A MULTIPLICATIVE MEAN VALUE AND ITS APPLICATIONS

CONSTANTIN P. NICULESCU

Abstract. We develop a parallel theory to that concerning the concept of integral mean value of a function, by replacing the additive framework with a multiplicative one. Particularly, we prove results which are multiplicative analogues of the Jensen and Hermite-Hadamard inequalities.

1. Introduction

Many results in Real Analysis exploits the property of convexity of the subintervals I of $\mathbb{R}$,

$$\text{(A)} \qquad x, y \in I,\ \lambda \in [0,1] \quad \text{implies} \quad (1-\lambda)x + \lambda y \in I$$

which is motivated by the vector lattice structure on $\mathbb{R}$. By evident reasons, we shall refer to it as the *arithmetic convexity*.

As well known, $\mathbb{R}$ is an ordered field and the subintervals J of $(0,\infty)$ play a multiplicative version of convexity,

$$\text{(G)} \qquad x, y \in J,\ \lambda \in [0,1] \quad \text{implies} \quad x^{1-\lambda} y^{\lambda} \in J.$$

which we shall refer to as the *geometric convexity.* Moreover, the pair $\exp - \log$ makes possible to pass in a canonical way from (A) to (G) and vice-versa.

As we noticed in a recent paper [8], this fact opens the possibility to develop several parallels to the classical theory involving (A), by replacing (A) by (G) (or, mixing (A) with (G)).

The aim of the present paper is to elaborate on the multiple analogue of the notion of mean value. In order to make our definition well understood we shall recall here some basic facts on the *multiplicatively convex functions* i.e., on those functions $f : I \to J$ (acting on subintervals of $(0,\infty)$) such that

$$\text{(GG)} \qquad x, y \in I \text{ and } \lambda \in [0,1] \quad \text{implies} \quad f(x^{1-\lambda} y^{\lambda}) \leq f(x)^{1-\lambda} f(y)^{\lambda};$$

the label (GG) is aimed to outline the type of convexity we consider on the domain and the codomain of f. Under the presence of continuity, multiplicative convexity means

$$f(\sqrt{xy}) \leq \sqrt{f(x)\, f(y)} \quad \text{for all } x, y \in I$$

which motivates the alternative terminology of *convexity according to the geometric mean* for (GG). Another equivalent definition of the multiplicative convexity (of a function$f$) is $\log f(x)$ *is a convex function of* $\log x$. See [8], Lemma 2.1. Modulo

1991 *Mathematics Subject Classification.* 26A51, 26D07, 26D15.

Key words and phrases. Keywords and phrases: Convex function, Mean value, $AM - GM$ Inequality.

Published in vol. *Theory of Inequalities and Applications*, vol. 1 (Y. J. Cho, S. S. Dragomir and J. Kim, editors), pp, 243-255, Nova Science Publishers, Huntington, New York, 2001. ISBN 1-59033-188-5.

this remark, the class of all multiplicatively convex functions was first considered by P. Montel [7], in a beautiful paper discussing the analogues of the notion of a convex function in n variables.

As noticed in [8], the class of multiplicatively convex functions contains a broad range of functions from the elementary ones, such as

$$\begin{aligned}
&\sinh,\ \cosh,\ \exp, \quad \text{on } (0,\infty)\\
&\tan,\ \sec,\ \csc,\ 1/x-\cot x, \quad \text{on } (0,\pi/2)\\
&\arcsin,\ \arccos, \quad \text{on } (0,1]\\
&-\log(1-x),\ \frac{1+x}{1-x}\,, \quad \text{on } (0,1)
\end{aligned}$$

to the special ones, such as $\Gamma|[1,\infty)$, Psi, L (the Lobacevski function), Si (the integral sine) etc.

The notion of a strictly multiplicatively convex function can be introduced in a natural way and we shall omit the details here. Notice that the *multiplicatively affine* functions are those of the form Cx^{α}, with $C>0$ and $\alpha\in\mathbb{R}$.

Some readers could be frustrated by the status of 0 (of being placed outside the theory of multiplicative convexity). This can be fixed each time we work with functions f such that $f(0)=0$ and $f(x)>0$ for $x>0$.

There is a functorial device to translate the results for the (A)-type of convexity to the (G)$-$type and vice-versa, based on the following remark:

Lemma 1.1. *Suppose that I is a subinterval of $(0,\infty)$ and $f: I\to(0,\infty)$ is a multiplicatively convex function. Then*

$$F=\log\circ f\circ\exp:\log(I)\to\mathbb{R}$$

is a convex function. Conversely, if J is an interval and $F: J\to\mathbb{R}$ is a convex function, then

$$f=\exp\circ F\circ\log:\exp(J)\to(0,\infty)$$

is a multiplicatively convex function.

In the standard approach, the *mean value* of an integrable function $f:[a,b]\to\mathbb{R}$ is defined by

$$M(f)=\frac{1}{b-a}\int_a^b f(t)\,dt$$

and the discussion above motivates for it the alternative notation $M_{AA}(f)$, as it represents the average value of f according to the arithmetic mean.

Taking into account Lemma 1 above, the *multiplicative mean value* of a function $f:[a,b]\to(0,\infty)$ (where $0<a<b$) will be defined by the formula

$$M_{GG}(f)=\exp\left(\frac{1}{\log b-\log a}\int_{\log a}^{\log b}\log f(e^t)dt\right)$$

equivalently,

$$\begin{aligned}
M_{GG}(f) &= \exp\left(\frac{1}{\log b-\log a}\int_a^b \log f(t)\,\frac{dt}{t}\right)\\
&= \exp\left(L(a,b)\,M\left(\frac{\log f(t)}{t}\right)\right)
\end{aligned}$$

where

$$L(a,b) = \frac{b-a}{\log b - \log a}$$

represents the *logarithmic mean* of a and b.

In what follows, we shall adopt for the multiplicative mean value of a function f the (more suggestive) notation $M_*(f)$.

The main properties of the multiplicative mean are listed below:

$$\begin{gathered} M_*(1) = 1 \\ m \leq f \leq M \ \Rightarrow m \leq M_*(f) \leq M \\ M_*(fg) = M_*(f)\, M_*(g). \end{gathered}$$

It is worth noticing that similar schemes can be developed for other pairs of types of convexity, attached to different averaging devices. See [11]. We shall not enter the details here, but the reader can verify easily that many other mean values come this way. For example, the *geometric mean* of a function f,

$$\exp\left(\frac{1}{b-a}\int_a^b \log f(t)dt\right)$$

is nothing but the mean value $M_{AG}(f)$, corresponding to the pair (A)$-$(G). The geometric mean of the identity of $[a,b]$,

$$I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)}.$$

(usually known as the *identric mean* of a and b) appears many times in computing the multiplicative mean value of some concrete functions.

Notice that the multiplicative mean value introduced here escapes the classical theory of integral $f-$means. In fact, it illustrates, in a special case, the usefulness of extending that theory for normalized weighted measures.

The aim of this paper is to show that two major inequalities in convex function theory, namely the Jensen inequality and the Hermite-Hadamard inequality, have multiplicative counterparts. As a consequence we obtain several new inequalities, which are quite delicate outside the framework of multiplicative convexity.

2. The multiplicative analogue of Jensen's Inequality

In what follows we shall be concerned only with the integral version of the Jensen Inequality.

Theorem 2.1. *Let $f : [a,b] \to (0,\infty)$ be a continuous function defined on a subinterval of $(0,\infty)$ and let $\varphi : J \to (0,\infty)$ be a multiplicatively convex continuous function defined on an interval J which includes the image of f. Then*

$$\varphi\left(M_*(f)\right) \leq M_*(\varphi \circ f).$$

Proof. In fact, using constant step divisions of $[a,b]$ we have

$$\begin{aligned} M_*(f) &= \exp\left(\frac{1}{\log b - \log a}\int_a^b \log f(t)\,\frac{dt}{t}\right) \\ &= \lim_{n\to\infty} \exp\left(\sum_{k=1}^{n} \log f(t_k)\frac{\log t_{k+1} - \log t_k}{\log b - \log a}\right) \end{aligned}$$

which yields, by the multiplicative convexity of φ,

$$\begin{aligned}
\varphi\left(M_*(f)\right) &= \lim_{n\to\infty} \varphi\left(\exp\left(\sum_{k=1}^{n} \log f(t_k) \frac{\log t_{k+1} - \log t_k}{\log b - \log a}\right)\right) \\
&\leq \lim_{n\to\infty} \left(\exp\left(\sum_{k=1}^{n} \log(\varphi\circ f)(t_k) \frac{\log t_{k+1} - \log t_k}{\log b - \log a}\right)\right) \\
&= M_*(\varphi\circ f). \quad \blacksquare
\end{aligned}$$

The multiplicative analogue of Jensen's Inequality is the source of many interesting inequalities. We notice here only a couple of them. First, letting $\varphi = \exp t^\alpha$ $(\alpha > 0)$, we are led to the following concavity type property of the log function:

$$\left(\frac{1}{\log b - \log a}\int_a^b \log f(t)\,\frac{dt}{t}\right)^\alpha \leq \log\left(\frac{1}{\log b - \log a}\int_a^b f^\alpha(t)\,\frac{dt}{t}\right)$$

for every $\alpha > 0$ and every function f as in the statement of Theorem 2.1 above.

Particularly, for $f = e^t$, we have

$$L(a,b)^\alpha \leq \log\left(\frac{1}{\log b - \log a}\int_a^b e^{\alpha\, t}\,\frac{dt}{t}\right)$$

whenever $\alpha > 0$.

Our second illustration of Theorem 2.1 concerns the pair $\varphi = \log t$ and $f = e^t$; φ is multiplicatively concave on $(1,\infty)$, which is a consequence of the AM-GM Inequality. The multiplicative mean of $f = e^t$ is $\exp\left(\frac{b-a}{\log b - \log a}\right)$, so that we have

$$L(a,b) \geq \exp\left(\frac{1}{\log b - \log a}\int_a^b \log\log t\,\frac{dt}{t}\right) = I(\log a, \log b)$$

for every $1 < a < b$. However, as J. Sándor pointed out to me, a direct application of the Hermite-Hadamard inequality gives us (in the case of the exp function) a better result:

$$L(a,b) > \sqrt{ab} > \log\sqrt{ab} > I(\log a, \log b).$$

The problem of estimating from above the difference of the two sides in Jensen's Inequality,

$$M_*(\varphi\circ f) - \varphi\left(M_*(f)\right)$$

can be discussed by adapting the argument in [10]. We leave the details to the reader.

3. The multiplicative analogue of the Hermite-Hadamard Inequality

The classical Hermite-Hadamard Inequality states that if $f : [a,b] \to \mathbb{R}$ is a convex function then

$$(HH) \qquad f\left(\frac{a+b}{2}\right) \leq M(f) \leq \frac{f(a)+f(b)}{2}\,,$$

which follows easily from the midpoint and trapezoidal approximation to the middle term. Moreover, under the presence of continuity, equality occurs (in either side) only for linear functions.

Interesting applications of (HH) are to be found in [12], pp. 137-151. Update information on the entire topics related to (HH) is available on the site of *Research Group in Mathematical Inequalities and Applications* (*http://rgmia.vu.au*).

The next result represents the multiplicative analogue of the Hermite-Hadamard Inequality and it is a translation (via Lemma 1.1) of the classical statement:

Theorem 3.1. *Suppose that* $0 < a < b$ *and let* $f : [a, b] \to (0, \infty)$ *be a continuous multiplicatively convex function. Then*

$$f(\sqrt{ab}) \leq M_*(f) \leq \sqrt{f(a)f(b)}. \tag{$*HH$}$$

The left side inequality is strict unless f *is multiplicatively affine, while the right side inequality is strict unless* f *is multiplicatively affine on each of the subintervals* $[a, \sqrt{ab}]$ *and* $[\sqrt{ab}, b]$.

As noticed L. Fejér [2] (see also [3]), the classical Hermite-Hadamard Inequality admits a weighted extension by replacing dx by $p(t)dt$, where p is a non-negative function whose graph is symmetric with respect to the center $(a + b)/2$. Of course, this fact has a counterpart in $(*HH)$, where dt/t can be replaced by $p(t)\,dt/t$, with p a non-negative function such that $p(t/\sqrt{ab}) = p(\sqrt{ab}/t)$.

In the additive framework, the mean value verifies the equality

$$M(f) = \frac{1}{2}\left(M(f\,|\,[a, \frac{a+b}{2}]) + M(f\,|\,[\frac{a+b}{2}, b])\right)$$

which can be checked by an immediate computation; in the multiplicative setting it reads as follows:

Lemma 3.2. *Let* $f : [a, b] \to (0, \infty)$ *be an integrable function, where* $0 < a < b$. *Then*

$$M_*(f)^2 = M_*(f\,|\,[a, \sqrt{ab}]) \cdot M_*(f\,|\,[\sqrt{ab}, b]).$$

Corollary 3.3. *The multiplicative analogue of the Hermite-Hadamard Inequality can be improved upon*

$$\begin{aligned} f(a^{1/2}b^{1/2}) \quad &< \quad \left(f(a^{3/4}b^{1/4})f(a^{1/4}b^{3/4})\right)^{1/2} < M_*(f) \\ &< \quad \left(f(a^{1/2}b^{1/2})\right)^{1/2} f(a)^{1/4}f(b)^{1/4} \\ &< \quad (f(a)f(b))^{1/2}. \end{aligned}$$

A moment's reflection shows that by iterating Corollary 3.3 one can exhibit approximations of $M_*(f)$ from below (or from above) in terms of (G)-convex combinations of the values of f at the *multiplicatively dyadic* points $a^{(2^n-k)/2^n}b^{k/2^n}$, $k = 0, ..., 2^n$, $n \in \mathbb{N}$.

For $f = \exp|\,[a, b]$ (where $0 < a < b$) we have $M_*(f) = \exp\left(\frac{b-a}{\log b - \log a}\right)$. According to the Corollary 3.3 above we obtain the inequalities

$$\frac{a^{3/4}b^{1/4} + a^{1/4}b^{3/4}}{2} < \frac{b-a}{\log b - \log a} < \frac{1}{2}\left(\frac{a+b}{2} + \sqrt{ab}\right),$$

first noticed by J. Sándor [13].

For $f = \Gamma\,|\,[a, b]$ (where $1 \leq a < b$) we obtain the inequalities

$$\log\Gamma\left(a^{1/2}b^{1/2}\right) < \frac{1}{\log b - \log a}\int_a^b \frac{\log\Gamma(x)}{x}dx < \frac{1}{2}\log\Gamma(a)\Gamma(b) \tag{Γ}$$

which can be strenghtened via Corollary 3.3.

The middle term can be evaluated by Binet's formula (see [16], p. 249), which leads us to

$$\frac{\log \Gamma(x)}{x} = \log x - 1 - \frac{1}{2} \cdot \frac{\log x}{x} + \frac{\log \sqrt{2\pi}}{x} + \frac{\theta(x)}{x}$$

where θ is a decresing function with $\lim_{x\to\infty} \theta(x) = 0$. In fact,

$$\begin{aligned} \theta(x) &= \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) e^{-xt} \frac{1}{t}\, dt \\ &= \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)x^{2k}} \\ &= \frac{1}{12x^2} - \frac{1}{360x^4} + \frac{1}{1260x^6} - \ldots \end{aligned}$$

where the B_{2k}'s denote the Bernoulli numbers. Then

$$\begin{aligned} \log M_*(\Gamma \mid [a,b]) &= \frac{1}{\log b - \log a} \int_a^b \frac{\log \Gamma(t)}{t} dt \\ &= \frac{-2(b-a)}{\ln b - \ln a} - \frac{1}{4} \ln ab + \ln \sqrt{2\pi} + \frac{(\ln b)\, b - (\ln a)\, a}{\ln b - \ln a} \\ &\quad + \frac{1}{\log b - \log a} \int_a^b \frac{\theta(x)}{x} dx \\ &= -L(a,b) - \frac{1}{4} \ln ab + \ln \sqrt{2\pi} + L(a,b) \log I(a,b) + \theta(c) \end{aligned}$$

for a suitable $c \in (a,b)$.

We pass now to the problem of estimating the precision in the Hermite-Hadamard Inequality. For, we shall need a preparation.

Given a function $f : I \to (0,\infty)$ (with $I \subset (0,\infty)$) we shall say that f is *multiplicatively Lipschitzian* provided there exist a constant $L > 0$ such that

$$\max \left\{ \frac{f(x)}{f(y)}, \frac{f(y)}{f(x)} \right\} \leq \left(\frac{y}{x} \right)^L$$

for all $x < y$ in I; the smallest L for which the above inequality holds constitutes the *multiplicative Lipschitz* constant of f and it will be denoted by $||f||_{\star Lip}$.

Remark 3.1. *Though the family of multiplicatively Lipschitz functions is large enough* (*to deserve attention in its own*), *we know the exact value of the multiplicative Lipschitz constant only in few cases:*

i) *If f is of the form $f(x) = x^\alpha$, then $||f||_{\star Lip} = \alpha$.*

ii) *If $f = \exp|[a,b]$ (where $0 < a < b$), then $||f||_{\star Lip} = b$.*

iii) *Clearly, $||f||_{\star Lip} \leq 1$ for every non-decreasing functions f such that $f(x)/x$ is non-increasing. For example, this is the case of the functions* sin *and* sec *on* $(0, \pi/2)$.

iv) *If f and g are two multiplicatively Lipschitzian functions* (*defined on the same interval*) *and $\alpha, \beta \in \mathbb{R}$, then $f^\alpha g^\beta$ is multiplicatively Lipschitzian too. Moreover,*

$$||f^\alpha g^\beta||_{\star Lip} \leq |\alpha| \cdot ||f||_{\star Lip} + |\beta| \cdot ||g||_{\star Lip}.$$

The following result can be easily derived (via Lemma 1.1) from the standard form of the Ostrowski Inequality for Lipschitzian functions as stated in [1], Corollary 2, p. 345:

Theorem 3.4. *Let* $f : [a, b] \to (0, \infty)$ *be a multiplicatively convex continuous function. Then*

$$f(\sqrt{ab}) \leq M_*(f) \leq f(\sqrt{ab}) \left(\frac{b}{a}\right)^{||f||_{\star Lip}/4}$$

and

$$M_*(f) \leq \sqrt{f(a)f(b)} \leq M_*(f) \left(\frac{b}{a}\right)^{||f||_{\star Lip}/4}.$$

A generalization of the second part of this result, based on Theorem 3.1 above, will make the subject of the next section.

For $f = \exp|[a, b]$ (where $0 < a < b$), we have $M_*(f) = \exp\left(\frac{b-a}{\log b - \log a}\right)$ and $||f||_{\star Lip} = b$. By Theorem 3.4, we infer the inequalities

$$\begin{aligned} 0 &< \frac{b-a}{\log b - \log a} - \sqrt{ab} < \frac{b}{4}(\log b - \log a) \\ 0 &< \frac{a+b}{2} - \frac{b-a}{\log b - \log a} < \frac{b}{4}(\log b - \log a). \end{aligned}$$

For $f = \sec$ (restricted to $(0, \pi/2)$) we have $||f||_{\star Lip} = 1$ and

$$\begin{aligned} M_*(\sec|[a,b]) &= \exp\left(\frac{-1}{\log b - \log a}\int_a^b \frac{\ln\cos x}{x}dx\right) \\ &= \exp\left(\frac{1}{\log b - \log a}\int_a^b \left(\frac{1}{2}x + \frac{1}{12}x^3 + \frac{1}{45}x^5 + \frac{17}{2520}x^7 + ...\right)dx\right) \\ &= \exp\left(\frac{1}{\log b - \log a}\left(\frac{b^2-a^2}{4} + \frac{b^4-a^4}{48} + \frac{b^6-a^6}{270} + ...\right)\right) \end{aligned}$$

for every $0 < a < b < \pi/2$. According to Theorem 3.4, we have

$$\sec(\sqrt{ab}) < M_*(\sec|[a,b]) < \sec(\sqrt{ab}) \cdot \left(\frac{b}{a}\right)^{1/4}$$

and

$$M_*(\sec|[a,b]) < \sqrt{\sec a \sec b} < M_*(\sec|[a,b]) \cdot \left(\frac{b}{a}\right)^{1/4}$$

for every $0 < a < b < \pi/2$.

4. Approximating $M_*(f)$ by geometric means

As the reader already noticed, computing (in a compact form) the multiplicative mean value is not an easy task. However, it can be nicely approximated. The following result, inspired by a recent paper of K. Jichang [6], outlines the possibility to approximate $M_*(f)$ (from above) by products $\left(\prod_{k=1}^{n} f(x_k)\right)^{1/n}$ for a large range of functions:

Theorem 4.1. *Let* $f : I \to (0, \infty)$ *be a function which is multiplicatively convex or multiplicatively concave.*

If $I=[1,a]$ (with $a>1$) and f is strictly increasing, then

$$\left(\prod_{k=1}^{n} f(a^{k/n})\right)^{1/n} > \left(\prod_{k=1}^{n+1} f(a^{k/(n+1)})\right)^{1/(n+1)} > M_*(f) \tag{A1}$$

for every $n=1,2,3,...$

The conclusion remains valid for $I=[a,1]$ (with $0<a<1$) and f a strictly decreasing function as above.

The inequalities $(A1)$ should be reversed in each of the following two cases:

$I=[1,a]$ (with $a>1$) and f is strictly decreasing;

$I=[a,1]$ (with $0<a<1$) and f a strictly increasing

Proof. Let us consider first the case of strictly increasing multiplicatively convex functions. In this case, for each $k\in\{1,...,n\}$ we have

$$\begin{aligned} f(a^{k/(n+1)}) &= f(a^{kn^2/(n+1)n^2}) < f(a^{(nk-k+1)/n^2}) \\ &= f(a^{\frac{k-1}{n}\cdot\frac{k-1}{n}+(1-\frac{k-1}{n})\cdot\frac{k}{n}}) \\ &\leq \left(f(a^{(k-1)/n})\right)^{(k-1)/n}\left(f(a^{k/n})\right)^{1-(k-1)/n}. \end{aligned}$$

By multiplying them side by side we get

$$\begin{aligned} \prod_{k=1}^{n} f(a^{k/(n+1)}) &< \prod_{k=1}^{n}\left(\left(f(a^{(k-1)/n})\right)^{(k-1)/n}\left(f(a^{k/n})\right)^{1-(k-1)/n}\right) \\ &= \left(\prod_{k=1}^{n} f(a^{k/n})\right)^{(n+1)/n} f(a), \end{aligned}$$

i.e., the left hand inequality in the statement of our theorem.

Consider now the case where f is strictly increasing multiplicatively concave. Then

$$\begin{aligned} f(a^{k/n}) &= f(a^{k(n+1)^2/n(n+1)^2}) > f(a^{k(n+2)/(n+1)^2}) \\ &= f(a^{\frac{k}{n+1}\cdot\frac{k+1}{n+1}+(1-\frac{k}{n+1})\cdot\frac{k}{n+1}}) \\ &\geq \left(f(a^{(k+1)/(n+1)})\right)^{k/(n+1)}\left(f(a^{k/(n+1)})\right)^{1-k/(n+1)} \end{aligned}$$

for each $k\in\{1,...,n\}$, which leads to

$$\begin{aligned} \prod_{k=1}^{n} f(a^{k/n}) &> \prod_{k=1}^{n}\left(\left(f(a^{(k+1)/(n+1)})\right)^{k/(n+1)}\left(f(a^{k/(n+1)})\right)^{1-k/(n+1)}\right) \\ &= \prod_{k=1}^{n}\left(\left(f(a^{(k+1)/(n+1)})\right)^{k/(n+1)}\left(f(a^{k/(n+1)})\right)^{(n-k+1)/(n+1)}\right) \\ &= (f(a))^{n/(n+1)}\cdot\left(\prod_{k=1}^{n} f(a^{k/(n+1)})\right)^{n/(n+1)} \\ &= \left(\prod_{k=1}^{n+1} f(a^{k/(n+1)})\right)^{n/(n+1)} \end{aligned}$$

i.e., again to the left hand inequality in (A1).

To end the proof of the first part of our theorem, notice that

$$\begin{aligned}
\lim_{n\to\infty}\left(\prod_{k=1}^{n} f(a^{k/n})\right)^{1/n} &= \exp\left(\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\log f(a^{k/n})\right)\\
&= \exp\left(\frac{1}{\log a}\int_0^{\log a}\log f(e^t)dt\right)\\
&= M_*(f).
\end{aligned}$$

As the sequence $\pi_n = \left(\prod_{k=1}^{n} f(a^{k/n})\right)^{1/n}$ is strictly decreasing we conclude that $\pi_n > M_*(f)$ for every $n = 1, 2, 3, ...$

The remainder of the proof follows by a careful inspection of the argument above. ■

As was noticed in [8], p.163, Γ is strictly multiplicatively convex on $[1,\infty)$. According to Theorem 4.1, for each $a > 1$ and each natural number n we have

$$\left(\prod_{k=1}^{n}\Gamma(a^{k/n})\right)^{1/n} > \left(\prod_{k=1}^{n+1}\Gamma(a^{k/(n+1)})\right)^{1/(n+1)} > \exp\left(\frac{1}{\log a}\int_1^a\frac{\log\Gamma(t)}{t}\,dt\right).$$

The same argument, applied to the multiplicatively concave functions $\sin\frac{\pi x}{2}$ and $\cos\frac{\pi x}{2}$ (cf. [8], p.159) gives us

$$\begin{aligned}
\left(\prod_{k=1}^{n}\sin(\frac{\pi}{2}\,a^{k/n})\right)^{1/n} &< \left(\prod_{k=1}^{n+1}\sin(\frac{\pi}{2}\,a^{k/(n+1)})\right)^{1/(n+1)}\\
&< \exp\left(\frac{1}{\log a}\int_1^a\frac{\log\sin(\pi t/2)}{t}\,dt\right)
\end{aligned}$$

and

$$\begin{aligned}
\left(\prod_{k=1}^{n}\cos(\frac{\pi}{2}\,a^{k/n})\right)^{1/n} &> \left(\prod_{k=1}^{n+1}\cos(\frac{\pi}{2}\,a^{k/(n+1)})\right)^{1/(n+1)}\\
&> \exp\left(\frac{1}{\log a}\int_1^a\frac{\log\cos(\pi t/2)}{t}\,dt\right)
\end{aligned}$$

for every $a \in (0,1)$; they should be added to a number of other curiosities noticed recently by G. J. Tee [14].

The following result answers the question how fast is the convergence which makes the subject of Theorem 4.1 above:

Proposition 4.2. *Let $f : [a,b] \to (0,\infty)$ be a strictly multiplicatively convex continuous function. Then*

$$\left(\frac{f(b)}{f(a)}\right)^{1/(2n)} < \left(\prod_{k=1}^{n} f(x_k)\right)^{1/n} / M_*(f) < \left(\frac{b}{a}\right)^{||f||_{*Lip}/(2n)}$$

where $x_k = a^{1-k/n}b^{k/n}$ for $k = 1, ..., n$

Proof. According to $(*HH)$, for each $k = 1, ..., n$, we have

$$f(\sqrt{x_{k-1}x_k}) < \exp\left(\frac{n}{\log{(b/a)}}\int_{x_{k-1}}^{x_k}\log f\,dt\right) < \sqrt{f(x_k)f(x_{k+1})}$$

which yields

$$\left(\prod_{k=1}^{n} f(\sqrt{x_{k-1}x_k})\right)^{1/n} < M_*(f) < \left(\prod_{k=1}^{n} f(x_k)\right)^{1/n} \Bigg/ \left(\frac{f(b)}{f(a)}\right)^{1/(2n)}$$

i.e.,

$$\begin{aligned}\left(\frac{f(b)}{f(a)}\right)^{1/(2n)} &< \left(\prod_{k=1}^{n} f(x_k)\right)^{1/n} / M_*(f) < \\ &< \left(\prod_{k=1}^{n} f(x_k)/f(\sqrt{x_{k-1}x_k})\right)^{1/n}.\end{aligned}$$

Or,

$$\left(\prod_{k=1}^{n} f(x_k)/f(\sqrt{x_k x_{k+1}})\right)^{1/n} \leq \prod_{k=1}^{n} \left(\frac{x_k}{x_{k-1}}\right)^{||f||_{\star Lip}/(2n)}. \quad \blacksquare$$

For $f(x) = e^x$, $x \in [1, a]$, the last result gives

$$\frac{a-1}{2n} < \frac{1}{n} \sum_{k=1}^{n} a^{k/n} < \frac{a}{\log a} + \frac{a}{2n}$$

for all $n = 1, 2, 3, ...$

References

[1] Lj. Dedić, M. Matić and J. Pečarić, *On generalizations of Ostrowski inequality via some Euler-type identities*, Math. Inequal. Appl., **3** (2000), 337-353.

[2] L. Fejer, *Über die Fouriereichen* II, Gesammelte Arbeiten **I**, Budapest, 1970, pp. 280-297.

[3] A. M. Fink, *A best possible Hadamard Inequality,* Math. Inequal. Appl., **1** (1998), 223-230.

[4] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*, Academic Press, 1996.

[5] G. Hardy, J. E. Littlewood and G. Polya, *Inequalities,* Cambridge Mathematical Library, 2nd Ed.., 1952, Reprinted 1988.

[6] K. Jichang, *Some extensions and refinements of Minc-Sathre inequality,* The Math. Gazette, March 1999, Note 83.17, pp. 123-127.

[7] P. Montel, *Sur les fonctions convexes et les fonctions sousharmoniques*, Journal de Math. (9), **7** (1928), 29-60.

[8] C. P. Niculescu, *Convexity according to the geometric mean,* Math. Inequal. & Appl., **3** (2000), 155-167.

[9] C. P. Niculescu, *A Note on the Dual Hermite-Hadamard Inequality,* The Math. Gazette, July 2001, Note 85.42, pp. 48-50.

[10] C. P. Niculescu, *An extension of Chebyshev's Inequality and its Connection with Jensen's Inequality,* J. of Inequal. & Appl., **6** (2001), 451-462.

[11] C. P. Niculescu, *Comparative convexity (*Preprint)

[12] J. E. Pečarić, F. Proschan and Y. L. Tong, *Convex Functions, Partial Orderings and Statistical Applications,* Academic Press, 1992.

[13] J. Sándor, *Some integral inequalities*, Elem. Math., **43** (1988), 177-180.

[14] G. J. Tee, *Further generalizations of a curiosity that Feynman remembered all his life,* Math. Mag., **72** (1999), no. 1, p. 44.

[15] A. Wayne Roberts and Dale E. Varberg, *Convex Functions,* Academic Press, New York and London, 1973.

[16] E. T. Whittaker and G. N. Watson, *A Course on Modern Analysis,* 4th Edition, Cambridge, 1927.

University of Craiova, Department of Mathematics, Street A. I. Cuza 13, Craiova 1100, ROMANIA
E-mail address: tempus@oltenia.ro