of Intelligent Material Systems and Structures, 4 (1993), pp. 98-116.

[2] H. T. Banks, R. C. Smith and Y. Wang, *Smart Material Structures. Modeling, estimation and control*, RAM, John Wiley & Sons, Masson, 1996.

[3] C. Bardos, G. Lebeau and J. Rauch, *Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary*, SIAM J. Control Optim., 30 (1992), pp. 1024-1065.

[4] F. Browder and W. Petryshyn, *The solution by iteration of nonlinear functional equations in Banach spaces*, Bull. A.M.S. 72 (1966), pp. 571-575.

[5] T. Cazenave and A. Haraux, *Introduction aux problèmes d'évolution semi-linéaires*, Mathématiques et Applications, 1, Ellipses, Paris, 1990.

[6] P. Grisvard, *Elliptic Problems in Non-smooth Domains*, Pitman, 1985.

[7] S. Hansen and E. Zuazua, *Exact Controllability and Stabilization of a Vibrating String with an Interior Point Mass*, SIAM J. Control Optim., 33, 5 (1995), pp. 1357-1391.

[8] A. Haraux, *Semi-linear wave equations in bounded domains*, Mathematical Reports, Vol. 3(1), Harwood Academic Publishers, Gordon and Breach, London, 1987.

[9] L. Hörmander, *Linear Partial Differential Operators*, Springer-Verlag, 1976.

[10] J. L. Lions, *Contrôlabilité exacte, perturbations et stabilization de systèmes distribués. Tome 1. Contrôlabilité exacte*, Masson RMA 8, Paris, 1988.

[11] W. Littman and L. Marcus, *Exact Boundary Controllability of a Hybrid System of Elasticity*, Archive Rat. Mech. Anal., 103, 3 (1988), pp. 193-236.

[12] S. Micu, *Análisis de un modelo híbrido bidimensional fluido-estructura*, Ph. D. dissertation at Universidad Complutense de Madrid, 1996.

[13] S. Micu and E. Zuazua, *Propriétés qualitatives d'un modèle hybride bi-dimensionnel intervenant dans le cotrôle du bruit*, C. R. Acad. Sci. Paris, 319 (1994), pp. 1263-1268.

[14] S. Micu and E. Zuazua, *Stabilization and Periodic Solutions of a Hybrid System Arising in the Control of Noise*, Proceedings of the IFIP TC7/WG-7.2 International Conference, Laredo, España, Lecture Notes in Pure and Applied Mathematics, Vol. 174, Marcel Dekker, New York, 1996, pp. 219-230.

[15] S. Micu and E. Zuazua, *Boundary controllability of a linear hybrid system arising in the control of noise*, SIAM J. Control Optim., to appear.

[16] S. Micu and E. Zuazua, *On a weakly damped system arising in the control of noise*, preprint.

[17] S. Micu and E. Zuazua, *Asymptotics for the spectrum of a fluid/structure hybrid system arising in the control of noise*, preprint.

[18] F. W. J. Olver, *Asymptotics and Special Functions*, Academic Press, 1974.

[19] A. Pazy, *Semi-groups of Linear Operators and Applicatios to Partial Differential Equations*, Springer-Verlag, 1983.

[20] J. Ralston, *Solutions of the wave equation with localized energy*, Comm. Pure Appl. Math. 22 (1969), pp. 807-823.

[21] B. Rao. *Uniform Stabilization of a Hybrid System of Elasticity*, SIAM J. Cont. Optim., 33, 2 (1995), pp. 440-454.