A NOTE ON THE KY FAN INEQUALITY

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Abstract. The Ky Fan inequality is essentially the assertion that $t/(1-t)$ is log-concave. We study its weighted form in the context of signed weights.

The inequality of Ky Fan (cf. [3]) represents a counterpart to the arithmetic-geometric mean inequality. It says that

$$\text{(F)} \qquad \left[\prod_{k=1}^{n} x_k / \prod_{k=1}^{n}(1-x_k)\right]^{1/n} \leq \sum_{k=1}^{n} x_k / \sum_{k=1}^{n}(1-x_k)$$

for every family of n numbers $x_k \in (0,1/2]$, with equality only if the x_k are equal.

During the last two decades this inequality has received a lot of attention and many extensions and refinements were done. See the references at the end of this paper.

Letting $\mathbf{x} = (x_1, ..., x_n)$ the given n-tuple of numbers, we can associate to it the arithmetic, geometric and harmonic means of weight $\mathbf{w} = (w_1, ..., w_n)$ ($\mathbf{w}_k \geq 0$ for each k and $\sum_{k=1}^{n} w_k = 1$) :

$$A_n(\mathbf{x}, \mathbf{w}) = \sum_{k=1}^{n} w_k x_k, \quad G_n(\mathbf{x}, \mathbf{w}) = \prod_{k=1}^{n} x_k^{w_k}, \quad H_n(\mathbf{x}, \mathbf{w}) = \frac{1}{\sum_{k=1}^{n} w_k/x_k}.$$

Put $1-\mathbf{x} = (1-x_1, ..., 1-x_n)$ for the "complementary" n-tuple. The inequality (F) means precisely that $t/(1-t)$ is mid log-concave on $(0,1/2]$. As it is continuous, this fact is equivalent to log-concavity. In other words, the inequality (F) works also in the weighted case:

$$\text{(wF)} \qquad \frac{G_n(\mathbf{x}, \mathbf{w})}{G_n(1-\mathbf{x}, \mathbf{w})} \leq \frac{A_n(\mathbf{x}, \mathbf{w})}{A_n(1-\mathbf{x}, \mathbf{w})}.$$

W.-L. Wang and P.-F. Wang [16] noticed a complementary inequality,

$$\text{(WW)} \qquad \frac{H_n(\mathbf{x}, \mathbf{w})}{H_n(1-\mathbf{x}, \mathbf{w})} \leq \frac{G_n(\mathbf{x}, \mathbf{w})}{G_n(1-\mathbf{x}, \mathbf{w})}.$$

We don't know the argument in [16] (as the paper is in Chinese), but we can indicate a simple proof along the ideas in [6]. The basic remark is the following variation to the existence of support lines for the graph of a convex function:

Lemma 1. *Let I be an interval, let $f : I \to \mathbb{R}$ be a continuous convex function and let φ be in the subdifferential of f ($\varphi = f'$ when f is differentiable). Then*

$$f\left(\sum_{k=1}^{n} w_k x_k\right) \leq f(t) + \sum_{k=1}^{n} w_k (x_k - t)\varphi(x_k)$$

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for every $t, x_1, ..., x_n \in I$ *and every* $w_1, ..., w_n \in [0,1]$, *with* $\sum_{k=1}^n w_k = 1$.

This follows by adding up the inequalities $f(t) \geq f(x_k) + (t - x_k)\varphi(x_k)$, each multiplied by w_k.

By Lemma 1, applied to $f(t) = \ln(1-t) - \ln t$ (under the hypotheses of (W)), we get

$$\ln \frac{G_n(1-\mathbf{x}, \mathbf{w})}{G_n(\mathbf{x}, \mathbf{w})} \leq \ln \frac{1-t}{t} - \frac{1}{H_n(1-\mathbf{x}, \mathbf{w})} + t\left(\frac{1}{H_n(\mathbf{x}, \mathbf{w})} + \frac{1}{H_n(1-\mathbf{x}, \mathbf{w})}\right)$$

so taking the infimum for $t \in (0, 1/2]$ in the right hand side we arrive at (WW).

Papers by M. Matić and J. Pečarić [6] and S.S. Dragomir and F.P. Scarmozzino [4] contains a number of valuable refinements of (wF).

One might think that (F) and (WW) extends to the whole family of power inequalities, but a counterexample by H. Alzer [1] disproves that. Instead, J. Sandor and T. Trif [15] noticed that the identric mean can be squeezed between the two members of (wF).

The aim of the present paper is to extend the Ky Fan inequality for a larger class of weights, by allowing some components to be negative. The basic fact we need is the discrete case of the Choquet theory for signed measures (as developed by the second author in [8]).

Suppose that $x_1 \leq ... \leq x_n$ are points in an interval I. An n-tuple $\mathbf{w} = (w_1, ..., w_n)$ of real numbers is said to be a *Popoviciu weight* for the n-tuple $\mathbf{x} = (x_1, ..., x_n)$ if

$$\text{(P)} \qquad \sum_{k=1}^n w_k > 0, \quad \sum_{k=1}^m w_k(x_m - x_k) \geq 0 \quad \text{and} \quad \sum_{k=m}^n w_k(x_k - x_m) \geq 0$$

for every $m \in \{1, ..., n\}$. In the terminology of [8], this means that $\sum_{k=1}^n w_k \delta_{x_k}$ is a Popoviciu measure.

A special case when (P) holds is the following, used by Steffensen in his extension of Jensen's inequality:

$$\text{(St)} \qquad \sum_{k=1}^n w_k > 0, \quad \text{and} \quad 0 \leq \sum_{k=1}^m w_k \leq \sum_{k=1}^n w_k, \quad \text{for every } m \in \{1, ..., n\}.$$

Theorem 1. (The Hermite-Hadamard inequality for Popoviciu weights [11], [8]). *Suppose that* $\mathbf{x} = (x_1 \leq ... \leq x_n)$ *is an* n*-tuple of points in an interval* $[m, M]$ *and* $\mathbf{w} = (w_1, ..., w_n)$ *is a Popoviciu weight for* $\mathbf{x}$. *Then for every continuous convex function* $f : [m, M] \to \mathbb{R}$ *we have*

$$\begin{aligned} f\left(\sum_{k=1}^n w_k x_k\right) &\leq \sum_{k=1}^n w_k f(x_k) \\ &\leq \frac{M - \sum_{k=1}^n w_k x_k}{M - m} \cdot f(m) + \frac{\sum_{k=1}^n w_k x_k - m}{M - m} \cdot f(M). \end{aligned}$$

Theorem 1 is can be covered also from [10], Theorem 2.24, p. 60.

Theorem 1 will allow us to adapt an idea due to Dragomir and Scarmozzino [4] in the context of Popoviciu weights, so to prove the following general result:

Theorem 2. *Suppose that* $\mathbf{x} = (x_1 \leq ... \leq x_n)$ *is an* n*-tuple of points in an interval* $[m, M] \subset (0, 1/2]$ *and* $\mathbf{w} = (w_1, ..., w_n)$ *is a Popoviciu weight for* $\mathbf{x}$*. Then*

$$\frac{A_n(\mathbf{x},\mathbf{w})}{G_n(\mathbf{x},\mathbf{w})} \geq \left(\frac{A_n(\mathbf{x},\mathbf{w})}{G_n(\mathbf{x},\mathbf{w})}\right)^{M^2/(1-M)^2} \geq \frac{A_n(1-\mathbf{x},\mathbf{w})}{G_n(1-\mathbf{x},\mathbf{w})}$$
$$\geq \left(\frac{A_n(\mathbf{x},\mathbf{w})}{G_n(\mathbf{x},\mathbf{w})}\right)^{m^2/(1-m)^2} \geq 1.$$

Particularly,

$$\frac{A_n(\mathbf{x},\mathbf{w})}{A_n(1-\mathbf{x},\mathbf{w})} \geq \frac{G_n(\mathbf{x},\mathbf{w})}{G_n(1-\mathbf{x},\mathbf{w})}.$$

Proof. An easy computation shows that the function $f(t) = \ln(1-t) + (\alpha - 1)\ln t$ is strictly convex on $[m, M]$ provided

$$\alpha \leq \frac{1-2M}{(1-M)^2};$$

for, check the sign of the second derivative. Then by Theorem 1 we get

$$\sum_{k=1}^{n} w_k \ln(1-x_k) + (\alpha-1)\sum_{k=1}^{n} w_k \ln x_k \geq \ln\left(1 - \sum_{k=1}^{n} w_k x_k\right) + (\alpha-1)\ln\left(\sum_{k=1}^{n} w_k x_k\right)$$

i.e.,

$$\left(\frac{G_n(\mathbf{x},\mathbf{w})}{A_n(\mathbf{x},\mathbf{w})}\right)^{\alpha-1} \geq \frac{A_n(1-\mathbf{x},\mathbf{w})}{G_n(1-\mathbf{x},\mathbf{w})}.$$

The best possible selection of α is $\alpha = (1-2M)/(1-M)^2$, which gives us the second inequality in the statement of Theorem 2. The third inequality can be proved in a similar way, by considering the strictly convex function $g(t) = (\beta - 1)\ln t - \ln(1-t)$ on $[m, M]$ provided $\beta \geq (1-2m)/(1-m)^2$).

The other inequalities are straightforward. $\square$

It is not clear whether

$$\frac{H_n(\mathbf{x},\mathbf{w})}{H_n(1-\mathbf{x},\mathbf{w})} \leq \frac{G_n(\mathbf{x},\mathbf{w})}{G_n(1-\mathbf{x},\mathbf{w})}$$

works also in the general context of Popoviciu weights.

The second inequality in Theorem 1 can be used to obtain further refinements of Theorem 2.

Of course, the whole theory extends to the general case of Popoviciu measures.

The role of $1/2$ in the whole business can be replaced by any positive number γ, provided $t/(1-t)$ is replaced by $t/(2\gamma - t)$.

We end this paper with an example: Let $(x_k)_{k=1}^{2n+1}$ be a family of elements of $(0, \gamma]$ with $x_1 \leq ... \leq x_{2n+1}$. According to the second part of Theorem 2 we have the inequalities

$$\frac{\sum_{k=1}^{2n+1} x_k}{\sum_{k=1}^{2n+1} (2\gamma - x_k)} \geq \prod_{k=1}^{2n+1} \left(\frac{x_k}{2\gamma - x_k}\right)^{1/(2n+1)}$$

and

$$\frac{\sum_{k=1}^{2n+1} (-1)^{k+1} x_k}{\sum_{k=1}^{2n+1} (-1)^{k+1}(2\gamma - x_k)} \geq \prod_{k=1}^{2n+1} \left(\frac{x_k}{2\gamma - x_k}\right)^{(-1)^{k+1}/(2n+1)},$$

In particular, if $c < b < a$ are the side lenghts in a triangle (of semiperimeter s), then

$$\frac{1}{8} \geq \frac{abc}{(b+c)(c+a)(a+b)} \tag{T}$$

and

$$\frac{s-b}{b} \geq \frac{ac\,(2s-b)}{(2s-a)(2s-c)}.$$

In general, the first part of Theorem 2 leads us to better results (at the cost of some sophistication). For example, (T) can be straightened as

$$\frac{1}{8}\left(\frac{2s}{3\sqrt[3]{abc}}\right)^{3\left[a^2/(b+c)^2-1\right]} \geq \frac{abc}{(b+c)(c+a)(a+b)}.$$

Added in Proofs (December 16, 2002). After this paper has been completed, we were informed by Prof. Jozsef Sándor that Theorem 1 above goes back to K. Knopp, *Über die maximalen Abstande und Verhaltnisse verschiedener Mittelwerte*, Math. Zeit. **39** (1935), 768-776. See also: J.E. Pecaric and P.R. Beesack, *On Knopp's inequality for convex functions*, Canad. Math. Bull. **30** (3) (1987), 267-272.

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