A GENERALIZATION OF A THEOREM OF BERNARD CONCERNING THE FRONTAL SETS

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1. Introduction

The notion of a frontal set with respect to a vector subspace of a space $C(X)$ was introduced by Alain Bernard [1] as a generalization of the concept of intersection of peak sets with respect to a subalgebra of $C(X)$.

Based on a new characterization of frontal sets (see Theorem 2.10 below) we shall further extend some of his fundamental results to ideals in locally convex lattices. To be more specific, let E be a real, metrizable, locally convex, locally solid vector lattice and let $\mathcal{V}_0$ be a basis of convex and solid neighborhoods of the origin. Typical examples of such spaces are the metrizable weighted spaces, as described in section 4 below. See also [7].

A closed ideal $\mathcal{I}$ of E is said to be a $\mathcal{V}_0-$*frontal ideal* (with respect to a vector subspace F of E) if for every pair of elements $x \in F$ and $y \in E_+$, with $(|x|-y))_+ \in \mathcal{I}$, and every $V \in \mathcal{V}_0$ there exists an $\bar{x} \in F$ such that

$$x - \bar{x} \in \mathcal{I} \quad \text{and} \quad (|\bar{x}| - y))_+ \in V.$$

Our main result is the following generalization of the Theorem 4 in [1]:

Theorem 1.1. *Let F be a complete vector subspace of E, let $\mathcal{I}$ be a $\mathcal{V}_0-$frontal ideal with respect to F and let p be a continuous seminorm of $AM-$type such that*

$$p_{\mathcal{I}}(x) = \inf \{p(x+u) : u \in \mathcal{I}\} > 0 \quad \textit{for every } x \in E.$$

Then for every $x \in F$ and $y \in E_+$ with $(|x|-y))_+ \in \mathcal{I}$ and every $V \in \mathcal{V}_0$ there exists an $\bar{x} \in F$ such that

$$x - \bar{x} \in \mathcal{I}\ , \quad (|\bar{x}| - y))_+ \in V \quad \textit{and} \quad p(\bar{x}) = p_{\mathcal{I}}(x).$$

Applications in the framework of weighted spaces are given in section 4.

2. Generalities

Throughout this section X will denote a compact Hausdorff space and $C(X)$ will denote the Banach space of all continuous complex-valued functions on X, equipped with the sup norm.

We shall denote by A° the polar set of any subset A of $C(X)$ and by B_r the open ball

$$B_r = \{f \in C(X) : ||f|| < r\}, \quad r > 0.$$

Also, for every subset K of X, we shall denote by χ_K the characteristic function of K and by $\mathcal{I}_K$ the ideal of all functions $f \in C(X)$ such that $f|K = 0$.

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Our first result concerns the *restriction* operator,

$$R_K : C(X) \to C(K), \quad R_K(f) = f|K.$$

2.1 Lemma. *Let F and G be two vector subspaces of $C(X)$ and let K be a closed subset of X such that $\chi_K F^\circ \subset G^\circ$. Then*

$$(B_r|K) \cap (G|K) \subset \overline{(B_r \cap F)\,|K}.$$

Proof. According to the bipolar theorem it is sufficient to prove that

$$[(B_r \cap F)\,|K]^\circ \subset [(B_r|K) \cap (G|K)]^\circ\,.$$

For, let $\mu \in [(B_r \cap F)\,|K]^\circ$. Then $R'_K(\mu) \in (B_r \cap F)^\circ \subset B_r{}^\circ + F^\circ$, where R'_K denotes the adjoint of R_K. Thus there exists a $\nu \in B_r{}^\circ$ such that $\nu - R'_K(\mu) \in F^\circ$. Since $\chi_K F^\circ \subset G^\circ$, it follows that $\chi_K \nu - R'_K(\mu) \in G^\circ$. Let $f \in [(B_r|K) \cap (G|K)]$ and chose $g_1 \in B_r$ and $g_2 \in F$ such that

$$f = g_1|K = g_2|K.$$

Then we have

$$\begin{aligned} \mu(f) &= \mu(g_1|K) = (\chi_K \nu)\,(g_1) - [\chi_K \nu - R'_K(\mu)]\,(g_1) \\ &= (\chi_K \nu)\,(g_1) - [\chi_K \nu - R'_K(\mu)]\,(g_2) = (\chi_K \nu)\,(g_1). \end{aligned}$$

Since $\nu \in B_r{}^\circ$ and $B_r{}^\circ$ is a solid set, it follows that $\chi_K \nu \in B_r{}^\circ$, hence $\operatorname{Re}\mu(f) \leq 1$. Consequently, $\mu \in [(B_r|K) \cap (G|K)]^\circ$, which ends the proof. ■

2.2 Corollary. *Let F be a vector subspace of $C(X)$ and let K be a closed subset of X such that $\chi_K F^\circ \subset F^\circ$. Then*

$$(B_r|K) \cap (F|K) \subset \overline{(B_r \cap F)\,|K}.$$

2.3 Corollary. *Let F be a vector subspace of $C(X)$ and let K be a closed subset of X such that $\chi_K F^\circ \subset \{0\}$. Then*

$$B_r|K \subset \overline{(B_r \cap F)\,|K}.$$

2.4 Lemma. *Let F be a vector subspace of $C(X)$ and let K be a closed subset of X such that $\chi_K F^\circ \subset F^\circ$. If $F/F \cap \mathcal{I}_K$ is complete then $(B_r|K) \cap (F|K) \subset (B_1 \cap F)|K$ for every $0 < r < 1$.*

Proof. Let $0 < r < t < 1$ be fixed and put

$$r_n = \begin{cases} r, & \text{if } n = 1 \\ \frac{t-r}{2^{n-1}}, & \text{if } n \geq 2. \end{cases}$$

Obviously, $t = \sum_{n=1}^{\infty} r_n$. Let $f \in (B_r|K) \cap (F|K)$. According to Corollary 2.2, $f \in \overline{(B_r \cap F)\,|K}$. Since $(B_{r_2}|K) \cap (F|K)$ is a neighborhood of f with respect to the relative topology of $F|K$, there exists a $g_2 \in B_{r_2} \cap F$ such that

$$f - g_1|K - g_2|K \in (B_{r_3}|K) \cap (F|K)\,.$$

By induction, we find a $g_n \in B_{r_n} \cap F$ such that

$$f - \sum_{i=1}^{n} g_i|K \in \left(B_{r_{n+1}}|K\right) \cap (F|K)\,.$$

Now consider the commutative diagram

$$\begin{array}{ccc} F & \overset{R_K}{\rightarrow} & F|K \\ \varphi_K \searrow & & \nearrow \rho_K \\ & F/F \cap \mathcal{I}_K & \end{array}$$

where φ_K is the canonical mapping and ρ_K is defined by the formula

$$\rho_K(\hat{h}) = h|K.$$

It is easily seen that ρ_K is well defined, one-to-one and continuous.

Since $||\widehat{g_i}|| \leq ||g_i|| < r_i$ for every i, it follows that

$$\left\|\sum_{i=1}^{p} \hat{g}_{n+i}\right\| < \frac{t-r}{2^{n-1}}, \quad \text{for every } p \in \mathbb{N}^\star,$$

hence the series $\sum_{n=1}^{\infty} \hat{g}_n$ is a Cauchy series in the complete space $F/F \cap \mathcal{I}_K$. Let $g \in F$ be such that

$$\hat{g} = \sum_{n=1}^{\infty} \hat{g}_n.$$

By the continuity of ρ_K we have $\rho_K(\hat{g}) = \sum_{n=1}^{\infty} \rho_K(\hat{g}_n)$ and thus $g|K = \sum_{n=1}^{\infty} \hat{g}_n|K$. Since $||f - \sum_{i=1}^{n} g_i|K|| < r_{n+1}$ it follows that

$$f = \sum_{n=1}^{\infty} g_n|K = g|K.$$

On the other hand

$$\left\|\sum_{i=1}^{n} \hat{g}_i\right\| < \sum_{i=1}^{n} r_i < t$$

which yields $||\hat{g}|| \leq t < 1$.

Let $\bar{u} \in F \cap \mathcal{I}_K$ be such that $||g + \bar{u}|| < 1$. Since $g + \bar{u} \in F \cap B_1$ and $(g + \bar{u})|K = g|K = f$, it results that $f \in (F \cap B_1)|K$ and so the proof is complete. ■

2.5 Corollary. *Let F be a vector subspace of $C(X)$ and let K be a closed subset of X such that $\chi_K F^\circ = \{0\}$. If $F/F \cap \mathcal{I}_K$ is complete, then $B_r|K \subset (F \cap B_1)|K$ for every $r \in (0, 1)$.*

Corollary 2.5 extends Bishop's lemma [2].

The following notion was introduced by Alain Bernard [1] in connection with the diagram which appeared in the proof of Lemma 2.4.

2.6 Definition. A closed subset K of X is said to be an *interpolating set* (respectively a *strictly interpolating set*) with respect to the vector subspace F of $C(X)$ if ρ_K is an isomorphism (respectively, an isometric isomorphism).

Clearly, if $F/F \cap \mathcal{I}_K$ is complete and $F|K$ is closed in $C(K)$, then K is an interpolating set with respect to F.

2.7 Theorem. *Let F be a vector subspace of $C(X)$ and let K be a closed subset of X such that $F/F \cap \mathcal{I}_K$ is complete and $\chi_K F^\circ \subset F^\circ$. Then given $f \in F$, $g \in C(X)$, $g \geq 0$, with $|f| \leq g$ and $\varepsilon > 0$, there exists a $\overline{f} \in F$ with the following properties:*

$$\overline{f}|K = f|K \quad \textit{and} \quad \left|\overline{f}\right| \leq g + \varepsilon.$$

Proof. Let $r = \sup\left\{\frac{|f(x)|}{g(x)+\varepsilon} \,;\, x \in K\right\}$ and choose $r' \in (r, 1)$.

The vector space

$$H = \{h \in C(X);\ (g + \varepsilon)h \in F\}$$

verifies the conditions: $H^\circ = (g + \varepsilon)F^\circ$, $\chi_K H^\circ \subset H^\circ$ and $H/H \cap \mathcal{I}_K$ is complete. According to Lemma 2.2 above,

$$(B_{r'}|K) \cap (H|K) \subset (B_1 \cap H)|K.$$

As $f/(g+\varepsilon)|K \in (B_{r'}|K) \cap (H|K)$, it follows that there exists a $h \in B_1 \cap H$ such that $f/(g+\varepsilon)|K = h|K$.

Then $\overline{f} = h(g+\varepsilon)$. has all properties as required in the statement of Theorem 2.7. ■

The following result is a slight generalization of Bishop's interpolation theorem [2]:

2.8 Theorem. *Let F be a vector subspace of $C(X)$ and let K be a closed subset of X such that $F/F \cap \mathcal{I}_K$ is complete and $\chi_K F^\circ = \{0\}$. Then, given $f \in C(X)$, $g \in C(X)$, $g \geq 0$, with $|f| \leq g$ and $\varepsilon > 0$, there exists a $\overline{f} \in F$ with the following properties:*

$$\overline{f}|K = f|K \quad \textit{and} \quad \left|\overline{f}\right| \leq g + \varepsilon.$$

Particularly, $F|K = C(K)$.

The proof is similar to the proof of Theorem 2.7, except for using Corollary 2.5 instead of Lemma 2.4.

Let us now recall the concept of a frontal set as introduced by Alain Bernard [1]:

2.9 Definition. A closed subset K of X is said to be a *frontal set* with respect to the vector subspace F of $C(X)$ if for every $f \in F$, every pair (ε, η) of strictly positive numbers and every neighborhood V of K there exists a $\overline{f} \in F$ such that

$$\overline{f}|K = f|K, \ ||\overline{f}||_X \leq ||f||_K + \eta \quad \text{and} \quad ||\overline{f}||_{X \setminus V} \leq \varepsilon.$$

2.10 Theorem. *Let F be a vector subspace of $C(X)$ and let K be a closed subset of X such that $F/F \cap \mathcal{I}_K$ is complete. Then the following assertions are equivalent:*

Theorem 2.1. (i) K is a frontal set with respect to F;

(ii) $\chi_K F^\circ \subset F^\circ$;

(iii) K is a strictly interpolating set with respect to the space hF, for every $h \in C(X)$, with $h \neq 0$ on X :

(iv) Given $f \in F$, $g \in C(X)$, with $g \geq 0$ and $|f| \leq g$, then for every $\varepsilon > 0$ there exists a $\overline{f} \in F$ such that

$$\overline{f}|K = f|K \quad \text{and} \quad \left|\overline{f}\right| \leq g + \varepsilon.$$

Proof. According to Theorem 3 in [1], we know that $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$.

Suppose there given $f \in F$, $\varepsilon > 0$, $\eta > 0$ and V a neighborhood of K. Put $\delta = \min\{\varepsilon, \eta\}$. According to Uryson's lemma, there exists a continuous function $g : X \to [\delta, ||f||_K + \delta]$ such that

$$g(x) = \begin{cases} ||f||_K + \delta, & \text{if } x \in K \\ \delta, & \text{if } x \in X \setminus V. \end{cases}$$

Obviously, $g - \delta \geq 0$ on X and $|f| \leq g - \delta$ on K.

From (iv) it follows that there exists an $\bar{f} \in F$ with the properties

$$\bar{f}\,|K = f|K \text{ and } |\bar{f}| \leq (g-\delta) + \delta = g \text{ on } X.$$

Consequently, for $x \in K$ we have $|\bar{f}(x)| \leq ||f||_K + \delta \leq ||f||_K + \eta$. On the other hand, if $x \in X \setminus V$, then $|f(x)| \leq \delta \leq \varepsilon$, hence $||f||_{X \setminus V} \leq \varepsilon$ and this concludes the proof. ■

3. $\mathcal{V}_0$-FRONTAL IDEALS

Our generalization of the concept of a frontal ideal is motivated by Theorem 2.10 above.

Let E be a real metrizable, locally convex, locally solid vector lattice and let $\mathcal{V}_0$ be a basis of convex and solid neighborhoods of the origin.

3.1 Definition. A closed ideal $\mathcal{I}$ of E is said to be a $\mathcal{V}_0$-*frontal ideal* with respect to the vector subspace F of E if for every pair of elements $x \in F$ and $y \in E_+$, with $(|x| - y))_+ \in \mathcal{I}$, and every $V \in \mathcal{V}_0$ there exists an $\bar{x} \in F$ such that

$$x - \bar{x} \in \mathcal{I} \quad \text{and} \quad (|\bar{x}| - y))_+ \in V.$$

If $E = C(X)$, then a closed subset K of X is a frontal set with respect to the vector subspace F of $C(X)$ if and only if the closed ideal $\mathcal{I}_K = \{f \in C(X) : f|K = 0\}$ is a frontal ideal in the sense of the Definition 3.1. Indeed, it is sufficient to notice that $(|f| - g)_+ \in \mathcal{I}_K$ if and only if

$$|f| \leq g \text{ on } K \text{ and } ||(|\bar{f}| - g)_+|| < \varepsilon \Longleftrightarrow |\bar{f}| \leq g + \varepsilon \text{ on } X.$$

3.2 Theorem (A. Bernard [1]). *Let F be a closed vector subspace of $C(X)$ and let K be a closed subset of X. Then the following assertions are equivalent:*

i) *K is a frontal set with respect to F;*

ii) *For every $f \in F$, every $\varepsilon > 0$ and every neighborhood V of K there exists an $\bar{f}$ such that*

$$\bar{f}|K = f|K, \ ||\bar{f}||_X \leq ||f||_K \quad \textit{and} \quad ||\bar{f}||_{X \setminus V} \leq \varepsilon.$$

Now we return to the general case. Let E be a real, metrizable, locally convex, locally solid vector lattice, and let $\mathcal{I}$ be a closed ideal of E. Given a continuous seminorm p on E we associate to it the quotient seminorm

$$p_{\mathcal{I}}(x) = \inf \{p(x + u) \ : \ u \in \mathcal{I}\}, \ x \in E.$$

The following theorem is a generalization of Theorem 3.2:

3.3 Theorem. *Let F be a complete vector subspace of E and let $\mathcal{V}_0$ be a basis of convex and solid neighborhoods of the origin. Suppose there are given a $\mathcal{V}_0-$frontal ideal $\mathcal{I}$ (with respect to F), two elements $x \in F$, $y \in E_+$ such that $(|x| - y)_+ \in \mathcal{I}$, a neighborhood $V \in \mathcal{V}_0$ and a continuous seminorm p of $(AM)-$type on E such that $p_{\mathcal{I}}(x) > 0$. Then there exists an $\bar{x} \in F$ with the following properties:*

$$\bar{x} - x \in \mathcal{I}, \ (|\bar{x}| - y)_+ \in V \quad \textit{and} \quad p(x) = p_{\mathcal{I}}(x).$$

Proof. Let $(V_n)_{n \geq 1}$ be a sequence of elements of $\mathcal{V}_0$ such that $\bar{V}_1 \subset V$ and $V_{n+1} + V_{n+1} \subset V_n$ for every $n \geq 1$. We shall exhibit two sequences $(x_n)_{n \geq 1} \subset F$ and $(\varepsilon_n)_{n \geq 1} \subset \mathbb{R}_+^\star$ with the following properties:

a) $x_n - x \in \mathcal{I}$ for every $n \geq 1$;

b) $p(x_n) \leq (1 + \varepsilon_n) p_{\mathcal{I}}(x)$ and $\varepsilon_n \leq 1/n$ for every $n \geq 1$;

c) $(|x_n| - y)_+ \in V_2 + ... + V_{n+1}$ for every $n \geq 1$;

d) $x_n - x_{n-1} \in V_n$ for every $n \geq 2$;

e) $\varepsilon_n x_n \in \frac{1}{3} V_{n+1}$ and $\varepsilon_n y \in \frac{1}{9} V_{n+1}$ for every $n \geq 0$.

From d) we can infer that $(x_n)_{n \geq 1}$ is a Cauchy sequence. As F is complete, there must exist $\bar{x} = \lim\limits_{x \to \infty} x_n$. From a) it follows that $\bar{x} - x \in \mathcal{I}$, while form b) it follows that $p(\bar{x}) \leq p_{\mathcal{I}}(x)$. On the other hand,

$$p_{\mathcal{I}}(x) \leq p(x + \bar{x} - x) = p(\bar{x})$$

and thus $p(\bar{x}) = p_{\mathcal{I}}(x)$.

From c) it follows that $(|x_n| - y)_+ \in V_2 + V_2 \subset \bar{V}_1 \subset V$.

We pass now to the construction of the sequences $(x_n)_{n\geq 1}$ and $(\varepsilon_n)_{n\geq 1}$. Put $u_0 = 0$ and choose $\varepsilon_0 > 0$ such that $\varepsilon_0 y \in \frac{1}{9}V_1$. Then choose $\varepsilon_1 \in (0,1]$ such that $\varepsilon_1 y \in \frac{1}{9}V_2$. According to the definition of $p_{\mathcal{I}}(x)$, we can find a $u_1 \in \mathcal{I}$ such that

$$p(x+u_1) \leq (1+\frac{\varepsilon_1}{2})\, p_{\mathcal{I}}(x).$$

Put $y_1 = |x+u_1| \wedge y$. We shall show that

$$(|x| - y_1)_+ \in \mathcal{I}.$$

Let $\pi : E \to E/\mathcal{I}$ be the canonical morphism. Proving the above relation amounts to show that $\pi\left((|x| - y_1)_+\right) = 0$. It is well known that π is a lattice homomorphism, therefore $0 = \pi\left((|x| - y)_+\right) = \left(|\pi(x)| - \pi(y)\right)_+$, hence $|\pi(x)| \leq \pi(y)$ which in turn implies $\pi(y_1) = |\pi(x)| \wedge \pi(y) = |\pi(x)|$ concluding thus the proof of the assertion.

As $p_{\mathcal{I}}(x) > 0$ and p is a continuous seminorm on E, it follows that there exists a $W \in \mathcal{V}_0$ such that

$$p(z) \leq \frac{\varepsilon_1}{2}\, p_{\mathcal{I}}(x) \quad \text{for every } z \in W.$$

Let $W' \in \mathcal{V}_0$ such that $W' \subset W \cap \frac{1}{9}V_2$. As $\mathcal{I}$ is a frontal ideal with respect to F, $x \in F$ and $(|x| - y_1)_+ \in \mathcal{I}$, we infer the existence of a $v_1 \in F$ with the following properties:

$v_1 - x \in \mathcal{I}$;

$(|v_1| - y_1)_+ \in W'$;

$p\left((|v_1| - y_1)_+\right) \leq \frac{\varepsilon_1}{2}\, p_{\mathcal{I}}(x)$.

Letting $x_1 = v_1$, we can verify easily that x_1 and ε_1 verify the conditions a)-e) above. In fact,

$x_1 - x = v_1 - x \in \mathcal{I}$;

$|x_1| = |v_1| = (|v_1| - y_1) + y_1 \leq (|v_1| - y_1)_+ + |x + u_1|$.

As p is a solid seminorm, the last relation yields

$$\begin{aligned} p(x_1) &\leq p\left((|v_1| - y_1)_+\right) + p(x+u_1) \\ &\leq \frac{\varepsilon_1}{2}\, p_{\mathcal{I}}(x) + (1+\frac{\varepsilon_1}{2})\, p_{\mathcal{I}}(x) \\ &= (1+\varepsilon_1)\, p_{\mathcal{I}}(x). \end{aligned}$$

For c), notice that $(|x_1| - y)_+ \in V_2$ is a consequence of the fact that V_2 is a solid set and

$$(|x_1| - y)_+ \leq (|v_1| - y_1)_+ + (y_1 - y)_+ = (|v_1| - y_1)_+ \in \frac{1}{9}V_2 \subset V_2.$$

The condition e) has a similar motivation, due to the following relations:

$$\begin{aligned} \varepsilon_1 |x_1| &\leq (|v_1| - y_1)_+ + \varepsilon_1 y_1 \\ &\leq (|v_1| - y_1)_+ + \varepsilon_1 y \in \frac{1}{9}V_2 + \frac{1}{9}V_2 \subset \frac{1}{3}V_2. \end{aligned}$$

Suppose now $x_0, ..., x_{n-1}$ and $\varepsilon_0, ...\varepsilon_{n-1}$ are already chosen with the properties a)-e). Then pick an $\varepsilon_n \in (0, 1/n]$ such that

$$\varepsilon_n x_{n-1} \in \frac{1}{9}V_{n+1} \text{ and } \varepsilon_n y \in \frac{1}{9}V_{n+1}.$$

Next, choose $u_n \in \mathcal{I}$ such that

$$p(x+u_n) \leq (1+\frac{\varepsilon_n}{2})\ p_{\mathcal{I}}(x).$$

The element y_n defined by

$$y_n = \left(|x+u_n| - \frac{|x_{n-1}|}{1+\varepsilon_{n-1}}\right)_+ \wedge \frac{\varepsilon_{n-1}}{1+\varepsilon_{n-1}}\, y$$

verifies the relation

$$\left(\frac{\varepsilon_{n-1}}{1+\varepsilon_{n-1}}\,|x| - y_n\right)_+ \in \mathcal{I}. \tag{1}$$

which can be seen by using a similar argument based on the lattice homomorphism π as above.

From the continuity of p and the fact that $p_{\mathcal{I}}(x) > 0$ we can derive the existence of a $U \in \mathcal{V}_0$ such that

$$p(z) \leq \frac{\varepsilon_n}{2}\, p_{\mathcal{I}}(x) \text{ for every } z \in U.$$

Choose $U' \in \mathcal{V}_0$ such that $U' \subset U \cap \frac{1}{9}\, V_{n+1}$. Since $\mathcal{I}$ is a frontal ideal with respect to F and

$$x \in F \quad \text{and} \quad \left(\frac{\varepsilon_{n-1}}{1+\varepsilon_{n-1}}\,|x| - y_n\right)_+ \in \mathcal{I}$$

we infer the existence of a $v_n \in F$ with the following properties:

(3.3) $v_n - \frac{\varepsilon_{n-1}}{1+\varepsilon_{n-1}}\, x \in \mathcal{I}$;
(3.4) $(|v_n| - y_n)_+ \in U'$;
(3.5) $p\left((|v_n| - y_n)_+\right) \leq \frac{\varepsilon_n}{2}\, p_{\mathcal{I}}(x)$.

Letting $x_n = \frac{x_{n-1}}{1+\varepsilon_{n-1}} + v_n$, we have completed a pair of sequences $(\varepsilon_k)_{k=1}^n$ and $(x_k)_{k=1}^n$ which verify the conditions a)-e) above. In fact,

$$x_n - x = \frac{x_{n-1}}{1+\varepsilon_{n-1}} + v_n - x = \frac{x_{n-1}-x}{1+\varepsilon_{n-1}} + \left(v_n - \frac{\varepsilon_{n-1}}{1+\varepsilon_{n-1}}\, x\right) \in \mathcal{I}$$

which yields a). As for b),

$$|x_n| \leq \frac{|x_{n-1}|}{1+\varepsilon_{n-1}} + (|v_n| - y_n)_+ + y_n$$

and

$$\begin{aligned} y_n &\leq \left(|x+u_n| - \frac{|x_{n-1}|}{1+\varepsilon_{n-1}}\right) \vee 0 \\ y_n + \frac{|x_{n-1}|}{1+\varepsilon_{n-1}} &\leq |x+u_n| \vee \frac{|x_{n-1}|}{1+\varepsilon_{n-1}} \end{aligned}$$

so that

$$|x_n| \leq \frac{|x_{n-1}|}{1+\varepsilon_{n-1}} \vee |x+u_n| + (|v_n| - y_n)_+ .$$

As p is a solid seminorm of (AM)-type, we are led to

$$p(x_n) \leq \max\left\{p(x+u_n),\ \frac{p(x_{n-1})}{1+\varepsilon_{n-1}}\right\} + p\left((|v_n| - y_n)_+\right).$$

Taking into account (3.1) and (3.5) and the fact that $p(x_{n-1}) \leq (1+\varepsilon_{n-1})\, p_{\mathcal{I}}(x)$, it follows that

$$p(x_n) \leq (1+\frac{\varepsilon_n}{2})\, p_{\mathcal{I}}(x) + \frac{\varepsilon_n}{2}\, p_{\mathcal{I}}(x) = (1+\varepsilon_n)\ p_{\mathcal{I}}(x).$$

For c), notice first that

$$\begin{aligned} (|x_n| - y)_+ &\leq \left(\frac{|x_{n-1}|}{1+\varepsilon_{n-1}} + |v_n| - y\right)_+ \\ &= \left(\frac{|x_{n-1}| - y}{1+\varepsilon_{n-1}} + |v_n| - \frac{\varepsilon_{n-1}}{1+\varepsilon_{n-1}}\, y\right)_+ \\ &\leq \frac{1}{1+\varepsilon_{n-1}}\,((|x_{n-1}| - y)_+ + \left(|v_n| - \frac{\varepsilon_{n-1}}{1+\varepsilon_{n-1}}\, y\right)_+) \\ &\leq \frac{1}{1+\varepsilon_{n-1}}\,((|x_{n-1}| - y)_+ + (|v_n| - y_n)_+). \end{aligned}$$

Since $(|x_{n-1}| - y)_+ \in V_2 + \cdots + V_n$ and $(|v_n| - y_n)_+ \in V_{n+1}$ it follows that

$$(|x_n| - y)_+ \in V_2 + \cdots + V_n + V_{n+1}.$$

The verification of condition d) is a consequence of the fact that V_n is solid. In fact,

$$\begin{aligned} |x_n - x_{n-1}| &= |v_n - \frac{\varepsilon_{n-1}}{1+\varepsilon_{n-1}}\, x_{n-1}| \\ &\leq \varepsilon_{n-1}|x_{n-1}| + y_n + |v_n| - y_n \\ &\leq \varepsilon_{n-1}|x_{n-1}| + \varepsilon_{n-1}y + (|v_n| - y_n)_+ \in \frac{1}{3}\, V_n + \frac{1}{9}\, V_n + \frac{1}{9}\, V_{n+1} \subset V_n. \end{aligned}$$

For e), we have to notice that

$$\begin{aligned} \varepsilon_n|x_n| &\leq \varepsilon_n|x_{n-1}| + \varepsilon_n(|v_n| - y_n) + \varepsilon_n y_n \\ &\leq \varepsilon_n|x_{n-1}| + \varepsilon_n(|v_n| - y_n)_+ + \varepsilon_n y. \end{aligned}$$

Taking into account the definition of ε_n and the relation (3.4), we obtain that

$$\varepsilon_n|x_{n-1}| + \varepsilon_n(|v_n| - y_n)_+ + \varepsilon_n y \in \frac{1}{9}\, V_{n+1} + \frac{1}{9}\, V_{n+1} + \frac{1}{9}\, V_{n+1} \subset \frac{1}{3} V_{n+1}$$

which yields $\varepsilon_n x_n \in \frac{1}{3} V_{n+1}$. ■

It is worth to explain how Theorem 3.3 extends Theorem 3.2.

For, let $E = C(X)$, F a closed vector subspace of E, $K \subset X$ a frontal set (with respect to F), $f \in F$, $\varepsilon > 0$, V a neighborhood of K and $p = ||\cdot||_X$. According to Uryson's lemma, there exists a continuous function $g : X \to [\varepsilon/2, ||f||_K]$ such that

$$g(x) = \begin{cases} ||f||_K, & \text{for } x \in K \\ \varepsilon/2, & \text{for } x \in X \setminus V. \end{cases}$$

Clearly, $|f| \leq g$ on K, hence $(|f| - g)_+ \in \mathcal{I}_K$. Letting

$$U = \{h \in C(K) \,:\, ||h|| < \varepsilon/2\}\,,$$

then U is a neighborhood of the origin of E. Since $\mathcal{I}_K$ is a frontal ideal with respect to F, Theorem 3.3 yields a $\bar{f} \in F$ such that

$$\bar{f} - f \in \mathcal{I}_K,\ \left(|\bar{f}| - g\right)_+ \in U \text{ and } ||\bar{f}||_X = p_{\mathcal{I}_K}(f).$$

Accordingly,

$$\bar{f}|K = f|K \text{ and } |\bar{f}(x)| \leq g(x) + \varepsilon/2 \text{ for every } x \in X.$$

In particular, for $x \in X \setminus V$ we have $|\bar{f}(x)| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$, hence $||\bar{f}||_{X \setminus V} < \varepsilon$.

As K is a frontal set, from Theorem 2.10 we can infer that K is also a strictly interpolating set (i.e. ρ_K is an isometric isomorphism). Then

$$\begin{aligned} ||f|K|| &= ||\rho_K(\varphi_K)|| = ||\rho_K(\hat{f})|| \\ &= \inf\{||f+h|| : h \in \mathcal{I}_K \cap F\} \\ &\geq \inf\{||f+h|| : h \in \mathcal{I}_K\} \\ &= p_{\mathcal{I}_K}(f) = ||\bar{f}||_X \end{aligned}$$

and the conclusion of Theorem 3.2 holds true.

3.4 Theorem. *Suppose there are given a vector subspace F of E, a $\mathcal{V}_0$-frontal ideal $\mathcal{I}$ (with respect to F), an element $x \in F$, a closed ideal $\mathcal{J}$ of E such that $x \in \overline{\mathcal{I} + \mathcal{J}}$, a continuous and solid seminorm p on E and $\varepsilon > 0$. Then there exists a $\overline{x} \in F$ with the following properties:*

$$\bar{x} - x \in \mathcal{I},\ p(\bar{x}) \leq p_{\mathcal{I}}(x) + \varepsilon \quad \textit{and} \quad p_{\mathcal{J}}(\bar{x}) \leq \varepsilon.$$

If in addition F is complete, p is an (AM)-type seminorm and $p(x) > 0$, then $p(\bar{x}) = p_{\mathcal{I}}(x)$.

Proof. Since $x \in \overline{\mathcal{I} + \mathcal{J}}$, and p is a continuous seminorm on E, one can find a $u \in \mathcal{I}$ and a $v \in \mathcal{J}$ such that

$$p(x - u - v) \leq \varepsilon/2.$$

According to the definition of $p_{\mathcal{I}}$, one can also find a $w \in \mathcal{I}$ with $p(x + w) \leq p_{\mathcal{I}}(x) + \varepsilon/2$. Put

$$y = (|v| + |x - u - v|) \wedge |x + w|.$$

Then $(|x| - y)_+ \in \mathcal{I}$, which can be proved by the same argument as in the proof of Theorem 3.3, based on the lattice homomorphism $\pi : E \to E/\mathcal{I}$.

As $\mathcal{I}$ is a $\mathcal{V}_0$-frontal ideal with respect to F, one can find a $\bar{x} \in F$ such that

$$\bar{x} - x \in \mathcal{I} \text{ and } p\left((|\bar{x}| - y)_+\right) \leq \varepsilon/2.$$

On the other hand,

$$(|\bar{x}| - y)_+ + |x + w| \geq (|\bar{x}| - y)_+ + y = |\bar{x}| \vee y \geq |\bar{x}|.$$

Since p is a solid seminorm, it results

$$\begin{aligned} p(\bar{x}) &\leq p\left((|\bar{x}| - y)_+\right) + p(x + w) \leq \frac{\varepsilon}{2} + p_{\mathcal{I}}(x) + \frac{\varepsilon}{2} \\ &= p_{\mathcal{I}}(x) + \varepsilon. \end{aligned}$$

In a similar way,

$$\begin{aligned} p_{\mathcal{J}}(\bar{x}) &\leq p_{\mathcal{J}}\left((|\bar{x}| - y)_+\right) + p_{\mathcal{J}}(x + w) \\ &\leq p_{\mathcal{J}}\left((|\bar{x}| - y)_+\right) + p_{\mathcal{J}}(|v| + |x - u - v|) \\ &\leq p_{\mathcal{J}}\left((|\bar{x}| - y)_+\right) + p_{\mathcal{J}}(x - u - v) \\ &\leq \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

The second part of our statement is a consequence of Theorem 3.3 above. ■

4. The case of weighted spaces

The aim of this section is to indicate an application of Theorem 3.3.

Let X be a locally compact Hausdorff space and let V be a *Nachbin family* on E i.e., a set of nonnegative upper semicontinuous functions on X, such that for every $v_1, v_2 \in V$ and every $\lambda > 0$ there is $v \in V$ such that $v_1, v_2 \leq \lambda v$. We shall denote by $CV_0(X)$ the corresponding *weighted space*,

$$CV_0(X) = \{f \in C(X) : \ fv \text{ vanishes at infinity for every } v \in V\}.$$

The *weighted topology* on $CV_0(X)$, denoted by ω_V, is determined by the seminorms $(p_v)_{v \in V}$, where

$$p_v(f) = \sup\{|f(x)|v(x) : \ x \in X\} \quad \text{for } f \in CV_0(X);$$

ω_V is a locally convex topology and a basis of open neighborhoods of the origin consists of the sets

$$D_v = \{f \in CV_0(X) : \ p_v(f) < 1\}.$$

This way, $CV_0(X)$ appears as a locally convex locally solid vector lattice of (AM)-type.

A result due to Summers [10] asserts that there is a linear isomorphism between the topological dual of $CV_0(X)$ and the vector subspace $VM_b(X)$, where $M_b(X)$ denotes the space of all bounded Radon measures on X. According to Proposition 3.8 of [4], $CV_0(X)$ is metrizable if and only if there exists a countable subset $W \subset V$ with the property that for every $v \in V$ there are $w \in W$ and $r > 0$ such that $v \leq rw$. Notice also that a metrizable weighted space is complete if and only if for every $x \in X$ there are $w \in W$ and $r > 0$ such that $v \geq r$ on a neighborhood of x. Cf. [4], Corollary 3.11.

On the other hand, a result due to C. Partenier (see [4], Lemma 3.8) asserts that for every closed ideal $\mathcal{I}$ of $CV_0(X)$ there exists a closed subset Y of X such that

$$\mathcal{I} = \{f \in CV_0(X) : \ f|Y = 0\}.$$

Therefore, there exists a one-to-one between the family of all closed ideals of $CV_0(X)$ and the family of all closed subsets of X.

If X is a compact Hausdorff space and V is the family of all positive constants, then $CV_0(X) = C(X)$ and the weighted topology coincides with the uniform topology of $C(X)$.

4.1 Definition. Let $CV_0(X)$ be a metrizable weighted space and let F be a vector subspace of $CV_0(X)$. A closed subset Y of X is said to be a *strictly interpolating set* with respect to F if

$$(D_v|Y) \cap (F|Y) = (D_v \cap F)\,|Y \quad \text{for every } v \in V.$$

From Theorem 1 of [8] we infer that a closed subset Y of X is strictly interpolating (with respect to F) if

$$\mathbf{1}_Y F^\circ \subset F^\circ \quad \text{and} \quad F/(F \cap \mathcal{I}_Y) \text{ is complete.}$$

4.2 Definition. Let $CV_0(X)$ be a metrizable weighted space and let F be a vector subspace of $CV_0(X)$. A closed subset Y of X is said to be a *frontal set* (with respect to F) if it verifies the following condition:

Given $f \in F$, $g \in CV_0(X)$ with $g \geq |f|$ on Y, $\varepsilon > 0$ and $v \in V$, there exists a $\bar{f} \in F$ such that

$$\bar{f}|Y = f|Y \quad \text{and} \quad |\bar{f}(x)|v(x) \leq g(x)v(x) + \varepsilon \text{ for every } x \in X.$$

According to Theorem 3.3, the following result works:

4.3 Proposition. *Let $CV_0(X)$ be a metrizable weighted space and let F be a complete vector subspace of $CV_0(X)$. Suppose there are given a closed subset Y of X, and the functions $f \in F$, $g \in CV_0(X)$ with $g \geq |f|$ on Y, and $v \in V$ with $p_v(f) > 0$. Then for every $\varepsilon > 0$ there exists a $\bar{f} \in F$ such that*

$$\bar{f}|Y = f|Y, \quad |\bar{f}|v \leq gv + \varepsilon \quad \textit{and} \quad ||\bar{f}v||_X \leq ||fv||_Y.$$

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