WHAT IS CHAOS AND WHY SHOULD WE MIND OF IT?

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Introduction

After the publication of Principia, Newton's calculus-based dynamics became rapidly the standard model for most scientific theories. At the beginning of the 20th Century, the ideas of Einstein refined Newtonian dynamics and Bohr's quantum theories were intended to remove Newton's classical determinism from the heart of the Physics. Despite this apparent setback, the differential equations of classical mechanics are still widespread in physical modelling of the macroscopic world. In fact, studies of the time evolution of mathematical models are allowing the Newtonian ideal of differential causality to grow in many unexpected new directions such as economy, ecology, physiology etc.

It might have been supposed that, with the arrival of computers, the mathematical theory of dynamical systems would simply come to an end. A quick glance at the current research journals show that in fact the contrary is true, namely the nonlinear dynamics is one of the fastest growing fields of applied mathematics. The explanation is the recognition of complex, seemingly irregular motions of the deterministic systems, characterized by a sensitive dependence on the initial conditions i.e., by the possibility that two adjacent trajectories starting close together to diverge and eventually to become uncorrelated. That makes long term predictability impossible, a picture that contrasts with the well-behaved systems of classical analysis.

The story received a great impetus in the early 60s, when the meteorologist E. N. Lorenz [20], interested in the limits of predictability of weather prediction, found and plotted a chaotic attractor which exhibited sensitive dependence on initial conditions. It arouses from a simple looking quadratic system in three variables which was a truncation of convection flow. The physical origin and simple mathematical nature of this system make it important for study.

The system considered by E. N. Lorenz was

$$\begin{cases} \dot{x} &= -\sigma x + \sigma y \\ \dot{y} &= rx - y - xz \\ \dot{z} &= -bz + xy \end{cases}$$

where σ, r, b are three positive parameters (from which r represents Rayleigh's number); he took $\sigma = 10$ and $b = \frac{8}{3}$, observing the structural change of dynamics while increasing r.

A natural (and very popular) way to deal with continuous dynamical systems of the type

$$\text{(CDS)} \qquad \dot{x} = F(x)$$

is the use of a discrete time integration scheme (the simplest one being that due to Euler). For a small time step, Δt, we can write $\Delta x = F(x)\Delta t$, allowing us to make a small finite step from point n to the next step $n+1$ using

$$\text{(DDS)} \qquad x_{n+1} = x_n + F(x_n)\Delta t$$

i.e., a recurrence of the form

$$x_{n+1} = f(x_n).$$

So we encounter the problem of describing a continuous dynamical systems through a discrete approximation of it.

The purpose of this paper is to argue that in the absence of a serious qualitative analysis no conclusion valid for (DDS) can be extended to (CDS). While solving problems by iteration on a computer is now very popular, the reader should be aware of the pitfalls of such an approach. The recent paper by Lanford [18] reporting computer experiments on the orbit structure of some discrete maps is very instructive in this respect.

The terminology and notation used in our paper are in full agreement with *Dictionary of Mathematical Analysis* [35]. Of course, some basic concepts will be recalled here.

First, few words about the discrete dynamical systems.

Technically, a *discrete dynamical system* acting on set M is the sequence $(f^n)_n$ of all iterates of a function $f : M \to M$. Recall that the *iterates* of f are given by

$$f^0 = id_M \quad \text{and} \quad f^n = \underbrace{f \circ ... \circ f}_{n \text{ times}}, \text{ if } n \in \mathbb{N}^\star.$$

Most of the authors use the formula "let $f : M \to M$ be a dynamical system" for referring to the discrete dynamical system generated by f.

The main problem concerning these systems is as follows: *Given a function f and an initial value a, what happen with the sequence*

$$x_0 = a, \quad x_n = f(x_{n-1}) \quad \text{for } n \in \mathbb{N}^\star,$$

of all iterates of f computed at a?

Call the sequence $(f^n(a))_n$ the *trajectory* of *a,* and the set of its values, the *orbit* of *a.* Usually the orbit of a is denoted $\mathcal{O}(a)$.

The theory of recurrent sequences considers separately each such an object, while the theory of dynamical systems deals with the *ensemble* of all trajectories of a system. It was H. Poincaré [29] who came first to the idea of taking into consideration the global study of the iterates (and thus started the qualitative study of dynamical systems).

Understanding the structure of orbits is not an easy task. However, that proved basic at the computational level of mathematics. Not surprisingly, many questions with an apparent innocent statement, puzzled the most gifted mathematicians of the 20th Century. Steve Smale [34], whose numerous distinctions and honors include the Fields Medal, has recently formulated a list of outstanding problems left open for the 21th Century. Some of them concern the dynamical systems, a fact which led us to discuss here some basic facts on the contemporary vision of recurrent sequences.

1. Invariant sets. Attractors

If there exists a number $n \in \mathbb{N}^\star$ such that $f^n(a) = a$, then a is called a *periodic point* (of period n); the *principal period* of a periodic point a is the smallest $n \in \mathbb{N}^\star$ such that $f^n(a) = a$. The orbit of any periodic point is a finite set. It reduces to a singleton if a is a *fixed point* of f i.e., if

$$f(a) = a.$$

The identity of $\mathbb{R}$ admits all points of $\mathbb{R}$ as fixed points. The mapping $f(x) = -x$, $x \in \mathbb{R}$, has a unique fixed point (which is the origin), all others being periodic, of principal period 2.

The fixed points and the periodic orbits are examples of invariant sets: Given a mapping $f : M \to M$ as above, a subset A of M is said to be *invariant* (for f) if

$$f(A) = A$$

and *positively invariant* if

$$f(A) \subset A;$$

in both cases, the orbits leaving at points of A remain in A.

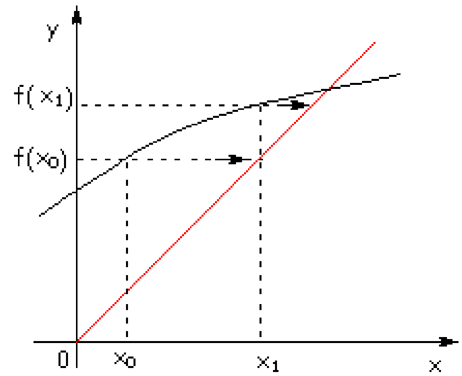


FIGURE 1

When the phase space M is an interval, we can outline the trajectories of a discrete dynamical system $f : M \to M$, via the *graphical analysis* of f (also called the *step diagram*).

The trajectory of a point a can be constructed as follows: Take a perpendicular at $a_0 = a$ to the $0x$ axis until it meets the graph of f in the point of coordinates $(a_0, f(a_0))$. The parallel through this point to $0x$ meets the first bisectrice in $(f(a_0), f(a_0))$. Then the parallel at $0y$ through the last point meet the graph of f in $(f(a_0), f^2(a_0))$ and so on. The intersection of the graph of f with the first bisectrice reveals the *fixed points*, i.e. the points whose trajectories are themselves. The arrows show orientation, while the ascendant / descendent steps show the monotonicity. See Fig. 1.

Calculus contributes significantly to the graphical analysis and thus to the study of dynamical properties.

1.1. Examples. i) Consider the mapping

$$f : [-1, \infty) \to [-1, \infty), \quad f(x) = \sqrt{1+x}.$$

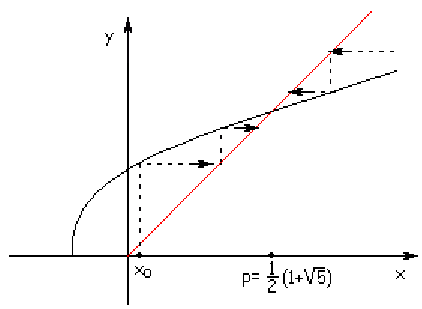
f has a unique fixed point, which is $p = (1 + \sqrt{5})/2$. The graphical analysis reveals the convergence of all trajectories to p; more precisely, if $x_0 < p$ then $f^n(x_0) \nearrow p$, while for $x_0 > p$ we have $f^n(x_0) \searrow p$. See Fig. 2.

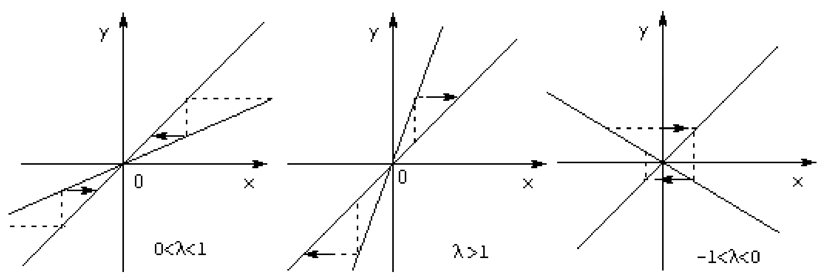
ii) Consider the case of the family of mappings

$$f_\lambda : \mathbb{R} \to \mathbb{R}, \quad f_\lambda(x) = \lambda x.$$

The origin is a fixed point for all of them, but varying the parameter λ, the dynamics near the origin can change drastically. See Fig. 3.

If $\lambda = 0$, then $f^n(x_0) = 0$ for every $x_0 \in \mathbb{R}$ and every $n \geq 1$.





If $\lambda \in (0,1)$, then all trajectories go monotonically to 0 (increasingly, if $x_0 < 0$, decreasingly, if $x_0 > 0$).

If $\lambda = 1$, then $\mathbb{R}$ consists only of fixed points.

If $\lambda > 1$, then the trajectory of each point $x_0 > 0$ goes (increasingly) to ∞, while of each point $x_0 < 0$, goes (decreasingly) to $-\infty$.

If $\lambda \in (-\infty, -1)$, then the trajectories go off the origin.

If $\lambda = -1$, then all trajectories are periodic, of period 2.

If.$\lambda \in (-1, 0)$, then all trajectories wrap 0.

The examples above outlined several types of behavior of fixed points. It is worth to formulate them in an abstract setting:

1.2. Definition. A fixed point p is said to be *attractive* (or, an *attractor*) for the dynamical system $f : M \to M$ if there exists a neighborhood U of p in M such that

$$f^n x_0 \to p \quad \text{for every} \quad x_0 \in U.$$

Call the set U which appears in Definition 1.2, a *basin of attraction* of p. If U can be chosen as the whole space M, then p is said to be the *global attractor* of the dynamical system $f : M \to M$.

1.3. Definition. A fixed point p is said to be a *repellor* for the dynamical system $f : M \to M$, if there exists a neighborhood U of p

in M such that for every $x_0 \in U$, $x_0 \neq p$, there exists a natural number n for which $f^n x_0 \notin U$.

However, we have to notice the existence of *indifferent* fixed points, which are not either attractive or repelling. This is the case for the origin, with respect to the dynamical system generated by the identity of $\mathbb{R}$.

An important source of global attractors is the Contraction Mapping Theorem. Recall that a mapping $f : M \to M$ (defined on a metric space $M = (M, d)$) constitutes a *contraction* if there exists a constant $C \in [0, 1)$ such that

$$d(f(x), f(y)) \leq Cd(x, y)$$

for every $x, y \in M$.

1.4. The Contraction Mapping Theorem (also known as the Banach-Cacciopoli Theorem). *Let M be a complete metric space (e.g., a closed subset of the Euclidean space $\mathbb{R}^n$) and let $f : M \to M$ be a contraction. Then f admits a unique fixed point p, which is also the global attractor of f (viewed as a discrete dynamical system on M).*

Moreover, p can be determined via the *successive approximation method*: given an *initial approximation* x_0, we consider the sequence of *successive approximations*,

$$x_n = f(x_{n-1}), \quad n \geq 1.$$

Then $x_n \to p$, regardless the choice of x_0.

The proof of the Contraction Mapping Theorem is well known. See our monograph [27], pp. 124-126. The property of the unique fixed point p to be the global attractor is equivalent with the fact that $x_n \to p$, for every $x_0 \in M$.

Exercises

(1) Use graphical analysis to describe the dynamics of the mapping

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = -x + 3x^2 - x^3.$$

(2) The method of successive approximations is more general then the principle of contraction. In other words, this method still works for mappings which are not necessarily contractions. In this respect, prove that for every $a \in \mathbb{R}$, the recurrent sequence defined by the formula

$$\begin{aligned} x_0 &= a \\ x_{n+1} &= \sin x_n, \quad \text{for } n \geq 0 \end{aligned}$$

converges to 0, the unique fixed point of the sine function.

(3) (B. P. Hillam [14]). Let $f:[a,b]\to[a,b]$ be a mapping which satisfies an estimate of the form

$$|f(x)-f(y)|\leq C|x-y| \quad \text{for every } x,y\in[a,b];$$

such a mapping is usually called a *Lipschitz mapping.* Put

$$F:[a,b]\to[a,b], \quad F(x)=(1-\lambda)x+\lambda f(x)$$

where $\lambda=1/(1+C)$. Show that F is nondecreasing and for every $x_0\in[a,b]$, the sequence defined by the formula

$$x_{n+1}=F^n(x_0), \quad n\geq 0$$

converges to one of the fixed points of f. See the paper of D. F. Bailey [3], for recent results in this connection.

(4) (B. P. Hillam [15]). Let $f:[0,1]\to[0,1]$ be a continuous function. Let x_0 be a point in $[0,1]$ and $(x_n)_n$ denote the resulting sequence of successive approximations. Then the sequence $(x_n)_n$ converges (necessarily to a fixed point of f) if and only if $\lim_{n\to\infty}(x_{n+1}-x_n)=0$.

2. Sharkovski's Theorem

In this section we will prove a remarkable theorem due A. N. Sharkovski, which came as a surprise by its simplicity and strong conclusions. In its simplest form it reads as follows:

2.1. Theorem (A. N. Sharkovski). *Every continuous mapping $f:\mathbb{R}\to\mathbb{R}$,which admits periodic points of principle period 3, admits also periodic points of any other period..*

Proof. Follow the following four steps:

i) Let $f:[a,b]\to\mathbb{R}$ be a continuous function. Show that for each compact interval B included in $f([a,b])$, there exists a compact interval A, included in $[a,b]$, such that $f(A)=B$.

ii) Let $f:[a,b]\to\mathbb{R}$ be a continuous function and let A be a non-empty compact interval included in $[a,b]$, with $A\subset f(A)$. Prove there exists a p in A such that $f(p)=p$.

iii) Let $f:[a,b]\to\mathbb{R}$ be a continuous function for which there exists a point c such that

$$f(a)=c,\ f(c)=b,\ f(b)=a.$$

For $n\in\mathbb{N}$, $n\geq 2$ consider the intervals $I_0=...=I_{n-2}=I_n=[a,b]$ and $I_{n-1}=[a,c]$. Show there exists a decreasing family $A_0\supset A_1\supset ...\supset A_n$, of compact subintervals of $[c,b]$, such that $f^k(A_k)=I_k$ for every k in $\{0,1,...,n\}$. Then, notice that by statement ii) above the function f^n admits a fixed point in A_n i.e., a periodic point of principal period n for f.

iv) Prove that every continuous function $f : \mathbb{R} \to \mathbb{R}$, with periodic points of principal period 3, has periodic points of any other principal period. ■

Actually the result proved by Sharkovski is much stronger (though specialized for intervals). Consider on $\mathbb{N}^\star$ the so-called *Sharkovski's ordering*

$$3 \rhd 5 \rhd 7 \rhd ... \rhd 2 \cdot 3 \rhd 2 \cdot 5 \rhd 2 \cdot 7 \rhd ... \rhd 2^2 \cdot 3 \rhd ...$$
$$\rhd 2^3 \rhd 2^2 \rhd 2 \rhd 1.$$

That is, first list all odd numbers except 1, followed by 2 times the odds, 2^2 times the odds, 2^3 times the odds and so on. That exhausts all the natural numbers with the exception of the power of 2, which we list last, in the decreasing order.

2.2. Theorem (A. N. Sharkovski [22]). *Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and f has a point of principal period m. If $m \rhd n$ in the above ordering, then f also has a periodic point of principal period n.*

The interested reader can find the details of Theorem 2.2 in the book of R. Devaney [8], pp. 60-68. Theorem 2.1 was rediscovered by Li and Yorke [19] in a celebrated paper where for the first time the term of *chaos* was used as a label for intricate behavior. Their main result is as follows:

2.3. Theorem (Li and Yorke [19]). *Let I be an interval and let $f : I \to I$ be a continuous mapping which admits points of period 3. Then there exists a uncountable subset S of I such that every orbit issued from S is aperiodic and unstable.*

Recall that an orbit $\mathcal{O}(a)$ is said to be *aperiodic* if it has infinitely many accumulation points.

The orbit $\mathcal{O}(a)$ is called *unstable* if there exist a number $\delta > 0$ such that for every neighborhood V of a there are $x \in V$ and $n \in \mathbb{N}^\star$ for which

$$d\left(f^n(x), f^n(a)\right) > \delta.$$

Notice that the critical set S in Theorem 2.3 can be very "thin", even of Lebesgue measure 0, as shows the following example due to M.

Martelli, M. Dang and T. Seph [21]:

$$F(x)=\begin{cases} 0, & \text{if } x\in[0,1/4)\\ 4x-1, & \text{if } x\in[1/4,1/2)\\ -4x+3, & \text{if } x\in[1.2,3/4)\\ 0, & \text{if } x\in[3/4,1]\end{cases}$$

A stronger concept of chaotic behavior will be described in section 6 below.

3. The irrational rotation

The *unit circle* is the set

$$S^1=\{z:\ z\in\mathbb{C},\ |z|=1\}.$$

It admits a natural structure of an abelian compact group, with respect to the metric topology

$$d(u,v)=|u-v|.$$

Given $\theta\in\mathbb{R}$, we define the *rotation of angle* θ on the unit circle as the mapping

$$R_\theta:S^1\to S^1,\quad R_\theta(z)=e^{i\theta}\cdot z.$$

3.1. Lemma. *If $\theta/2\pi\in\mathbb{Q}$, then all orbits of R_θ are periodic.*

Proof. Assuming that $\theta=2\pi\cdot\frac{p}{q}$, with $p,q\in\mathbb{N}^\star$ and p,q relatively prime, then

$$R_\theta^q(z)=e^{i\,q\theta}\cdot z=e^{i\,p2\pi}\cdot z=z,$$

for every $z\in S^1$. ∎

Apropos of Sharkovski's Theorem limitations, notice that the rotation R_θ, with $\theta=2\pi/3$, has periodic points of period 3 but no periodic points of any other principal period.

3.2. Theorem (C. G. J. Jacobi)**.** *If $\theta/2\pi\notin\mathbb{Q}$, then every orbit of R_θ is dense in S^1.*

Proof. As $\theta/2\pi\notin\mathbb{Q}$, the elements of the sequence z, $R_\theta(z)$, $R_\theta^2(z), \ldots$ are pairwise distinct.

Because S^1 is compact, every sequence of elements of S^1 has a convergent subsequence. Particularly, a subsequence $\left(R_\theta^{k(n)}(z)\right)_n$ is convergent. Given $\varepsilon>0$, there exists an $N_\varepsilon\in\mathbb{N}$ such that

$$|R_\theta^{k(m)}(z)-R_\theta^{k(n)}(z)|<\varepsilon$$

for every $m,n\geq N_\varepsilon$. Let $N=k(N_\varepsilon+1)-k(N_\varepsilon)$. Then, $N>0$. As rotations are isometries, we have

$$\begin{aligned} |R_\theta^N(z) - z| &= |R_\theta^{k(N_\varepsilon)}\left(R_\theta^N(z) - z\right)| = \\ &= |R_\theta^{k(N_\varepsilon+1)}(z) - R_\theta^{k(N_\varepsilon)}(z)| < \varepsilon \end{aligned}$$

so that by iterating this argument we infer that the points z, $R_\theta^N(z)$, $R_\theta^{2N}(z)$, ... cut S^1 into arcs of diameters $< \varepsilon$. Or, every point w of S^1 belongs to such an arc, which yields the existence of an $n = n(w, \varepsilon) \in \mathbb{N}$ for which $|w - R_\theta^n(z)| < \varepsilon$. ■

The points with dense orbits are usually called *topologically transitive points* (while the systems which admit topologically transitive points are called *topologically transitive systems*). Their dynamics is typically complicated. See section 6 below.

More details on the dynamics of mappings on the unit circle are to be found in [8], [17].

4. Hyperbolicity

The aim of this section is to describe a condition (due to O. Perron), under which the fixed points are either attractive or repelling. As above, we shall restrict ourselves to the case where M is an interval I and the dynamical system under attention is associated to a C^1-mapping $f : I \to I$. Recall that the membership in the class C^r, with $r \geq 1$, means the existence and the continuity of the rth derivative.

4.1. Definition. A fixed point p of the dynamical system $f : I \to I$ is said to be *hyperbolic* if

$$|f'(p)| \neq 1.$$

The usefulness of Definition 4.1 is outlined by the following result:

4.2. Theorem. *Suppose that $f : I \to I$ is a C^1-mapping.*

i) *If $|f'(p)| < 1$, then there exists a neighborhood U of p such that $f(U) \subset U$ and for every x of U we have*

$$\lim_{n\to\infty} f^n(x) = p.$$

ii) *If $|f'(p)| > 1$, then there exists a neighborhood V of p such that whenever x is in $V \setminus \{p\}$, one can find an n in $\mathbb{N}$ for which $f^n(x) \notin V$.*

Proof. i) As f has continuous derivative and $|f'(p)| < C < 1$, there exists an $\varepsilon > 0$ so that

$$|f'(x)| < C \quad \text{on} \quad U = [p - \varepsilon, p + \varepsilon] \cap I.$$

According to the Mean Value Theorem, for every x in U we have

$$|f(x) - p| = |f(x) - f(p)| \leq C|x - p| \leq |x - p| \leq \varepsilon.$$

Therefore $f(x)$ belongs to U, and by iterating the above argument, we are led to

$$|f^n(x) - p| \leq C^n \, |x - p|,$$

which yields $f^n(x) \to p$.

The assertion ii) can be argued in a similar manner. ■

It is important to extend the above considerations to the case of positively invariant sets.

4.3 Definition. A positively invariant set A, for the C^1-mapping $f : I \to I$, is said to be *hyperbolic* if

$$|f'(a)| \neq 1 \quad \text{for every } a \in A.$$

A special case is that of periodic orbits. Let p be a periodic point, of principal period m, for the mapping $f : I \to I$, of class C^1. Then the hyperbolicity condition for $\mathcal{O}(p)$ is

$$|(f^m)'(p)| \neq 1.$$

In fact,

$$(f^m)'(x) = (f^m)'(p) \quad \text{for every } x \in \mathcal{O}(p)$$

since for $x \in \mathcal{O}(p)$ we have $x = f^k(p)$ for some $k \in \{0, ..., m-1\}$, and thus

$$\begin{aligned} (f^m)'(f^k(p)) &= f'(f^{m-1}(f^k(p))) \cdot f'(f^{m-2}(f^k(p))) \cdot ... \cdot f'(f^k(p)) = \\ &= f'(f^{m-1}(p)) \cdot f'(f^{m-2}(p)) \cdot ... \cdot f'(p). \end{aligned}$$

Replacing f by f^m in Theorem 4.2, we get the behavior of the hyperbolic periodic orbits (in the case of mappings acting on intervals), which can be either an attractor or a repellor.

For example, the mapping

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = -\left(x^3 + x\right)/2$$

admits the origin as a hyperbolic attractor, with basin $U = \mathbb{R} \setminus \{-1, 1\}$; the periodic orbit $\mathcal{O}(1) = \{-1, 1\}$ is also hyperbolic, but it constitutes a repellor.

The theory exposed above can be easily generalized to the case where f is a differentiable mapping on a differentiable manifold (particularly, on an open subset of the Euclidean space $\mathbb{R}^n$). In that case the derivative should be replaced by the differential, and the hyperbolicity of a fixed point p will mean that the spectrum of $df(p)$ doesn't intersect the unit circle.

See [1], [6] and [7] for more details.

Exercises

(1) Classify the periodic points of the following mappings (acting on $\mathbb{R}$) :

$$\begin{aligned} f_1(x) &= \sin x \\ f_2(x) &= x^3 - x \\ f_3(x) &= \arctan x \\ f_4(x) &= e^x. \end{aligned}$$

(2) Call a diffeomorphism $f : [a, b] \to [a, b]$ a *Morse-Smale diffeomorphism* if all its periodic orbits are hyperbolic. Prove that $f(x) = x^3 + 3x/4$, $x \in [-1/2, 1/2]$, is such an example. Prove that a Morse-Smale diffeomorphism can have only finitely many periodic points.

5. An open problem concerning the 1-dimensional dynamics

In the theory of dynamical systems the points of interest are the so called nonwandering points.

5.1. Definition. Given a continuous mapping $f : M \to M$, call a point $a \in M$ *nonwandering* for f if for every neighborhood U of a there exists a $k \in \mathbb{N}^\star$ such that $f^k(U) \cap U \neq \emptyset$.

The set $\Omega(f)$, of all nonwandering points for f, includes $Per\,(f)$, the set of all periodic points for f, as well as the set of all topologically transitive points for f.

The set $\Omega(f)$ is closed and positively invariant. For the first assertion, notice that $M \setminus \Omega(f)$ is open, because together with a point a it contains an entire neighborhood of a.

5.2. Definition. A continuous mapping $f : M \to M$ is said to be *hyperbolic* if it verifies the following two conditions:

i) The set $\Omega(f)$ is hyperbolic;

ii) The set of all periodic points of f is dense in $\Omega(f)$.

Letting $C^r([0, 1], [0, 1])$ the space of all mappings $f : [0, 1] \to [0, 1]$ of class C^r, endowed with the C^r*−metric*,

$$d_{C^r}(f, g) = \sum_{k=0}^{r} \sup_{x \in [0,1]} |D^k f(x) - D^k g(x)|,$$

the following problem naturally arises:

5.3. Problem. Can every mapping $f \in C^r([0, 1], [0, 1])$ be approximated in the C^r−metric by hyperbolic mappings of class C^r ?

This problem was settled in the affirmative by M. Jakobson [16], in the case where $r = 1$; see also [9]. Despite a lot of work, the case $r > 1$ is still open.

6. The sensitive dependence on the initial conditions

Iterating continuous mappings $f : M \to M$ (acting on metric spaces) we can encounter bad surprises at computational level. The reason is the initial conditions of a real system may be imprecise or even totally unknown. Typically, instead of dealing with the true trajectory of a point x_0, we deal with the trajectory of a certain approximation $\bar{x}_0$ of it. The continuity of f assures us that for a given $\varepsilon > 0$ we can find a $\delta > 0$ so that

$$d(x_0, \bar{x}_0) < \delta \quad \text{implies} \quad d(f(x_0), f(\bar{x}_0)) < \varepsilon.$$

However, nothing assures that

$$\sup_{n \geq 0} d(f^n(x_0), f^n(\bar{x}_0)) < \varepsilon.$$

On the contrary, simple examples such as the *doubling angle mapping*,

$$f : S^1 \to S^1, \quad f(z) = z^2,$$

show that exactly the opposite can happen! This phenomenon (much stronger than that of the existence of unstable trajectories) imposes the following definition:

6.1. Definition (J. Guckenheimer [12]). A discrete dynamical system $f : M \to M$ shows sensitive *dependence on the initial conditions* if there exists a number $\delta > 0$ such that for every point $x \in M$ and every neighborhood V of x one can find $y \in V$ and $n \in \mathbb{N}^\star$ with

$$d\left(f^n(x), f^n(y)\right) > \delta.$$

Nowadays, terms such as *chaotic behavior*, *chaotic dynamics*, or *chaos* are very popular. While there is no unanimously recognized definition for the mathematical concept of chaos, all people agree that the sensitive dependence on the initial conditions should be at the heart of this concept.

6.2. Definition (R. Devaney). A dynamical system f is called *chaotic* if it satisfies the following three conditions:

(T) f is topologically transitive;
(P) The set of all periodic points is dense;
(S) f shows sensitive dependence on the initial conditions.

Actually, the third condition in Definition 6.2 proved later extraneous:

6.3. Theorem (J. Banks, J. Brooks, G. Cairns, G. Davis and P. Stacey [4]; see also [11], or [28]).

$$(T)\,\&\,(P) \Rightarrow (S).$$

The main result of topological dynamics on intervals is as follows:

6.4. Theorem (L. S. Block and W. A. Coppel [6], p. 156). *Let I be an interval. Then every topologically transitive continuous mapping $f : I \to I$ is chaotic.*

Theorem 6.4 still works for finite unions of intervals. See I. Melbourne, M. Dellnitz and M. Golubitsky [23] and C. Niculescu [26].

Another case of the appearance of chaos was noticed by E. Glasner and B. Weiss [11], Lemma 1.2: If K is a compact metric space and $F : K \to K$ is a continuous, transitive and not an one-to-one mapping, then F shows sensitive dependence to initial conditions.

A simple technique to prove the chaotic behavior of certain systems is that by *semiconjugation*:

6.5. Theorem. *Let X and Y be two metric spaces and let the commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ h\downarrow & & \downarrow h \\ Y & \xrightarrow[g]{} & Y \end{array}$$

consisting of continuous mappings. We assume that f is chaotic and h is onto. Then g is chaotic too.

Proof. One verifies immediately the conditions (T) and (P) (and thus (S), according to Theorem 6.3). ■

How difficult is to verify that a certain mapping is chaotic and how common are those mappings?

No doubt the most surprising case is that of the Chebyshev polynomial of order 2,

$$T_2(x) = 2x^2 - 1, \quad x \in \mathbb{R}.$$

In order to simplify the computation is worth to notice that the substitution $x = \cos\theta$ gives us $T_2(x) = \cos 2\theta$.

The points of $\mathbb{R}$ are either topologically transitive or eventually periodic for T_2; recall that a point p is *eventually periodic* if $T_2^n(p)$ is periodic for some n. The eventually periodic points of T_2 are precisely the points p of the form $p = \cos\theta$ where

$$\theta = \frac{k\pi}{2^n - 1}\,, \quad k \in \mathbb{Z},\, n \in \mathbb{N}.$$

The chaotic behavior of the polynomial T_2 follows from Theorem 6.5, applied to $X = S^1$, $Y = [-1,1]$, f =doubling angle mapping, $h(e^{i\theta}) = \cos\theta$ (the projection on the first coordinate) and $g = T_2$.

From T_2 one can infer in a similar manner (using the function $h(x) = \frac{1}{2}(1-x)$) the chaotic behavior of the *logistic mapping,*

$$F_4 : [0,1] \to [0,1], \quad F_4(x) = 4x(1-x).$$

A recent paper by M. Martelli, M. Dang and T. Seph [21] calls the attention to many other possible definitions of the mathematical concept of chaos, which might play a role in the future.

Exercises

(1) Show that in general all Chebyshev polynomials
$$T_n(x) = \cos(n \arccos x), \quad x \in [-1,1]$$
of order $n \geq 2$ are chaotic.

(2) Indicate a direct argument (i.e., independent of Theorem 6.5 above) for the chaotic behavior of the polynomial T_2.

(3) Prove that the function $f(x) = \pi \sin x$, $x \in [0,\pi]$, is chaotic.

7. Sensitivity on $\omega-$limit sets

The purpose of this section is to prove a generalization of Theorem 6.3 at the level of $\omega-$limit sets. The importance of this result is that the post transient dynamics of a dissipative system is concentrated on such sets.

As above, M will denote a (perfect) metric space and $F : M \to M$ will denote a continuous mapping.

Given $x \in M$, its $\omega-$*limit set* is the set $\omega(x)$, consisting of all points $y \in M$ for which there is an increasing sequence $(k(n))_n$ of positive integers such that $F^{k(n)}x \to y$. Among the basic properties of $\omega(x)$ we mention:

$\omega1)$ $\omega(x)$ is closed;
$\omega2)$ $\omega(F(x)) = \omega(x)$;
$\omega3)$ $F(\omega(x)) \subset \omega(x)$, with equality if $\omega(x)$ is compact.

Notice that $\omega(x) \neq \emptyset$ (and $\omega(x)$ is compact) if the orbit of x is relatively compact.

An important remark about $\omega(x)$ is the following property of topological transitivity: *If $y, z \in \omega(x)$ and U and V are open neighborhoods of y and respectively z, then $F^n(U) \cap V \neq \emptyset$ for some $n \geq 1$.*

An *attractor* for F is any $\omega-$limit set $\mathcal{A}$ such that for any open neighborhood U of $\mathcal{A}$ there is a smaller open neighborhood V of $\mathcal{A}$ such that

$$F^n(V) \subset U \quad \text{for all } n \in \mathbb{N}.$$

The following result raised in a discussion with Prof. Benjamin Weiss, at the Hebrew University, Jerusalem, in November 1998.

7.1 Theorem. *Suppose that $F : M \to M$ is a continuous mapping and $x \in M$ is a point whose $\omega-$limit set $\omega(x)$ has the following two properties*:

(R) $\omega(x)$ *is contained in the closure of all regular points of F*;

(NM) $\omega(x)$ *contains a proper positively invariant compact subset K.*

Then there exists an $\varepsilon > 0$ such that for every $y \in \omega(x)$ and every $\delta > 0$ one can find a $z \in M$ with $d(z, y) < \delta$ such that

$$\sup_{n\geq 1} d(F^n z, F^n y) \geq \varepsilon.$$

Recall that a point $a \in M$ is said to be *regular* for F (or *generic,* in the terminology of [10]) if there exists an $F-$invariant probability measure μ (on the Borel subsets of M) such that $\mu(U) > 0$ for every neighborhood U of a.

Regular points include the uniformly recurrent (equivalently, the almost periodic) points and thus the periodic points.

Proof. Suppose that the contrary is true i.e., for every $\varepsilon > 0$ there exist a point $y_\varepsilon \in \omega(x)$ and a number $\delta > 0$ such that

$$z \in M, \quad d(z, y_\varepsilon) < \delta \Rightarrow \sup_{n\in\mathbb{N}} d(F^n z, F^n y_\varepsilon) < \varepsilon.$$

There are two possibilities:

1) $y_\varepsilon \in K$.

2) $y_\varepsilon \notin K$. Choose $\varepsilon > 0$ with $\varepsilon < \ diam\,(\omega(x) \setminus K)\,/10$. Let $d(F^{n_0}x, y_\varepsilon) < \delta$ and let $n_1 > n_0$ with $d(F^{n_1}x, K) \geq 5\varepsilon$. Put $y_1 = F^{n_1-n_0}y_\varepsilon$ and choose a regular point q in $B_\delta(y_\varepsilon)$ such that

$$d(z, q) < \eta \Rightarrow d(F^{n_1-n_0}z, F^{n_1-n_0}q) < \varepsilon$$

Because q is regular, there must exist some $F-$invariant measure μ with $\mu\,(B_\delta(y_\varepsilon)) > 0$. Then

$$J = \{m \,:\, \mu\left(F^{-m}(B_\delta(y_\varepsilon)) \cap B_\delta(y_\varepsilon)\right) > 0\}$$

is a syndetic subset (i.e., a set with bounded gaps) of $\mathbb{N}$. See [10] for more information on this type of subsets. If $m \in J$, then

$$F^m z \in B_\delta(y_\varepsilon) \quad \text{for some } z \in B_\delta(y_\varepsilon).$$

so that

$$d(F^{m+n_1-n_0}z,\, y_1) < \varepsilon.$$

Then

$$(*) \quad d(F^{m+n_1-n_0}y_\varepsilon,\, y_1) < d(F^{m+n_1-n_0}y_\varepsilon, F^{m+n_1-n_0}z\,) + \\ + d(F^{m+n_1-n_0}z,\, y_1) < 2\varepsilon.$$

Or,

$$d(y_1,\, K) \geq d(F^{n_1}x,\, K) - d(F^{n_1-n_0}F^{n_0}x, F^{n_1-n_0}y_\varepsilon) > 5\varepsilon - \varepsilon = 4\varepsilon$$

and

$$d(F^{m+n_1-n_0}y_\varepsilon,\, K) \geq d(y_1, K) - d(F^{m+n_1-n_0}y_\varepsilon, y_1) \geq 4\varepsilon - 2\varepsilon = 2\varepsilon$$

on a syndetic set of m's, which leads to a contradiction with (*). ■

8. Numerical algorithms and chaotic behavior

As well known, most evolution equations can be solved only by numerical methods. Recently, ideas from dynamical system theory have begun to open new avenues in numerical analysis. See the proceedings volume [30]. In fact, the accuracy of the numerical approximation of the solutions can be determined by analyzing the dynamics of a scalar mapping depending on one parameter, the step size.

We shall consider here the simplest case of Euler's algorithm of approximating solutions of differential equations of the form

$$\frac{dx}{dt} = f(x).$$

Under this algorithm we take a discrete set of points $t_0 = 0,\ t_1 = h,\ t_2 = 2h, \ldots$ of constant step size h. Next, we compute inductively approximate values of the solution $x = x(t)$ at these points. For, $\frac{dx}{dt}(t_n)$ is replaced by $\frac{x_{n+1} - x_n}{h}$ and we are led to the recurrent sequence $(x_n)_n$, where

$$\begin{aligned} x_0 &= x(0) \\ x_{n+1} &= x_n + h \cdot f(x_n). \end{aligned}$$

Consider now the particular case of logistic equation

$$\frac{dx}{dt} = ax(1-x)$$

where a is a positive parameter. Its trajectories are of the form

$$x(t) = \frac{x(0) \cdot e^{at}}{1 - x(0) + x(0) \cdot e^{at}}$$

which makes $x = 1$ an attracting fixed point.

Letting $b = ha$, the Euler's algorithm applied to the logistic equation becomes

$$x_{n+1} = bx_n(\frac{1+b}{b} - x_n).$$

The study of this discrete dynamical system can be subordinated to that of logistic mapping. In fact, letting $x_n = \frac{1+b}{b} \cdot y_n$, we are led to the iterative process

$$y_{n+1} = (1+b)y_n(1-y_n).$$

For $a = 1000.000$ and $h = 0.003$ we get $1 + b = 4$, so the iterates behave chaotically. In particular, for this choice of parameters, the existence of the attracting point $x = 1$ cannot be revealed numerically.

Recently, O. E. Lanford III [18], called the attention to yet another subtleties of numerical treatment of discrete dynamical systems.

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